An iterative method and its application to stable inversion

Faik Gürsoy · Johannes Jacobus Arnoldi Eksteen · Abdul Rahim Khan · Vatan Karakaya

Abstract In this paper, we study convergence and data dependence of SP and normal-S iterative methods for the class of almost contraction mappings under some mild conditions. The validity of these theoretical results is confirmed with numerical examples. It has been observed that a special case of SP iterative method, namely, normal-S iterative method, performs better and so the latter is implemented in the stable inversion of nonlinear discrete time dynamical systems to yield convergence results when Picard iterative method diverges. This is also illustrated with a numerical example. Our work extends and improves upon many results existing in the literature.

Keywords Iterative method · Convergence · Rate of convergence · Data dependency · Stable inversion.

Mathematics Subject Classification (2000) 47H09 · 47H10 · 47J25 · 54H25 · 65J15 · 65J22 · 65H · 65P · 93A

F. Gürsoy
Department of Mathematics, Adiyaman University, 02040, Adiyaman, Turkey
Tel.: +90-416-2233800-4018
Fax: +90-416-2231774
E-mail: faikgursoy02@hotmail.com

J. J. A. Eksteen
Department of Mechanical and Aeronautical Engineering, University of Pretoria, Pretoria, South Africa

A. R. Khan
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

V. Karakaya
Department of Mathematical Engineering, Yildiz Technical University, Istanbul 34210, Turkey

1 Introduction and preliminaries

A quick look at the current and vast literature of fixed point theory reveals that iterative approximation of fixed points has become a rapidly growing area of research and it has been successfully applied to solve problems arising in many branches of science and engineering. So the iterative methods have become important and useful computational tools for solving certain problems and consequently, iterative methods proposed by many researchers, serve the afore mentioned purposes. In this dynamic area of study, extremely useful research as well as discovery of new methods is taking place to meet some of the challenges of mathematics today.

Throughout the paper, $\mathcal{S}$ is a closed and convex subset of a Banach space $\mathcal{B}$, $\mathcal{T}:\mathcal{S}\to\mathcal{S}$ is an operator, $\mathcal{F}_{\mathcal{T}}$ denotes the set of fixed points of $\mathcal{T}$ and $\left\{\beta_n^{(i)}\right\}_{n=0}^{\infty}\subset[0,1]$, $i=1,2,3$ are real sequences satisfying certain control conditions.

The Xu-Noor iterative method (Xu and Noor 2002) is defined as:

$$\begin{cases} a_0^{(1)}\in\mathcal{S}, \\ a_{n+1}^{(1)}=\left(1-\beta_n^{(1)}\right)a_n^{(1)}+\beta_n^{(1)}\mathcal{T}a_n^{(2)} \\ a_n^{(2)}=\left(1-\beta_n^{(2)}\right)a_n^{(1)}+\beta_n^{(2)}\mathcal{T}a_n^{(3)} \\ a_n^{(3)}=\left(1-\beta_n^{(3)}\right)a_n^{(1)}+\beta_n^{(3)}\mathcal{T}a_n^{(1)},\ n\in\mathbb{N}. \end{cases} \tag{1}$$

Remark 1 Xu-Noor iterative method (1) reduces to the well-known (i) Ishikawa iterative method (Ishikawa 1974) if $\beta_n^{(3)}=0$ for all $n\in\mathbb{N}$, (ii) Mann iterative method (Mann 1953) if $\beta_n^{(i)}=0$, $i=2,3$ for all $n\in\mathbb{N}$, (iii) Picard iterative method (Picard 1890) if $\beta_n^{(1)}=1$, $\beta_n^{(i)}=0$, $i=2,3$ for all $n\in\mathbb{N}$, and (iv) Normal$-S$ iterative method (Sahu 2011) if $\beta_n^{(1)}=1$, $\beta_n^{(3)}=0$ for all $n\in\mathbb{N}$.

Osilike (1995/96) considered a class of operators satisfying the following inequality

$$\left\|\mathcal{T}a-\mathcal{T}a'\right\|\leq\delta\left\|a-a'\right\|+\epsilon\left\|a-\mathcal{T}a\right\|,\ \forall a,a'\in\mathcal{B}, \tag{2}$$

where $\epsilon\geq 0$, $\delta\in[0,1)$. He established stability of Mann, Ishikawa and Kirk (Kirk 2004) iterative methods for this class of operators. It is worth noting that contractive condition (2) is not sufficient to guarantee the existence of fixed point of the operator; $\mathcal{T}:[0,0.5]\to[0,0.5]$ defined by $\mathcal{T}a=0.5$, $a\in[0,0.25]$ and $\mathcal{T}a=0.1$, $a\in(0.25,0.5]$ constitutes an example to this negative claim. In order to guarantee the existence of fixed points of the operator satisfying condition (2), Berinde (2004a), by making a small but subtle modification in condition (2), has introduced the so-called almost contractive operators satisfying the following contractive type condition

$$\left\|\mathcal{T}a-\mathcal{T}a'\right\|\leq\delta\left\|a-a'\right\|+\epsilon\left\|a'-\mathcal{T}a\right\|,\ \forall a,a'\in\mathcal{B}, \tag{3}$$

where $\epsilon \geq 0$, $\delta \in [0,1)$. He also showed that this class of nonlinear operators is wider than the class of well-known Zamfiresu operators (Zamfirescu 1972), and an operator $\mathcal{T}$ satisfying both conditions (2) and (3) has a unique fixed point.

In 2005, Berinde (2005) used the Ishikawa iterative method to approximate fixed point of the operators satisfying contractive conditions (2) in a normed space setting. More precisely, he proved the following:

Theorem 1 *Let $\mathcal{B}$ be a normed linear space, $\mathcal{S}$ a closed and convex subset of $\mathcal{B}$, and $\mathcal{T}:\mathcal{S}\to\mathcal{S}$ an operator with $\mathcal{F}_{\mathcal{T}}=\emptyset$, satisfying (2). Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative method with the sequence $\left\{\beta_n^{(1)}\right\}_{n=0}^{\infty}$ in $[0,1]$ satisfying*

$$\text{(i)} \qquad \sum_{n=0}^{\infty} \beta_n^{(1)} = \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of $\mathcal{T}$.

Recently, Sintunavarat and Pitea (2016) introduced the following iterative method:

$$\begin{cases} s_0^{(1)} \in \mathcal{S}, \\ s_{n+1}^{(1)} = \left(1-\beta_n^{(1)}\right)\mathcal{T}s_n^{(2)} + \beta_n^{(1)}\mathcal{T}s_n^{(3)} \\ s_n^{(2)} = \left(1-\beta_n^{(2)}\right)s_n^{(1)} + \beta_n^{(2)}s_n^{(3)} \\ s_n^{(3)} = \left(1-\beta_n^{(3)}\right)s_n^{(1)} + \beta_n^{(3)}\mathcal{T}s_n^{(1)},\ n\in\mathbb{N}. \end{cases} \tag{4}$$

Remark 2 The iterative method (4), also denoted by SP, is independent of Xu-Noor, Ishikawa and Mann iterative methods. However, it reduces to the well-known (i) Picard iterative method if $\beta_n^{(2)}=1$, $\beta_n^{(3)}=0$ for all $n\in\mathbb{N}$, (ii) Normal$-S$ iterative method if $\beta_n^{(2)}=1$ for all $n\in\mathbb{N}$, and (iii) $S-$iterative method (Agarwal et al. 2007) if $\beta_n^{(2)}=0$ for all $n\in\mathbb{N}$.

When two or more fixed point iterative methods are known to be convergent to the same fixed point of a certain class of mappings, it is of greatest importance from both theoretical and practical point of view to choose the iterative method with highest rate of convergence and lightest workload so that the number of arithmetic and logical operations for finding the solution is a minimum (or the computer time is a minimum), from a set of available iterative methods for solving the given problem. In order to take such a decision objectively, one needs the following concepts of rate of convergence for the considered fixed point iterative methods.

Definition 1 (Knopp 1956) Let $\left\{s_n^{(i)}\right\}_{n=0}^{\infty}$ $(i=1,2)$ be two iterative methods converging to the same fixed point s_*. We say that $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{s_n^{(2)}\right\}_{n=0}^{\infty}$ to s_* if

$$\lim_{n\to\infty}\frac{\left\|s_n^{(1)}-s_*\right\|}{\left\|s_n^{(2)}-s_*\right\|}=0.$$

Definition 2 (Berinde 2004b) Suppose that there are two fixed point iterative methods $\left\{s_n^{(i)}\right\}_{n=0}^{\infty}$ $(i=1,2)$ both converging to the same fixed point s_*. Assume further that the following error estimates

$$\left\|s_n^{(1)}-s_*\right\| \le \epsilon_n^{(1)},$$
$$\left\|s_n^{(2)}-s_*\right\| \le \epsilon_n^{(2)}, \text{ for all } n\in\mathbb{N},$$

are available (and these estimates are the best possible, see (Berinde 2016)), where $\epsilon_n^{(i)}$ $(i=1,2)$ are two sequences of positive numbers (converging to zero). If $\left\{\epsilon_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{\epsilon_n^{(2)}\right\}_{n=0}^{\infty}$ (in the sense of above definition), then we shall say that $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ converges faster than $\left\{s_n^{(2)}\right\}_{n=0}^{\infty}$ to s_*.

Sintunavarat and Pitea (2016) established some theoretical and numerical comparison results for various iterative methods including the iterative method (4) to approximate fixed point of the operators satisfying contractive condition (2) in Banach spaces. More precisely, they proved the following:

Theorem 2 *Let $\mathcal{S}$ be a nonempty closed and convex subset of a Banach space $(\mathcal{B},\|\cdot\|)$ and $\mathcal{T}:\mathcal{S}\to\mathcal{S}$ be a mapping satisfying the contractive condition (2), with the fixed point s_*. Suppose that the sequence $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ is defined by the iterative method (4) and the sequences $\left\{\beta_n^{(1)}\right\}_{n=0}^{\infty}$, $\left\{\beta_n^{(2)}\right\}_{n=0}^{\infty}$, and $\left\{\beta_n^{(3)}\right\}_{n=0}^{\infty}$ are in $\left[\beta^{(1)},1-\beta^{(1)}\right]$, $\left[\beta^{(2)},1-\beta^{(2)}\right]$, and $\left[\beta^{(3)},1-\beta^{(3)}\right]$ respectively, with $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}\in\left(0,\frac{1}{2}\right)$. If $\beta^{(1)}\left(2-\beta^{(2)}\right)<\beta^{(2)}$, then the iterative method (4) converges strongly to the fixed point s_* of $\mathcal{T}$ faster than $S-$iterative method.*

This result naturally gives rise to the following question.

