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Nonlinear Acoustics: What happens when the constraint is challenged?

by

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Declaration of Authorship

I, Simbarashe K. Dziwa, declare that the dissertation, which I hereby submit for the degree Master of Science (Applied Mathematics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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"Wealth, if you use it, comes to an end; learning, if you use it, increases."

Swahili Proverb

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Abstract

The Lagrangian formulation for the propagation of sonic disturbances consists of a system of first order partial differential equations and an inequality constraint under which the system is hyperbolic. We study the behaviour of solutions when the system is initially at rest but strained under an initial pressure field that challenges the constraint. The result is that the challenge is transferred to the solutions, that the pressure decays, the rest state changes violently, and shock discontinuities appear in velocity as well as pressure. The main tool of investigation is the notion of *inverse characteristics* in which a chosen point in the time-space plane is associated with points on the initial manifold where characteristics through the given point emanate from. They also lead to the introduction of an alternative measure of time in terms of which explicit expressions for the onset of shocks are derived.

Key words. Nonlinear acoustics, hyperbolic systems, shock phenomena, inverse characteristics.

Acknowledgements

I am extremely grateful to my supervisor Professor Niko Sauer for his friendship, guidance, super human patience and commitment to my studies; it has been a long road travelled together. I learned a lot of mathematics from him and some useful perspectives on life as well. Professor Sauer taught me that the thinking behind mathematics matters more than mathematical complexity. As a result I found myself re-learning good old-fashioned calculus and classical analysis from a refreshingly different angle; mathematics has now become a daily part of my life. I now have reasonably developed insights into how research in mathematics is conducted, how its results are presented and communicated to the broader audience. These insights will no doubt be crucial as I continue with my studies.

I would also like to express my sincere appreciation of my parents for instilling in me the value of education as a lifelong pursuit and for constantly encouraging me to complete my studies and continue. I am sure that a bound copy of this thesis document will have pride of place on their library at their home in Norton, Zimbabwe. May they continue to be blessed with good health as they enjoy their retirement.

Last but not least, I would like to thank the examiners for their valuable comments.

Contents

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Symbols

$\frac{\partial}{\partial t}$	partial derivative with respect to t
$\frac{\partial}{\partial x}$	partial derivative with respect to x
ϕ_t	$\frac{\partial \phi}{\partial t}$
v_t	$\frac{\partial v}{\partial t}$
v_x	$\frac{\partial v}{\partial x}$
p_t	$\frac{\partial p}{\partial t}$
p_x	$\frac{\partial p}{\partial x}$

As I was effecting the finishing touches to this dissertation, I received news of the transitioning of Morgan Tsvangirai to the ancestral realm on 14 February 2018. He was a brave and committed true son of the soil who fought, to the last breath of his life, a ruthless, selfish and kleptocratic authoritarian regime for a better life for all Zimbabweans. While we scattered all over the globe, he remained behind in the hazardous trenches of Zimbabwean politics. He fought the good fight, he finished the race and he kept the faith. Hambani kahle Comrade. See you when we get there.

Chapter 1

Introduction

1.1 Background to the research

Sauer in [21] derives the following system of quasilinear hyperbolic partial differential equations and an inequality constraint for the equations of motion of sonic disturbances in an ideal gas under isentropic assumptions

$$\left.\begin{aligned} v_t(x,t) + p_x(x,t) = 0; \\ p_t(x,t) + [1 + p(x,t)]^2 v_x(x,t) = 0, \end{aligned}\right\} \tag{1.1}$$

to which is added the constraint

$$1 + p(x,t) > 0. \tag{1.2}$$

Here v is the velocity of the gas and p is the ambient pressure exerted on the gas.

These equations follow from the so-called *Lagrangian* or material description of motion in which a point x in a reference configuration is followed in time. This is in contrast to the *Eulerian* (field-theoretical) description in which motion is composed from observations at fixed points in space. The variables v and p represent dimensionlessly scaled velocity of the point x at time t and pressure experienced by x at time t.

Based on rudimentary numerical computations, Sauer in [21] observed that under the initial conditions

$$\left.\begin{aligned} &v_0(x) = v(x,0) = 0 \\ &p_0(x) = p(x,0) = \exp\{-x^2\} - 1 \end{aligned}\right\}$$

a strange behaviour is manifested by the solution of (1.1). Sauer ascribes this to the effect of the initial pressure "pushing" the constraint (1.2) when x is large (positive or negative). Sauer's observations are depicted in Figure 1.1 below.

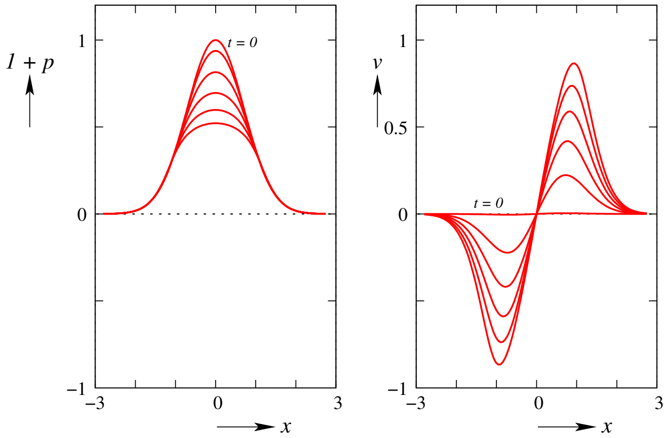

FIGURE 1.1: Pressure and velocity when constraint is challenged

The purpose of this research is to add theoretical substance to Sauer's observations.

1.2 Structure of dissertation

An overview of the dissertation is as follows. Chapter 2 gives an historical overview of gas dynamics from the 18th century to mid 20th century, emphasizing the key players and their most important works. This is followed by a derivation of Sauer's equations in one dimension. Chapter 4 presents some relevant results from the theory of hyperbolic partial differential equations. Thereafter we present the Cauchy problem and its 'simplification', followed by an exploration of some of its interesting qualitative characteristics such as asymptotic behaviour and shock phenomena. We round of the dissertation with some examples to illustrate the results and answer the research question: what happens when the constraint is pushed?

Chapter 2

A short history of acoustics

2.1 Introduction

Acoustics (Greek *akouein*, to hear) as a mathematical field is the study of the description of sound. The field can be divided in two parts, linear acoustics and nonlinear acoustics. Linearity results when the simplifying assumption of small disturbances in the medium in which sound is propagated is made. The resultant wave propagation is mathematically described by the wave equation. Nonlinearity occurs otherwise and its mathematical description is complex.

Acoustics has its origins in ancient Greece, in particular with Pythagoras who was interested in vibrating strings and musical sounds. It was however in the 18th and 19th centuries C.E. when the field flourished and the foundations of fluid mechanics and wave motion were laid. A chronological account of the history of the field will be unwieldy. Since scientific ideas and knowledge develop organically as scientists build on each other's work, we focus on the contributions of a few mathematicians whose results were instrumental in the field.

It is worth mentioning that an historical overview of acoustics would be incomplete without an account of the experiments to determine the the speed of sound in air. This is due in part to the story's own intrinsic interest and to the fact that the core of the terminology and concepts of acoustics (and indeed fluid dynamics) originated from these experiments. So this chapter begins with a brief account of these experiments. We follow this up with brief reviews of the works of representative mathematicians whose contributions to the field were foundational. The review is limited to the period mid 1800s to the early 1900s as this is the period that is generally considered as the field's most fruitful.

The main source of the material in this chapter and its presentation is the text by Blackstock [1, Chapter 1, pp.1–23].

2.2 Speed of sound in air

The story usually begins with the French philosopher, priest, scientist, astronomer and mathematician Pierre Gassend who in 1635 measured the speed of sound as 478m/s. He used firearms and assumed that the muzzle flash is transmitted instantaneously. In a more careful experiment in 1640, another French mathematician Marin Mersenne improved on this and determined the speed of sound to be 450m/s.

The Italians, G.I. Baroilli and V. Vivian of the Academia del Cimento (Academy of Experiment) Florence Academy of Experiment, refined this further in 1650 with a value of 350m/s.

The first attempt to calculate the speed of sound through air was apparently made by Sir Isaac Newton in 1686. He used Boyle's law,

$$P/P_0 = \rho/\rho_0, \tag{2.1}$$

where P_0 and ρ_0 are the ambient pressure and the ambient density, respectively, to calculate the speed of sound by the following formula:

$$b = \sqrt{P_0/\rho_0}. \tag{2.2}$$

The symbol b denotes the association with Boyle's law. Newton's formula is now known to be the speed of sound in an isothermal gas.

Newton's prediction was about 16% lower than the measured values at the time and the discrepancy took well over a century to resolve. It was finally resolved by Pierre Simon de Laplace in 1816 who proferred the correct explanation. He argued that heat does not flow when sound propagates; that is, sound propagation is an adiabatic process. Instead, the local temperature changes in accordance with the compressions and expansions of the air. He concluded that Newton's formula (2.2) should be corrected by multiplying it by $\sqrt{\lambda}$, the ratio of the specific heats. Thus Laplace obtained the speed of sound c through the formula:

$$c_0 = \sqrt{\lambda P_0/\rho_0}. \tag{2.3}$$

The conventional symbol c denotes the association of the speed of sound with an isentropic fluid. Laplace's formula (2.3) is now universally accepted and intepreted as the adiabatic speed of sound.

2.3 The Classical Era: 1759-1880s

According to Blackstock [1, p.3] theoretical linear acoustics has its origins in Leonhard Euler's formulation of the equations that bear his name [6]:

$$\begin{aligned} \text{Continuity} : \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} &= 0 \\ \text{Momentum} : \rho\frac{Du}{Dt} + \nabla P &= \mathbf{F} \end{aligned}$$

where ρ is density, $\mathbf{u}$ is particle velocity, P is total pressure, $\mathbf{F}$ is an external body force per unit volume, and t is time. The material derivative D/Dt represents the combination $\partial/\partial t + \mathbf{u}\cdot\nabla$.

