A differential evolution copula-based approach for a multi-period cryptocurrency portfolio optimisation

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Abstract

Recent years have seen a growing interest among investors in the new technology of blockchain and cryptocurrencies and some early investors in this new type of digital assets have made significant gains. The heuristic algorithm, differential evolution has been advocated as a powerful tool in portfolio optimization. We propose in this study two new approaches derived from the traditional Differential Evolution (DE) method: the GARCH Differential Evolution (GARCH-DE) and the GARCH Differential Evolution t-copula (GARCH-DE-t-copula). We then contrast these two models with DE (benchmark) in single and multi-period optimization on a portfolio consisting of five cryptoassets under the coherent risk measure CVaR constraint. Our analysis shows that the GARCH-DE-t-copula outperforms the DE and GARCH-DE approaches in both single and multi-period frameworks. For these notoriously volatile assets, the GARCH-DE-t-copula have shown risk-control ability, hereby confirming the ability of t-copula to capture the dependence structure in the fat tail.

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1. Introduction

The increasing adoption of cryptocurrencies has contributed to the global dependence between cryptoassets. Due to both contagion effects and volatility spillovers, modelling this dependence is important for asset allocation and management. Cryptocurrency which still in its infancy of development and adoption is known to be highly volatile and susceptible to important changes driven by speculations and institutional regulations. These extreme dynamics can results in dependence shifts and portfolio losses (see e.g. [7, 28, 15, 3]). It is therefore important for cryptocurrency portfolio optimization, to find technical tools that are able to deal with such underlying interactions. For portfolio optimization, several models have been developed following the traditional approach based on the bilateral correlation coefficient algorithm. However, these models appear to be restrictive. An example of these models is given by the class of multivariate GARCH models developed to alleviate the normality assumption sustaining the pioneered meanvariance approach. Deterministic by nature, these models rely on parametric multivariate distribution likely to be erroneously specified when the distribution of all the variables are not the same. This is likely the case for financial assets in general and for cryptoassets returns in particular, exhibiting various underlying distribution properties such as non-normality, asymmetric correlations, volatility clustering, heavy tail behaviour (see e.g. [30, 31, 1]). A positive response to these deterministic and restrictive shortcomings of the traditional models can be obtained through the copula approach, particularly the t-copula (see e.g. [12, 14]) which represents implicitly the dependence structure in a multivariate t-distribution. It has recently received much attention in the context of modeling multivariate financial time series. It has also been advocated that its empirical fit is in general superior to that of the dependence structure of the multivariate normal distribution, given by the Gaussian copula (see e.g., [27, 4]). This is explained by the ability of the t-copula to successfully capture the extreme values dependence phenomenon, which is generally perceived in financial data return.

Markowitz original approach provides a fundamental basis for portfolio selection in a single period model of investment. In this model of investment, at the beginning of a selected period, the investor make once and for all allocation decisions which remain unchanged until the end of the period, disregarding the market behavior during that locked period. This is why, single-period models lead to what is named myopic policies.

Because markets are risky, as changes arise in financial markets and create imbalances in the portfolio allocations, investors need to respond to these changes by rebalancing/realigning their portfolios. This leads to what is called multi-period models. The problem of multi-period portfolio allocation have been studied by Hakansson [18], Sahalia and W. Brandt [35] and Calafiore [8] and many others. Boudt *et al.* in [5] make use of the differential evolution (DE) algorithm in a single and multi-period settings to numerically solve portfolio optimization problems under complex constraints and objectives. DE was introduced by Storn and Price in [37] and they found that DE was more efficient than genetic algorithms and simulated annealing. It has become a powerful and flexible tool to solve optimization problems arising in finance. Similar to classic genetic algorithms, DE algorithm is an evolutionary technique which can be used to solve global optimization problems. This algorithm has shown remarkable performance on continuous numerical problems [32] and optimizing portfolios under non-convex settings, see [2, 21, 22, 24, 38].

In the literature, GARCH models have been applied to analyze and understand the dynamics of cryptocurrencies price movement. For example, Dyhrberg [11] used the asymmetric GARCH methodology to explore the Bitcoin hedging capabilities.

In this paper, we investigate the performance of the GARCH-Differential Evolution (GARCH-DE) and GARCH-Differential Evolution t-copula (GARCH-DE-t-copula) models; and compare them to the existing DE model in single and multi-period optimization. These models are performed on a portfolio consisting of 5 cryptocurrencies, namely, Bitcoin , Ripple , Litecoin, Dash and Dogecoin representing at the time of this writing, nearly 50% market share of the cryptocurrency market.

Our optimisation problem is subject to the minimisation of the Conditional Value at Risk (CVaR) introduced by Rockafellar and Uryasev [33] as a coherent risk measure.
Contrasting our two approaches in Multi-period portfolio optimization under the above-mentioned risk measure, we are aiming in this way in identifying under which methodology, cryptocurrency portfolio may be more profitable or risky than the other. Though innovative, our approach is closely related to that of Bekiros *et al.* in [3] who estimated the multivariate dependence using pair vine copula to optimise a portfolio consisting of the Australian mining stocks, subject to the minimisation of five risk measures including CVaR. Since the extreme value distribution (EVT) is able to accurately model tails risk, we used in this study, the Generalized Pareto Distribution (GPD) based t-copula (GARCH-DE-t-copula) instead of Pair Vine Copula-GARCH as in [3], to explicitly capture the tail events.
The rest of the paper is organised as follows: Section 2 presents introduces the new optimization models. Section 3 discusses the empirical findings and the last section 4 concludes the work.

2. Optimization periods mathematical formalism

In this section, we review the mathematical formalism of portfolio optimization methods used in this current study, i.e., single period optimization, multi-period optimization, DE, GARCH-DE, GJR-GARCH, GARCH-DE-t-copula.

2.1. Single period optimization

In portfolio optimisation process, the main challenge resides in designing a proper model that empirically best fits the data and at the same time feasible and robust enough to generate simulation-based inference for risk evaluation.
The basic Mean-Variance optimization can be formulated as follows (see Markowitz [26]):

$$\begin{aligned}
&\min_{\omega} \sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ik}\omega_i\omega_k \\
&\text{subject to} \\
&\sum_{i=1}^{n} \omega_i = 1 \\
&\sum_{i=1}^{n} E[r_i]\omega_i = \mu_p \\
&\omega_i \geq 0,\ i = 1, 2, \cdots, n.
\end{aligned} \tag{2.1}$$

where ω_i are portfolio weights, r_i is the rate of return of asset i and $E[r_i]$ its expectation, $\sigma_{ik} = cov(r_i, r_k)$ is the covariance between r_i and r_k, μ_p is the portfolio expected return.
Among various risk measures, Value-at-Risk (VaR) is a popular measure of risk that represents the percentile of the loss distribution with a specified confidence level. Let $\omega \in \mathbb{R}^n$ denote a portfolio vector indication a proportion of investment of a given budget in each of

the n financial assets. Let $\alpha \in (0,1)$ and $f(\omega, \mathbf{r})$ denoting respectively a confidence level and a loss function for the portfolio ω and the return vector $\mathbf{r} \in \mathbb{R}^n$. Then the VaR function, $\xi(\omega, \alpha)$ results to the smallest number satisfying $\psi(\omega, \xi(\omega, \alpha)) = \alpha$, where $\psi(\omega, \xi) = \Pr[f(\omega, \mathbf{r}) \leq \xi]$ is the probability that the loss $f(\omega, \mathbf{r})$ does not exceed the threshold value ξ. However, VaR does not satisfy the sub-additivity axiom. Furthermore, VaR is nondifferentiable as well as non-convex when using scenarios analysis. Hence, it is difficult to find a global minimum using conventional optimization techniques.

