

Dysfunctional Functions: The Case of Zambian Mathematics Education Students

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ABSTRACT

This study investigated student mathematics teachers' ability to recognise and explain their understanding of school level functions. We challenged the assumption that studying advanced mathematics automatically develops an understanding of school mathematics that is sufficient to explain concepts and justify reasoning. This case study tested this assumption by exploring the depth of pre-service mathematics student teachers' understanding of school function concepts at the University of Zambia. The test items required calculation, as well as justification of the answers, and an explanation of the concepts. Of the 22 participants, all final year mathematics education students, 18 student teachers scored below the 50% pass mark. The average mark was 8 out of a possible 28 (27%). The majority of the participants found it difficult to explain and justify their reasoning. This study resulted in the development of a new school mathematics module for prospective mathematics teachers at the University of Zambia.

Keywords: mathematics education students, secondary school functions, mathematical knowledge for teaching

INTRODUCTION

The development of student teachers lies at the very core of the pursuit of quality in mathematics education. In fact, their training usually includes high level mathematics. However, there are researchers who, over the last thirty years, have become convinced that the notion that a teacher who has studied advanced mathematics automatically understands secondary school mathematics is a faulty one (Akkoc, 2008; Ball, 1990; Even, 1990, 1993; Fi, 2003, 2006; Wilburne & Long, 2010; Wilson, 1994; Wood, 1993). While the study of advanced mathematics with future engineers and mathematicians can create a foundation for understanding school mathematics, this is not a guarantee that the students will *know* and understand the school mathematics they have to teach (Bryan, 1999; Cooney, 2003; Hiebert, 2013). It is argued that secondary school mathematics has its 'own life', which is different from mathematics as it is taught in universities (Bromme, 1994). Yet this notion persists in southern Africa, where the training of pre-service mathematics teachers often does not include training to teach the fundamentals (Henning, 2014). This is because it is thought that such principles should form part of the 'cognitive wallpaper' in the mind of these pre-service mathematics teachers, who themselves did well in mathematics at school. This notion is exemplified at the University of Zambia (UNZA) (Malambo, 2015), where mathematics courses are offered by the School of Natural Sciences, which is not focused on teacher preparation, while methodology courses are taught in the School of Education. Thus, mathematics courses studied by student teachers are the same ones studied by future mathematicians.

How then would these fledgling teachers reacquaint themselves with the school mathematics that they would be required to teach? The mathematics textbooks used in schools provide both structure and content, theoretically making it relatively easy for the freshly university-trained mathematics teachers to refresh the understanding and teach the basic principles that did not form part of tertiary training. There is little evidence to support this theory. Kambilombilo and Sakala (2015) found that the ability and competence of nascent teachers to answer questions that

Contribution of this paper to the literature

- The research provides evidence that in the African context, the assumption that studying advanced mathematics automatically develops an understanding of school mathematics that is sufficient to explain concepts and justify reasoning, is refutable.
- This study shows that final year mathematics student teachers may hold misconceptions regarding key aspects of functions that they will have to teach, such as the difference between functions and non-functions and inverses of functions.
- This paper provides evidence that final year mathematics student teachers may possess procedural knowledge while lacking conceptual knowledge of the mathematics required to teach.

reach even slightly beyond the scope of the textbook are lacking. Additionally, textbooks are in short supply in Zambia, to the extent that the government mandate of no more than two students per textbook is not achieved (Lee & Zuilowski, 2015).

This study investigates mathematics student teachers' understanding of key school function concepts after completing advanced mathematics courses. This was done with a view to informing teacher educators about the aspects of functions that need to be specifically revisited in tertiary training. Ball (1990) states that, "Examining a specific topic also makes more vivid the contrast between some key characteristics of what pre-service teachers have learned as students and what they need to know as teachers" (p. 451). Functions are central to many topics in mathematics (Dubinsky & Wilson, 2013; Nyikahadzoyi, 2013) and are fundamental to the study of further mathematics and related subjects such as Physics (Even, 1998; Watson & Harel, 2013). However, research has shown that students do not find functions easy (Bloch, 2003; Spyrou & Zagorianakos, 2010), and while they may understand procedures and structures, conceptual insight is scarce.