Not to be left out of the speed-of-sound problem, Euler derived, using Lagrangian coordinates, the following equation for aerial plane waves [7]

$$b^2\frac{\partial^2\xi}{\partial^2 a} - \left(1 + \frac{\partial\xi}{\partial a}\right)^2\frac{\partial^2\xi}{\partial^2 t} = 0.$$

where ξ is particle displacement and a is particle position. The coefficient b^2 identifies the air as having the properties of Boyle' law and is given by (2.2). Euler proposed that if the nonlinear terms were taken into account, the predicted propagation of speed of sound would be higher than the Newtonian value b, that is closer to the experimentally measured speed ([1], p.5).

2.4 Siméon Poisson (1781-1840)

Siméon Poisson's important contribution [17] was to find an exact solution for progressive waves of finite amplitude. He assumed Boyle' law and used Eulerian coordinates coordinates x, t, in contrast to Leonard Euler who used Lagrangian coordinates a, t to derive

$$b^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x}\right),$$

where b is defined by (2.2). Poisson's exact solution is

$$u = g[x - (u+b)t],$$

where g is an arbitrary function. This solution is valid for outgoing waves. For incoming waves, Poisson determined the exact solution to be

$$u = G[x - (u - b)t]$$

where G is an arbitrary function.

The full significance of Poisson's solutions were not recognized until 40 years later by the British mathematician and physicist George Stokes.

2.5 George Stokes (1819-1903)

Following Poisson's definitive but uninterpreted solutions, a lengthy period of inactivity ensued. This ended in 1848 when a squabble erupted between British physicist James Challis (1803-1882) and mathematician George Airy (1801-1892) over the existence of plane waves of sound. Challis [2] invoked Poisson's exact solution for a plane wave of initially sinusoidal shape $u = u_0 \sin k[x-(b+u)t]$, where $k = \omega/b$ is a wave number, ω is angular frequency, and u_0 is amplitude. He showed that at time $t = \pi/2ku_0$, the wave peak is predicted to be at the same point in space as a zero. In Challis's words, "the points of no velocity are also points of maximum velocity. This is manifest absurdity". He concluded, "Plane waves are thus shown to be physically impossible".

This set the stage for Stokes contribution [23]. He picked up Challis's "manifest absurdity" and gave a clear description of the waveform distortion implied by Poisson's solution. Points on the waveform for which u is positive travel faster than points for which u is negative. Recognition of the distortion implied by Poisson's solution opened up a whole new frontier. First, Stokes determined the minimum time t_0 required for a continuous wave to develop a vertical slope. Thereafter the wave motion would have to be qualitatively different. In prescient remarks that foretold future developments 40 or so years later, Stokes remarked: "Of course, after the instant at which the [slope] becomes infinite, some motion or other will go on, and we might wish to know... the nature of that motion". Thus the concept of a shock wave was born, called by Stokes a "surface of discontinuity".

Stokes derived two conservation laws, conservation of mass and conservation of momentum, that must hold across the discontinuity. Later these would come to be known as two of the three Rankine-Hugoniot shock relations (2.4) but Stokes received no credit for being the first to derive them. His analysis lacked the necessary tools of thermodynamics which were not well developed at the time and in particular Stokes did not realize that shock propagation is accompanied by energy dissipation and that expansion shocks are impossible (Rayleigh, [19]). Stokes [23] concluded that shock formation destroys the progressive wave nature of the wave motion: "Apparently something like reflexion must take place". He also saw that viscosity would limit the formation of shocks and prevent true discontinuities from forming or "render the motion continuous again if it were for an instant discontinuous".

2.6 Samuel Earnshaw (1805-1888)

If Stokes [23] was the turning point in the development of nonlinear acoustics, Earnshaw [5] represents the high-water mark of the era on the subject of progressive waves ([1], page 10). Earnshaw considered the now-classic problem of the wave motion generated by arbitrary movement of a piston in a lossless tube, a problem later to be revisited by Sauer in [22]. His approach was exhaustive: he began by considering waves in a gas obeying Boyle's law, which was the equation of state most often used by previous investigators, moved on to the case of an adiabatic gas, and finally generalized the analysis to cover an arbitrary pressure-density relation $P = P(\rho)$. If the piston, located initially at $x = 0$, has displacement $X = X(t)$ and velocity, $U(t) = dX/dt$, Earnshaw's solution for an adiabatic gas is:

$$
\begin{aligned}
u &= U(\phi), t > \pm x/c_0, \\
\phi &= t - \frac{x - X(\phi)}{\beta U(\phi) \pm c_0},
\end{aligned}
$$

where $\beta = \frac{1}{2}(\gamma + 1)$ is today called the coefficient of nonlinearity. It is assumed that the piston starts from rest and that the gas is initially quiet. The parameter ϕ in Earnshaw's solution represents the time a given point on a waveform — e.g., peak, trough, or zero crossing — left the piston and c_0 represents the speed of sound in adiabatic gas.

Earnshaw's other contributions in this long work were:

1. A discussion of the formation of shocks (he called them bores), including a calculation of time and place at which a shock first occurs.
2. An argument that shock speed would exceed c_0.
3. A calculation of the speed at which a piston must be withdrawn to create a vacuum.
4. A deduction that waves of permanent shape are possible only if the pressure-density relation has the form $P = A - B/\rho$. This was to become known as "Earnshaw's law" [19].

2.7 Bernhard Riemann (1826-1886)

Up to this point, all the results presented are limited to progressive waves or simple waves as they are sometimes called. The influential German mathematician Bernard Riemann [20] found a way to deal with compound wave fields, that is, fields in which waves travelling in both directions are present, as for example in reflection. In compound wave fields, propagation speeds $u \pm c$ are still appropriate, but in a more general sense. Form the following two linear combinations of u and λ:

$$J_+ = \frac{1}{2}(\lambda + u)$$
$$J_- = \frac{1}{2}(\lambda - u),$$

which are now called Riemann invariants (Riemann used the symbols r and s instead of J_+ and J_-, respectively). The generalization obtained by Riemann is

$$\left.\frac{dx}{dt}\right|_{J_+} = u + c$$
$$\left.\frac{dx}{dt}\right|_{J_-} = u - c,$$

which means that the Riemann invariants J_+ and J_- (not u or λ by themselves,) are propagated with speeds $u + c$ and $u - c$, respectively.

By the end of the early 1860s the intense unanswered question was what happens after shocks form. Stokes had already mused on this. A theory based on lossless equations of motion had been developed that was quite successful. Inherent in this theory is the assumption of continuous functions. Yet, the prediction is that discontinuities always form. Although it might be possible to modify the theory to account for discontinuities, Stokes had observed that when discontinuities form, reflected waves are generated. So the challenge after 1860 was to find ways to deal with shocks that develop in the waveform. Specifically the issue was how to deal with the neglect of dissipation because shock propagation is always accompanied by energy loss. So success in addressing this question depended on how dissipation was accounted for.

2.8 William Rankine (1820-1872) and Pierre-Henri Hugoniot (1851-1887)

During the half century after Earnshaw and Riemann, the focus was on shock waves and substantial gains were made in this regard. The theory of shock waves was given a firm foundation by the establishment of what are now called the Rankine-Hugoniot shock relations ([18], [12]). At the same time, the problem of the profile of the steady shock wave in a viscous, heat-conducting gas was found to be solvable ([18], [19], [24]).

William Rankine, a Scottish physicist and mathematician and Pierre-Henri Hugoniot, a French physicist and mathematician, arrived independently and from quite different starting points at the shock relations named after them. Rankine [18] wanted to find the heat transfer within the gas necessary for a waveform not to change. He ended up with not only the shock relations but also the profile of a steady shock in a heat-conducting, but inviscid, gas. Hugoniot [12], on the other hand, simply sought the relations necessary for steady discontinuities to exist. Not realizing that dissipation is essential to shock propagation, Hugoniot assumed an inviscid, thermally non-conducting gas at the outset.

The Rankine-Hugoniot relations are conservation equations that connect the flow field behind a shock (P_b, ρ_b, T_b, u_b) with that ahead of it (P_a, ρ_a, T_a, u_a). Let the shock be moving with constant velocity U_{sh}. The conservation equations are most simply expressed in a reference frame in which the shock is at rest, since then the flow is steady. In this frame the particle velocity behind the shock is $v_b = u_b - U_{sh}$ and that ahead is $v_a = u_a - U_{sh}$. Conservation of mass and momentum for this case lead to

$$\begin{aligned} \rho_a v_a &= \rho_b v_b, \\ P_a + \rho_a v_a^2 &= P_b + \rho_b v_b^2. \end{aligned} \tag{2.4}$$

As already mentioned, Stokes [23] was the first to obtain a form of these equations. The mistake he made, as did Riemann [20] after him, was to close the system by adding a lossless pressure-density relation, such as Boyle's law.

The key, found by both Rankine (for perfect gases) and Hugoniot was to use an energy equation in which losses may be included. For the general case (not limited to perfect gases), the third Rankine-Hugoniot relation ([1], page 16) is

$$Energy : \frac{v_a^2}{2} + e_a + \frac{P_a}{\rho_a} = \frac{v_b^2}{2} + e_b + \frac{P_b}{\rho_b},$$

where e is the internal energy per unit mass.

2.9 The Modern Era

Following the 1910 papers of Rayleigh [19] and Taylor [24], nonlinear acoustics went into retreat. The retreat ended in the 1930s with renewed interest stimulated by the war effort. International symposia devoted exclusively to nonlinear acoustics have since been held. The first was held in 1968 in New London, Connecticut, USA. The most recent was the 20th International Symposium on Nonlinear Acoustics held in Lyon, France in July 2015. The 21st edition will be held in July 2018 in Santa Fe, New Mexico, USA. The scope of the symposia cover

- General theory of nonlinear acoustics : Analytical methods, numerical methods, ray theory, scattering theory, shocks, solitons, chaos, bifurcation, localization, phase conjugation, etc
- linear acoustics in fluids : Sound beams, parametric arrays, resonators, acoustic streaming, radiation pressure, acoustic levitation, etc.
- Nonlinear acoustics in multiphase and porous media, and cavitation phenomena : Bubbly liquid, cavitation, sonoluminescence, sonochemistry, etc.
- Nonlinear acoustics in solids and structures : Elastic waves, viscoelastic waves, surface waves, nonlinear acousto-electronics, non-destructive evaluation and testing, etc.
- Nonlinear acoustics of reacting, relaxing media, and physical kinetics : Nonlinear acoustics in superfluid helium, waves in rarefied gases,micro-acoustics, nano-acoustics, quantum effects, sonic crystals, metamaterials, etc.
- Nonlinear acoustics in medicine and biology : Shock wave therapy, diagnostic ultrasound, ultrasound propagation in bone and biological tissue, nonlinear acoustics in speech, etc.
- Thermoacoustics : Energy conversion and their devices, aero-thermoacoustics, combustion noise and oscillations, etc.
- Nonlinear acoustics of atmosphere, ocean, and earth, and nonlinear underwater acoustics : Shock wave, sonic boom, aircraft noise, intense noise generated by ground transportation, infrasound, acoustic-gravity waves, explosions, earthquakes, etc.
- Nonlinear aero- and hydroacoustics : Vortex sound, jets, turbulence, flow-induced sound, etc.
- Nonlinear acoustics and optics : Laser generation of acoustic waves, optoacoustical spectroscopy, magneto-acoustics, etc.
- General experimental methods : Measurements, instrumentations, etc.
- Devices and industrial applications of nonlinear acoustics : Musical acoustics is included here.