Question: Is it possible to approximate fixed point of almost contractions by a simpler and faster iterative method under some mild conditions on the parametric sequences $\left\{\beta_n^{(i)}\right\}_{n=0}^{\infty}$ $(i=1,2,3)$?

It is our aim in this paper to answer the above question in the affirmative. Thus, motivated by the above mentioned results of Berinde (2005) (i.e., Therorem 1) and Sintunavarat and Pitea (2016) (i.e., Therorem 2), as a complement of these works, we prove some convergence and data dependence results in a Banach space for the iterative method (4) and normal-S iterative method of almost contractive type operators. We also prove that these iterative methods are equivalent when converging to the same fixed point of the operators. We compare rate of convergence of the iterative method (4) with Xu-Noor, Ishikawa, Mann, S, and normal-S iterative methods. Finally, an application of normal-S iterative method to stable inversion of discrete-time dynamical systems is presented where Picard iterative method fails to converge. A numerical example in support of the proposed application is also given.

For the development of our main results, the following lemmas will be needed:

Lemma 1 *(Berinde 2007) Let* $\left\{\sigma_n^{(i)}\right\}_{n=0}^{\infty}$, $i = 1, 2$ *be non-negative real sequences satisfying the following inequality:*

$$\sigma_{n+1}^{(1)} \leq \sigma\sigma_n^{(1)} + \sigma_n^{(2)},$$

where $\sigma \in [0, 1)$ *and* $\lim_{n\to\infty} \sigma_n^{(2)} = 0$. *Then* $\lim_{n\to\infty} \sigma_n^{(1)} = 0$.

Lemma 2 *(Şoltuz and Grosan 2008) Let* $\left\{\sigma_n^{(i)}\right\}_{n=0}^{\infty}$, $i = 1, 2, 3$ *be sequences of non-negative numbers with* $\left\{\sigma_n^{(2)}\right\}_{n=0}^{\infty} \subset (0, 1)$ *and* $\sum\limits_{n=0}^{\infty} \sigma_n^{(2)} = \infty$. *Assume there exists* $n_0 \in \mathbb{N}$, *such that for all* $n \geq n_0$ *one has the inequality:*

$$\sigma_{n+1}^{(1)} \leq \left(1 - \sigma_n^{(2)}\right)\sigma_n^{(1)} + \sigma_n^{(2)}\sigma_n^{(3)}.$$

Then the following inequality holds

$$0 \leq \limsup_{n\to\infty} \sigma_n^{(1)} \leq \limsup_{n\to\infty} \sigma_n^{(3)}.$$

2 Convergence Analysis

In the rest of this paper, we assume that $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ satisfies (2) and (3).

Theorem 3 *The iterative method* $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ *in (4) converges to a unique fixed point* s_* *of* $\mathcal{T}$.

Proof Using (2), (4) and triangle inequality of norm, we have

$$\begin{aligned}\left\|s_{n+1}^{(1)} - s_*\right\| &\leq \left(1 - \beta_n^{(1)}\right)\left\|\mathcal{T}s_n^{(2)} - s_*\right\| + \beta_n^{(1)}\left\|\mathcal{T}s_n^{(3)} - s_*\right\| \\ &\leq \left(1 - \beta_n^{(1)}\right)\delta\left\|s_n^{(2)} - s_*\right\| + \beta_n^{(1)}\delta\left\|s_n^{(3)} - s_*\right\|, \qquad (5)\end{aligned}$$

$$\left\|s_n^{(2)} - s_*\right\| \leq \left(1 - \beta_n^{(2)}\right)\left\|s_n^{(1)} - s_*\right\| + \beta_n^{(2)}\left\|s_n^{(3)} - s_*\right\|, \qquad (6)$$

$$\begin{aligned}\left\|s_n^{(3)} - s_*\right\| &\leq \left(1 - \beta_n^{(3)}\right)\left\|s_n^{(1)} - s_*\right\| + \beta_n^{(3)}\left\|\mathcal{T}s_n^{(1)} - s_*\right\| \\ &\leq \left[1 - \beta_n^{(3)}(1 - \delta)\right]\left\|s_n^{(1)} - s_*\right\| \\ &\leq \left\|s_n^{(1)} - s_*\right\| \text{ as } 1 - \beta_n^{(3)}(1 - \delta) < 1. \qquad (7)\end{aligned}$$

Combining (5)-(7), we get

$$\|s_{n+1}-s_*\| \leq \delta \left\| s_n^{(1)} - s_* \right\|$$

which further implies

$$\|s_{n+1}-s_*\| \leq \delta^{n+1} \left\| s_0^{(1)} - s_* \right\|. \tag{8}$$

By passing to the limit in (8), we obtain

$$\lim_{n\to\infty} \|s_{n+1}-s_*\| = 0.$$

Thus we have $s_n \to s_*$ as $n \to \infty$.

We have the following result as a direct consequence of Theorem 3.

Theorem 4 *Let $\{q_n\}_{n=0}^{\infty}$ be defined by normal–S iterative method as follows:*

$$\begin{cases} q_0 \in S, \\ q_{n+1} = \mathcal{T} r_n, \\ r_n = \left(1-\beta_n^{(3)}\right) q_n + \beta_n^{(3)} \mathcal{T} q_n,\ n \in \mathbb{N}, \end{cases} \tag{9}$$

where $\left\{\beta_n^{(3)}\right\}_{n=0}^{\infty} \subset [0,1]$. Then $\{q_n\}_{n=0}^{\infty}$ converges to a unique fixed point s_* of $\mathcal{T}$.

Proof Put $\beta_n^{(1)} = 0$ and $\beta_n^{(2)} = 1$ for all $n \in \mathbb{N}$ in the proof Theorem 3.

Theorem 5 *Let $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be the iterative methods in (4) and (9), respectively. Assume that the sequence $\left\{\beta_n^{(1)}\right\}_{n=0}^{\infty} \subset [0,1]$ satisfies the condition $0 < \beta^{(1)} = \inf_{n\in\mathbb{N}} \beta_n^{(1)}$. Then the following statements are equivalent:*

(i) $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ converges to the unique fixed point s_* of $\mathcal{T}$;

(ii) $\{q_n\}_{n=0}^{\infty}$ converges to the unique fixed point s_* of $\mathcal{T}$.

Proof (i)⇒(ii): It is established in the proof of Theorem 4.

(ii)⇒(i): Suppose that $q_n \to s_*$ as $n \to \infty$. Using (2), (4) and (9), we obtain the following estimates:

$$\begin{aligned} \left\| q_{n+1} - s_{n+1}^{(1)} \right\| &= \left\| \left(1-\beta_n^{(1)}\right) \left(\mathcal{T} r_n - \mathcal{T} s_n^{(2)}\right) + \beta_n^{(1)} \left(\mathcal{T} r_n - \mathcal{T} s_n^{(3)}\right) \right\| \\ &\leq \left(1-\beta_n^{(1)}\right) \left\| \mathcal{T} r_n - \mathcal{T} s_n^{(2)} \right\| + \beta_n^{(1)} \left\| \mathcal{T} r_n - \mathcal{T} s_n^{(3)} \right\| \\ &\leq \left(1-\beta_n^{(1)}\right) \left\{ \delta \left\| r_n - s_n^{(2)} \right\| + \epsilon \left\| r_n - \mathcal{T} r_n \right\| \right\} \\ &\quad + \beta_n^{(1)} \left\{ \delta \left\| r_n - s_n^{(3)} \right\| + \epsilon \left\| r_n - \mathcal{T} r_n \right\| \right\} \\ &= \left(1-\beta_n^{(1)}\right) \delta \left\| r_n - s_n^{(2)} \right\| + \beta_n^{(1)} \delta \left\| r_n - s_n^{(3)} \right\| \\ &\quad + \epsilon \left\| r_n - \mathcal{T} r_n \right\|, \end{aligned} \tag{10}$$

$$\begin{aligned}\left\|r_n - s_n^{(3)}\right\| &= \left\|\left(1-\beta_n^{(3)}\right)\left(q_n - s_n^{(1)}\right) + \beta_n^{(3)}\left(\mathcal{T}q_n - \mathcal{T}s_n^{(1)}\right)\right\| \\ &\le \left(1-\beta_n^{(3)}\right)\left\|q_n - s_n^{(1)}\right\| + \beta_n^{(3)}\left\|\mathcal{T}q_n - \mathcal{T}s_n^{(1)}\right\| \\ &\le \left(1-\beta_n^{(3)}\right)\left\|q_n - s_n^{(1)}\right\| + \beta_n^{(3)}\delta\left\|q_n - s_n^{(1)}\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\| \\ &= \left[1-\beta_n^{(3)}(1-\delta)\right]\left\|q_n - s_n^{(1)}\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\|. \end{aligned} \tag{11}$$

In a similar fashion, again using (2), (4) and (9), we get

$$\begin{aligned}\left\|r_n - s_n^{(2)}\right\| &= \left\|\left(1-\beta_n^{(3)}\right)q_n + \beta_n^{(3)}\mathcal{T}q_n - \left(1-\beta_n^{(2)}\right)s_n^{(1)} - \beta_n^{(2)}s_n^{(3)}\right\| \\ &\le \left\|\left(1-\beta_n^{(3)}\right)q_n + \beta_n^{(3)}\mathcal{T}q_n - s_*\right\| + \left\|s_* - \left(1-\beta_n^{(2)}\right)s_n^{(1)} - \beta_n^{(2)}s_n^{(3)}\right\| \\ &\le \left(1-\beta_n^{(3)}\right)\left\|q_n - s_*\right\| + \beta_n^{(3)}\left\|\mathcal{T}q_n - s_*\right\| \\ &\quad + \left(1-\beta_n^{(2)}\right)\left\|s_* - s_n^{(1)}\right\| + \beta_n^{(2)}\left\|s_* - s_n^{(3)}\right\| \\ &\le \left[1-\beta_n^{(3)}(1-\delta)\right]\left\|q_n - s_*\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\| \\ &\quad + \left(1-\beta_n^{(2)}\right)\left\|s_* - s_n^{(1)}\right\| + \beta_n^{(2)}\left[1-\beta_n^{(3)}(1-\delta)\right]\left\|s_* - s_n^{(1)}\right\| \\ &\le \left\|q_n - s_*\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\| + \left\|s_* - s_n^{(1)}\right\| \;\text{ as } 1-\beta_n^{(3)}(1-\delta) < 1 \\ &\le \left\|q_n - s_n^{(1)}\right\| + 2\left\|q_n - s_*\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\|. \end{aligned} \tag{12}$$