CONCEPTUAL FRAMEWORK

In our investigation into UNZA student teachers' understanding of secondary school functions at the end of their training, it was important to access 'ability to explain and justify reasoning' and how they would 'un-pack' concepts. For guidance, the Framework of Mathematical Knowledge for Teaching was used, as developed by Ball, Thames and Phelps (2008), who divided subject matter knowledge into three domains: Common Content Knowledge, Specialised Content Knowledge (SCK), and Horizon Content Knowledge. This study focuses on SCK as "the unique mathematical content knowledge needed for teaching mathematics with understanding" (Bair & Rich, 2011, p. 295). Bair and Rich (2011) identified four components of SCK: the ability to explain and justify reasoning; the ability to use multiple representations; the ability to recognise, use and generalise relationships among conceptually similar problems; and the ability to pose problems. The first component is the focus of this study. In the conceptual framework, this study used the following descriptors of what the students should be able to do as an adaptation of the works of Bair and Rich (2011), Ball, Thames and Phelps (2008), Nyikahadzoyi (2013), and Steele, Hillen and Smith (2013). Students should be able to:

- Provide mathematical explanations for common rules, and procedures;
- Recognise and explain conceptual differences and relationships;
- Explain understanding of concepts (definitions, properties, theorems, and formulas); and
- Justify rules, algorithms, and the use of specific methods.

METHODOLOGY

A qualitative case study was conducted with 22 final-year mathematics student teachers who wrote a mathematics test that assessed their content knowledge of functions with open-ended items. The students were training to teach Grades 8 to 12 and had studied advanced mathematics in the following areas: mathematics methods, algebra, real analysis, and statistics. They had also completed mathematics education courses that focus on generic aspects of mathematics teaching. While the students had conducted peer teaching, none of them had been out on teaching practice at schools.

In this paper, the 10 items that assessed the student teachers' ability to explain and justify reasoning are discussed, including items that assessed the student teachers' ability to recognise and explain or justify conceptual relationships and differences. In order to align this with the conceptual framework, analyses were conducted according to the descriptors: the ability to explain and justify reasoning, and the ability to use different representations.

The possible total marks per item ranged from 1 to 8, and the possible total score was 28. The test items were aligned with the learning outcomes of the Zambian secondary school curriculum. The items were also developed to be consistent with the descriptors of SCK in the study’s conceptual framework. Some items were adapted from mathematics textbooks and past Zambian Grade 12 national examinations papers. The rationale behind this was to ensure that the test items reflect the contents of functions as taught in Zambian secondary schools. A document analysis was conducted of the Zambian secondary school mathematics curriculum, examination syllabuses, and mathematics textbooks (Bostock, 2000; Buckwell, 1996; Channon, 1994, 1996; Fuller, 1986; Kalimukwa, 1995; Laridon, 1995; Nkhata, 1995; Redspot, 2013; Talbert, 1995). These sources indicated that the following function topics are currently taught: relations, function, domain, range, one-to-one functions, inverse functions, linear functions, quadratic functions, and composite functions. Learners are expected to evaluate functions, calculate inverse functions, and simplify composite functions. They are supposed to identify an inverse of a function, represent composite functions, and even solve problems involving linear functions. They are also expected to identify relations that are functions and those that are non-functions.

VALIDITY

A draft test was developed, and an item by item analysis was conducted with the help of mathematics and mathematics education lecturers, curriculum specialists and secondary school mathematics teachers, who were requested to comment on the suitability of the test. The test was piloted with a view to making sure that the items were clear, well phrased and measured the relevant secondary school content. Expert judgement was then sought to ensure that the final version of the test had content and face validity. The improved version of the draft test, together with the learning outcomes in the Zambian secondary school curriculum, were presented to two experts in mathematics in the department of Mathematics and Applied Mathematics at the University of Pretoria, in order to ensure that the wording as well as the mathematical content of the test were suitable and valid. The items were found to be mathematically correct and the test was declared generally suitable for the intended purpose.

RESULTS

Figure 1 shows the student teachers’ total scores in the ‘ability to explain and justify reasoning’ category. Pseudonyms were used.