- Other topics in nonlinear acoustics.

As can be seen, the scope of nonlinear acoustics has broadened significantly from its mathematical roots to become a multidisciplinary field. The conference proceedings are published under the auspices of the American Institute of Physics Conference Proceedings.

Chapter 3

One-dimensional gas motion

3.1 Introduction

In this chapter we derive from first principles the governing equations of motion of gas considered as a one-dimensional continuum. Physically, this can be thought of as motion of gas or fluid in a cylindrical tube. The flow variables of interest are pressure, velocity and density. The lateral wall of the pipe is assumed to have no effect on these flow parameters. Further, it is also assumed that each cross-section remains planar and moves longitudinally down the cylinder and that there is no variation of any of the flow parameters in any cross section. This assumption of longitudinal variation gives the description its one-dimensional character. The derivation will show that the motion is governed by a quasilinear hyperbolic system of first order partial differential equations and an inequality constraint.

The presentation here follows the original 3-dimensional version of Sauer [21].

3.2 Eulerian and Lagrangian descriptions

There are two basic coordinate systems for keeping track of motion of a fluid in one dimension. These are the Eulerian and Lagrangian coordinate systems.

Suppose one wishes to measure the temperature of water in a stream. The experimenter can, for instance, stand on the bank at a fixed location x, measured from some reference position $x = 0$, and insert a thermometer, thereby measuring the emperature $\theta(x, t)$ as a function of position x and time t. The coordinate x is called a Euler coordinate. The variable x is a fixed spatial coordinate, and the representation of the physical variables (temperature, density, pressure, etc) as functions of t and x gives a Eulerian description of the flow.

On the other hand, the experimenter can measure the temperature from a boat drifting with the flow. In this case, at time $t = 0$ each particle (or section) is labelled with particle label x, and each particle retains its label as it moves downstream. The result of the measurement is the temperature $\Theta(x,t)$ as a function of t and Lagrangian or material coordinate x. A representation in terms of t and the material variable x gives a Lagrangian description of the flow.

Now take points on the real line as the reference configuration for the 'particles' in the tube. Let $X = X(t)$ be the position of particle x at time t. Suppose that at time t particle x is in position X. By a *fluid motion*, or flow, we mean a twice continuously differentiable function $\phi : \mathbb{R} \times [0,t_1] \longrightarrow \mathbb{R}$ defined by

$$X = \phi(x,t) \tag{3.1}$$

which for each $t \in [0,t_1]$ is invertible in $\mathbb{R}$.

This mapping describes the motion of every particle in the tube. Since ϕ is a function, two particles cannot be in the same place at the same time. This is illustrated in Figure 3.1 below.

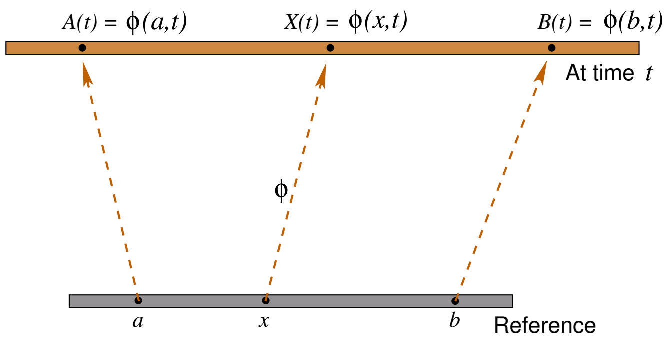

FIGURE 3.1: Illustration of fluid motion.

The (one-dimensional) Jacobian is defined by

$$J(x,t) \equiv \phi_x(x,t). \tag{3.2}$$

Also, since we expect particles to maintain the numerical placement of the reference configuration, that is, if $a < b$ in the reference configuration, then $A = \phi(a,t) < B = \phi(b,t)$, it is necessary to require that $J(x,t) > 0$.

If the velocity of particle x at time t is defined by

$$V(X) = v(x,t) := \phi_t(x,t); \quad X = \phi(x,t), \tag{3.3}$$

we see that

$$J_t(x,t) = \frac{\partial}{\partial t}\phi_x(x,t) = \frac{\partial}{\partial x}\phi_t(x,t) = v_x(x,t), \tag{3.4}$$

which is Euler's expansion formula in one-dimensional form.

3.3 Conservation of mass

The equations that govern the motion of a gas or fluid in a one-dimensional continuous medium essentially express conservation of mass and momentum and are universal in that they are valid for any medium. Our derivation of these equations is based on a Lagrangian approach. Our first governing equation states mathematically that the mass in an arbitrary material portion of the cylinder does not change as that portion of material moves in time. Consider in the reference state, an arbitrary portion of fluid of cross-sectional area $\mathcal{A}$ between $x = a$ and $x = b$ that after time t has moved to the region between $A(t) \equiv \phi(a,t)$ and $B(t) \equiv \phi(b,t)$. Let $\rho(x)$ denote the mass density function of the fluid in the reference state and let $\sigma(X,t)$ denote the mass density function at time t. Then mass conservation is expressed in the form

$$\int_a^b \rho(x)\mathcal{A}\, dx = \int_{A(t)}^{B(t)} \sigma(X,t)\mathcal{A}\, dX.$$

A change of variables according to $X = \phi(x,t)$ results in

$$\int_a^b \rho(x) = \int_a^b \sigma(\phi(x,t),t)\phi_x(x,t)\, dx.$$

Since the interval $[a,b]$ was arbitrary, it follows that the integrands are equal and we arrive at the equation

$$\rho(x) = \sigma(\phi(x,t),t)\phi_x(x,t),$$

or, from (3.2),

$$\rho(x) = \sigma(\phi(x,t),t)J(x,t), \tag{3.5}$$

which expresses conservation of mass as a simple equation.

3.4 Balance of linear momentum

In classical mechanics, Newton's second law asserts that the time rate of change of momentum of the particle is equal to the net external force acting upon it. This is extended to continuous media, by the balance of linear momentum principle which states that the time rate of change of linear momentum of any part of the continuum equals the resultant of forces acting upon it. In the present case the linear momentum at time t of a material of cross-sectional area $\mathcal{A}$ in the material region $A(t) \leq X \leq B(t)$ moving with velocity V is defined as

$$\mathcal{A} \int_{A(t)}^{B(t)} \sigma(X,t)V(X)\ dX.$$

The forces acting on a material region in a continuous medium can be characterized into two types: body forces and forces due to internal stress. Body forces are forces such as gravity, electric or magnetic fields. We shall ignore body forces.

We consider an ideal gas in which pressure is the only stress. This means that the internal force on every surface element of the fluid is pressure times the area of the element directed inward along the normal to the surface. At time t this takes place in the current state where position is measured by X. Thus the resultant force on a section $[A(t), B(t)]$ of the cylinder is $\mathcal{A}[P(A(t),t) - P(B(t),t)]$ with $\mathcal{A}$ the cross-sectional area of the tube. The principle of balance of linear momentum may now be stated mathematically as follows:

$$\frac{d}{dt}\int_{A(t)}^{B(t)} \sigma(X,t)V(X)\mathcal{A}\,dX = \mathcal{A}[P(A(t),t) - P(B(t),t)] = -\mathcal{A}\int_{A(t)}^{B(t)} P_X(X,t)\,dX.$$

We revert to the reference configuration by using (3.1) as transformation. At the same time we set $p(x,t) = P(X,t) = P(\phi(x,t),t)$ for the pressure "experienced" by x at time t. With $p_x = P_X\,\phi_x$, the result is

$$\frac{d}{dt}\int_a^b v(x,t)\sigma(\phi(x,t),t)\phi_x(x,t)\ dx = -\int_a^b p_x(x,t)\ dx, \tag{3.6}$$

having made use of the definition (3.3) of velocity. Substituting expression (3.5) into (3.6), we obtain the simplification

$$\frac{d}{dt}\int_a^b \rho(x)v(x,t)\ dx + \int_a^b p_x(x,t)\ dx = 0. \tag{3.7}$$