Alternatively, Conditional VaR (CVaR), introduced by Rockafellar and Uryasev [33] is a coherent risk measure with more interesting features such as sub-additivity and convexity. Moreover, it is more appropriate to the loss function of the tail distribution. CVaR is given by

$$\psi_\alpha(\omega) = (1-\alpha)^{-1} \int_{f(\omega,\mathbf{r}) > \xi(\omega,\alpha)} f(\omega, \mathbf{r}) p(\mathbf{r}) \, d\mathbf{r} \tag{2.2}$$

To avoid complications resulting from the implicitly defined function $\xi(\omega, \alpha)$, Rockafellar and Uryasev [33] provided an alternative function given by

$$F_\alpha(\omega, \xi) = \xi + (1-\alpha)^{-1} \int_{f(\omega,\mathbf{r}) > \xi} [f(\omega, \mathbf{r}) - \xi] p(\mathbf{r}) \, d\mathbf{r} \tag{2.3}$$

for which they show the minimum of CVaR can be found by minimizing $F_\alpha(\omega, \xi)$ with respect to (ω, ξ).

Given returns data r_j for $j = 1, \cdots, n$, $F_\alpha(\omega, \xi)$ can be approximated by

$$\tilde{F}_\alpha(\omega, \xi) = \xi + [(1-\alpha)n]^{-1} \sum_{j=1}^{n} max\{f_j(\omega) - \xi, 0\} \tag{2.4}$$

where $f_j(\omega) = f(\omega, r_j)$.

In this study, our three models are designed to solve the following CVaR-optimization problem

$$\begin{aligned} &\min_{\omega} \xi + [(1-\alpha)n]^{-1} \sum_{i=1}^{n} max\{f_i(\omega) - \xi, 0\} \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \sum_{i=1}^{n} E[r_i]\omega_i = \mu_p \\ \omega_i \geq 0, \; i = 1, \cdots, n. \end{cases} \end{aligned} \tag{2.5}$$

We intend to find the portfolio that minimizes CVaR under 90% confidence level subject to the following weight constraints: Weights must sum to 1 ($\sum_i^n \omega_i = 1$) and no short-selling is allowed ($\omega_i \geq 0$).

2.2. Multi-period optimization

In multi-period portfolio optimization, the portfolio optimization problem is to choose a sequence of transactions/trades to perform over a chosen set of periods. One of the advantage of multi-period portfolio optimization is its ability to naturally handle multiple return estimates on different time scales (see for example, [16, 29]).

Consider a universe of n assets $\{a_1, a_2, \cdots, a_n\}$ and an investment planning horizon that extends T periods of equal duration δ ($\delta = 1$ month or $\delta = 1$ quarter). Let $s_i(t)$ be the dollar value of the total wealth portion invested in asset a_i at time t. Let $s(t) = [s_1(t) \cdots s_n(t)]^T$, then the total wealth invested at time t is given by

$$v(t) = \sum_{i=1}^{n} s_i(t) = \mathbf{1}^T s(t) \tag{2.6}$$

where $\mathbf{1}$ denotes a $n \times 1$ column matrix of ones.
The investor has the opportunity at the end of each period to adjust the portfolio composition. Let $u(t) = [u_1(t) \cdots u_n(t)]^T$ be the vector of adjustments. A value of $u_i(t) > 0$ means that the value of asset a_i is increased by $u_i(t)$ dollars (by buying more of the asset a_i), whereas $u_i(t) < 0$ means that the value of asset a_i is decreased by $u_i(t)$ dollars (by selling part or all of the asset a_i).
Let $s^+(t)$ be the portfolio composition after the adjustment $u(t)$ is made at time t.

$$s^+(t) = s(t) + u(t) \tag{2.7}$$

Without loss of generality, we assume in this study, a self-financing portfolio, i.e., $\sum_{i=1}^{n} u_i(t) = 0$, for all t. The weight corresponding to asset a_i at time t is $\omega_i(t) = s_i(t)/v(t)$ and $\omega(t) = [\omega_1(t) \cdots \omega_n(t)]^T$ is the vector of weights with $\omega_i(t) \geq 0$ and $\sum_{i=1}^{n} \omega_i(t) = 1$.

Let $p_i(t)$ be the price of the asset a_i at time t. Let $r_i(t)$ be the log-return given by $r_i(t) = \ln[\dfrac{p_i(t+1)}{p_i(t)}]$, then $r(t) = [r_1(t) \cdots r_n(t)]^T$ will denote the return vector.

Portfolio dynamics

Let $\omega(0)$ be the initial portfolio allocation at time $t = 0$. If at time $t = 0$, transactions are conducted on the market, then the portfolio will be adjusted, either by increasing or decreasing the amount invested in each asset. So, the rebalanced portfolio will be

$$\omega^+(0) = \omega(0) + \widetilde{u}(0) \tag{2.8}$$

where $\widetilde{u}_i(t) = \dfrac{u_i(t)}{v(t)}$ and $\widetilde{u}(t) = [\widetilde{u}_1(t) \cdots \widetilde{u}_n(t)]^T$.
Assume that the portfolio remains unchanged for the first period of time δ. The portfolio allocation at the end of the first period is $\omega(1) = R(1)\omega^+(0) = R(1)\omega(0) + R(1)\widetilde{u}(0)$, where $R(t) = \mathrm{diag}(r(t))$ is a diagonal matrix of asset returns over the interval period $[t-1, t]$, for $t \geq 1$. Iteratively, the composition of the portfolio at the end of period $t+1$ is:

$$\omega(t+1) = R(t+1)\omega(t) + R(t+1)\widetilde{u}(t),\ t = 0, 1, \cdots, T-1 \tag{2.9}$$

Since the asset returns $r_i(t)$ are random, the iterative equations in 2.9 defines a stochastic process $\omega(t),\ t = 1, \cdots, T$

$$\omega(t) = \psi(1,t)\omega(0) + [\psi(1,t)\,\psi(2,t)\ \cdots\ \psi(t-1,t)\,\psi(t,t)][\widetilde{u}(0)\,\widetilde{u}(1)\ \cdots\ \widetilde{u}(t-2)\,\widetilde{u}(t-1)]^T \tag{2.10}$$

where $\psi(k,t),\ k \leq t$, is the compounded return matrix from the start of period k to the end of period t:
$\psi(k,t) = R(t)R(t-1)\cdots R(k)$, $\psi(t,t) = R(t)$, so that
$\psi(k,t+1) = R(t+1)\psi(k,t)$. From 2.10, we have the weights constraint

$$\psi^T(1,t)\omega(0) + \sum_{j=1}^{t} \psi^T(j,t)\widetilde{u}(j-1) = 1, \tag{2.11}$$

where $\psi^T(j,t) = 1^T\psi(j,t)$.

The multi-period optimization problem is formulated as follows:

$$\min_{\widetilde{u}_1(t),\cdots,\widetilde{u}_n(t)} \sum_{t=1}^{T} \lambda(t)(\xi(\omega(t),\alpha) + [(1-\alpha)n]^{-1}\sum_{i=1}^{n} max\{f_i(\omega(\mathbf{t})) - \xi(\omega(t),\alpha),0\})$$

subject to

$$\begin{cases} \omega(t) = \psi(1,t)\omega(0) + [\psi(1,t)\,\psi(2,t)\ \cdots\ \psi(t-1,t)\,\psi(t,t)][\widetilde{u}(0)\,\widetilde{u}(1)\ \cdots\ \widetilde{u}(t-2)\,\widetilde{u}(t-1)]^T; \\ \psi^T(1,t)\omega(0) + \sum_{j=1}^{t}\psi^T(j,t)\widetilde{u}(j-1) = 1; \\ \lambda(t) \geq 0; \\ \sum_{i=1}^{n} E[r_i(t)]\omega_i(t) \geq 0 \end{cases} \tag{2.12}$$

where $r_i(t)$ are the returns or the standardized residuals filtered from GARCH model.