Substituting (11) and (12) into (10) and applying the inequalities $\left(1-\beta_n^{(1)}\right)\delta < 1-\beta_n^{(1)}$, $1-\beta_n^{(3)}(1-\delta) < 1$ for all $n \in \mathbb{N}$ in the resulting inequality, we obtain

$$\begin{aligned}\left\|q_{n+1} - s_{n+1}^{(3)}\right\| &\le \left(1-\beta_n^{(1)}\right)\delta\left\{\left\|q_n - s_n^{(1)}\right\| + 2\left\|q_n - s_*\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\|\right\} \\ &\quad + \beta_n^{(1)}\delta\left\{\left[1-\beta_n^{(3)}(1-\delta)\right]\left\|q_n - s_n^{(1)}\right\| + \beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\|\right\} \\ &\quad + \epsilon\left\|r_n - \mathcal{T}r_n\right\| \\ &\le \left[1-\beta_n^{(1)}(1-\delta)\right]\left\|q_n - s_n^{(1)}\right\| + 2\delta\left(1-\beta_n^{(1)}\right)\left\|q_n - s_*\right\| \\ &\quad + \delta\beta_n^{(3)}\epsilon\left\|q_n - \mathcal{T}q_n\right\| + \epsilon\left\|r_n - \mathcal{T}r_n\right\|. \end{aligned} \tag{13}$$

Using the fact $Ts_* = s_*$, triangle inequality of norm and (2), we have

$$\begin{aligned}\left\|q_n - \mathcal{T}q_n\right\| &\le \left\|q_n - s_*\right\| + \left\|\mathcal{T}s_* - \mathcal{T}q_n\right\| \\ &\le (1+\delta)\left\|q_n - s_*\right\|, \end{aligned} \tag{14}$$

and

$$\begin{aligned}\left\|r_n - Tr_n\right\| &\le \left\|r_n - s_*\right\| + \left\|\mathcal{T}s_* - \mathcal{T}r_n\right\| \\ &\le (1+\delta)\left\|r_n - s_*\right\| \\ &= (1+\delta)\left\|\left(1-\beta_n^{(3)}\right)(q_n - s_*) + \beta_n^{(3)}(\mathcal{T}q_n - \mathcal{T}s_*)\right\| \\ &\le (1+\delta)\left[1-\beta_n^{(3)}(1-\delta)\right]\left\|q_n - s_*\right\|. \end{aligned} \tag{15}$$

Inserting (14) and (15) into (13) and using the assumption $0 < \beta^{(1)} = \inf_{n\in\mathbb{N}} \beta_n^{(1)}$, we have

$$\begin{aligned}\left\|q_{n+1} - s_{n+1}^{(1)}\right\| &\leq \left[1 - \beta^{(1)}(1-\delta)\right]\left\|q_n - s_n^{(1)}\right\| \\ &\quad + \left\{2\delta\left(1 - \beta_n^{(1)}\right) + \delta\beta_n^{(3)}\epsilon(1+\delta)\right. \\ &\quad \left. + \epsilon(1+\delta)\left[1 - \beta_n^{(3)}(1-\delta)\right]\right\}\|q_n - s_*\| \\ &= \left[1 - \beta^{(1)}(1-\delta)\right]\left\|q_n - s_n^{(1)}\right\| + \left\{2\delta\left(1 - \beta_n^{(1)}\right)\right. \\ &\quad \left. + \epsilon(1+\delta)\left[1 - \beta_n^{(3)}(1-2\delta)\right]\right\}\|q_n - s_*\|. \qquad (16)\end{aligned}$$

Set

$$\begin{aligned}\sigma_n^{(1)} &= \left\|q_n - s_n^{(1)}\right\| \geq 0, \\ \sigma &= 1 - \beta^{(1)}(1-\delta) \in (0,1), \\ \sigma_n^{(2)} &= \left\{2\delta\left(1 - \beta_n^{(1)}\right) + \epsilon(1+\delta)\left[1 - \beta_n^{(3)}(1-2\delta)\right]\right\}\|q_n - s_*\|.\end{aligned}$$

Now all the conditions of Lemma 1 are satisfied by (16) and so by its conclusion, we have that $\lim_{n\to\infty}\left\|q_n - s_n^{(1)}\right\| = 0$. Also, since $\left\|s_n^{(1)} - s_*\right\| \leq \left\|q_n - s_n^{(1)}\right\| + \|q_n - s_*\|$, we conclude that $\lim_{n\to\infty}\left\|s_n^{(1)} - s_*\right\| = 0$.

Theorem 6 *Let $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be the iterative methods in (4) and (9), respectively. Assume the sequences $\left\{\beta_n^{(i)}\right\}_{n=0}^{\infty} \subset [0,1]$ $(i = 1,2,3)$ satisfy $0 < \beta^{(i)} = \inf_{n\in\mathbb{N}} \beta_n^{(i)}$ $(i = 1,2,3)$. Then $\{q_n\}_{n=0}^{\infty}$ converges to $s_* \in \mathcal{F}_\mathcal{T}$ faster than $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$, provided that $s_0^{(1)} = q_0$.*

Proof It follows from (2), (4) and (9) that

$$\begin{aligned}\|q_{n+1} - s_*\| &\leq \delta\left[1 - \beta_n^{(3)}(1-\delta)\right]\|q_n - s_*\| \\ &\leq \cdots \\ &\leq \prod_{k=0}^{n}\delta\left[1 - \beta_k^{(3)}(1-\delta)\right]\|q_0 - s_*\|,\end{aligned}$$

$$\begin{aligned}\left\|s_{n+1}^{(1)} - s_*\right\| &\leq \delta\left\{\left(1 - \beta_n^{(1)}\right)\left[1 - \beta_n^{(2)}\beta_n^{(3)}(1-\delta)\right] + \beta_n^{(1)}\left[1 - \beta_n^{(3)}(1-\delta)\right]\right\}\left\|s_n^{(1)} - s_*\right\| \\ &\leq \delta\left\{\left(1 - \beta_n^{(1)}\right)\left[1 - \beta_n^{(2)}\beta_n^{(3)}(1-\delta)\right] + \beta_n^{(1)}\left[1 - \beta_n^{(2)}\beta_n^{(3)}(1-\delta)\right]\right\}\left\|s_n^{(1)} - s_*\right\| \\ &= \delta\left[1 - \beta_n^{(2)}\beta_n^{(3)}(1-\delta)\right]\left\|s_n^{(1)} - s_*\right\| \\ &\leq \cdots \\ &\leq \prod_{k=0}^{n}\delta\left[1 - \beta_k^{(2)}\beta_k^{(3)}(1-\delta)\right]\left\|s_0^{(1)} - s_*\right\|.\end{aligned}$$

By the assumption on $\left\{\beta_n^{(i)}\right\}_{n=0}^{\infty}$ $(i = 1, 2, 3)$, we have

$$\begin{aligned}\|q_{n+1} - s_*\| &\leq \prod_{k=0}^{n} \delta \left[1 - \beta_k^{(3)} (1-\delta)\right] \|q_0 - s_*\| \\ &\leq \prod_{k=0}^{n} \delta \left[1 - \beta^{(3)} (1-\delta)\right] \|q_0 - s_*\| \\ &= \delta^{n+1} \left[1 - \beta^{(3)} (1-\delta)\right]^{n+1} \|q_0 - s_*\|,\end{aligned}$$

$$\begin{aligned}\left\|s_{n+1}^{(1)} - s_*\right\| &\leq \prod_{k=0}^{n} \delta \left[1 - \beta_k^{(2)} \beta_k^{(3)} (1-\delta)\right] \left\|s_0^{(1)} - s_*\right\| \\ &\leq \prod_{k=0}^{n} \delta \left[1 - \beta^{(2)} \beta^{(3)} (1-\delta)\right] \left\|s_0^{(1)} - s_*\right\| \\ &= \delta^{n+1} \left[1 - \beta^{(2)} \beta^{(3)} (1-\delta)\right]^{n+1} \left\|s_0^{(1)} - s_*\right\|.\end{aligned}$$

Now, let

$$\begin{aligned}\epsilon_n^{(1)} &= \delta^{n+1} \left[1 - \beta^{(3)} (1-\delta)\right]^{n+1} \|q_0 - s_*\|, \\ \epsilon_n^{(2)} &= \delta^{n+1} \left[1 - \beta^{(2)} \beta^{(3)} (1-\delta)\right]^{n+1} \left\|s_0^1 - s_*\right\|.\end{aligned}$$

Define

$$\kappa_n = \frac{\epsilon_n^{(1)}}{\epsilon_n^{(2)}} = \frac{\delta^{n+1} \left[1 - \beta^{(3)} (1-\delta)\right]^{n+1} \|q_0 - s_*\|}{\delta^{n+1} \left[1 - \beta^{(2)} \beta^{(3)} (1-\delta)\right]^{n+1} \left\|s_0^{(1)} - s_*\right\|}.$$

Since $q_0 = s_0^{(1)}$, therefore we have

$$\kappa_n = \frac{\epsilon_n^{(1)}}{\epsilon_n^{(2)}} = \frac{\left[1 - \beta^{(3)} (1-\delta)\right]^{n+1}}{\left[1 - \beta^{(2)} \beta^{(3)} (1-\delta)\right]^{n+1}}.$$

Thus

$$\frac{\kappa_{n+1}}{\kappa_n} = \frac{\frac{\left[1-\beta^{(3)}(1-\delta)\right]^{n+2}}{\left[1-\beta^{(2)}\beta^{(3)}(1-\delta)\right]^{n+2}}}{\frac{\left[1-\beta^{(3)}(1-\delta)\right]^{n+1}}{\left[1-\beta^{(2)}\beta^{(3)}(1-\delta)\right]^{n+1}}} = \frac{1 - \beta^{(3)} (1-\delta)}{1 - \beta^{(2)} \beta^{(3)} (1-\delta)} < 1. \tag{17}$$

Taking limit on both sides of (17) as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{\kappa_{n+1}}{\kappa_n} = \frac{1 - \beta^{(3)} (1-\delta)}{1 - \beta^{(2)} \beta^{(3)} (1-\delta)} < 1.$$

Hence, $\lim_{n\to\infty} \kappa_n = 0$ which implies that $\{q_n\}_{n=0}^{\infty}$ converges to s_* faster than $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$.