If we assume that the time derivative and the integral can be interchanged and that v has an integrable time derivative, we obtain from (3.7) the equation

$$\int_a^b \left[\rho(x)v_t(x,t) + p_x(x,t)\right]\,dx = 0. \tag{3.8}$$

Since the interval $[a, b]$ is arbitrary, it follows that the integrand in (3.8) is zero. Thus we obtain the first equation of motion

$$\rho(x)v_t(x,t) + p_x(x,t) = 0. \tag{3.9}$$

3.5 Constitutive relations

Equation (3.9) is a nonlinear partial differential equation in two unknowns v and p. An additional dynamic equation of motion is therefore required. We can intuitively expect that the particular physical properties of the medium must play a role in the derivation of the second equation because the properties are not included in (3.9). The equations that specify properties of the medium are known as equations of state or *constitutive relations*. A particular constitutive relation that we will use is the barotropic equation of state in which the uniform density σ of a gas in a closed container depends on the pressure, or

$$\sigma = f(p); \quad f'(p) > 0. \tag{3.10}$$

3.5.1 The Acoustic assumption

The *acoustic assumption* due to Laplace is that the derivative $f'(p)$ is constant for an isentropic gas. It is convenient to introduce the *static sound speed* c defined by

$$\frac{d\sigma}{dp} = f'(p) = \frac{1}{c^2} \tag{3.11}$$

It is expedient to introduce the idea of *compression* of a gas in a closed container. Following Sauer [21], consider a closed container of compressible gas in a 'reference state' where the volume of gas is V_0, its pressure is p_0 and its density is ρ. If we compare this to another state in which the volume is V and the density is σ, we define the compression r of the gas (relative to the reference state) as the dimensionless quantity

$$r := \frac{V - V_0}{V_0} = \frac{V}{V_0} - 1.$$

Since *density = mass per unit volume*, we obtain from from the principle of conservation of mass, valid since the container is closed, that

$$\frac{V}{V_0} = \frac{\rho}{\sigma} = r + 1, \tag{3.12}$$

From the barotropic equation of state (3.10) and the acoustic assumption (3.12) it follows that

$$\frac{\rho}{1+r} = \sigma = f(p).$$

We will now derive a result relating compression to pressure only. If in the reference state, $p = p_0$ and $V = V_0$, it follows that the reference state compression $r(p_0) = 0$. From the acoustic assumption (3.11)

$$\frac{d}{dp}\left[\frac{\rho}{1+r}\right] = \frac{1}{c^2}.$$

Since the reference density ρ is constant, this equation may be integrated directly to obtain

$$\frac{1}{1+r} = \frac{p}{\rho c^2} + C$$

But in the reference state, $p = p_0$ and $r = r(p_0) = 0$, so that

$$C = 1 - \frac{p_0}{\rho c^2}.$$

This results in the useful expression

$$\frac{1}{1+r} = 1 + \frac{p - p_0}{\rho c^2}. \tag{3.13}$$

3.5.2 The dynamics of compression

As we are operating in a Lagrangian framework, we localize the notion of compression set out above in the following way. The compression $r(x,t)$ experienced by a particle x at time t is defined by

$$\begin{aligned} r(x,t) &:= \lim_{b\to a} \frac{\mathcal{A}[(B(t) - A(t)] - \mathcal{A}[b-a]}{\mathcal{A}[b-a]} \\ &= \lim_{b\to a} \frac{B(t) - A(t)}{b-a} - 1 \\ &= \lim_{b\to a} \frac{\phi(b,t) - \phi(a,t)}{b-a} - 1 \\ &= \phi_x(x,t) - 1 \\ &= J(x,t) - 1, \end{aligned} \tag{3.14}$$

provided that $a < x < b$.

From (3.5)

$$r(x,t) = \frac{\rho(x)}{\sigma(\phi(x,t),t)} - 1.$$

This is the localized version of (3.12).

Next we use the expression (3.13) in localized form to obtain the *localized acoustic assumption*

$$\frac{1}{1+r(x,t)} = 1 + \frac{p(x,t) - p_0}{\rho c^2} \tag{3.15}$$

where the static sound speed c and the pressure p_0 are assumed constant.

3.6 Evolution equations

When we differentiate (3.15) with respect to time, the result is

$$\frac{p_t(x,t)}{\rho c^2} = -\frac{r_t(x,t)}{(1+r(x,t))^2}. \tag{3.16}$$

Euler's expansion formula (3.4) together with (3.14) shows that

$$r_t(x,t) = J_t(x,t) = v_x(x,t).$$

We can use this result and the localized acoustic assumption (3.15) to simplify (3.16). The result is

$$\frac{p_t(x,t)}{\rho c^2} = -\left[\frac{\rho c^2 + p(x,t) - p_0}{\rho c^2}\right]^2 v_x(x,t).$$

This gives us our second dynamic equation to accompany (3.9)

$$p_t(x,t) + \frac{1}{\rho c^2}\left[\rho c^2 + p(x,t) - p_0\right]^2 v_x(x,t) = 0. \tag{3.17}$$

We can deduce a constraint on the lower bound of the pressures that can develop. The Jacobian $J(x,t)$ is positive so $1 + r(x,t) > 0$. Therefore from (3.15) we have

$$1 + \frac{p(x,t) - p_0}{\rho c^2} > 0. \tag{3.18}$$

The next section considers dimensionalization and scaling issues in which ρc^2 is the unit of pressure. This will express the constraint (3.18) in a very simple form.

3.7 Dimensional considerations

The construction of a mathematical model of physical processes is not complete without dimensional analysis. A mathematical model is a functional relationship of relevant variables and parameters that describe a particular problem. In our instance, the variables are p, v, x, t and the parameters are ρ, c. The equations (3.9) and (3.17) functionally relate them. Each variable and parameter has a relationship with the basic dimensions of mass, length and time. A complete model is one which is dimensionally correct. Dimensional correctness ensures that apples do not equal oranges. Dimensional methods involve two techniques: *dimensional analysis* and *scaling*.

In this section we determine the dimensions of the variables and parameters of our problem. We employ the techniques of dimensional analysis and scaling to construct dimensionless variables that enable us to nondimensionalize (3.9), (3.17) and (3.18). For reference material, Chapter 1 of the text by Logan [16] provides a thorough and balanced treatment of dimensional methods. The classic book by Lin and Siegel [15] devotes Chapter 6 to nondimensionalizing, albeit from a more formal standpoint. Fowler [8] and Howison [11] are oriented towards applications and provide examples and problems using a variety of mathematical models.

3.7.1 Units and dimensions

Most physical quantities can be expressed in terms of combinations of primary dimensions. These are mass [M], length [L], time [T], electrical current [I] and temperature [Θ]. The standard notation is to denote the dimensions of a quantity in brackets around the quantity. Given the primary dimensions of a quantity, one can derive all its secondary dimensions. In some cases, this is direct. For example, for speed u, we have $[u] = [L][T]^{-1}$. In other cases, the dimensional structure of a physical law is used as in force F = mass × acceleration, so $[F] = [M][L][T]^{-2}$ and pressure, p = force per unit area so $[p] = [M][L][T]^{-2}[L]^{-2} = [M][L]^{-1}[T]^{-2}$.

Table (3.1) below summarizes what we know about the variables and parameters of our model. From the table $\left[\rho c^2\right] = [M][L]^{-3}[L]^2[T]^{-2} = [M][L]^{-1}[T]^{-2} = [p]$. This shows that ρc^2 has the same dimensions as pressure p so that the inequality (3.18) makes sense.

3.7.2 Dimensional analysis

Dimensional analysis entails verifying that each term in the equations of our model, (3.9), (3.17) and (3.18), has the same dimensions, that is, the model is dimensionally

TABLE 3.1: Variables and their dimensions

Variable	Physical quantity	Meaning	Dimension
Independent variable x	Distance	-	$[L]$
Independent variable t	Time	-	$[T]$
Dependent variable v	Velocity	Distance per unit time	$[L][T]^{-1}$
Dependent variable p	Pressure	Force per unit area	$[M][L]^{-1}[T]^{-2}$
Parameter ρ	Density	Mass per unit volume	$[M][L]^{-3}$
Parameter c	Velocity	Distance per unit time	$[L][T]^{-1}$

homogeneous. Dimensional analysis permits us to understand the dimensional relationships (meaning mass, length, time etc) of the quantities and the resulting implications to dimensional homogeneity. The cornerstone result in dimensional analysys is known as the *Pi theorem*. The Pi theorem [16, Chapter 1, pp.6–17] states that if there is a physical law that gives a relation among a certain number of dimensioned quantities, then there is an equivalent law that can be expressed as a relation among dimensionless quantities (often noted by $\pi_1, \pi_2, \ldots$, and hence the name). In the early 1900s, E. Buckingham gave a proof of the Pi theorem for special cases, and now the theorem often carries his name. Lin and Siegel [15, Chapter 6, exercise 12, p.207] outline, by way of a guided, step-by-step exercise, a proof of the Buckingham Pi theorem.

3.7.3 Scaling

Scaling on the other hand, is a technique that helps us to understand the magnitude of the terms that appear in the model equations by comparing the quantities to intrinsic reference quantities that appear naturally in the physical situation. Scaling facilitates the construction of dimensionless variables. These are key to nondimensionalizing a model and consequently, to dimensional homogeneity. Fowler [8, Chapter 2, pp.19–34] describes this process very simply: if a model has a variable w, then we can nondimensionalize that variable by writing, for example,

$$u = \bar{w}w^*,$$

where $\bar{w}$ is the chosen scale and w^* is the corresponding dimensionless variable.

3.7.4 Dimensionless parameters

Equations (3.9) and (3.17) contain four variables —p, v, x, t— and two parameters, ρ, c. The presence of the parameters in particular, render equations (3.9) and (3.17) somewhat intractable to work with. To dispense with this potential irritation, we nondimensionalize (3.9) and (3.17). The Buckingham π theorem then guarantees us that the parameters will be eliminated altogether. We follow Lin and Siegel's two-step procedure for nondimensionalization [15, Chapter 6, pp.195–203].

STEP A. *List all parameters and variables, together with their dimensions.*

In the present instance, this is presented in Table (3.1).

STEP B. *For each variable, select an intrinsic reference quantity and form an associated dimensionless variable.*

An intrinsic reference quantity is a "standard of measurement formed from the parameters of a given problem" ([15], p.196). In our case, if we imagine a length parameter L (which could possibly be the length of the tube in our model) in terms of which we can scale position, then we can scale time in terms of $T = L/c$, velocity in terms of c and pressure in terms of ρc^2. This allows us to introduce the dimensionless variables

$$\tau = t/T, \quad y = x/L, \quad w = v/c, \quad f = (p - p_0)/\rho c^2. \tag{3.19}$$

The variables are dimensionless because their numerical value is the same whatever standard of measurement is used.