2.3. Optimization methods mathematical formalism

2.3.1. Differential Evolution (DE)

Differential Evolution (DE), introduced by Storn and Price [37], is a stochastic, population-based evolutionary algorithm for solving nonlinear optimization problems. This algorithm uses biology-inspired operations of **initialization**, **mutation**, **recombination**, and **selection** on a population to minimize an objective function through successive generations (see [20]). Similar to other evolutionary algorithms, to solve optimization problems, DE uses alteration and selection operators to evolve a population of candidate solutions.
Let N denote the population size. To create the initial generation, the optimal guess for N is made, either by using values input by the user or random values selected between lower and upper bounds (choosing by the user).
Consider the optimization problem 2.5 and let
$\xi + [(1-\alpha)n]^{-1}\sum_{i=1}^{n} max\{f_i(\omega) - \xi, 0\} = h(\omega)$. where $\omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$.
Given the population

$$\omega_{ki}^g = \{\omega_{k1}^g, \omega_{k2}^g, \cdots, \omega_{kn}^g$$

where g is the generation and $k = 1, 2, \cdots, N$. The process is achieved through the following stages:

1) **Initial population**
The initial population is randomly generated as

$$\omega_{ki} = \omega_{ki}^L + \text{rand}()(\omega_{ki}^U - \omega_{ki}^L)$$

where ω_i^L and ω_i^U represents the lower and upper bounds of ω_i respectively and $i = 1, 2, \cdots, n$.

2) **Mutation**
The differential mutation is accomplished as follows: A random selection of three members of the population $\omega_{r_1k}^g, \omega_{r_2k}^g$ and $\omega_{r_3k}^g$ to create an initial mutant vector parameter $\mathbf{u}_k^{g+1}$, called donor vector, which is generated as

$$\mathbf{u}_k^{g+1} = \omega_{r_1k}^g + F(\omega_{r_2k}^g - \omega_{r_3k}^g)$$

where F is the scale vector and $k = 1, 2, \cdots, N$.
3) **Recombination**
Let ω_{ki}^g denotes the target vector.
From the target vector and the donor vector, a trial vector $\mathbf{v}_{ki}^{g+1}$ is selected as follows

$$\mathbf{v}_{ki}^{g+1} = \begin{cases} \mathbf{u}_{ki}^{g+1}, \text{ if rand}() \leq C_p \text{ or } i = I_{\text{rand}} & i = 1, 2, \cdots, n; \\ \omega_{ki}^g, \text{ if rand}() > C_p \text{ and } i \neq I_{\text{rand}} & k = 1, 2, \cdots, N \end{cases}$$

where I_{rand} is a random integer in $[1, n]$ and C_p the recombination probability.
4) **Selection**
At this stage, the target vector is compared with the trial vector and the one with the smallest function value is the candidate for the next generation

$$\omega_{ki}^{g+1} = \begin{cases} \mathbf{v}_{ki}^{g+1}, \text{ if } h(\mathbf{v}_{ki}^{g+1}) < h(\omega_{ki}^g); \\ \omega_{ki}^g, \text{ Otherwise.} \end{cases}$$

where $k = 1, 2, \cdots, N$.

2.3.2. GARCH Differential Evolution (GARCH-DE)

Jeffrey et al. [9] fitted twelve GARCH models to each of the seven most popular cryptocurrencies and realized that the IGARCH (Integrated GARCH) of Engle and Bollerslev [13], and the GJR-GARCH of Glosten, Jagannathan and Runkle [17] models provide the best fits, in terms of modelling of the volatility in the largest and most popular cryptocurrencies. In this study, we use the GJR-GARCH (1,1).

The implementation of the GARCH-DE is as follows.
Let r_t be the log-return at time t.

a) Fit the mean model ARMA(1,1) and the variance model GJR-GARCH(1,1) of the log-returns as follows

i) The mean model

$$r_t = \mu + \theta_1(r_{t-1} - \mu) + \theta_2\varepsilon_{t-1} + \varepsilon_t \tag{2.13}$$

where $\varepsilon_t = \sigma_t h_t$, $h_t \sim N(0, 1)$.

ii) The variance model

$$\sigma_t^2 = \omega + \alpha_1 h_{t-1}^2 + \gamma_1 I_{t-1} h_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{2.14}$$

Alberg, Shalit and Yosef [**?**] showed that the GARCH models with fat-tail distributions are relatively better suited for analyzing returns on stocks. So, student t-distribution have been chosen for our analysis.

b) From the predicted log-returns $\widehat{r_t} = \mu + \theta_1(r_{t-1} - \mu)$, obtain the residuals

$$D_t = r_t - \widehat{r_t} \tag{2.15}$$

c) Solve the following optimization problem using DE,

$$\begin{aligned} &\min_{\omega} \xi + [(1-\alpha)n]^{-1} \sum_{i=1}^{n} max\{f_i(\omega) - \xi, 0\} \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \sum_{i=1}^{n} E[d_i]\omega_i \geq 0 \\ \omega_i \geq 0,\ i = 1, \cdots, n. \end{cases} \end{aligned} \tag{2.16}$$

where $\alpha = 0.1$, $f_i(\omega) = f(\omega, d_i)$ with d_i is the standardized residuals.

2.3.3. GARCH Differential Evolution t-copula (GARCH-DE-t-copula)

Modeling statistical dependence using linear correlation is deeply embedded in financial risk management practice in such a way that many practitioners are not aware of any other alternatives. Its attractiveness are its simplicity and the fact that, in terms of dependence in an elliptical world, the correlations provide us with all information we need to know. But, it is just one measure of dependence among others. It does have some limitations. For example, it is not invariant to transformations of variables (e.g., the correlation between two random variables X and Y is not generally the same as the one between $\ln(X)$ and $\ln(Y)$). Moreover, correlation is not defined in some circumstances, especially when the variance of one of the variables is not finite or when the two variables are not cointegrated. In other circumstances, especially in the tails, it tells us very little about the dependence. So, linear correlation will often appear to be limited or even of no use in non-elliptical situations. Copulas provide a better way to model dependence.

Statistically, given a collection of marginal distributions, copula is a function that joins these marginals to form the multivariate distribution that captures how they all move together. Conversely, copula is able to take a multivariate distribution and separate its dependence structure from the marginal distribution functions.

t-copula theory

The t-copula (see e.g. [12, 14]) is the copula type of the multivariate t-distribution use in representing the dependence structure. Much attention has been given to t-copula, especially in modelling financial time series and has be shown that its empirical fit is generally superior to that of Gaussian copula, the dependence structure of the multivariate normal distribution (see e.g. [27, 4]). This is due to the ability of the t-copula to better capture the dependency of fat tails displayed by financial data.