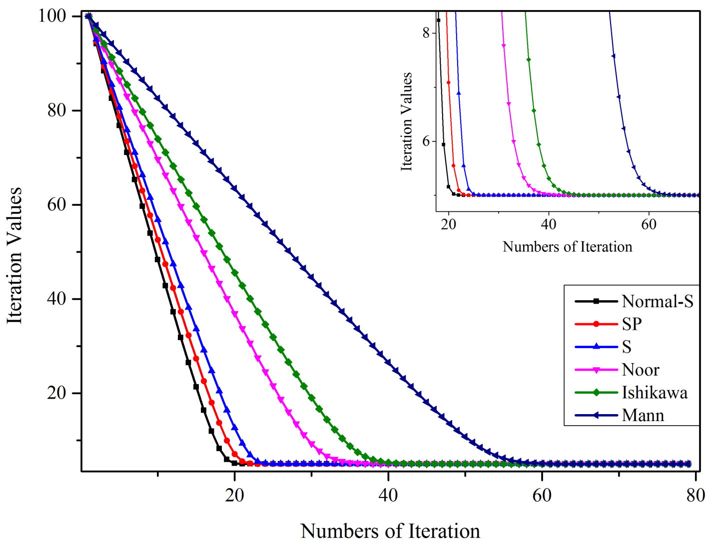

Fig. 1: Comparison (rate of convergence among various iterative methods for Example 1 with initial guess 100).

Example 1 Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual norm, $\mathcal{S} = [1, 100]$ and $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ an operator defined by $\mathcal{T}x = \sqrt{x^2 - 8x + 40}$ for all $x \in \mathcal{S}$. Clearly, $\mathcal{T}$ satisfies conditions (2) and (3) with $\delta \in [0.5222, 0.9987]$ and for any $\epsilon \geq 0$ and it has a unique fixed point $s_* = 5$. Let $\left\{s_n^{(1)}\right\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be generated by (4) and (9), respectively. Take $\beta_n^{(i)} = \frac{1}{2}$ $(i = 1, 2, 3)$ for all $n \in \mathbb{N}$ and initial guess $s_0^{(1)} = q_0 = 100$. Now Figure 1 and Table 1 below show that normal$-S$ iterative method converges to $s_* = 5$ faster than Mann, Ishikawa, Noor, S, and SP iterative methods.

3 Data Dependency

In some cases, it is difficult or may be impossible to find a fixed point of an operator. In such cases, instead of computing the fixed point of that operator, we approximate it with the help of another one whose fixed points are available. In the literature, this approach is referred to as "Data Dependence". This topic has recently received a lot of attention by virtue of its promis-

Table 1: Comparison (rate of convergence among various iterative methods for Example 1 with initial guess 100).

Iter. No.	Normal-S	SP	S	Noor	Ishikawa	Mann
0	100	100	100	100	100	100
1	94.1900	94.6737	95.1575	96.6119	97.0950	98.0625
2	88.3924	89.3576	90.3233	93.2281	94.1930	96.1262
3	82.6088	84.0532	85.4985	89.8486	91.2942	94.1913
⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	5.00000	5.00028	5.01818	21.6024	31.9249	54.0134
25	⋮	5.00004	5.00294	18.7565	29.2590	52.1331
26		5.00000	5.00048	16.0426	26.6293	50.2575
⋮		⋮	⋮	⋮	⋮	⋮
29			5.00000	9.29078	19.0506	44.6622
⋮			⋮	⋮	⋮	⋮
52				5.00000	5.00018	7.57905
⋮				⋮	⋮	⋮
57					5.00000	5.32335
⋮					⋮	⋮
78						5.00000
⋮						⋮

ing and interesting applications (Şoltuz and Grosan 2008; Gürsoy, Karakaya and Rhoades 2013; Karakaya, Gürsoy and Ertürk 2016; Khan, Gürsoy and Karakaya 2016; Khan, Gürsoy and Kumar 2016; Khan, Kumar and Hussain 2014; Gürsoy 2014; Ozturk Celiker 2014).

Definition 3 (Berinde 2007) Let $\mathcal{T},\widetilde{\mathcal{T}} : \mathcal{B} \to \mathcal{B}$ be operators. We say that $\widetilde{\mathcal{T}}$ is an approximate operator of $\mathcal{T}$ if for all $a \in \mathcal{B}$ and for a fixed $\varepsilon > 0$, we have

$$\left\| \mathcal{T}a - \widetilde{\mathcal{T}}a \right\| \leq \varepsilon.$$

Theorem 7 *Let $\widetilde{\mathcal{T}} : \mathcal{S} \to \mathcal{S}$ be an approximate operator of $\mathcal{T}$ and $\widetilde{s}_*$ be the fixed point of $\widetilde{\mathcal{T}}$. Let $\left\{ s_n^{(1)} \right\}_{n=0}^{\infty}$ be the iterative method defined by (4) with $\lim_{n\to\infty} \beta_n^{(3)} = 0$ and $\left\{ \widetilde{s}_n^{(1)} \right\}_{n=0}^{\infty}$ an iterative method generated by*

$$\begin{cases} \widetilde{s}_0^{(1)} \in \mathcal{S}, \\ \widetilde{s}_{n+1}^{(1)} = \left(1 - \beta_n^{(1)}\right) \widetilde{\mathcal{T}}\widetilde{s}_n^{(2)} + \beta_n^{(1)} \widetilde{\mathcal{T}}\widetilde{s}_n^{(3)} \\ \widetilde{s}_n^{(2)} = \left(1 - \beta_n^{(2)}\right) \widetilde{s}_n^{(1)} + \beta_n^{(2)} \widetilde{s}_n^{(3)} \\ \widetilde{s}_n^{(3)} = \left(1 - \beta_n^{(3)}\right) \widetilde{s}_n^{(1)} + \beta_n^{(3)} \widetilde{\mathcal{T}}\widetilde{s}_n^{(1)},\ n \in \mathbb{N}. \end{cases} \tag{18}$$

Suppose that $\widetilde{s}_n^{(1)} \to \widetilde{s}_*$ *as* $n \to \infty$. *Then we have*

$$\|s_* - \widetilde{s}_*\| \leq \frac{\varepsilon}{1-\delta}.$$

Proof It follows from (2), (4) and (18) that

$$\begin{aligned}\left\|s_{n+1}^{(1)} - \widetilde{s}_{n+1}^{(1)}\right\| &\leq \left(1-\beta_n^{(1)}\right)\left\|\mathcal{T}s_n^{(2)} - \widetilde{\mathcal{T}}\widetilde{s}_n^{(2)}\right\| + \beta_n^{(1)}\left\|\mathcal{T}s_n^{(3)} - \widetilde{\mathcal{T}}\widetilde{s}_n^{(3)}\right\| \quad (19)\\ &\leq \left(1-\beta_n^{(1)}\right)\left\{\left\|\mathcal{T}s_n^{(2)} - \mathcal{T}\widetilde{s}_n^{(2)}\right\| + \left\|\mathcal{T}\widetilde{s}_n^{(2)} - \widetilde{\mathcal{T}}\widetilde{s}_n^{(2)}\right\|\right\}\\ &\quad +\beta_n^{(1)}\left\{\left\|\mathcal{T}s_n^{(3)} - \mathcal{T}\widetilde{s}_n^{(3)}\right\| + \left\|\mathcal{T}\widetilde{s}_n^{(3)} - \widetilde{\mathcal{T}}\widetilde{s}_n^{(3)}\right\|\right\}\\ &\leq \left(1-\beta_n^{(1)}\right)\left\{\delta\left\|s_n^{(2)} - \widetilde{s}_n^{(2)}\right\| + \epsilon\left\|s_n^{(2)} - \mathcal{T}s_n^{(2)}\right\| + \varepsilon\right\}\\ &\quad +\beta_n^{(1)}\left\{\delta\left\|s_n^{(3)} - \widetilde{s}_n^{(3)}\right\| + \epsilon\left\|s_n^{(3)} - \mathcal{T}s_n^{(3)}\right\| + \varepsilon\right\},\end{aligned}$$

$$\left\|s_n^{(2)} - \widetilde{s}_n^{(2)}\right\| \leq \left(1-\beta_n^{(2)}\right)\left\|s_n^{(1)} - \widetilde{s}_n^{(1)}\right\| + \beta_n^{(2)}\left\|s_n^{(3)} - \widetilde{s}_n^{(3)}\right\|, \quad (20)$$

$$\begin{aligned}\left\|s_n^{(3)} - \widetilde{s}_n^{(3)}\right\| &\leq \left(1-\beta_n^{(3)}\right)\left\|s_n^{(1)} - \widetilde{s}_n^{(1)}\right\| + \beta_n^{(3)}\left\|\mathcal{T}s_n^{(1)} - \widetilde{\mathcal{T}}\widetilde{s}_n^{(1)}\right\| \quad (21)\\ &\leq \left[1-\beta_n^{(3)}(1-\delta)\right]\left\|s_n^{(1)} - \widetilde{s}_n^{(1)}\right\| + \beta_n^{(3)}\epsilon\left\|s_n^{(1)} - \mathcal{T}s_n^{(1)}\right\| + \beta_n^{(3)}\varepsilon.\end{aligned}$$

On the other hand, we have the following estimates (as in the proof of Theorem 5):

$$\begin{aligned}\left\|s_n^{(2)} - \mathcal{T}s_n^{(2)}\right\| &\leq (1+\delta)\left\|\left(1-\beta_n^{(2)}\right)\left(s_n^{(1)} - s_*\right) + \beta_n^{(2)}\left(s_n^{(3)} - s_*\right)\right\|\\ &\leq (1+\delta)\left[\left(1-\beta_n^{(2)}\right)\left\|s_n^{(1)} - s_*\right\| + \beta_n^{(2)}\left\|s_n^{(3)} - s_*\right\|\right]\\ &\leq (1+\delta)\left(1-\beta_n^{(2)} + \beta_n^{(2)}\left[1-\beta_n^{(3)}(1-\delta)\right]\right)\left\|s_n^{(1)} - s_*\right\|\\ &\leq (1+\delta)\left\|s_n^{(1)} - s_*\right\|, \quad (22)\end{aligned}$$

$$\begin{aligned}\left\|s_n^{(3)} - \mathcal{T}s_n^{(3)}\right\| &\leq (1+\delta)\left[1-\beta_n^{(3)}(1-\delta)\right]\left\|s_n^{(1)} - s_*\right\|\\ &\leq (1+\delta)\left\|s_n^{(1)} - s_*\right\|, \quad (23)\end{aligned}$$

$$\left\|s_n^{(1)} - \mathcal{T}s_n^{(1)}\right\| \leq (1+\delta)\left\|s_n^{(1)} - s_*\right\|. \quad (24)$$

Combining (19)-(24) and using the facts $\beta_n^{(2)} < 1$, $1-\beta_n^{(3)}(1-\delta) < 1$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\left\|s_{n+1}^{(1)} - \widetilde{s}_{n+1}^{(1)}\right\| &\leq (1-\theta)\left\|s_n^{(1)} - \widetilde{s}_n^{(1)}\right\| \quad (25)\\ &\quad +\theta\frac{\left(1+\delta\beta_n^{(3)}\right)(1+\delta)\epsilon\left\|s_n^{(1)} - s_*\right\| + \delta\beta_n^{(3)}\varepsilon + \varepsilon}{\theta},\end{aligned}$$

where $\theta = 1 - \delta \in (0,1)$.