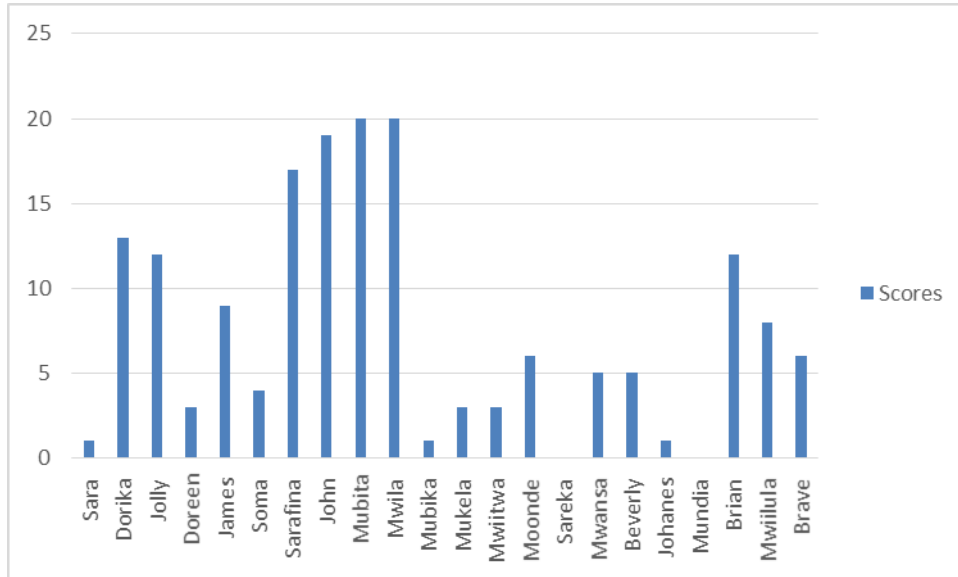


Figure 1. Student teachers’ total scores in ‘ability to explain and justify reasoning’ category)

The minimum score was zero, while the maximum was 20. This gives a range of 20, which suggests a large disparity in the students’ performance. An analysis of the data indicates that 18 student teachers (about 82% of the group) scored below the 50% pass mark, while only four student teachers (18% of the group) obtained scores that were above 50%. The mean mark representing the student teachers’ achievement in the category is almost 8 (27%). These results suggest that the majority of the student teachers found it difficult to explain and justify any reasoning for the topic of functions, and that there was a lack of competence to recognise and explain conceptual relationships and differences.

Item 1

This item required the student teachers to recognise and explain the relationships that existed among the domains and ranges of R and R_1 . The relation R was defined on set $X = \{3; 4; 5; 6\}$ by the rule 'is greater than', while R_1 was defined on set X by the rule 'is less than'. The student teachers were expected to recognise and explain that the domain of R is equal to the range of R_1 , while the range of R is equal to the domain of R_1 . On the one hand, thirteen student teachers (59% of the group) could neither recognise the relationships nor explain any relationship of any kind among the domains and ranges of R and R_1 . Nine student teachers (41% of the group) recognised and appropriately explained the relationships. The concepts of Domain and Range are clearly not as well understood as they should be.

Item 2(a)

In this item, the student teachers were asked to confirm whether or not they thought there was a difference between a relation and a function, and then they had to provide an explanation in support of their view. Ten student teachers (45% of the group) could do neither. Five student teachers (about 23% of the group) correctly stated that a difference exists between a general relation and a function without providing a reason to support their view. Five student teachers (about 23% of the group) correctly stated that a difference exists between a relation and a function, but provided explanations that were flawed. The remaining two student teachers correctly stated that a difference exists and also supported their position with good explanations. While 12 student teachers (about 55%) stated that a difference exists between a relation and a function, only two (17%) of these students provided sound explanations for this difference. This means that 20 student teachers (about 91% of the group) could not explain why a relation is different from a function.

Item 2(b)

The item 2(b) required the student teachers to provide justifications for why they thought a particular figure represented a function or a non-function.