Let $\mathbf{Y} = (Y_1, \cdots, Y_n)$ be a n-dimensional random vector. The vector $\mathbf{Y}$ is said to follow a multivariate t distribution (non-singular), denoted $\mathbf{Y} \sim t_n(d, \mu, \Sigma)$, if its density has the form

$$f(\mathbf{y}) = \frac{\Gamma(\frac{d+n}{2})}{\Gamma(\frac{d}{2}\sqrt{(\pi d)^n|\Sigma|})}\left(1 + \frac{(\mathbf{y}-\mu)'\Sigma^{-1}(\mathbf{y}-\mu)}{d}\right)^{-\frac{d+n}{2}} \tag{2.17}$$

where d is the degrees of freedom, μ is the mean vector and Σ) is a positive-definite dispersion matrix. It is to note here that in these parameterizations

$\mathrm{cov}(\mathbf{Y}) = \dfrac{d}{d-2}\Sigma.$

Given that marginal distributions of financial return data are generally not normally distributed, one can use the Sklar's Theorem [36] to associate these distributions with a copula. Recent developments emerged several type of copulas from two families: Elliptical and Archimedean copula. In our study, we use the functional forms of Student t-copula (i.e., t-copula) which derives from multivariate elliptical distributions. It is to be noted that the copula is invariant under a standardization of the marginal distributions. As such, the copula of a t-distribution $t_n(d, \mu, \Sigma)$ is the same as that of a $t_n(d, 0, P)$ distribution, where P is the correlation matrix implied by the scatter matrix Σ. The unique t-copula is thus given by

$$C^t_{d,P}(\mathbf{u}) = \int_{-\infty}^{t_d^{-1}(u_1)} \cdots \int_{-\infty}^{t_d^{-1}(u_n)} \frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\sqrt{(\pi d)^n|P|}}\left(1 + \frac{\mathbf{x}'P^{-1}\mathbf{x}}{d}\right)^{\frac{d+n}{2}} d\mathbf{x}, \tag{4}$$

where t_d^{-1} denotes the quantile function of a standard univariate t_d distribution. In the bivariate case, the notation $C^t_{d,\rho}$ is used, where ρ is the off-diagonal entry of P. In contrast, the unique copula of a multivariate Gaussian distribution may be thought as the limit of t-copula as d tends to infinity. It is denoted by C_P^{Ga} (see, [12]).

To simulate the t-copula, a multivariate t-distributed random vector $\mathbf{X} \sim t_n(d, 0, P)$ is generated using the representation $X \overset{n}{=} \mu + \sqrt{W}\mathbf{Z}$, (where $\mathbf{Z} \sim N_n(0, \Sigma)$ and W is independent of $\mathbf{Z}$ and has an inverse gamma distribution $W \sim I_g(\frac{d}{2}, \frac{d}{2})$), and then return a vector $U = (t_d(X_1), \cdots, t_d(X_n))'$, where t_d denotes the distribution function of a standard univariate t. The density of the t-copula can be obtained from (3) and has the following form

$$c^t_{d,P}(\mathbf{u}) = \frac{f_{d,P}(t_d^{-1}(u_1), \cdots, t_d^{-1}(u_n))}{\prod_{i=1}^n f_d(t_d^{-1}(u_i))}, \qquad u \in (0,1)^n, \tag{5}$$

where $f_{d,P}$ is the joint density function of a $t_n(d, 0, P)$-distributed random vector and f_d represents the density function of the univariate standard t-distribution with d degrees of

Table 1: Multivariate Diagnostic tests

			Normality	Serial Correlation	Arch effect
	Skewness test	Kurtosis test	JB-Test	Portmanteau Test	Arch test
Chi-squared	2410.5***	103850***	106260***	303.89***	4241.3***
p-value	$2.2e^{-16}$	$2.2e^{-16}$	$2.2e^{-16}$	$2.969e^{-06}$	$2.2e^{-16}$

freedom.
The GARCH Differential Evolution t-copula (GARCH-DE-t-copula) method is implemented as follows:

1) Obtain the standardized residuals d_i from GJR-GARCH.

2) Simulate using t-copula a sample data c_i from the standardized residuals d_i
3) Solve the following optimization problem using DE

$$\begin{aligned}&\min_{\omega} \xi + [(1-\alpha)n]^{-1} \sum_{i=1}^{n} max\{f_i(\omega) - \xi, 0\}\\ &\text{subject to}\\ &\begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \sum_{i=1}^{n} E[c_i]\omega_i \geq 0 \\ \omega_i \geq 0,\ i = 1, \cdots, n. \end{cases}\end{aligned} \tag{2.18}$$

where $\alpha = 0.1$ and $f_i(\omega) = f(\omega, c_i)$.

3. Results and Analysis

3.1. Data and Preliminary analysis

The data consists of the daily returns (100 times the difference in logarithms of Crypt/USD exchange rates) of 5 cryptocurrencies representing at the time of this writing close to 50% market share of the cryptocurrency market and was traded as early as 2014 with at least $300 millions market capitalization. The data spanning the period 01 March 2014 to 28 February 2018 comprises the following cryptocurrency assets: Bitcoin (BTC) , Ripple (XRP) , Litecoin (LTC), Dash (DASH) and Dogecoin (DOGE). These assets exhibit evidence of high volatility clustering (see Figure 1) and the assumptions of serial correlation, non-normality, and arch effect could not be rejected across the returns series (Table 1). The skewness and kurtosis tests results in Table 1 and values in Table 2 point to the leptokurtic skewed type of distribution for these returns, suggesting that large fluctuations are more likely on the fat tails. Dash appear to be the most riskier among the 5 cryptocurrencies and as expected offers the highest return. Though Bitcoin is the least riskier, it offers return higher than that of Litecoin and Dogecoin (See Figure 2). More interestingly, they all display a significantly positive correlation. (see Table 3).

Table 2: Fat tails parameters

	α	κ	ω	δ	ς	ξ
BTC	2.909	2.986	3.068	-0.009	-0.388	6.339
XRP	2.221	2.056	1.914	0.036	2.452	37.633
LTC	2.446	2.358	2.276	0.015	0.645	13.875
DASH	3.072	2.742	2.476	0.039	1.454	12.755
DOGE	2.801	2.646	2.507	0.021	0.749	10.214

Note: ς is the skewness parameter, ξ is the kurtosis parameter, α is the left tail parameter, ω is the right tail parameter, κ is the harmonic mean of α and ω and describes a global tail parameter and δ is the distortion parameter between the right tail parameter ω and the left tail parameter α, satisfying the inequality $-\kappa < d < \kappa$. A negative value $\delta < 0$ (resp. positive value $\delta > 0$) implies $\alpha < \omega$ (resp. $\alpha > \omega$) and indicates that the left tail (resp. the right tail) is heavier than the right tail (resp. the left tail).

3.2. Empirical findings

To account for the observed characteristics of the returns series in modelling their true dependence, we consider a multivariate t-distribution.

3.2.1. Dependence estimation

The dependence properties are of great importance for portfolio selection and/or risk evaluation. Though the Pearsons correlation coefficient[4] could provide a substantial statistical power even for distributions with moderate skewness or excess kurtosis, its sensitivity to extreme values makes it a less powerful statistical test for distributions with extreme skewness or excess of kurtosis (where the data with extreme values are more likely). Thus, our choice of a Student t-copula with extreme value distribution in assessing the dependence structure of the sample returns. We estimates the three main measures of dependence, Pearson, Spearman and Kendall[5]. The obtained coefficients are displayed in Table3.
Following Table3[6], unlike Dash, which exhibit a positive weak relationship with Bitcoin and Ripple according to Pearson's measure, there is a moderate positive relationship accross the studied cryptocurrencies. This suggests that the prices of these cryptocurrencies move in the same direction. The dependence structure between Dogecoin and Ripple seems to be the strongest across the three measures.

[4]Pearson's correlation coefficients provide the degree of linear relationship between two variables.

[5]It is a non-parametric test that measures the strength of dependence between two variables. It is given by: $\tau = \dfrac{n_c - n_d}{\frac{1}{2}n(n-1)}$, where n_c is the number of concordant (Ordered in the same way) pairs and n_d is the number of discordant (Ordered differently) pairs.