Define

$$\sigma_n^{(1)} = \left\| s_n^{(1)} - \widetilde{s}_n^{(1)} \right\| \geq 0,$$
$$\sigma_n^{(2)} = \theta \in (0,1),$$
$$\sigma_n^{(3)} = \frac{\left(1 + \delta \beta_n^{(3)}\right)(1+\delta)\,\epsilon \left\| s_n^{(1)} - s_* \right\| + \delta \beta_n^{(3)} \varepsilon + \varepsilon}{\theta} \geq 0.$$

It can be easily seen that $\sigma_n^{(1)}$, $\sigma_n^{(2)}$ and $\sigma_n^{(3)}$ satisfy all the conditions of Lemma 2. On the other hand, we know by Theorem 3 that $\lim_{n\to\infty} \left\| s_n^{(1)} - s_* \right\| = 0$. Hence, keeping in mind the assumption $\lim_{n\to\infty} \beta_n^{(3)} = 0$, an application of Lemma 2 to (25) gives

$$\| s_* - \widetilde{s}_* \| \leq \frac{\varepsilon}{1-\delta}.$$

Example 2 Let $\mathcal{S} = [0, 1/2]$ be endowed with the usual norm. Let the operators $\mathcal{T}, \widetilde{\mathcal{T}} : \mathcal{S} \to \mathcal{S}$ be given by

$$\mathcal{T}x = \frac{1}{6}\sin^2 x + \frac{1}{8}\left(e^x - 1\right),$$

$$\widetilde{\mathcal{T}}x = \frac{1}{50} + \frac{x}{10} + \frac{x^2}{4} + \frac{x^3}{36} - \frac{3x^4}{58} + \frac{x^5}{1000} + \frac{27x^6}{3460} + \frac{x^7}{40200}.$$

Applying the mean value theorem, one can easily verify that $\mathcal{T}$ satisfies condition (2) with $\delta = \frac{1}{3} + \frac{\sqrt{e}}{8} \approx 0.539 \in [0,1)$ and for any $\epsilon \geq 0$ and it has a unique fixed point $s_* = 0$.

Now, by using Wolfram Mathematica 9 software package, we have

$$\max_{x\in[0,1/2]} \left| \mathcal{T} - \widetilde{\mathcal{T}} \right| = 0.02,$$

which implies

$$\left| \mathcal{T}x - \widetilde{\mathcal{T}}x \right| \leq \varepsilon \text{ for all } x \in [0, 1/2]$$

where $\varepsilon = 0.02$. Hence, $\widetilde{\mathcal{T}}$ is an approximate operator of $\mathcal{T}$ in the sense of Definition 3. On the other hand, the operator $\widetilde{\mathcal{T}}$ has a fixed point $\widetilde{s}_* = 0.022361$ and thus the distance between the fixed points x_* and $\widetilde{x}_*$ is 0.022361.

If we put $\beta_n^{(1)} = \frac{n+1}{n+2}$, $\beta_n^{(2)} = \frac{2n+1}{3n+100}$, $\beta_n^{(3)} = \frac{1}{n+2}$ for all $n \in \mathbb{N}$ and $\widetilde{\mathcal{T}}x = \frac{1}{50} + \frac{x}{10} + \frac{x^2}{4} + \frac{x^3}{36} - \frac{3x^4}{58} + \frac{x^5}{1000} + \frac{27x^6}{3460} + \frac{x^7}{40200}$ in (18), then we have

$$\begin{cases} \widetilde{s}_0^{(1)} \in \mathcal{S}, \\ \widetilde{s}_{n+1}^{(1)} = \frac{1}{n+2}\left(\frac{1}{50} + \frac{\widetilde{s}_n^{(2)}}{10} + \frac{\left(\widetilde{s}_n^{(2)}\right)^2}{4} + \frac{\left(\widetilde{s}_n^{(2)}\right)^3}{36} - \frac{3\left(\widetilde{s}_n^{(2)}\right)^4}{58} + \frac{\left(\widetilde{s}_n^{(2)}\right)^5}{1000} + \frac{27\left(\widetilde{s}_n^{(2)}\right)^6}{3460} + \frac{\left(\widetilde{s}_n^{(2)}\right)^7}{40200}\right) \\ \quad + \frac{n+1}{n+2}\left(\frac{1}{50} + \frac{\widetilde{s}_n^{(2)}}{10} + \frac{\left(\widetilde{s}_n^{(3)}\right)^2}{4} + \frac{\left(\widetilde{s}_n^{(3)}\right)^3}{36} - \frac{3\left(\widetilde{s}_n^{(3)}\right)^4}{58} + \frac{\left(\widetilde{s}_n^{(3)}\right)^5}{1000} + \frac{27\left(\widetilde{s}_n^{(3)}\right)^6}{3460} + \frac{\left(\widetilde{s}_n^{(3)}\right)^7}{40200}\right) \\ \widetilde{s}_n^{(2)} = \frac{n+99}{3n+100}\widetilde{s}_n^{(1)} + \frac{2n+1}{3n+100}\widetilde{s}_n^{(3)} \\ \widetilde{s}_n^{(3)} = \frac{n+1}{n+2}\widetilde{s}_n^{(1)} \\ + \frac{1}{n+2}\left(\frac{1}{50} + \frac{\widetilde{s}_n^{(1)}}{10} + \frac{\left(\widetilde{s}_n^{(1)}\right)^2}{4} + \frac{\left(\widetilde{s}_n^{(1)}\right)^3}{36} - \frac{3\left(\widetilde{s}_n^{(1)}\right)^4}{58} + \frac{\left(\widetilde{s}_n^{(1)}\right)^5}{1000} + \frac{27\left(\widetilde{s}_n^{(1)}\right)^6}{3460} + \frac{\left(\widetilde{s}_n^{(1)}\right)^7}{40200}\right), n \in \mathbb{N}. \end{cases} \tag{26}$$

Table 2 shows that iterative method (26) converges to the fixed point $\widetilde{s}_* = 0.022361$. As a matter of fact, without knowing the fixed point of $\widetilde{\mathcal{T}}$ and

Table 2: Convergence behaviour of iterative method (26).

Iter. No.	Iterative Method (26)
0	0.500000
1	0.104650
2	0.031991
3	0.023386
4	0.022470
5	0.022372
6	0.022362
7	0.022361
⋮	⋮

without computing it, we can find the following upper bound for the error in approximating $\widetilde{s}_*$ by s_* from the conclusion of Theorem 7:

$$0.022361 = |s_* - \widetilde{s}_*| \leq \frac{0.02}{1 - 0.539} = 0.043384.$$

4 An Application of The Normal-S Iterative Method to Stable Inversion

The final validation of the structural integrity of complex mechanical engineering structures against, for example, metal fatigue, is usually done experimentally in the laboratory through structural integrity testing (SIT). In SIT a test specimen is repeatedly subjected to a load history that is typically representative of normal usage loads. In control terms the combined system (of specimen, actuators, sensors and power supply) is subjected to repeated output tracking of the same desired output trajectory. The repetitive nature of the control process allows the incorporation of a learning mechanism in the process through a control procedure called *iterative learning control* (ILC). While ILC in its simplest form can be a purely data driven control procedure (thus not utilizing a system model in any way), its application in SIT has almost always been inverse model-based. In fact, the use of an inverse model based approach allows the most favourable transient convergence properties of the achieved output trajectory. While the models originally (and in many cases still) deployed in ILC in SIT were frequency domain transfer functions, the prominence of time domain model identification and analysis in the latter part of the twentieth century has seen the implementation of parametric time domain models also in model based ILC. In particular time domain models may be relatively easily extended to nonlinear form both in their identification and simulation. Whereas ILC in the SIT setting is usually done with linear inverse models, Eksteen and Heyns (2016b) have recently performed ILC using the inversion of nonlinear parametric models with polynomial basis functions. The inversion of nonlinear parametric (time domain) models was not solved until the last decade of the twentieth century, when it was discovered how to solve the instability of the inverse model that results in case of nonminimum phase systems (i.e., systems with unstable zeros), which often result in case of sample data systems (especially when employing a large sample frequency) or due to system delays. The process of obtaining the stable inverse of a nonminimum phase system is referred to as *stable inversion.*

Consider the square, nonlinear discrete-time system

$$x(k+1) = f\left(\left(x(k), u(k)\right)\right) \tag{27}$$

$$y(k) = h\left(x(k)\right) , \tag{28}$$

where $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^m$, f and h analytic in their domains, and k represents discrete time steps. Let $(x^\circ, u^\circ) = ([0],[0])$ be an equilibrium pair of the system. This is the description of a multi-input multi-output (MIMO) dynamic system in the state variable formulation, where (27) is called the state equation, and (28) is called the output equation. The system is referred to as square because it has the same number of inputs as outputs (namely m).

Assuming the system has *vector relative degree* it can be inverted (see Eksteen and Heyns 2016a) and the references therein for details) to obtain the (smooth) inverse system as:

$$\eta(k+1) = \overline{f}\left(\eta(k), \Xi(k)\right) \tag{29}$$

$$u(k) = \overline{h}\left(\eta(k), \Xi(k)\right) , \tag{30}$$

where $\eta, \Xi \in \mathbb{R}^{\hat{n}}$. Ξ may be determined from y.

Solving this inverse system for the bounded solution η (and thus u) is referred to as the stable inversion of (27) and (28). The stable inversion of nonlinear discrete-time systems (such as (27) and (28)) was first developed by

Zeng and Hunt (2000) as a fixed point problem through the Picard iterative method. Eksteen and Heyns (2016a), however, have implemented the Mann and Ishikawa iterative methods for solution of the stable inverse, aimed particularly at cases when Picard iterations fail to converge. In this section the use of another fixed-point iterative method that is independent of Mann and Ishikawa iterations, namely the normal-S iterative method, is used for the solution of a bounded inverse.