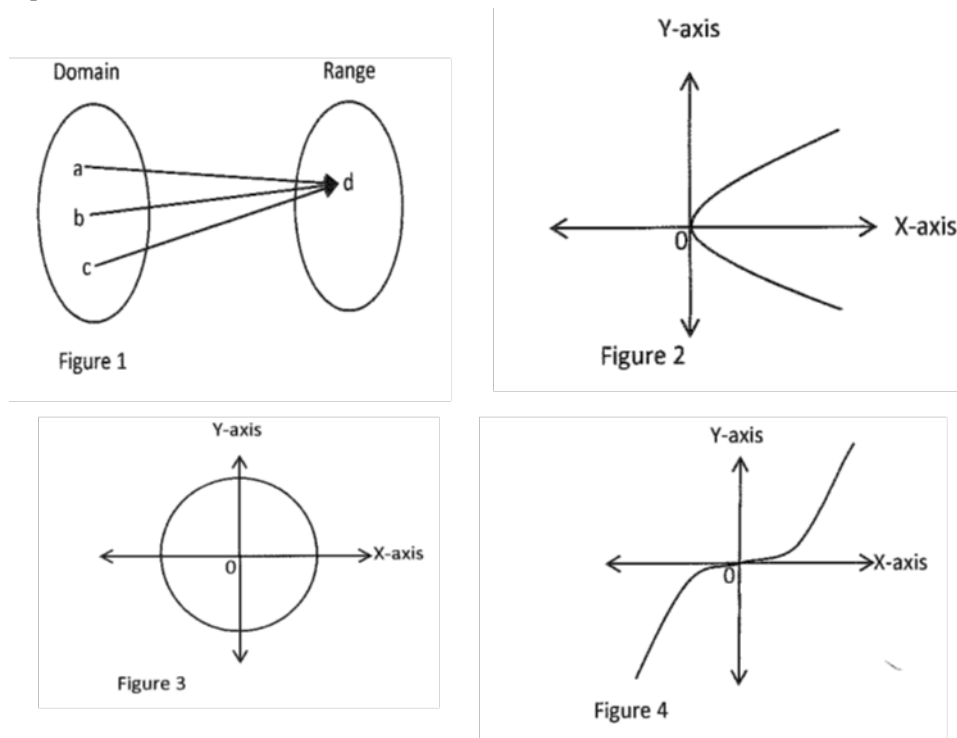


Figure 2. Item 2(b)

A summary of the results per figure is provided in [Table 1](#).

Table 1. Summary of the results per figure for item 2(b)

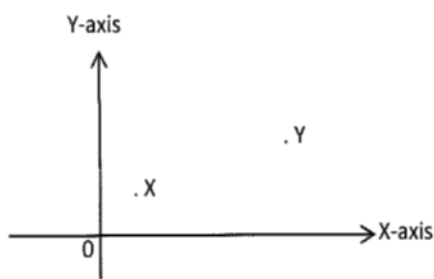
Figures	NJP	FJP	SJP	Totals
1	13	0	9	22
2	13	2	7	22
3	11	5	6	22
4	13	4	5	22
Total	50	11	27	88

Key: NJP represents 'no justification provided', FJP implies 'flawed justification provided', and SJP means 'sound justification provided'

Table 1 shows that 50 (57%) of the students provided no justification, while 11 (13%) of the students provided flawed reasons. Only in 27 (31%) instances did the student teachers give sound justifications. Interestingly, for the first sketch, there were 13 student teachers who provided no justification while nine students provided sound justification. This result shows that the majority of the student teachers (59%) could not justify why a many-to-one relation is a function. The second sketch delivered much the same result, with 59% of the group not being able to provide a justification for their answer, while in two instances (9% of the group), the students provided flawed justifications. Only in seven instances (32%) did student teachers provide correct justifications. These results suggest that the majority of the student teachers could also not provide sound justifications as to why a one-to-many Cartesian graph is not a function. For **Figure 3**, 11 students could not explain their answer, while five (23% of the group) produced flawed justifications. Only in six instances (27%) did the student teachers give correct justifications. Thus, 73% of the students were either unable to justify why a Cartesian graph of a circle is not a function or when they did, their justifications were not accurate. In the case of **Figure 4**, 13 student teachers could not provide justifications for why a one-to-one relation is a function. In four instances (18% of the group), flawed justifications were provided, while only in five instances (23% of the group) did the student teachers provide sound justifications. It would appear from these results that the majority of the student teachers were not competent at justifying why a one-to-one Cartesian graph is a function.

Item 2(c)

In the context of the 'ability to explain and justify reasoning' category, the student teachers were first required to draw any graph that passed through the points X and Y in Diagram D. They were then asked whether there were other graphs that could also go through those two points. The aspects assessed here were a follow-up on what was initially assessed in Item 2(c).

**Figure 3.** Diagram D

Eighteen student teachers (82% of the group) could not think of other functions whose graphs pass through X and Y, but they could not explain the reason. Nevertheless, four student teachers (18%) confirmed that there were indeed other functions whose graphs pass through X and Y. We realised that these students had a very limited understanding of the graphs of functions. They could only think of a straight-line graph passing through the two given points.

Item 2(d)

The statement that $y^2 = x + 9$ with domain $\{x: 0 \leq x < 2 \text{ and } x \in \mathbb{Z}\}$ was an example of a function. The students were required to say whether or not this was true, and why they thought so. Thirteen student teachers (59%) were unable to justify their opinion. Two student teachers (9%) attempted to provide explanations in support of their view, but those explanations were flawed: they did not, for example, refer to the fact that this was the inverse of a parabolic function. The other seven student teachers (about 32%) explained that for each member of the domain $\{x: 0 \leq x < 2 \text{ and } x \in \mathbb{Z}\}$ the relation $y^2 = x + 9$ would produce more than one image.