Next, we substitute with the help of the chain rule the new variables (3.19) in (3.9) and (3.17). The results are

$$\begin{aligned}
v(x,t) &= cw(y,\tau), \\
v_x(x,t) &= cw_y(y,\tau)\frac{dy}{dx} = \frac{c}{L}w_y(y,\tau), \\
v_t(x,t) &= w_\tau(y,\tau)\frac{d\tau}{dt} = \frac{c}{T}w_\tau(y,\tau), \\
p(x,t) &= \rho c^2 f(y,\tau) + p_0, \\
p_x(x,t) &= \rho c^2 f_y(y,\tau)\frac{dy}{dx} = \frac{\rho c^2}{L}f_y(y,\tau), \\
p_t(x,t) &= \rho c^2 f_\tau(y,\tau)\frac{d\tau}{dt} = \frac{\rho c^2}{T}f_\tau(y,\tau).
\end{aligned} \tag{3.20}$$

We then substitute the respective dimensionless components from (3.20) in (3.9). We note that $T = L/c$. The result is

$$\frac{\rho c^2}{L} w_\tau(y,\tau) + \frac{\rho c^2}{L} f_y(y,\tau) = 0.$$

By assumption, ρ is constant. Hence the scaled and dimensionless form of (3.9) is

$$w_\tau(y,\tau) + f_y(y,\tau) = 0.$$

In a similar manner, we make relevant substitutions from (3.20) in (3.17) to obtain

$$\frac{\rho c^2}{T} f_\tau(y,\tau) + \frac{\rho c^2}{T} \left[1 + f(y,\tau)\right]^2 w_y(y,\tau) = 0.$$

This results in the scaled and dimensionless form of (3.17)

$$f_\tau(y,\tau) + \left[1 + f(y,\tau)\right]^2 w_y(y,\tau) = 0.$$

We can also scale and nondimensionalize the constraint (3.18). This is

$$1 + f(y,\tau) > 0.$$

Note that the constant pressure p_0 can be absorbed in the pressure term and will therefore be ignored. In the pages to follow, we shall simply identify w with v and f with p. Thus we derived a system of scaled equations that govern the motion of a fluid in one dimension

$$\left.\begin{aligned} v_t(x,t) + p_x(x,t) = 0; \\ p_t(x,t) + [1 + p(x,t)]^2 v_x(x,t) = 0. \end{aligned}\right\} \tag{3.21}$$

To this is added the constraint

$$1 + p(x,t) > 0.$$

3.8 Summary

In this chapter we formulated in the Lagrangian framework the equations of motion of a fluid in one-dimension. We employed techniques from dimensional analysis to scale and nondimensionalize the equations. In the next Chapter we digress a little and present key results from the theory of hyperbolic partial differential equations that we will need in later chapters. In particular, we will use the results to reduce the system of partial differential equations (3.21) to a system of ordinary differential equations that is amenable to further analysis.

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Chapter 4

On hyperbolic partial differential equations

4.1 Introduction

In this chapter we pause to collate some relevant concepts and results from the general theory of first order hyperbolic Partial Differential Equations (PDEs). The material is sourced from the texts by Jeffrey [13, Chapters 1–3, pp. 1–80], Courant and Hilbert [4, Chapter 1, pp.28–56, Chapter 2, pp.62–68], Courant and Friedrichs [3, Chapter 2, pp.37–59, Chapter 3, pp.79–88] and Toro [25, Chapter 2, pp.41–86].

4.2 First-order quasi-Linear equations

General systems of first-order partial differential equations are of the form

$$u_{i,t} + \sum_{j=1}^{m} a_{ij}(x, t, u_1, \dots, u_m) u_{j,x} + b_i(x, t, u_1, \dots, u_m) = 0, \tag{4.1}$$

for $i = 1, \dots, m$. This is a system of m equations in m unknowns u_i that depend on space x and a time-like variable t. Here u_i are dependent variables and x, t are the independent variables, expressed via the notation $u_i = u_i(x, t)$. The partial derivative of $u_i(x, t)$ with respect to t is denoted by $u_{i,t}$; similarly ($u_{i,x}$ denotes the partial derivative of $u_i(x, t)$ with respect to x. System (4.1) can be written in matrix form as

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B} = 0,$$

with

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

If the entries a_{ij} of the matrix $\mathbf{A}$ are all constant and the components b_j of the vector $\mathbf{B}$ are also constant then the system (4.1) is *linear with constant coefficients.* If $a_{ij} = a_{ij}(x,t)$ and $b_i = b_i(x,t)$ the system is *linear with variable coefficients.* The system is called quasi-linear if the coefficient matrix $\mathbf{A}$ is a function of the vector $\mathbf{U}$, that is $\mathbf{A} = \mathbf{A}(\mathbf{U})$. For a system of PDEs of the form system (4.1) the range of variation of the independent variables x and t needs to be specified. Usually x lies in a subinterval of the real line, namely $x_1 < x < x_r$ and this subinterval is called the *domain* of the PDEs. At the values of x_1, x_r, one also needs to specify the *boundary conditions (BCs).* In this research, we assume the domain is the full real line, $-\infty < x < \infty$, and thus no boundary conditions need to be specified. As to the variation of time t we assume $t_0 < t < \infty$. An *initial condition (IC)* needs to be specified at the initial time, which is usually chosen to be $t_0 = 0$.

The system (3.21) derived in Chapter 3 is an example of system (4.1) with $m = 2, u_1 = u_1(t,x) = v(t,x)$ and $u_2 = u_2(t,x) = p(t,x)$.

4.3 Characteristics and the solution of hyperbolic PDEs

Consider the special case of a homogeneous linear differential equation

$$\sum_{i=1}^{m} a_i u_{x_i} = 0. \tag{4.2}$$

In the n-dimensional space of the variables $x_1, x_2, \ldots, x_m$ we determine the curves $x_i = x_i(s)$ in terms of a parameter s by means of the system of ordinary differential equations

$$\frac{dx_i}{ds} = a_i(x_1, x_2, \ldots, x_m). \tag{4.3}$$

for $i = 1, 2, \ldots, m$. These curves are called the *characteristic curves.* Application of the chain rule for the values $u(s) = u(x_1(s), x_2(s), \ldots, x_m(s))$ (a solution u of (4.2)) along the characteristic curves of ordinary differential equations, shows that

$$\frac{du}{ds} = \sum_{i=1}^{m} u_{x_i} \frac{dx_i}{ds} = \sum_{i=1}^{m} a_i u_{x_i} = 0$$

holds. Thus along each characteristic curve of the system (4.3) every solution of the partial differential equation (4.2) has a constant value. Every solution of the partial differential equation is an integral of the system of ordinary differential equations. For the converse and an argument extending the result to the more general case of a non-homogeneous nonlinear PDE, see [4, Chapter 1, pp.29–32].

Thus through its characteristics a hyperbolic PDE is reduced to a system of ordinary differential equations, which may be more tractable to solve in most cases.

Related to the concept of characteristics is the concept of domain of dependence, the subject of the next section.

4.3.1 Domain of dependence

The characteristic directions are determining factors in discussing the dependance of the solutions on the given Cauchy data. Consider any point P in the (t, x)-plane. If we draw the the two characteristic curves C_1 and C_2 through the point(t, x) until they intersect the x-axis in two points a_1 and a_2, the interval $[a_1, a_2]$ on the x-axis is called the *domain of dependence* of the point (t, x) and is illustrated in Figure 4.1 below. Its significance is that if any two solutions to a PDE share the same domain of dependence then they are necessarily identical [3, p.51].

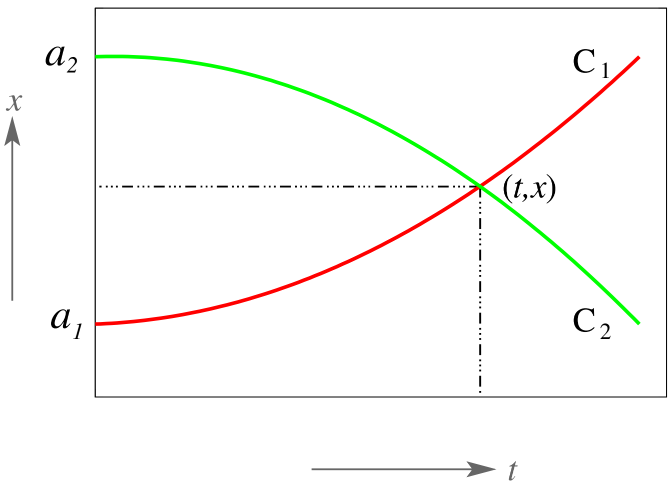

Figure 4.1: Domain of dependence

4.4 Well-posedness of quasilinear hyperbolic systems

The well-posedness of the Cauchy problem of hyperbolic systems is determined, in part, through the concept of *hyperbolicity*. This, in turn, is expressed through the concepts of *eigenvalues* and *eigenvectors*.

Definition 4.1. (**Eigenvalues**). *The eigenvalues λ_i of a matrix $\mathbf{A}$ are the solutions of the characteristic polynomial*

$$|\mathbf{A} - \lambda\mathbf{I}| = \det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

where $\mathbf{I}$ is the identity matrix. The eigenvalues of the coefficient matrix $\mathbf{A}$ of system (4.1) *are called the eigenvalues of the system.*

Physically, eigenvalues represent speeds of propagation of information. Positive speeds are measured in the direction of increasing x and negative speeds otherwise.

Definition 4.2. (**Eigenvectors**). *A right eigenvector of a matrix $\mathbf{A}$ corresponding to an eigenvalue λ_i of $\mathbf{A}$ is a vector $\mathbf{K}^i = \left[k_1^i, k_2^i, \ldots, k_m^i\right]^T$ satisfying $\mathbf{A}\mathbf{K}^i = \lambda_i \mathbf{K}^i$. Similarly, a left eigenvector of a matrix $\mathbf{A}$ corresponding to an eigenvalue λ_i of $\mathbf{A}$ is a vector $\mathbf{L}^i = \left[l_1^i, l_2^i, \ldots, l_m^i\right]$ satisfying $\mathbf{L}^i\mathbf{A} = \lambda_i \mathbf{L}^i$.*

Definition 4.3. (**Hyperbolic System**). *A system* (4.1) *is said to be hyperbolic at a point (x, t) if $\mathbf{A}$ has m real eigenvalues $\lambda_1, \ldots, \lambda_m$ and a corresponding set of m linearly independent right eigenvectors $\mathbf{K}^1, \ldots, \mathbf{K}^m$. The system is said to be strictly hyperbolic if the eigenvalues λ_i are all distinct.*

The basic idea underlying the hyperbolicity of a system is that the Cauchy problem for hyperbolic problems is well-posed, that is, a unique solution that depends continuously on the data in the domain, exists. This is encapsulated in the following existence theorem [13, Chapter 2, p.71].