We first briefly show how the bounded solution of the inverse system is formulated. Define

$$\mathcal{U}\left(\eta\left(k\right),\varXi\left(k\right)\right):=\overline{f}\left(\eta\left(k\right),\varXi\left(k\right)\right)-A\eta\left(k\right)$$

where

$$A:=\left.\frac{\partial\overline{f}\left(\eta\left(k\right),\varXi\left(k\right)\right)}{\partial\eta\left(k\right)}\right|_{([0],[0])}.$$

The state dynamics, (29), may be restated as

$$\eta\left(k+1\right)=A\eta\left(k\right)+\mathcal{U}\left(\eta\left(k\right),\varXi\left(k\right)\right), \tag{31}$$

which is structured as a linear system. The linear structure of (31) is used to construct a bounded solution for the system. Assuming that $y_d(k)$, and thus $\varXi\left(k\right)$, $k\in\mathbb{N}$, is bounded, and that $\overline{f}\left(\eta\left(k\right),\varXi\left(k\right)\right)$ is bounded if $\eta\left(k\right)$ and $\varXi\left(k\right)$ are bounded, it can be shown that the bounded solution of (31) (and thus (29)) is equivalent to the bounded solution $\eta\left(k\right)$, $k\in\mathbb{N}$, of

$$\eta\left(k\right)=\sum_{i=-\infty}^{\infty}\phi\left(k-i\right)\mathcal{U}\left(\eta\left(i-1\right),\varXi\left(i-1\right)\right), \tag{32}$$

where $\phi(k)$ is an $\hat{n}\times\hat{n}$ matrix (for details see the work of Eksteen and Heyns 2016a).

Define a non-causal linear operator G to represent the system in (31) as

$$\eta=G\mathcal{U}\left(\eta,\varXi\right)\ . \tag{33}$$

Finding the bounded solution, $\eta\left(k\right)$, for (31), by using the bounded solution of $\phi\left(k\right)$ in (32) (equivalently (33)) lies at the core of stable inversion. This solution is obtained iteratively by recasting (32) as a fixed point problem. Iterative methods that may be used for this purpose include the Picard, Mann and Ishikawa iterative methods. For example, in case of the Picard iterative method, we obtain the sequence $(\eta_m : m\geq 0)$, $\eta_0=[0]$, with

$$\eta_{m+1}=\mathcal{T}\eta_m\ =G\mathcal{U}\left(\eta_m,\varXi\right). \tag{34}$$

To adapt (33) for the normal$-S$ iterative method, assuming $\alpha_m=\alpha$ for all m, gives the sequence $(\eta_m : m\geq 0)$, $\eta_0=[0]$, as:

$$\eta_{m+1}=G\mathcal{U}\left(\mu_m,\varXi\right) \tag{35}$$

$$\mu_m=\left(1-\alpha\right)\eta_m+\alpha G\mathcal{U}\left(\eta_m,\varXi\right)\ . \tag{36}$$

The following definitions will be helpful for the subsequent analysis. As in (Zeng and Hunt 2000), let $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the l^1 and l^∞ norm on $\mathbb{N}$ respectively. Also, let

$$\|\eta\|_\infty = \max_i \|\eta_i\|_\infty \ ,$$

where i denotes i-th component of the state vector.

Now $\mathcal{U}(\eta(k), \Xi(k))$ is uniformly Lipschitz in a closed s neighbourhood of $([0],[0])$ in (η, Ξ) space with positive real constants (K_1, K_2) if there exists an $s > 0$ such that for all $\eta_1(k), \eta_2(k), \Xi_1(k)$ and $\Xi_2(k)$, having $\|\cdot\|_\infty$ norms $\leq s$, so we have

$$\begin{gathered}\|\mathcal{U}(\eta_1(k), \Xi_1(k)) - \mathcal{U}(\eta_2(k), \Xi_2(k))\|_\infty \leq \\ K_1 \|\eta_1(k) - \eta_2(k)\|_\infty + K_2 \|\Xi_1(k) - \Xi_2(k)\|_\infty\end{gathered}$$

$\forall k \in \mathbb{N}$.

Furthermore, define

$$\|\phi\|_1 = \widehat{n} \max_{i,j} \|\phi_{i,j}(k)\|_1$$

where $\phi_{i,j}(k)$ is the (i,j) $-$th element of $\phi(k)$.

Finally, we note that

$$\begin{aligned}\|G\mathcal{U}(\eta, \Xi)\|_\infty &= \left\| \sum_{i=-\infty}^{\infty} \phi(k-i)\, \mathcal{U}(\eta(i-1), \Xi(i-1)) \right\|_\infty \\ &\leq \|\phi\|_1 \|\mathcal{U}(\eta, \Xi)\|_\infty \ .\end{aligned}$$

Define

$$\begin{aligned}H_m &= \eta_{m+1} - \eta_m \\ &= G\mathcal{U}(\mu_m, \Xi) - G\mathcal{U}(\mu_{m-1}, \Xi) \\ &= G[\mathcal{U}(\mu_m, \Xi) - \mathcal{U}(\mu_{m-1}, \Xi)] \ .\end{aligned}$$

Taking the norm of H_m, we have

$$\begin{aligned}\|H_m\|_\infty &= \|G[\mathcal{U}(\mu_m, \Xi) - \mathcal{U}(\mu_{m-1}, \Xi)]\|_\infty \\ &\leq \|\phi\|_1 \|\mathcal{U}(\mu_m, \Xi) - \mathcal{U}(\mu_{m-1}, \Xi)\|_\infty \\ &\leq \|\phi\|_1 K_1 \|\mu_m - \mu_{m-1}\|_\infty \ .\end{aligned} \tag{37}$$

By (36), we have

$$\begin{aligned}\mu_m - \mu_{m-1} &= (1-\alpha)\,\eta_m + \alpha G\mathcal{U}(\eta_m, \Xi) - (1-\alpha)\,\eta_{m-1} - \alpha G\mathcal{U}(\eta_{m-1}, \Xi) \\ &= (1-\alpha)(\eta_m - \eta_{m-1}) + \alpha G[\mathcal{U}(\eta_m, \Xi) - \mathcal{U}(\eta_{m-1}, \Xi)] \ .\end{aligned} \tag{38}$$

An application of the process in (37) and (38) leads to

$$\begin{aligned}
\|\mu_m-\mu_{m-1}\|_\infty &= \|(1-\alpha)(\eta_m-\eta_{m-1})+\alpha G[\mathcal{U}(\eta_m,\Xi)-\mathcal{U}(\eta_{m-1},\Xi)]\|_\infty \\
&\le (1-\alpha)\|\eta_m-\eta_{m-1}\|_\infty+\alpha\|G[\mathcal{U}(\eta_m,\Xi)-\mathcal{U}(\eta_{m-1},\Xi)]\|_\infty \\
&\le (1-\alpha)\|\eta_m-\eta_{m-1}\|_\infty+\alpha\|\phi\|_1\|\mathcal{U}(\eta_m,\Xi)-\mathcal{U}(\eta_{m-1},\Xi)\|_\infty \\
&\le (1-\alpha)\|\eta_m-\eta_{m-1}\|_\infty+\alpha\|\phi\|_1 K_1\|\eta_m-\eta_{m-1}\|_\infty \\
&= [1-\alpha+\alpha\|\phi\|_1 K_1]\|\eta_m-\eta_{m-1}\|_\infty \ . \qquad (39)
\end{aligned}$$

Substituting (39) into (37), we obtain

$$\begin{aligned}
\|H_m\|_\infty &\le \|\phi\|_1 K_1[1-\alpha(1-\|\phi\|_1 K_1)]\|\eta_m-\eta_{m-1}\|_\infty \\
&= \|\phi\|_1 K_1[1-\alpha(1-\|\phi\|_1 K_1)]\|H_{m-1}\|_\infty \ , \qquad (40)
\end{aligned}$$

where $\|H_{m-1}\|_\infty=\|\eta_m-\eta_{m-1}\|_\infty$.

Theorem 8 *A solution $\eta(k)\in l^\infty$, $k\in\mathbb{N}$, of (29) exists and is obtained by the normal-S iterative method of (33) if*

1. $\overline{f}$ is uniformly Lipschitz in an s neighbourhood of $([0],[0])$ with Lipschitz constants (K_1,K_2),

2. $\|\phi\|_1 K_1<1$ (which implies $\|\phi\|_1 K_1[1-\alpha(1-\|\phi\|_1 K_1)]<1$ as $\alpha\in[0,1]$),

3. $\Xi(k)\in l^\infty$ with $\|\Xi\|_\infty\le s$, and

4. $\frac{\|\phi\|_1 K_2\|\Xi\|_\infty}{1-\|\phi\|_1 K_1}\le s$.

Proof Clearly, $\eta_0\in l^\infty$ and $\|\eta_0\|_\infty\le s$. Assume that $\eta_m\in l^\infty$ and $\|\eta_m\|_\infty\le s$. Then, using conditions 1 and 4, it follows from (35) that

$$\begin{aligned}
\|\eta_{m+1}(k)\|_\infty &= \|G\mathcal{U}(\mu_m,\Xi)\|_\infty \\
&\le \|\phi\|_1\|\mathcal{U}(\mu_m,\Xi)\|_\infty \\
&\le \|\phi\|_1\{K_1\|\mu_m\|_\infty+K_2\|\Xi\|_\infty\} \\
&= \|\phi\|_1\{K_1\|(1-\alpha)\eta_m+\alpha G\mathcal{U}(\eta_m,\Xi)\|_\infty+K_2\|\Xi\|_\infty\} \\
&\le \|\phi\|_1\{K_1\{(1-\alpha)\|\eta_m\|_\infty+\alpha\|G\mathcal{U}(\eta_m,\Xi)\|_\infty\}+K_2\|\Xi\|_\infty\} \\
&\le \|\phi\|_1\{K_1\{(1-\alpha)\|\eta_m\|_\infty+\alpha\|\phi\|_1\|\mathcal{U}(\eta_m,\Xi)\|_\infty\}+K_2\|\Xi\|_\infty\} \\
&\le \|\phi\|_1\{K_1\{(1-\alpha)\|\eta_m\|_\infty+\alpha\|\phi\|_1\{K_1\|\eta_m\|_\infty+K_2\|\Xi\|_\infty\}\} \\
&\quad +K_2\|\Xi\|_\infty\} \\
&\le \|\phi\|_1\{K_1\{(1-\alpha)s+\alpha\|\phi\|_1\{K_1 s+K_2\|\Xi\|_\infty\}\}+K_2\|\Xi\|_\infty\} \\
&\le \|\phi\|_1\{K_1\{(1-\alpha)s+\alpha s\}+K_2\|\Xi\|_\infty\} \\
&= \|\phi\|_1\{K_1 s+K_2\|\Xi\|_\infty\}\le s \ .
\end{aligned}$$

Thus, by induction $\eta_m\in l^\infty$ and $\|\eta_m\|_\infty\le s$ for all m. Furthermore, from condition 2 and (40), it follows that $\|H_m\|_\infty<\|H_{m-1}\|_\infty$, and so by the ratio test, the series $\sum_{m=0}^{\infty}\|H_m(k)\|_\infty$ is convergent. Hence

$$\eta_m(k)=\sum_{j=1}^{m-1}H_j(k) \ , \qquad (41)$$

is a Cauchy sequence in l^∞, and so converges to an element of l^∞. Denote this limit element by $\eta\,(k)$., $k \in \mathbb{Z}$. Thus, $\eta(k)$ is the fixed point of Normal-S iterative method described by (35) and (36), and therefore also of the Picard iterative method (34). Hence, $\eta(k)$ satisfies (33) and thus represents the bounded solution of the inverse system.