Item 3

The student teachers were asked to explain the relationship that exists between the maximum value and the range of the function $f(x) = -2x^2 - x + 8$. This item assessed the student teachers' ability to recognise and explain that the maximum value of the function f is the largest value in the range of f . Fifteen student teachers (68% of the group) could neither recognise nor explain that there is any relationship between the maximum value of f and its range. Six student teachers (27%) provided flawed explanations, while only one was able to present a correct explanation.

Item 4

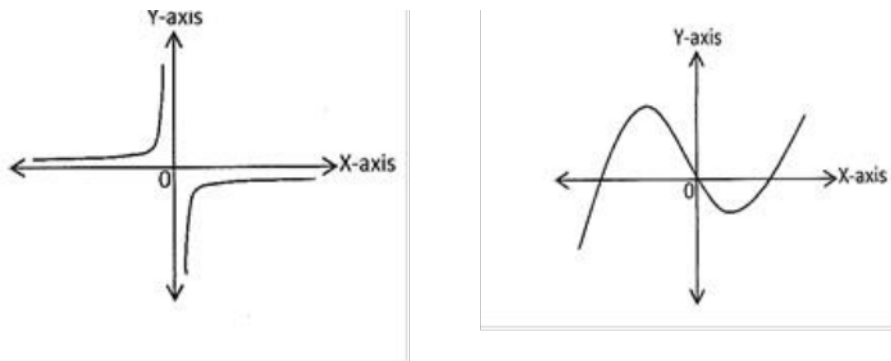


Figure 4. Hyperbola and cubic graph

In this item, two graphs are presented: one a hyperbola and the other a cubic graph. The question was whether they were examples of one-to-one functions or not. The students had to justify their answer. None of the 22 students could provide a correct explanation for their answer for either graph. Four students (18% of the group) could provide a flawed explanation for each answer. The students could clearly not explain the difference between the representation of a one-to-one function and that of a many-to-one relation.

Item 5(a)

This item consisted of two functions expressed in symbolic form without any graphic representations: $h(x) = x^2 + 1$ for $-2 \leq x \leq 2$ and $f(x) = x^2 + 1$ for $0 \leq x \leq 2$. The student teachers were required to recognise and state two differences between h and f , for example that h is a many-to-one function and that f is a one-to-one function, and that h is not invertible for $[-2,2]$, whereas f is invertible for $[0,2]$. Six student teachers (27%) failed to recognise and explain any difference between the two functions. However, 13 student teachers (59% of the group) did recognise and state one difference between the two functions h and f . Only three student teachers (about 14%) could describe two differences between h and f . These three student teachers were also able to recognise that h was an example of a many-to-one function while f was a one-to-one function. The majority of the student teachers performed fairly well in terms of recognising conceptual differences between functions.

Item 5(b)

Item 5(b) was a continuation of 5(a). The students were asked to describe the relationship that exists between the range of h and the domain of f^{-1} . In other words, the student teachers were assessed on their ability to recognise and describe that the range of the function h is equal to the domain of f^{-1} . Fifteen student teachers (68% of the group) could not provide any description of the relationship between the two functions. One student teacher provided a description that was flawed. Six student teachers were able to present descriptions that were considered to be sound. These results suggest that most of the student teachers (about 73%) did not seem to have a good understanding of functions and their inverses.

Item 6

For this item, the student teachers were required to state and justify two of the domains in which the function $z: x \rightarrow x^2 - 2x$ has an inverse. The idea was to assess whether the student teachers could relate inverse functions to one-to-one functions. Twenty student teachers (91% of the group) could not answer this question at all. Two student teachers managed to state one appropriate domain, but without giving any justification. These results confirmed the results of the previous question: the students were not able to work comfortably with inverses and restricted domains.