[6]Although, there are no absolute standards, many analysts view coefficients, in absolute values, of less that 0.25 as describing weak relationships; coefficients between 0.25 and 0.50 as moderate relationships and those greater that 0.50 as strong relationships

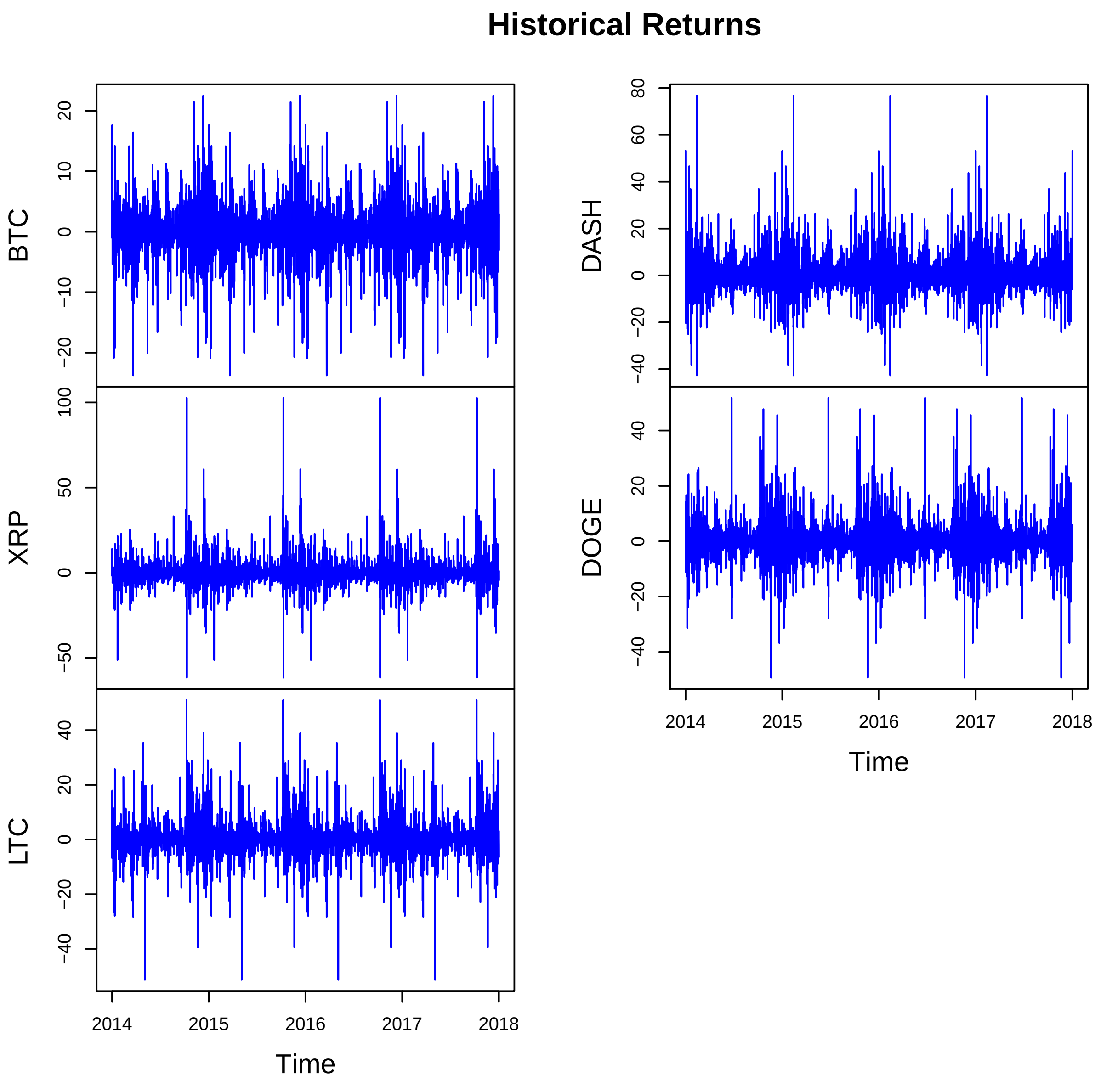

Figure 1: Historical Returns

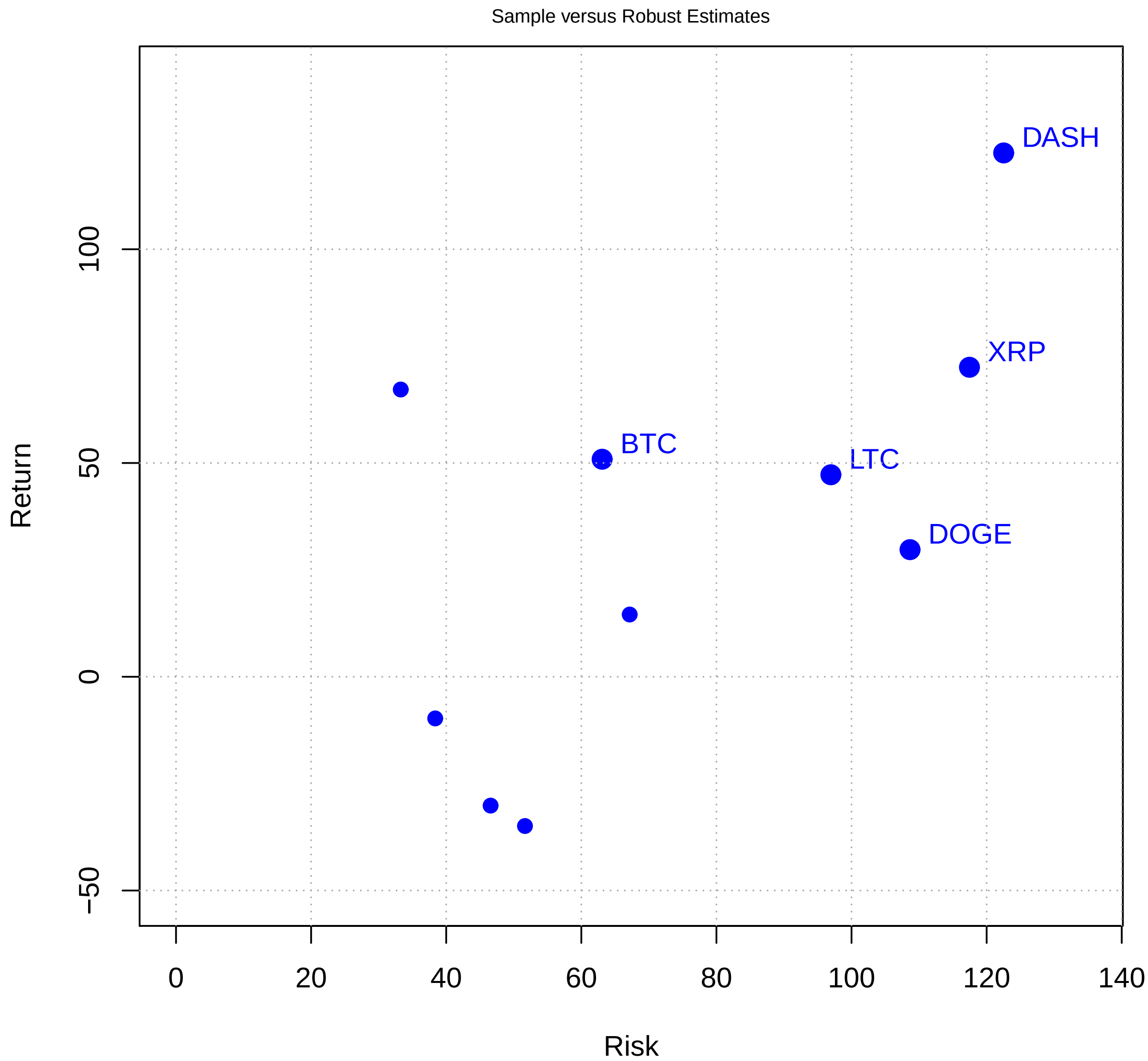

Figure 2: Risk vs Return

Table 3: Correlation coefficients

	Pearson	Spearman	Kendall
BTC vs XRP	0.42	0.41	0.27
BTC vs LTC	0.43	0.42	0.28
BTC vs DASH	0.17	0.42	0.29
BTC vs DOGE	0.41	0.43	0.29
XRP vs LTC	0.34	0.40	0.26
XRP vs DASH	0.17	0.42	0.28
XRP vs DOGE	0.50	0.41	0.25
LTC vs DASH	0.34	0.42	0.27
LTC vs DOGE	0.41	0.43	0.31
DASH vs DOGE	0.31	0.42	0.28