Example 3 In this example (from Eksteen and Heyns 2016a) the use of Picard, and Mann iteration is supplemented with Normal-S iterative method for the stable inversion of a dynamic system for a short-duration random signal. Consider the following polynomial NARX (nonlinear autoregressive with exogenous input) system:

$$\begin{aligned} y(k) = {} & \theta_1 u(k-4) + \theta_2 u(k-5) + \theta_3 u(k-6) + \theta_4 y(k-4) \\ & + \theta_5 u(k-5) y(k-4) + \theta_6 u(k-5) u(k-6) y(k-2) \\ & + \theta_7 u(k-5)^2 u(k-6) y(k-1) \ , \end{aligned} \tag{42}$$

where

$$(\theta_1, \ldots, \theta_7) = (0.150,\ 0.50,\ 0.50,\ 1/6,\ -2.0,\ 6.0,\ 11.0) \ .$$

This type of nonlinear system typically results when nonlinear system identification methods (Chen and Billings 1989a, b) are applied to physical systems, e.g. structural integrity test systems of mechanical structures in the laboratory fitted with actuators.

We can convert (42) to the state variable formulation (Eqs. (27) and (28)) followed by inversion to obtain the inverse system as in Example 2 in Eksteen and Heyns (2016a). For demonstration purposes a random reference (desired) input trajectory, $u_d(k)$, is constructed having a bandwidth of approximately 50 Hz (assuming a sample frequency of 250 Hz). A corresponding reference response (output) trajectory $y_d(k)$ is then obtained from (42) for the given $u_d(k)$ (cf. Fig. 2).

In this example we demonstrate the stable inversion of the system (42) for the given $y_d(k)$. This involves finding the bounded solution of the inverse system for the given reference response $y_d(k)$. The bounded solution is searched through the Picard, Mann and Normal-S iterative methods, ideally converging to the desired input $u_d(k)$ (which is easily obtained from $\eta(k)$). In case of divergence an approximation of $u_d(k)$ may be obtained as the best result prior to divergence. Let the input calculated during iteration m of stable inversion be represented by $u_m(k)$. The error between $u_m(k)$ and $u_d(k)$ is defined here as the following distance measure:

$$\mathrm{err}_1(u_m) = d_1(u_m, u_d) := 100 \|u_m - u_d\|_1 / \|u_d\|_1 \ .$$

Similarly,

$$\mathrm{err}_1(y_m) = d_1(y_m, y_d) \ .$$

It is found that stable inversion using the Picard, Mann and Normal-S iterative methods is divergent. This is referred to here as case 1. (See Table 3 for the best results of case 1.) The iteration gains (i.e., step size) of Mann and Normal-S iteration were chosen so as to result in the most accurate calculated input, and gave much more accurate results prior to divergence than Picard iteration. The accuracy of Normal-S iteration is intermediate between Mann and Picard iteration, and represents a considerable improvement over the latter. The convergence behaviour for the calculated input is shown in Fig. 3 for case 1.

In order to improve the results of case 1, we employ a data dependence approach, where we slightly modify the operator used in the fixed point iterative methods. To this end we apply a zero phase low pass filter (designated F in operator form) with a cut frequency of 50 Hz to the operator $G\mathcal{U}$ in (33), resulting in a new formula for the stable inversion solution as

$$\eta = FG\mathcal{U}(\eta, \Xi) \; . \tag{43}$$

We can adapt (43) for various iterative methods in a similar way as is done for (33), for example it results in the Picard iterative method as:

$$\eta_{m+1} = FG\mathcal{U}(\eta_m, \Xi) \tag{44}$$

The extension to Mann, Ishikawa and Normal-S iterative methods is obvious. The decision to use a low pass filter was based on observations that the divergence observed in case 1 was mainly due to high frequency oscillations. In view of the association of F with G, it follows that $FG\mathcal{U}$ impacts the convergence conditions in Theorem 8 in the same way as $G\mathcal{U}$ does.

After performing stable inversion with the filter in place (referred to here as case 2) it is found that Picard iteration still diverges. However, the Mann and Normal-S iterative methods (the latter using a constant step size) now converge. The convergence behaviour for the calculated input is shown in Fig. 4 for case 2. The convergence of the Mann and Normal-S iterative methods is not to zero error with respect to the correct solution of the $G\mathcal{U}$ operator due to the modification of the operator by the filter. Together with case 1, this example shows that Normal-S iterative method not only improves on Picard iteration, but can be on par with Mann iterative method from the point of view of securing convergence when Picard iteration diverges, provided an appropriate choice of step size is made.

Table 3: Example 3: Results of stable inversion. M is the iteration resulting in $\min_m \text{err}_1(u_m)$, i.e., $M = \text{argmin}_m \text{err}_1(u_m)$. Here it is assumed $m \geq 1$.

Case	Iteration type	α_n	β_n	$\min_m \text{err}_1(u_m)$ [%]	$\text{err}_1(y_M)$ [%]	M = Iter. no.	Comment
1	Picard	1.0	0	36.4	196.0	1	No filter
1	Mann	$\frac{1.2}{0.8m+1}$	0	8.7	5.7	4	No filter
1	Normal-S	1	$\frac{0.8}{0.6m+0.9}$	13.1	20.1	1	No filter
2	Picard	1.0	0	36.0	199.2	1	50 Hz LPF
2	Mann	$\frac{1}{0.4m+1}$	0	0.5	0.2	100+	50 Hz LPF
2	Normal-S	1	0.3	0.5	0.2	22	50 Hz LPF

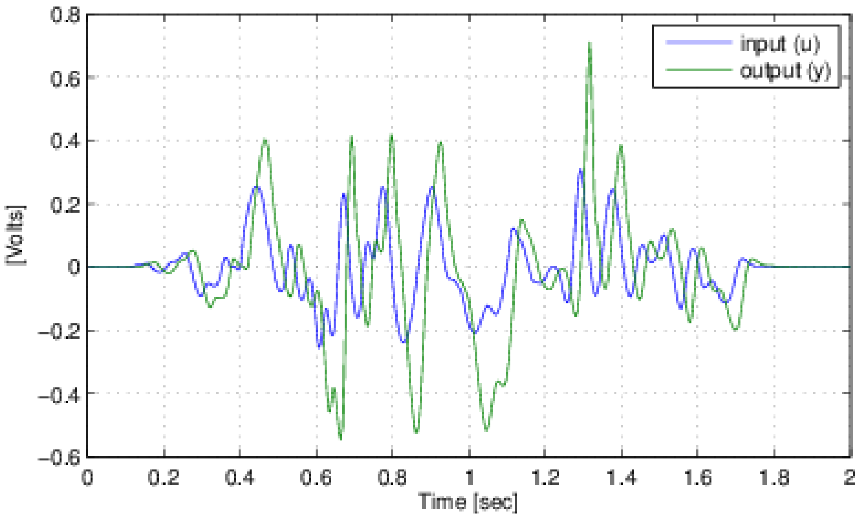

Fig. 2: Example 3: $u_d(t)$ and $y_d(t)$.

5 Conclusions and Possible Future Works

1. All the results obtained in this paper complement the corresponding results of Sintunavarat and Pitea (2016).

2. Theorems 3-5 show that Normal-S and SP iterative methods do exactly the same job of approximating fixed point of almost contraction operators. However, Theorem 6 shows that Normal-S iterative method is faster than the SP iterative method. So, we conclude that the Normal-S iterative method is better than the SP iterative method in approximating the fixed point and thus, in the presence of simpler and faster Normal-S iterative method, one does not need to go to complex SP iterative method.

3. Our Theorems 3 and 4 are improvements of Theorem 1 in the sense that we do not impose conditions on the parametric sequences.

4. Theorem 6 has been proved under much weaker conditions on the parametric sequences and so improves Theorem 2 substantially.

5. Theorem 7 provides an affirmative answer to an open question raised by Sintunavarat and Pitea (2016) about stability of SP iterative method.

6. In 2014, Sakurai and Iiduka (2014) pointed out that, to guarantee stability and reliability of practical systems and networks (see, for example, Iiduka 2012, 2013), the fixed point has to be found as quickly as possible. So, there are increasing interests in the study of fast algorithms for approximating fixed points of nonexpansive mappings (i.e., a mapping $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ satisfy-

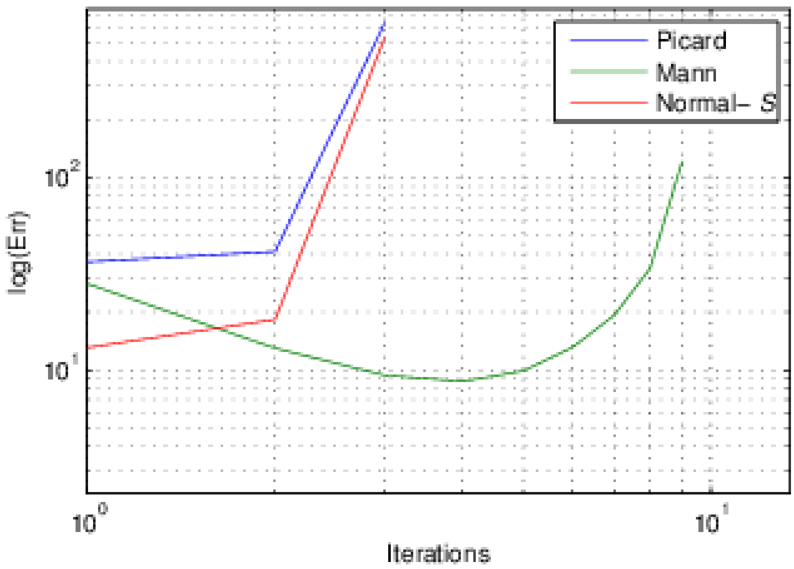

Fig. 3: Example 3, case 1: Convergence behaviour of calculated inputs for the Picard, Mann and Normal-S iterative methods.

ing $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$ for all $x, y \in \mathcal{S}$). Hence, it will be an interesting task to find which one of the iterative methods (Mann, Ishikawa, Xu-Noor, SP, S and Normal-S) for the class of nonexpansive mappings will converge faster than the others.