DISCUSSION

The literature (Even, 1993; Spyrou & Zagorianakos, 2010) tells us that difficulties in understanding functions are neither new nor uncommon. Spyrou and Zagorianakos (2010) indicate that most of the students in their study did not consider many-to-one relations as functions and battled to distinguish relations from functions. While students may use the univalence condition to define and identify functions and non-functions, they have limited understanding of this property. Likewise, students misconstrue the graph of a circle to be a function despite their ability to explain the vertical line test. They also do not easily understand that constant functions are functions (Tall & Bakar, 1992). The definition of a function seems to be easily confused with the one-to-one correspondence condition (Leinhardt, Zaslavsky, & Stein, 1990; Markovits, Eylon, & Bruckheimer, 1986) as is the definition of a one-to-one function with the univalence condition (Dubinsky & Harel, 1992). Another study found that students' knowledge was compartmentalised in terms of functions, function notation and periodicity (Gerson, 2008). Moreover, Jojo, Brijlall, and Maharaj (2011) report that students do not demonstrate comprehensive understanding of composite functions. In fact, Chesler (2012) indicates that reasoning with and about mathematical definitions of functions remains problematic.

Researchers suggest that any valid definition of a function should include the univalence and arbitrariness properties of functions (Even, 1990; Lloyd et al., 2010; Nyikahadzoyi, 2013). This study shows that the univalence property characterised the students' explanations of a function, but that even though they frequently referred to this property, they were unable to explain it clearly. There was ambiguity as the students attempted to explain that for a function, every element of the domain should be associated with one image in the range. Likewise, the arbitrariness property was overlooked in most of the students' explanations of a function. Some students said that functions should be accompanied by algebraic formulas. This is not an uncommon way of thinking (Bayazit, 2011; Clement, 2001).

While the majority of the students could explain that a function relates members of two sets, their explanations lacked depth, and at times were restrictive, confining their thinking to the one-to-one correspondence property. Evangelidou, Spyrou, Elia, and Gagatsis (2004) also found that a function is normally narrowly viewed to be a relation that only satisfies the one-to-one correspondence property. In this study, some students could not recognise a many-to-one relation, while others could not correctly describe a many-to-one function, and several others did not understand that a many-to-one relation is a function. The students struggled to explain a one-to-one function, which they were actually only expressing in terms of the univalence condition. Other studies have also found that students have difficulties differentiating between the definition of one-to-one functions and the univalence property (Dubinsky & Harel, 1992; Leinhardt et al., 1990; Markovits et al., 1986).

When explaining inverses, our students spoke of the procedure of 'reversing' or 'un-doing' that which is done by another function. Of course, the characteristic of 'un-doing' is considered to be a defining feature of the definition of an inverse function (Bayazit & Gray, 2004). Interestingly though, an inverse function was also explained by one student as 'a process of reverting the function to the original expression'. While the idea of a function being perceived as a 'process' is dealt with in research literature, an inverse function does not 'revert' to the initial expression, but relates images to their corresponding objects.

The students used the definitions of a function to justify their opinion that particular figures represent functions or non-functions, frequently demonstrating shallow conceptions of the univalence condition. Others used the vertical line test to justify that a circle is a non-function. While this strategy helped to make the correct determination, it did not amount to a justification. Some students stated that a many-to-one arrow diagram represents a non-function because there is no one-to-one association between the elements of the domain and the range. This suggested their lack of capacity to distinguish a one-to-one function from a function in general. A one-to-many graph was considered by some students as a function, but without giving any justification. Of course, a one-to-many Cartesian graph can be a function if the vertical axis is taken as the domain and the horizontal axis as the range, but none of those students demonstrated this understanding.

Students who held the misconception that a vertical line drawn in a Cartesian plane is a function could not justify their viewpoint. Nonetheless, those who concluded that vertical lines are non-functions understood that on a vertical line, a single x -value is associated with more than one y -value. One of them indicated that vertical lines have no gradient. Some students thought that horizontal lines are not functions, but were unable to explain why they thought so. Most of the students could not comprehensively justify their viewpoints on examples and non-examples of graphs of one-to-one functions. Those who correctly identified the graphs used definitions of a function to justify their positions. Even the students who incorrectly identified the graphs used their flawed conceptions of the univalence condition to provide justifications. This suggests that the students had no thorough understanding of the difference between the univalence condition and the definition of a one-to-one function.

Although the students adeptly explained the process involved in computing an inverse function, none of them could explain *why* they performed the specific procedures. This finding points to a lack of relational understanding.

Additionally, none of the students explained the relationship that exists between the leading coefficient of an algebraic quadratic function and its graph.