3.3. Portfolio Optimization

Let's recall here that in both our methods, the heuristic search algorithm, Differential Evolution (DE) is used. The efficiency and reliability of different heuristic optimization techniques in portfolio choice problems have been explored in [24, 25]. In the competition of simulated annealing, threshold accepting and stochastic differential equations, they find DE to be well-suited for non-convex portfolio optimization. It has shown efficient convergence to one (presumably global) optimum. These are the raison-d'être of our choice of incorporating DE into our portfolio optimization problems.

3.3.1. Single period optimization

We first implement the single period optimization with GARCH-DE and GARCH-DE-t-copula. The weights obtained for the two methods are displayed in Table4. We observe that GARCH-DE-t-copula seems to allow a well diversified portfolio through a well controlled risk as compared to GARCH-DE. This is not surprising, given the ability of the Generalized Pareto Distribution to model extreme events (in the heavy tails) and the t-copula which can better capture the complex dependence structure between the analysed variables. Moreover, the GARCH-DE-t-copula outperforms the GARCH-DE in terms of returns. it is important to note here that, while GARCH-DE allocates the largest weight to DASH, it appears in GARCH-DE-t-copula with the smallest weight.

3.3.2. Multi-period optimization

In our multi-period optimization, the 12 first months are used as training period after which the portfolio is re-balanced monthly. There are in total 37 rebalancing periods, starting on the 28 February 2015 and ending 27 February 2018.
The multi-period optimization using GARCH-DE-t-copula outperforms both the one using DE and GARCH-DE as illustrated in Figure3 and Table3.3.2. Across the entire investment period, the returns given by GARCH-DE-t-copula are higher than that of DE and GARCH-DE. Let's also stretch out that, with the exception of 5 periods, the returns in the multi-period GARCH-DE-t-copula are all greater than that in single period and nearly its double in some periods. So, in portfolio optimization, the power of the DE algorithm is

Table 4: Optimal weights and Target return for single period optimization of DE, GARCH-DE and GARCH-DE-t-copula models.

		DE	GARCH-DE	GARCH-DE-t-copula
Optimal Weights	Bitcoin	0.494	0.000	0.150
	Ripple	0.014	0.350	0.210
	Litecoin	0.480	0.000	0.224
	Dash	0.008	0.638	0.090
	Dogecoin	0.004	0.012	0.326
Target Return	Return	0.418	0.5949	1.004
Risk Measure	CVaR	1	1	0.4488

potential increased by combining it with t-copula.

Through our analysis, among our 5 studied crypto assets, it appears that the weights assigned to Bitcoin, the leading cryptocurrency, although the lowest in our portfolio remaind relatively constant across all the rebalancing periods. So, long term investor should consider including Bitcoin in their portfolio. Weights allocation reveal that DASH and DOGE move in opposite direction and share the highest weights in our portfolio. Cryptocurrencies investors should consider have them both in their portfolio.

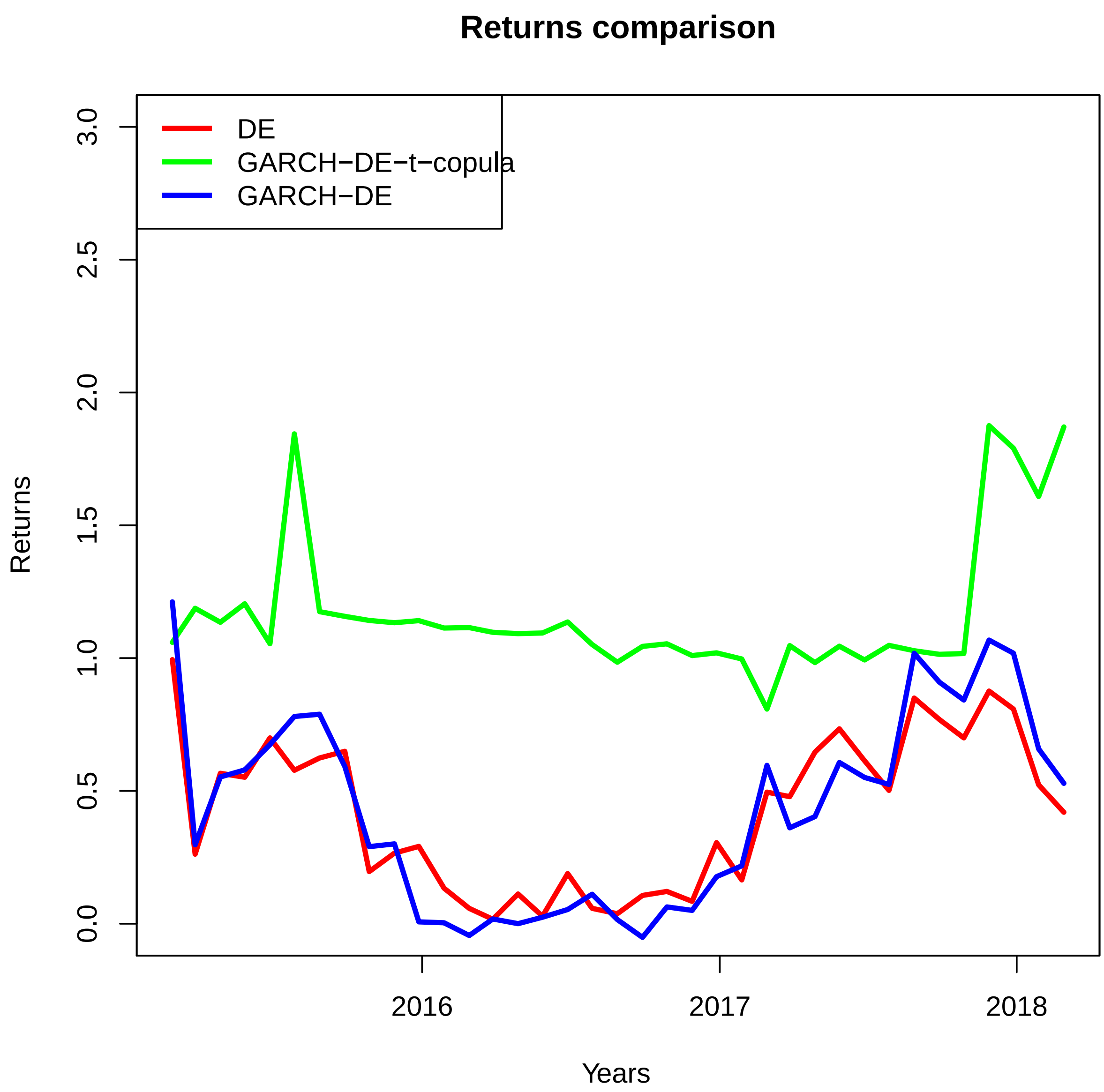

Figure 3: Returns comparison of the three methods

Table 5: Weights, CVaR and Returns from multi-period.