Theorem 4.4. (Existence and Uniqueness in the General Quasilinear Case). *Let the quasilinear system of partial differential equations*

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B} = 0, \tag{4.4}$$

be such that:

1. *It is hyperbolic throughout the (t, x)-plane;*

2. *The coefficient matrices $\mathbf{A}, \mathbf{B}$ have Lipschitz continuous partial derivatives;*

3. It satisfies an initial condition

$$\mathbf{U}(x,0) = \mathbf{U}_0(x), \quad a < x < b \tag{4.5}$$

for which $d\mathbf{U}_0/dx$ *is Lipschitz continuous.*

Then a unique solution to (4.4) *and* (4.5) *exists in a neighbourhood of the interval* $a < x < b$ *of the initial line* $t = 0$*, and furthermore, within this neighbourhood the solution has Lipschitz continuous partial derivatives.*

4.5 Summary

In this chapter we collated some concepts and results from the general theory of hyperbolic PDEs that are most relevant to our work. In the next chapter we continue from Chapter 3, and formulate our Cauchy problem and consider its well-posedness.

Chapter 5

The Cauchy problem

5.1 Introduction

In Chapter 3 we reformulated Sauer's equations for the transmission of sonic disturbances in one dimension. In this chapter we initiate an attempt to solve the equations. We formulate the Cauchy problem and consider its well-posedness. We then determine its characteristics and introduce the new concept of *inverse characteristic* which we use to reduce the partial differential equations to systems of nonlinear ordinary differential equations.

5.2 The Cauchy problem

Let $v(t, x)$ and $p(t, x)$ be continuously differentiable functions defined for real x and $t > 0$ that describe the velocity and pressure of fluid particles of an isentropic ideal gas in one-dimension which satisfy the following equations derived in Chapter 3:

$$\left.\begin{aligned} v_t(t, x) + p_x(t, x) = 0; \\ p_t(t, x) + [1 + p(t, x)]^2 v_x(t, x) = 0. \end{aligned}\right\} \tag{5.1}$$

To this is added the constraint

$$1 + p(t, x) > 0, \tag{5.2}$$

and initial conditions

$$\left.\begin{aligned} &v(0, x) = 0; \\ &p(0, x) = p_0(x). \end{aligned}\right\} \tag{5.3}$$

The equations (5.1),(5.3) is called the Cauchy problem. We must find conditions that will ensure that the system (5.1) is hyperbolic. Hyperbolicity is crucial to the well-posedness of the Cauchy problem we would like to formulate. See Chapter 4, Section 4.4 for a summary of the requisite notions and results. To this end, we write system (5.1) in matrix-vector form as

$$\mathbf{V}_t + \mathbf{A}\mathbf{V}_x = \mathbf{0},$$

with

$$\mathbf{V}(t,x) = \begin{bmatrix} v(t,x) \\ p(t,x) \end{bmatrix},$$

and

$$\mathbf{A}(t,x) = \begin{bmatrix} 0 & 1 \\ [1+p(t,x)]^2 & 0 \end{bmatrix}.$$

The right eigenvalue problem is: find numbers λ and nonzero vectors $\mathbf{k}$ such that $\mathbf{A}\mathbf{k} - \lambda\mathbf{k} = \mathbf{0}$. It turns out that the eigenvalues λ are $\lambda_1 = 1 + p(t,x)$ and $\lambda_2 = -[1 + p(t,x)]$. The constraint (5.2) therefore guarantees that the system is hyperbolic.

The left eigenvalue problem is: find μ and row vectors $\mathbf{k}$ such that $\mathbf{A}^T\mathbf{k}^T = \mu\mathbf{k}^T$. Because of the particular form of $\mathbf{A}$ the left and right eigenvalues are the same. The left eigenvectors are the rows of the matrix $\mathbf{K}$ defined as

$$\mathbf{K}(t,x) = \begin{bmatrix} 1+p & 1 \\ -[1+p] & 1 \end{bmatrix}.$$

These vectors are linearly independent and so because of the constraint, we conclude that the system is hyperbolic.

To accommodate the constraint we introduce the change of variable

$$1 + p(t,x) = \exp\{q(t,x)\} \tag{5.4}$$

with $q(t,x)$ a continuously differentible function defined on the real line. This transforms system (5.1) to the following system:

$$\left.\begin{aligned} v_t(t,x) + \exp\{q(t,x)\}q_x(t,x) = 0; \\ q_t(t,x) + \exp\{q(t,x)\}v_x(t,x), = 0 \end{aligned}\right\} \tag{5.5}$$

with initial conditions

$$\left.\begin{aligned} &v(0,x) = 0; \\ &q(0,x) = q_0(x) = \ln[1 + p_0(x)]. \end{aligned}\right\} \tag{5.6}$$

The matrix formulation now changes to

$$\mathbf{V} = \begin{bmatrix} v \\ q \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & \exp\{q\} \\ \exp\{q\} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1+p \\ 1+p & 0 \end{bmatrix}.$$

The left eigenvalue problem is: to find λ and row vectors $\mathbf{k}$ such that $\mathbf{A}^T\mathbf{k}^T = \lambda\mathbf{k}^T$. Let $\mathbf{k} = [k_1, k_2]$. We are left to solve the equations $k_2(1+p) = \lambda k_1$ and $k_1(1+p) = \lambda k_2$. It follows $\lambda_1 = (1+p)$ and $\lambda_2 = -(1+p)$, as before. For λ_1, we find that $k_1 = k_2$ and we can without loss of generality take $k_2 = 1$. Similarly, for λ_2, we find that $k_1 = -k_2$ and we can take $k_2 = 1$. Consequently the *canonical* matrix $\mathbf{K}$ now is:

$$\mathbf{K} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Notice that the matrices $\mathbf{A}$ and $\mathbf{K}$ are symmetric. So the effect of the transformation is to symmetrize the problem! But there is more. Since the eigenvectors of the canonical matrix $\mathbf{K}$ are orthogonal, the matrix $\mathbf{K}$ is almost its own inverse. In fact, $\mathbf{K}^{-1} = \frac{1}{2}\mathbf{K}$. Let us make one more transformation. Let $\mathbf{V} = \mathbf{KU}$ thus defining the vector $\mathbf{U}$ with $U^T = [u_1, u_2]$. Since $\mathbf{K}$ is a constant matrix, (5.1) becomes

$$\mathbf{KU}_t + \mathbf{AKU}_x = \mathbf{0}.$$

This equation can be put in standard form by operating on it by $\mathbf{K}^{-1}$ the result is

$$\mathbf{U}_t + \tfrac{1}{2}\mathbf{KAKU}_x = \mathbf{0}.$$

A direct calculation shows that

$$\tfrac{1}{2}\mathbf{KAK} = \exp\{q\}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The newly-transformed system in component form is

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} + \exp\{q\}\begin{bmatrix} u_{1,x} \\ -u_{2,x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (5.7)$$

Since $U = \mathbf{K}^{-1}\mathbf{V} = \frac{1}{2}\mathbf{K}\mathbf{V}$, the components of $\mathbf{U}$ can be expressed as follows

$$\left.\begin{aligned} u_1(t,x) &= \frac{1}{2}\Big[v(t,x) + q(t,x)\Big]; \\ u_2(t,x) &= \frac{1}{2}\Big[v(t,x) - q(t,x))\Big]. \end{aligned}\right\} \tag{5.8}$$

In component form (5.7) is

$$\left.\begin{aligned} u_{1,t}(t,x) + \exp\{q(t,x)\}u_{1,x}(t,x) = 0; \\ u_{2,t}(t,x) - \exp\{q(t,x)\}u_{2,x}(t,x) = 0, \end{aligned}\right\} \tag{5.9}$$

and from (5.8) we see that

$$q(t,x) = u_1(t,x) - u_2(t,x).$$

The equations (5.9) can be obtained directly by addition and subtraction of the two equations in (5.5). The transformed initial conditions are

$$\left.\begin{aligned} u_{10}(x) = u_1(0,x) = \frac{1}{2}\Big[v(0,x) + q(0,x)\Big] = \frac{1}{2}q_0(x); \\ u_{20}(x) = u_2(0,x) = \frac{1}{2}\Big[v(0,x) - q(0,x)\Big] = -\frac{1}{2}q_0(x). \end{aligned}\right\} \tag{5.10}$$

Thus the Cauchy problem implicit in system (5.1) - (5.3) is compounded of the equations

$$\left.\begin{aligned} u_{1,t}(t,x) + \exp\{q(t,x)\}u_{1,x}(t,x) = 0; \\ u_{2,t}(t,x) - \exp\{q(t,x)\}u_{2,x}(t,x) = 0; \\ q(t,x) = u_1(t,x) - u_2(t,x), \end{aligned}\right\} \tag{5.11}$$

and the initial conditions

$$\left.\begin{aligned} u_{10}(x) &= \frac{1}{2}q_0(x); \\ u_{20}(x) &= -\frac{1}{2}q_0(x). \end{aligned}\right\} \tag{5.12}$$

We remark that the problem is well-posed by courtesy of Theorem 4.4.

5.3 Characteristics

We consider the first equation from (5.11)

$$u_{1,t}(t,x) + \exp\{q(t,x)\}u_{1,x}(t,x) = 0. \tag{5.13}$$

We can use the chain rule to determine the rate of change of $U_1(t) := u_1(t, X_1(t))$ (measured by an observer moving along the curve $x = X_1(t)$). The result is

$$U_1'(t) = u_{1,t} + X_1' u_{1,x}. \tag{5.14}$$

The first term $u_{1,t}(t,x)$ represents the rate of change of $u_1(t,x)$ at a fixed position, while the term $X_1' u_{1,x}$ represents the change due to the fact that the observer moves into a region of possibly different u_1. If we compare (5.14) with (5.13), it is apparent that if the observer moves with velocity $\exp\{q(t,x)\}$, that is, if $X_1' = \exp\{q(t, X_1(t))\}$ then $U_1'(t) = 0$.