CONCLUSION

Was the University of Zambia's mathematics student teachers' specialised content knowledge of functions at secondary school level of such a nature that they would be able to justify their reasoning and explain their understanding of concepts? It has been established that this was not the case. The study found that the student teachers generally lacked proficiency in the SCK of functions at secondary school level. They demonstrated instrumental, and not relational, understanding of concepts in functions at secondary school level. At the same time, it appeared that most of the student teachers lacked the ability to work with different representations of functions, nor could they coherently explain concepts and provide comprehensive justifications for their reasoning. While the students had studied advanced university mathematics and were in the final phase of their training, they had not acquired relational understanding of functions for secondary school level. It is clear that the study of advanced mathematics does not automatically result in students' in-depth understanding of secondary school mathematics, with particular reference, in this study, to functions.

This study provided evidence that enabled us to argue that during training, student teachers should study the mathematics topics that they are expected to teach. Official approval has now been received for the development of a school mathematics course for mathematics student teachers during the course of 2018. This module will be submitted to the School Curriculum Committee for final approval by the university. This study can therefore lay claim to the enhancement of teaching and learning at UNZA in terms of the change to the model that will be implemented to train student mathematics teachers. They will now be learning the mathematics that they will be required to teach at school level. The literature has revealed that there are many institutions where school mathematics is not part of the training of mathematics student teachers. Our recommendation is that further research is conducted, particularly in terms of other topics in the school syllabus. This will allow for similar change to be brought about in training models in other institutions with a view to enhancing mathematics teacher training as a whole.

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REFERENCES

- Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science & Technology*, 39(7), 857-878. <https://doi.org/10.1080/00207390802054458>
- Bair, S. L., & Rich, B. S. (2011). Characterizing the Development of Specialized Mathematical Content Knowledge for Teaching in Algebraic Reasoning and Number Theory. *Mathematical Thinking & Learning*, 13(4), 292-321. <https://doi.org/10.1080/10986065.2011.608345>
- Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21(2), 132-144. <https://doi.org/10.2307/749140>
- Bayazit, I. (2011). Prospective Teachers' Inclination to Single Representation and Their Development of the Function Concept. *Educational Research and Reviews*, 6(5), 436-446. Retrieved from <https://academicjournals.org/journal/ERR/article-abstract/6C23F115280>
- Bayazit, I., & Gray, E. (2004). *Understanding inverse functions: the relationship between teaching practice and student learning*. Paper presented at the Proceedings of the 28th Conference of the International.
- Bell, J. (2014). *Doing Your Research Project: A guide for first-time researchers*. UK: McGraw-Hill Education.
- Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In R. Biehler, W. R. Scholz, R. StraBer, R., & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 73-88). The Netherlands: Kluwer Academic Publishers.
- Bryan, T. J. (1999). The conceptual knowledge of preservice secondary mathematics teachers: How well do they know the subject matter they will teach. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal. Volume 1: Content Knowledge*.
- Chesler, J. (2012). Pre-service Secondary Mathematics Teachers Making Sense of Definitions of Functions. *Mathematics Teacher Education and Development*, 14(1), 27-40.
- Clement, L. (2001). What do students really know about functions. *MatheMaticS teacher*, 94(9), 745-748.