	BTC			XRP			LTC			DASH			DOGE			CVaR			Returns		
Periods	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE
2015/02/28	0.036	0.030	0.288	0.316	0.032	0.000	0.494	0.000	0.010	0.148	0.000	0.682	0.006	0.000	0.020	0.0125	1	1	1.059	1.212	0.994
2015/03/31	0.002	0.426	0.302	0.242	0.028	0.008	0.084	0.000	0.130	0.388	0.536	0.480	0.284	0.010	0.080	0.126	1	1	1.188	0.297	0.262
2015/04/30	0.096	0.330	0.880	0.416	0.000	0.000	0.088	0.000	0.082	0.400	0.670	0.038	0.000	0.000	0.000	0.063	1	1	1.135	0.553	0.566
2015/05/31	0.026	0.066	0.006	0.212	0.000	0.004	0.080	0.044	0.030	0.680	0.862	0.928	0.002	0.028	0.032	0.016	1	1	1.205	0.579	0.551
2015/06/30	0.06	0.186	0.092	0.290	0.000	0.000	0.142	0.054	0.060	0.322	0.742	0.848	0.186	0.018	0.000	0.041	1	1	1.054	0.674	0.700
2015/07/31	0.000	0.028	0.200	0.016	0.004	0.030	0.030	0.000	0.000	0.936	0.950	0.760	0.018	0.018	0.010	1	1	1	1.844	0.780	0.578
2015/08/31	0.038	0.040	0.506	0.134	0.004	0.000	0.394	0.022	0.016	0.010	0.934	0.478	0.424	0.000	0.000	0.339	1	1	1.175	0.789	0.624
2015/09/30	0.000	0.354	0.208	0.056	0.002	0.004	0.302	0.014	0.012	0.000	0.628	0.772	0.642	0.002	0.004	0.315	1	1	1.157	0.592	0.649
2015/10/31	0.038	0.068	0.854	0.140	0.000	0.000	0.400	0.034	0.008	0.002	0.898	0.124	0.420	0.000	0.014	0.294	1	1	1.142	0.290	0.196
2015/11/30	0.016	0.060	0.960	0.210	0.000	0.000	0.366	0.062	0.000	0.002	0.866	0.028	0.406	0.012	0.012	0.310	1	1	1.133	0.301	0.266
2015/12/31	0.016	0.342	0.954	0.332	0.000	0.000	0.290	0.308	0.034	0.006	0.312	0.012	0.356	0.038	0.000	0.287	1	1	1.141	0.007	0.291
2016/01/31	0.024	0.882	0.884	0.402	0.000	0.002	0.188	0.112	0.100	0.008	0.006	0.012	0.378	0.000	0.002	0.280	1	1	1.113	0.004	0.134
2016/02/29	0.000	0.888	0.752	0.240	0.018	0.008	0.308	0.024	0.150	0.000	0.068	0.090	0.452	0.002	0.000	0.231	1	1	1.115	-0.044	0.058
2016/03/31	0.000	0.912	0.702	0.110	0.000	0.008	0.316	0.034	0.000	0.000	0.054	0.248	0.574	0.000	0.042	0.209	1	1	1.097	0.018	0.017
2016/04/30	0.024	0.886	0.832	0.004	0.014	0.016	0.438	0.002	0.148	0.000	0.098	0.000	0.534	0.000	0.004	0.223	1	1	1.092	0.0003	0.112
2016/05/31	0.012	0.686	0.714	0.194	0.012	0.010	0.320	0.242	0.146	0.014	0.058	0.000	0.460	0.002	0.130	0.173	1	1	1.094	0.024	0.028
2016/06/30	0.004	0.264	0.904	0.116	0.000	0.014	0.384	0.388	0.034	0.088	0.346	0.042	0.408	0.002	0.006	0.275	1	1	1.136	0.053	0.189
2016/07/31	0.000	0.220	0.850	0.096	0.000	0.000	0.312	0.002	0.000	0.000	0.774	0.078	0.592	0.004	0.072	0.186	1	1	1.051	0.111	0.058
2016/08/31	0.096	0.012	0.884	0.098	0.002	0.000	0.388	0.098	0.036	0.006	0.886	0.010	0.412	0.002	0.070	0.185	1	1	0.985	0.016	0.038
2016/09/30	0.030	0.844	0.966	0.164	0.006	0.006	0.430	0.138	0.010	0.000	0.000	0.002	0.376	0.012	0.016	0.202	1	1	1.044	-0.052	0.106
2016/10/31	0.008	0.766	0.702	0.194	0.006	0.000	0.434	0.194	0.086	0.000	0.032	0.210	0.364	0.002	0.002	0.208	1	1	1.054	0.063	0.122
2016/11/30	0.092	0.168	0.780	0.266	0.014	0.016	0.296	0.816	0.024	0.070	0.000	0.082	0.276	0.002	0.098	0.216	1	1	1.010	0.050	0.084
2016/12/31	0.002	0.302	0.738	0.012	0.000	0.002	0.530	0.678	0.252	0.002	0.000	0.000	0.454	0.020	0.008	0.288	1	1	1.020	0.177	0.305
2017/01/31	0.000	0.806	0.854	0.244	0.018	0.074	0.302	0.168	0.000	0.022	0.008	0.000	0.432	0.000	0.072	0.220	1	1	0.997	0.218	0.165
2017/02/28	0.212	0.020	0.958	0.178	0.000	0.006	0.118	0.004	0.032	0.044	0.954	0.004	0.448	0.022	0.000	0.164	1	1	0.808	0.596	0.495
2017/03/31	0.002	0.640	0.904	0.018	0.000	0.006	0.436	0.156	0.004	0.046	0.198	0.070	0.498	0.006	0.012	0.410	1	1	1.047	0.361	0.478
2017/04/30	0.094	0.430	0.978	0.262	0.000	0.000	0.316	0.306	0.000	0.032	0.264	0.022	0.296	0.000	0.000	0.368	1	1	0.983	0.403	0.646
2017/05/31	0.006	0.702	0.932	0.060	0.020	0.000	0.534	0.248	0.016	0.000	0.030	0.050	0.400	0.000	0.002	0.400	1	1	1.045	0.607	0.734
2017/06/30	0.000	0.898	0.810	0.186	0.002	0.002	0.226	0.096	0.116	0.002	0.004	0.072	0.586	0.000	0.000	0.303	1	1	0.993	0.551	0.613
2017/07/31	0.018	0.274	0.864	0.208	0.016	0.002	0.414	0.702	0.066	0.008	0.008	0.068	0.352	0.000	0.000	0.324	1	1	1.048	0.524	0.502
2017/08/31	0.014	0.016	0.034	0.176	0.022	0.000	0.412	0.952	0.934	0.014	0.000	0.018	0.384	0.010	0.018	0.341	1	1	1.028	1.018	0.850
2017/09/30	0.010	0.176	0.198	0.202	0.002	0.000	0.252	0.790	0.764	0.002	0.032	0.034	0.534	0.000	0.004	0.304	1	1	1.014	0.910	0.769
2017/10/31	0.020	0.062	0.212	0.040	0.002	0.002	0.358	0.890	0.766	0.042	0.028	0.018	0.540	0.018	0.002	0.328	1	1	1.017	0.842	0.700
2017/11/30	0.000	0.124	0.216	0.002	0.000	0.000	0.018	0.856	0.720	0.976	0.020	0.000	0.004	0.000	0.064	1	1	1	1.875	1.0678	0.876
2017/12/31	0.002	0.000	0.000	0.032	0.000	0.052	0.124	0.936	0.934	0.842	0.064	0.004	0.000	0.000	0.010	1	1	1	1.790	1.019	0.808
2018/01/31	0.050	0.018	0.966	0.026	0.034	0.000	0.356	0.916	0.018	0.562	0.000	0.004	0.006	0.032	0.012	1	1	1	1.608	0.658	0.523
2018/02/27	0.000	0.000	0.406	0.026	0.030	0.014	0.000	0.942	0.572	0.974	0.006	0.004	0.000	0.022	0.004	1	1	1	1.870	0.528	0.420