Thus u_1 is constant along the curve $x = X_1$ chosen as above. An observer moving with this special speed $\exp\{q(t, X_1(t))\}$ would measure no change in u_1. Similarly, from the second equation of (5.11), an observer moving along $x = X_2(t)$ with the special speed $-\exp\{q(t, X_2(t))\}$ would see no change in u_2.

The curves C_1 and C_2 defined by $x = X_1(t)$ and $x = X_2(t)$ respectively are called the *characteristics* of the system (5.11). Their significance is that along them u_1 and u_2 are constant. The characteristics satisfy the ordinary differential equations

$$X_1{}'(t) = \exp\{q_1(t)\}, \quad X_2{}'(t) = -\exp\{q_2(t)\}, \tag{5.15}$$

where

$$q_1(t) = q(t, X_1(t)), \quad q_2(t) = q(t, X_2(t)). \tag{5.16}$$

We note that C_1 is an upward sloping curve and C_2 is sloping downward.

5.4 Reduction to ordinary differential equations

We noted in Chapter 4 that a fundamental result in the theory of first order quasilinear PDEs is that they can be reduced to equivalent ODEs. The characteristics of the PDEs makes this possible. In this section we apply this idea differently to reduce the system (5.11) to an equivalent system of ODEs. An advantage of the reduction process is that we bypass the calculation of characteristics; a process that can be quite tricky as it involves complicated numerical integration techniques.

5.4.1 Inverse characteristics

Let (t,x) be a point in the tx-plane $\{t > 0;\ -\infty < x < \infty\}$ and consider the characteristics passing through this point. Suppose that C_1 originates from the point $(0, a_1)$ and

C_2 from the point $(0, a_2)$. Recall from Chapter 4, Figure 4.1, that the interval $[a_1, a_2]$ is the *domain of dependency* of the point (t, x). For our further purposes, we shall consider $a_1 = a_1(t, x)$ and $a_2 = a_2(t, x)$ as functions and refer to them as *inverse characteristics.* From the definitions of C_1 and C_2 we obtain,

$$\left.\begin{aligned} X_1(0) &= a_1, \quad X_1(t) = x; \\ X_2(0) &= a_2, \quad X_2(t) = x. \end{aligned}\right\} \tag{5.17}$$

We can integrate (5.15) and use (5.17) to obtain

$$X_1(t) - X_1(0) = x - a_1 = \int_0^t \exp\{q_1(s)\}\, ds \geq 0 \tag{5.18}$$

and

$$X_2(0) - X_2(t) = a_2 - x = \int_0^t \exp\{q_2(s)\}\, ds \geq 0. \tag{5.19}$$

From this we conclude that $a_1 \leq x \leq a_2$ and that a_1 and a_2 are differentiable with respect to t and x.

5.4.2 The reduction process

The main result of the chapter is the following:

Theorem 5.1. *Consider the Cauchy problem* (5.11), (5.12)*. Let* (t, x) *be a point on the* tx*- plane where the Cauchy problem is satisfied. Then the inverse characteristics* a_1, a_2 *associated with the point* (t, x) *satisfy, for fixed* x*,the system of ordinary differential equations*

$$\left.\begin{aligned} a_{1,t} &= -\frac{\exp\{\frac{1}{2}[q_0(a_1) + q_0(a_2)]\}}{1 + \frac{1}{2}(x - a_1)q_0'(a_1)}; \\ \\ a_{2,t} &= \frac{\exp\{\frac{1}{2}[q_0(a_1) + q_0(a_2)]\}}{1 - \frac{1}{2}(a_2 - x)q_0'(a_2)}, \end{aligned}\right\}$$

with initial conditions

$$a_1(0, x) = x, \quad a_2(0, x) = x.$$

If a_1 *and* a_2 *can be found, the Cauchy problem* (5.11), (5.12) *is solved.*

Proof. From (5.18) and (5.19) we see that

$$a_1 = x - \int_0^t \exp\{q_1(s)\}\, ds, \tag{5.20}$$

and

$$a_2 = x + \int_0^t \exp\{q_2(s)\}\ ds. \tag{5.21}$$

We analyze movement along the characteristic C_1 from the point $(0, a_1)$ to the point $(t, X_1(t))$. Since u_1 is constant along C_1, $u_1(s, X_1(s)) = u_1(0, a_1)$, or by (5.12),

$$u_1(s, X_1(s)) = \frac{1}{2}q_0(a_1). \tag{5.22}$$

From (5.16)

$$\begin{aligned} q_1(s) &= q(s, X_1(s)) \\ &= u_1(s, X_1(s)) - u_2(s, X_1(s)) \\ &= \frac{1}{2}q_0(a_1) - u_2(s, X_1(s)). \end{aligned} \tag{5.23}$$

We do the same for movement along C_2, from the point $(0, a_2)$ to the same point $(t, X_2(t))$. We find that $u_2(s, X_2(s)) = u_2(0, a_2)$, or by (5.12)

$$u_2(s, X_2(s)) = -\frac{1}{2}q_0(a_2). \tag{5.24}$$

From (5.16)

$$\begin{aligned} q_2(s) &= q(s, X_2(s)) \\ &= u_1(s, X_2(s)) - u_2(s, X_2(s)) \\ &= u_1(s, X_2(s) + \frac{1}{2}q_0(a_2). \end{aligned} \tag{5.25}$$

Substitution of (5.23) and (5.25) in (5.20) and (5.21) yields

$$\begin{aligned} a_1 &= x - \int_0^t \exp\left\{ \left[\frac{1}{2}q_0(a_1) - u_2(s, X_1(s))\right] \right\} ds \\ &= x - \exp\left\{\frac{1}{2}q_0(a_1)\right\} \int_0^t \exp\{-u_2(s, X_1(s))\}\, ds, \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} a_2(t, x) &= x + \int_0^t \exp\left\{ \left[u_1(s, X_2(s) + \frac{1}{2}q_0(a_2)\right] \right\}\ ds \\ &= x + \exp\left\{\frac{1}{2}q_0(a_2)\right\} \int_0^t \exp\{u_1(s, X_2(s))\}\ ds. \end{aligned} \tag{5.27}$$

Now we fix x and differentiate (5.26) with respect to t:

$$\begin{aligned}
-a_{1,t} &= \frac{1}{2}q_0'(a_1)a_{1,t}\exp\left\{\frac{1}{2}q_0(a_1)\right\}\cdot\int_0^t \exp\{-u_2(s,X_1(s))\}\,ds \\
&\quad + \exp\left\{\frac{1}{2}q_0(a_1)\right\}\cdot\frac{d}{dt}\int_0^t \exp\{-u_2(s,X_1(s))\}\,ds \\
&= \frac{1}{2}q_0'(a_1)a_{1,t}\exp\left\{\frac{1}{2}q_0(a_1)\right\}\cdot\int_0^t \exp\{-u_2(s,X_1(s))\}\,ds \\
&\quad + \exp\left\{\frac{1}{2}q_0(a_1)\right\}\cdot\exp\{-u_2(t,X_1(t))\} \\
&= \frac{1}{2}q_0'(a_1)a_{1,t}\left[\exp\left\{\frac{1}{2}q_0(a_1)\right\}\cdot\int_0^t \exp\{-u_2(s,X_1(s))\}\,ds\right] \\
&\quad + \exp\left\{\frac{1}{2}q_0(a_1)-u_2(t,X_1(t))\right\} \\
&= \frac{1}{2}q_0'(a_1)(x-a_1)a_{1,t} + \exp\left\{\frac{1}{2}[q_0(a_1)+q_0(a_2)]\right\}. \qquad (5.28)
\end{aligned}$$

Rearrangement of (5.28) yields the expression

$$a_{1,t} = -\frac{\exp\{\frac{1}{2}[q_0(a_1)+q_0(a_2)]\}}{1+\frac{1}{2}(x-a_1)q_0'(a_1)}. \qquad (5.29)$$

With (5.27) we follow the same procedure. This yields

$$\begin{aligned}
a_{2,t} &= \frac{1}{2}q_0'(a_2)a_{2,t}\exp\left\{\frac{1}{2}q_0(a_2)\right\}\cdot\int_0^t \exp\{u_1(s,X_2(s))\}\,ds \\
&\quad + \exp\left\{\frac{1}{2}q_0(a_2)\right\}\cdot\frac{d}{dt}\int_0^t \exp\{u_1(s,X_2(s))\}\,ds \\
&= \frac{1}{2}q_0'(a_2)a_{2,t}\left[\exp\{\frac{1}{2}q_0(a_2)\}\cdot\int_0^t \exp\{u_1(s,X_2(s))\}\,ds\right] \\
&\quad + \exp\left\{\frac{1}{2}q_0(a_2)\right\}\cdot\exp\{u_1(t,X_2(t))\} \\
&= \frac{1}{2}(a_2-x)q_0'(a_2)a_{2,t} + \exp\{\frac{1}{2}q_0(a_2)+u_1(t,X_2(t))\} \\
&= \frac{1}{2}q_0'(a_2)(a_2-x)a_{2,t} + \exp\{\frac{1}{2}[q_0(a_1)+q_0(a_2)]\}. \qquad (5.30)
\end{aligned}$$

Rearrangement of (5.30) yields the expression

$$a_{2,t} = \frac{\exp\{\frac{1}{2}[q_0(a_1)+q_0(a_2)]\}}{1-\frac{1}{2}(a_2-x)q_0'(a_2)}. \qquad (5.31)$$

At $t=0$, the C_1 characteristic passes through the point $(0,x)$ which is also the point $(0,a_1(0,x))$. Similarly, the C_2 characteristic passes through the same point $(0,x)$ which, however, is also the point $(0,a_2(0,x))$. Hence for a fixed x the equations (5.29) and

(5.31) is a system of ordinary differential equations with initial condition

$$a_1(0,x) = x, \quad a_2(0,x) = x. \tag{5.32}$$

From (5.8), $v(t,x) = u_1(t,x) + u_2(t,x)$ and $q(t,x) = u_1(t,x) - u_2(t,x)$. It follows from (5.22) and (5.24) that

$$v(t,x) = \frac{1}{2}\left[q_0(a_1) - q_0(a_2)\right]; \quad q(t,x) = \frac{1}{2}\left[q_0(a_1) + q_0(a_2)\right]. \tag{5.33}$$