- Cooney, T. J., & Wiegel, H. G. (2003). Examining the mathematics in mathematics teacher education. In A. J. C. M. A. K., C. Bishop; J. Kilpatrick, & F.K.S. Leung (Ed.), *Second International Handbook of Mathematics Education* (pp. 795). Netherlands: Kluwer Academic Publishers.
- Dede, Y., & Soybaş, D. (2011). Preservice mathematics teachers' experiences about function and equation concepts. *Eurasia Journal of Mathematics, Science & Technology Education*, 7(2), 89-102. <https://doi.org/10.12973/ejmste/75183>
- Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. *The concept of function: Aspects of epistemology and pedagogy*, 25, 85-106.
- Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, 32(1), 83-101. <https://doi.org/10.1016/j.jmathb.2012.12.001>
- Evangelidou, A., Spyrou, P., Elia, I., & Gagatsis, A. (2004). *University students' conceptions of function*. Paper presented at the Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education.
- Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, 21(6), 521-544. <https://doi.org/10.1007/BF00315943>
- Even, R. (1993). Subject-Matter Knowledge and Pedagogical Content Knowledge: Prospective Secondary Teachers and the Function Concept. *Journal for Research in Mathematics Education*, 24(2), 94-116. <https://doi.org/10.2307/749215>
- Fi, C. (2003). *Preservice secondary school mathematics teachers' knowledge of trigonometry: subject matter content knowledge, pedagogical content knowledge, & envisioned pedagogy* (PhD thesis). University of Iowa.
- Fi, C. (2006). *Pre-service secondary school mathematics teachers' knowledge of trigonometry: Cofunctions* Paper presented at the Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mérida, México.
- Gerson, H. (2008). David's Understanding of Functions and Periodicity. *School Science and Mathematics*, 108(1), 28-38. <https://doi.org/10.1111/j.1949-8594.2008.tb17937.x>
- Haambokoma, C., Nkhata, B., Kostyuk, V. S., Chabalengula, V., Mbewe, S., Tabakamulamu, M., Ndhlovu, Z. B., Mushanga, R., & Nthani, D. (2002). *Strengthening of Mathematics and Science Education in Zambian Secondary schools*. Lusaka: Ministry of Education.
- Henning, E. (2014). Matric begins in Grade 1. *The Con*, 13th January 2014.
- Hiebert, J. (2013). *Conceptual and procedural knowledge: The case of mathematics*. UK: Routledge. <https://doi.org/10.4324/9780203063538>
- Jojo, Z., Brijlall, D., & Maharaj, A. (2011). *The reliability of a research instrument used to measure mental constructs in the learning of the chain rule in calculus*. Paper presented at the Proceedings of the 16th Annual Congress of the Association of Mathematics Education of South Africa.
- Kambilombilo, D., & Sakal, W. (2015). An Investigation into the Challenges In-Service Student Teachers Encounter in Transformational Geometry, "Reflection and Rotation": The Case of Mufulira College of Education. *Journal of Education and Practice*, 6(2), 139-149.
- Lee, J., & Zuilowski, S. S. (2015). 'Making do': Teachers' coping strategies for dealing with textbook shortages in urban Zambia. *Teaching and Teacher Education*, (48), 117-128. <https://doi.org/10.1016/j.tate.2015.02.008>
- Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of educational research*, 60(1), 1-64. <https://doi.org/10.3102/00346543060001001>
- Lloyd, G., Beckmann, S., Zbiek, R. M., & Cooney, T. (2010). *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12*. ERIC.
- Malambo, P. (2015). *Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools* (PhD thesis). University of Pretoria, South Africa.
- Markovits, Z., Eylon, B. S., & Bruckheimer, M. (1986). Functions today and yesterday. *For the learning of mathematics*, 18-28. Retrieved from <https://www.jstor.org/stable/40247808>
- Ministry of Education. (2013a). *"O" Level Mathematics Syllabus (Grades 10 to 12)* Lusaka, Zambia: Zambia Curriculum Development Centre.
- Ministry of Education. (2013b). *Zambia Education Curriculum Framework*. Lusaka: Zambia Curriculum Development Centre.
- Nyikahadzoyi, M. R. (2013). Teachers' Knowledge of the concept of a function: A theoretical framework. *International Journal of Science and Mathematics Education*, 1-23. <https://doi.org/10.1007/s10763-013-9486-9>

- Spyrou, P., & Zagorianakos, A. (2010). Greek students' understandings of the distinction between function and relation. *Research in Mathematics Education*, 12(2), 163-164. <https://doi.org/10.1080/14794802.2010.496988>
- Steele, M. D., Hillen, A. F., & Smith, M. S. (2013). Developing mathematical knowledge for teaching in a methods course: the case of function. *Journal of Mathematics Teacher Education*, 16(6), 451-482. <https://doi.org/10.1007/s10857-013-9243-6>
- Thames, M. H., Ball, D. L., & Phelps, G. (2008). Content Knowledge for Teaching What Makes It Special? *Journal of Teacher Education*, 59(5), 389-407. <https://doi.org/10.1177/0022487108324554>
- Watson, A., & Harel, G. (2013). The role of teachers' knowledge of functions in their teaching: A conceptual approach with illustrations from two cases. *Canadian Journal of Science, Mathematics and Technology Education*, 13(2), 154-168. <https://doi.org/10.1080/14926156.2013.784826>
- Wilburne, J. M., & Long, M. (2010). Secondary Pre-Service Teachers' Content Knowledge for State Assessments: Implications for Mathematics Education Programs. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1. Retrieved from <https://eric.ed.gov/?id=EJ872484>
- Wilson, M. R. (1994). One preservice secondary teacher's understanding of function: The impact of a course integrating mathematical content and pedagogy. *Journal for Research in Mathematics Education*, 346-370. <https://doi.org/10.2307/749238>
- Wood, E. (1993). *Making the connections: The mathematical understandings of prospective secondary teachers* (PhD thesis). Michigan State University East Lansing, MI.

<http://www.ejmste.com>