4. Conclusion

Cryptocurrencies are new type of assets on the financial market with trade volumes reaching billions of dollars a day and market capitalizations reaching hundreds of billions of dollars. Highly volatile, they present investors with great opportunity of high returns. Though cryptocurrencies are still in their infancy, recent years have seen some savvy individuals making significant amount of money by speculating on cryptocurrencies. It is then important to develop portfolio optimization methods to assist cryptocurrencies investors in controlling their exposure risk while maximizing their returns. This study has confirmed the power of regular rebalancing of portfolio assets to adapt to market changes through GARCH Differential Evolution t-copula method (GARCH-DE-t-copula). Due to the high volatility that characterizes cryptocurrencies, the modeling of the tail dependence through t-copula and extreme value distribution (GPD) has shown significant positive impact on the returns of the portfolio across all multi-period optimization periods and also in the control of risk.

[1] J. Alvarez-Ramirez, E. Rodriguez and C. Ibarra-Valdez, Long-range correlations and asymmetry in the Bitcoin market, Physica A: Statistical Mechanics and its Applications, 492(2018), 948-955.

[2] D. Ardia, K. M. Mullen, B. G. Peterson and J. Ulrich, DEoptim: Differential Evolution Optimization in R, URL http://CRAN.R-project.org/package=DEoptim. R package version 2.1-0, (2011).

[3] D. Bekiros, J. A. Hernandez, S. Hammoudeh and D. K. Nguyen, Multivariate dependence risk and portfolio optimization: an application to mining stock portfolios, International journal of minerals policy and economics, 46(2)(2015), 111.

[4] W. Breymann, A. Dias and P. Embrechts, Dependence structures for multivariate high-frequency data in finance, Quant. Finance 3(2003), 114.

[5] K. Boudt, P. Carl and B.G. Peterson, PortfolioAnalytics: Portfolio analysis, including numeric methods for optimization of portfolios, R package version 0.8, (2012).

[6] E. Bouri, G. Azzi and A. H. Dyhrberg, On the return-volatility relationship in the Bitcoin market around the price crash of 2013, Economics: The Open-Access, Open-Assessment E-Journal, 11(2017), 116.

[7] M.K. Brunnermeier, Deciphering the 200708 liquidity and credit crunch, Journal of Economic Perspectives, 23(1)(2009), 77100.

[8] G. Calafiore, Ambiguous risk measures and optimal robust portfolios, SIAM J. Control and Optimiz., 18(3)(2007), 853-877.

[9] J. Chu, S. Chan, S. Nadarajah and J. Osterrieder, GARCH Modelling of Cryptocurrencies". J. Risk Financial Manag., 10(17)(2017), 1-15.

[10] A. H. Dyhrberg, Bitcoin, gold and the dollar A GARCH volatility analysis, Finance Res. Lett., 16(2016), 8592.

[11] A. H. Dyhrberg, Hedging capabilities of bitcoin. Is it the virtual gold? Finance Res. Lett., 16(2016), 139144.

[12] P. Embrechts, A. McNeil, D. Straumann, Correlation and dependency in risk management: properties and pitfalls. In Risk Management: Value at Risk and Beyond, M. Dempster & H. Moffatt, eds. http://www.math.ethz.ch/mcneil: Cambridge University Press, (2001), 176-223.

[13] R. F. Engle and T. Bollerslev, Modelling the persistence of conditional variances, Econometric Reviews, 5(1986), 150.

[14] H. Fang and K. Fang, The metaelliptical distributions with given marginals, J. Multivariate Anal., 82(2002), 116.

[15] C. Florackis, A. Kontonikas and A. Kostakis, Stock market liquidity and macro - liquidity shocks: Evidence from the 20072009 financial crisis, Journal of International Money and Finance, 44(2014), 97117.

[16] N. Gârleanu and L. Pedersen, Dynamic trading with predictable returns and transaction costs, Journal of Finance, 68(6)(2013), 23092340.

[17] L.R. Glosten, R. Jagannathan and D.E. Runkle, Relationship between the expected value and the volatility of the nominal excess return on stocks, The Journal of Finance, 48(5)(1993), 1779-1801.

[18] N. Hakansson, Multi-period mean-variance analysis: Toward a general theory of portfolio choice, J. Finance, 26(4)(1971), 857-884.

[19] B. Hagströmer and J. M. Binner, Stock portfolio selection with fullscale optimization and differential evolution, Applied Financial Economics, 19(19)(2009), 1559-1571.

[20] J. H. Holland, Adaptation in Natural Artificial Systems". University of Michigan Press, Ann Arbor (1975).

[21] T. Krink and S. Paterlini, Multiobjective optimization using differential evolution for real-world portfolio optimization, Computational Management Science, (2009).

[22] T. Krink, S. Mittnik and S. Paterlini, Differential evolution and combinatorial search for constrained index-tracking, Annals of Operations Research, 172(2009), 153-176.

[23] R.K.Y. Low, J. Alcock, R. Faff and T. Brailsford, Canonical vine copulas in the context of modern portfolio management: Are they worth it? Journal of Banking and Finance, 37(8)(2013), 3085-3099.

[24] D. G. Maringer and O. Oyewumi, Index tracking with constrained portfolios, Intelligent Systems in Accounting, Finance & Management, 15(12)(2007), 5771.

[25] D. G. Maringer and M. Meyer, Smooth transition autoregressive models: New approaches to the model selection problem, Studies in Nonlinear Dynamics& Econometrics, 12(1)(2008), 119. URL: http://www.bepress.com/snde/vol12/iss1/. Article nr. 5.

[26] H. Markowitz, Portfolio selection, J. Finance, 7(1)(1952), 77-91.

[27] R. Mashal and A. Zeevi, Beyond correlation: extreme comovements between financial assets, Unpublished, Columbia University, (2002).

[28] F. Moshirian, The global financial crisis and the evolution of markets institutions and regulation, Journal of Banking and Finance, 35(2011), 502511.

[29] P. Nystrup, H. Madsen and E. Lindstrm, Dynamic portfolio optimization across hidden market regimes, Working paper, Technical University of Denmark, (2016).

[30] J. Osterrieder and J. Lorenz, A statistical risk assessment of Bitcoin and its extreme tail behavior, Ann. Financ. Econ., 12(01)(2017), 1750003(1-19).

[31] A. Phillip, J. Chan and S. Peiris, A new look at cryptocurrencies, Econom. Lett., 163(2018), 69.

[32] K. Price, R. M. Storn and J.A. Lampinen, Differential evolution: a practical approach to global optimization, Springer Science & Business Media (2006).

[33] R.T. Rockafellar and S. Uryasev, Optimization of conditional value-at-risk, Journal of Risk, 2(2000), 493-517.

[34] R.T. Rockafellar and S. Uryasev, Conditional value-at-risk for general loss distributions, J. Bank. Finance, 26(7)(2002), 14431471.

[35] V. Sahalia and W. Brandt, Variable selection for portfolio choice, Journal of Finance, 56(4)(2001), 1297-1351.

[36] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publications de lInstitut de Statistique de lUniversité de Paris, 8(1959), 229231.

[37] R. Storn and K. Price, Differential evolution a simple and efficient heuristic for global optimization over continuous spaces, Journal of Global Optimization, 11(4)(1997), 341359.

[38] G. Yollin, R tools for portfolio optimization, In Presentation at R/Finance conference 2009, (2009).