7. Normal-S iterative method for stable inversion of discrete time nonlinear systems has provided much better results than Picard iterative method.

8. The implementation of advanced ILC methods in SIT along the lines of Eksteen and Heyns (2016b) results in the need for the stable inversion of smooth discrete-time nonminimum-phase nonlinear models. However, the inversion can potentially also be accomplished through a nested ILC or Gauss-Newton iteration phase that happens off-line and is done on the system model rather than the physical system. This off-line iteration on the system model is done during every iteration of the outer ILC loop on the physical system. The purpose of the nested iteration loop is purely to invert the system model as an alternative to stable inversion, and is along the lines suggested by, for example, Markusson et al. (2001). With the system model not needing to be stably inverted, this could allow a wider array of models to be used in system identification to potentially represent the physical system more accurately than smooth polynomial-based models. For example, system identification may be

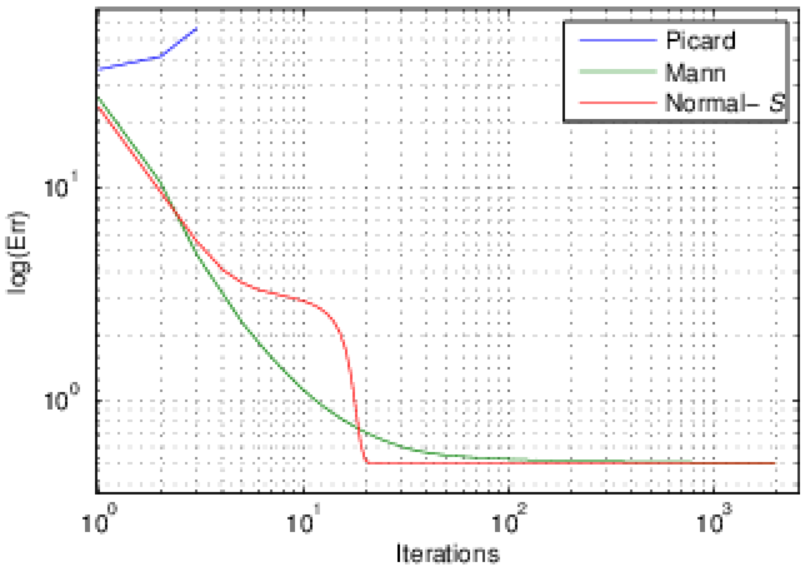

Fig. 4: Example 3, case 2 (low pass filter): Convergence behaviour of calculated inputs for the Picard, Mann and Normal-S iterative methods.

done of a wide range of artificial neural network models or piecewise affine (PWA) models. PWA models are ideal for describing hybrid systems, switched systems and nonlinear systems, and have been receiving a lot of attention from the point of view of their identification and control (Elhamifar et al. 2014; Qiu et al. 2017), stability analysis, and their utility in networked control systems. Future research will focus on the use of ILC as a model inversion method to invert neural network and PWA models, and the possible direct inversion of PWA models through stable inversion of the linear models on the subspaces.

9. The ability of stable inversion to successfully invert nonlinear models that are otherwise unstable in the inverse direction (due to being nonminimum phase) means the method may also find application in the network control system environment. See, for example, (Pang et al. 2014) for an interesting presentation of networked control systems. While to the best of the authors' knowledge stable inversion has not yet been implemented in the network control system environment, an implementation is conceivable where at discrete time k an observer on the sensor side provides a state estimate for time $k+1$, $\hat{x}(k+1|k)$, that is network-sent to the controller. The state estimate serves as an initial value in a control generator that is stable inversion based and employs preview of the desired output trajectory for a time, τ, that may

be informed by both the round trip time and the minimum requirements on the time horizon for stable inversion to be sufficiently accurate. The control trajectory is network-sent to the plant which stores the trajectory and employs it in a way that accommodates any possible subsequent packet losses. Note, however, that the calculation intensity of stable inversion is extensive when compared to, for example, a controller using just a gain matrix, because it is both iterative and for every iteration operates on a trajectory rather than an instantaneous value. As a result its implementation may be inhibited in case of high frequency control systems.

10. Investigation of a general systems based on uncertain dynamic model is another interesting research direction (Chen et al. 2017a,b,c).

Acknowledgements The authors would like to thank the anonymous reviewers for their constructive comments to improve quality and presentation of the paper.

Conflict of Interest: The authors declares that there is no conflict of interest regarding the publication of this article.
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Agarwal RP, O'Regan D, Sahu DR (2007) Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J Nonlinear Convex Anal 8:61–79.
2. Berinde V (2004a) Approximating fixed points of weak contractions using the Picard iterations. Nonlinear Analysis Forum 9(1):43–53.
3. Berinde V (2004b) Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. Fixed Point Theory Appl 2:97–105.
4. Berinde V (2005) A convergence theorem for some mean value fixed point iteration procedures. Demonstratio Math 38:177–184.
5. Berinde V (2007) Iterative approximation of fixed points. 2nd edn. Springer, Berlin.
6. Berinde V (2016) On a notion of rapidity of convergence used in the study of fixed point iterative methods. Creat Math Inform 25:29–40.
7. Chen L, Peng J, Zhang B, Rosyida I (2017a) Diversified models for portfolio selection based on uncertain semivariance. Internat J Systems Sci 48(3): 637–648.
8. Chen L, Peng J, Liu Z, Zhao R (2017b) Pricing and effort decisions for a supply chain with uncertain information. Int J Prod Res 55(1):264–284.
9. Chen L, Peng J, Zhang B (2017c) Uncertain goal programming models for bicriteria solid transportation problem. Appl Soft Comput 51:49–59.
10. Chen S, Billings SA (1989a) Representations of nonlinear systems: the NARMAX model. Internat J Control 49:1013–1032.
11. Chen S, Billings SA (1989b) Orthogonal least squares methods and their application to nonlinear system identification. Internat J Control 50(5):1873–1896.
12. Eksteen JJA, Heyns PS (2016a) Improvements in stable inversion of NARX models by using Mann Iteration. Inverse Probl Sci Eng 24(4):667–691.
13. Eksteen JJA, Heyns PS (2016b) An alternative update formula for non-linear model-based iterative learning control. Inverse Probl Sci Eng 24(5):860–888.
14. Elhamifar E, Burden SA, Sastry SS (2014) Adaptive piecewise-affine inverse modeling of hybrid dynamical systems. IFAC Proc 47(3):10844–10849.
15. Gürsoy F (2014) Applications of normal S-iterative method to a nonlinear integral equation. The Scientific World Journal 2014:943127, 5 pages.

16. Gürsoy F, Karakaya V, Rhoades BE (2013) Data dependence results of new multi-step and S-iterative schemes for contractive-like operators. Fixed Point Theory Appl 2013(1):1–12.
17. Iiduka H (2012) Iterative algorithm for triple-hierarchical constrained nonconvex optimization problem and its application to network bandwidth allocation. SIAM J Optim 22:862–878.
18. Iiduka H (2013) Fixed point optimization algorithms for distributed optimization in networked systems. SIAM J Optim 23, 1–26, (2013).
19. Ishikawa S (1974) Fixed points by a new iteration method. Proc Amer Math Soc 44(1):147–150.
20. Karakaya V, Gürsoy F, Ertürk M (2016) Some convergence and data dependence results for various fixed point iterative methods. Kuwait J Sci 43(1):112–128.
21. Khan AR, Gürsoy F, Karakaya V (2016) Jungck-Khan iterative scheme and higher convergence rate. Int J Comput Math 93(12):2092–2105.
22. Khan AR, Gürsoy F, Kumar V (2016) Stability and data dependence results for Jungck-Khan iterative scheme. Turkish J Math 40(3):631–640.
23. Khan AR, Kumar V, Hussain N (2014) Analytical and numerical treatment of Jungck-type iterative schemes. Appl Math Comput 231:521–535.
24. Kirk WA (2004) On successive approximations for nonexpansive mappings in Banach spaces. Glasgow Math J 12:6–9.
25. Knopp K (1956) Infinite sequences and series. Dover Publications, Inc., New York, pp 279.
26. Mann WR (1953) Mean value methods in iterations. Proc Amer Math Soc 4(3):506–510.
27. Markusson O, Hjalmarsson H, Norrlof M (2001) Iterative learning control of nonlinear non-minimum phase systems and its application to system and model inversion. Proc. 40th IEEE Conf. Decision Control 5:4481-4482.
28. Osilike MO (1995/96) Stability results for fixed point iteration procedures. J Nigerian Math Soc 14/15:17–29.
29. Ozturk Celiker F (2014) Convergence analysis for a modified SP iterative method. The Scientific World Journal 2014:840504, 5 pages.
30. Pang ZH, Liu GP, Zhou D, Chen M (2014) Output tracking control for networked systems: a model-based prediction approach. IEEE Trans Ind Elec 61(9):4867–4877.
31. Picard E (1890) Mémoire sur la théorie des é quations aux dérivées partielles et la méthode des approximations successives. J Math Pure Appl 6:145–210.
32. Qiu J, Wei Y, Wu L (2017) A novel approach to reliable control of piecewise affine systems with actuator faults. IEEE Trans Circuits Syst II. Express Briefs 64(8):957–961.
33. Sahu DR (2011) Applications of S iteration process to constrained minimization problems and split feasibility problems. Fixed Point Theory 12:187–204.
34. Sakurai K, Iiduka H (2014) Acceleration of the Halpern algorithm to search for a fixed point of a nonexpansive mapping. Fixed Point Theory Appl 2014:202.
35. Sintunavarat W, Pitea A (2016) On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis. J Nonlinear Sci Appl 9:2553–2562.
36. Şoltuz SM, Grosan T (2008) Data dependence for Ishikawa iteration when dealing with contractive like operators. Fixed Point Theory Appl 2008:1–7.
37. Xu B, Noor MA (2002) Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces. J Math Anal Appl 267(2): 444–453.
38. Zamfirescu T (1972) Fix point theorems in metric spaces. Arch Math (Basel) 23:292–298.
39. Zeng G, Hunt LR (2000) Stable inversion for nonlinear discrete-time systems. IEEE Trans Automat Control 45(6):1216–1220.