Thus, if a_1 and a_2 can be found, the Cauchy problem (5.11), (5.12) is solved. □

Remark 5.2. Theorem 5.1 suggests a useful interpretation of the inverse characteristics: *The functions a_1, a_2 define a coordinate system from which the solutions of the Cauchy problem* (5.11), (5.12) *can be read off.*

We can also find suitable expressions for the spatial derivatives. Indeed, differentiation of (5.26) with respect to x gives

$$\begin{aligned} a_{1,x} &= 1 - \frac{1}{2}q_0'(a_1)a_{1,x}\exp\left\{\frac{1}{2}q_0(a_1)\right\}\int_0^t \exp\{-u_2(s,X_1(s))\}\,ds \\ &= 1 - \frac{1}{2}q_0'(a_1)a_{1,x}[x - a_1]. \end{aligned}$$

Similarly, differentiation of (5.27) gives

$$\begin{aligned} a_{2,x} &= 1 + \frac{1}{2}q_0'(a_2)a_{2,x}\exp\left\{\frac{1}{2}q_0(a_1)\right\}\int_0^t \exp\{u_1(s,X_2(s))\}\,ds \\ &= 1 + \frac{1}{2}q_0'(a_2)a_{2,x}[a_2 - x]. \end{aligned}$$

Thus we have arrived at:

Theorem 5.3. *For fixed t, the inverse characteristics a_1, a_2 satisfy the system of ordinary differential equations*

$$a_{1,x} = \frac{1}{1 + \frac{1}{2}(x - a_1)q_0'(a_1)}; \tag{5.34}$$

$$a_{2,x} = \frac{1}{1 - \frac{1}{2}(a_2 - x)q_0'(a_2)}. \tag{5.35}$$

5.5 A preliminary to qualitative analysis

An investigation of some important qualititative properties of the Cauchy problem is presented in Chapter 6. For the moment a glance at Theorems 5.1 suggests the results to come. We concluded from Section 5.4.1 that $a_1 \leq x \leq a_2$. Hence the signs of the derivatives $a_{1,t}$ and $a_{2,t}$ depend on the behaviour of the derivative q_0'. For further analysis we define two functions:

$$\left.\begin{aligned} f_1(z;x) &= 1 + \tfrac{1}{2}(x-z)q_0'(z), \quad z \leq x; \\ f_2(z;x) &= 1 - \tfrac{1}{2}(z-x)q_0'(z). \quad z \geq x. \end{aligned}\right\} \tag{5.36}$$

We may now express the equations (5.29), (5.31) in the form

$$\left.\begin{aligned} f_1(a_1,x)a_{1,t} &= -\exp\{\tfrac{1}{2}\left[q_0(a_1)+q_0(a_2)\right]\}; \\ f_2(a_2,x)a_{2,t} &= \exp\{\tfrac{1}{2}\left[q_0(a_1)+q_0(a_2)\right]. \end{aligned}\right\} \tag{5.37}$$

The spatial derivatives (5.34), (5.35) can also be succintly expressed in the form

$$\left.\begin{aligned} f_1(a_1,x)a_{1,x} &= 1; \\ f_2(a_2,x)a_{2,x} &= 1. \end{aligned}\right\} \tag{5.38}$$

These uncoupled expressions show that singularities may develop at the zeros of f_1 or f_2.

5.6 Invariance under translation and reflection

In this last section we derive a useful result on invariance under certain substitutions. The substitutions of interest are *translations* and *reflections* and they are useful for the transfer of results obtained under apparently restrictive assumptions and for the understanding of the nature of the equations we deal with. The new variables are denoted by asterisks.

Definition 5.4. A *translation* is the change of variables

$$x^* = x - x_0; \quad v^*(t,x^*) = v(t, x-x_0); \quad p^*(t,x^*) = p(t,x-x_0). \tag{5.39}$$

Definition 5.5. A *reflection* is a change of the directional variables x and v defined by

$$x^* = -x; \quad v^*(t,x^*) = -v(t,-x); \quad p^*(t,x^*) = p(t,-x). \tag{5.40}$$

The main result is:

Theorem 5.6. *Under translation and reflection the system of equations and constraint* (5.1), (5.2) *retain their form. Under reflection the following holds:*

$$u_1^*(t, x^*) = -u_2(t, x) \tag{5.41}$$

$$u_2^*(t, x^*) = -u_1(t, x) \tag{5.42}$$

$$a_1^*(t, x^*) = -a_2(t, x) \tag{5.43}$$

$$a_2^*(t, x^*) = -a_1(t, x), \tag{5.44}$$

where u_k^* and a_k^* are derived from the respective transformed quantities.

Proof. Under translation,

$$v_t^*(t, x^*) + p_{x^*}^*(t, x^*) = v_t(t, x^*) + p_{x^*}(t, x^*) = 0,$$
$$p_t^*(t, x^*) + [1 + p^*(t, x^*)]^2 v_{x^*}^*(t, x^*) = p_t(t, x^*) + [1 + p(t, x^*)]^2 v_{x^*}(t, x^*) = 0,$$
$$\text{and} \quad 1 + p^*(t, x^*) = 1 + p(t, x^*) > 0.$$

Hence the system and constraint retain their original form under translation.

Under reflection,

$$v_t^*(t, x^*) = -v_t(t, -x),$$
$$p_t^* = p_t^*(t, -x),$$
$$v_{x^*}^*(t, x^*) = -v_{x^*}(t, -x) = v_x(t, -x),$$
$$\text{and} \quad p_{x^*}^*(t, x) = p_{x^*}(t, -x) = -p_x(t, -x).$$

So that

$$v_{x^*}^*(t, x^*) + p_{x^*}^*(t, x^*) = -v_t(t, -x) - p_{x^*}(t, -x) = 0,$$
$$p_t^*(t, x^*) + [1 + p^*(t, x^*)]^2 v_{x^*}^*(t, x^*) = p_t(t, -x) + [1 + p(t, -x)]^2 v_x(t, -x) = 0,$$
$$\text{and} \quad 1 + p^*(t, x^*) = 1 + p(t, -x) > 0.$$

Thus the system and constraint retain their original form under reflection. □

5.7 Summary

In this chapter we formulated the Cauchy problem (5.11), (5.12), implicit in the mathematical model that was derived in Chapter 3. We introduced the notion of inverse characteristics and utilized them to reduce the Cauchy problem for a system of quasilinear hyperbolic PDEs to an initial value problem for systems of nonlinear ODEs (Theorem 5.1). A preliminary analysis of the resultant system of ODEs shows the possibility of singularities occurring. In the next chapter, we investigate this further and present some important qualitative properties of the solutions of the resultant Cauchy problem, which as a consequence of Theorem 5.1, are also the properties of the solutions of the original Cauchy problem.

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Chapter 6

Qualitative results

6.1 Introduction

In the previous chapter we showed that to solve (5.11), (5.12) equates to solving the system (5.29), (5.31) and (5.32). Closed form solutions for these equations are in general not possible. So we need to make qualitative investigations to obtain some insights into the behaviour of their solutions. In this chapter we investigate the possible existence of asymptotes and discontinuities and in the process we shall discover the possibility of introducing a convenient alternative time variable.

6.2 Asymptotic behaviour

Consider the spatial derivatives (5.38). Our objective is to determine the asymptotic behaviour of the functions $a_1(t,x)$ and $a_2(t,x)$ for large x and fixed t. The definitions (5.36) imply that the curves $a_1(t,x) = x$ and $a_2(t,x) = x$ are solutions of the differential equations (5.38). This suggests an underlying 'asymptoticness'. We seek to confirm this intuition. To achieve this, we rephrase the expressions (5.26) and (5.27) in terms of the initial pressure p_0.

Let us consider the case for a_1. Let t be fixed. We notice from (5.24) and definition (5.4) that

$$\begin{aligned}
\exp\Big\{-u_2(s,X_1(s))\Big\} &= \exp\Big\{\frac{1}{2}q_0(a_2(s,X_1(s)))\Big\} \\
&= \exp\Big\{\frac{1}{2}\ln(1+p_0(a_2(s,X_1(s)))\Big\} \\
&= [1+p_0(a_2(s,X_1(s)))]^{\frac{1}{2}}.
\end{aligned}$$

This enables us to re-express (5.26),

$$x - a_1(t,x) = [1 + p_0(a_1)]^{\frac{1}{2}} \int_0^t [1 + p_0(a_2(s, X_1(s)))]^{\frac{1}{2}} \, ds. \tag{6.1}$$

The case for (5.27) is dealt with in a similar way. From (5.22) and (5.4)

$$\begin{aligned}
\exp\left\{u_1(s, X_2(s))\right\} &= \exp\left\{\frac{1}{2} q_0(a_1(s, X_2(s)))\right\} \\
&= \exp\left\{\frac{1}{2}\ln(1 + p_0(a_1(s, X_2(s)))\right\} \\
&= [1 + p_0(a_1(s, X_2(s)))]^{\frac{1}{2}}.
\end{aligned}$$

Hence (5.27) becomes

$$a_2(t,x) - x = [1 + p_0(a_2)]^{\frac{1}{2}} \int_0^t [1 + p_0(a_1(s, X_2(s)))]^{\frac{1}{2}} \, ds. \tag{6.2}$$

Theorem 6.1. *Suppose there are constants M and N such that for all x*

$$0 < 1 + p_0(x) \leq M. \tag{6.3}$$

and

$$|x p_0'(x)| \leq N. \tag{6.4}$$

Then for fixed t

$$\left.\begin{aligned}
a_1(t,x) &\sim x - [1 + p_0(x)]t \\
a_2(t,x) &\sim x + [1 + p_0(x)]t.
\end{aligned}\right\}$$

as $x \to \pm\infty$.

Proof. Case 1: $x \to \infty$ From (6.1), (6.2) and (6.3) we obtain

$$\begin{aligned}
x - a_1(t,x) &\leq M^{\frac{1}{2}} \int_0^t M^{\frac{1}{2}} ds = Mt, \\
a_2(t,x) - x &\leq M^{\frac{1}{2}} \int_0^t M^{\frac{1}{2}} ds = Mt.
\end{aligned}$$

It follows immediately that as $x \to \infty$,

$$\left.\begin{aligned}
\nu_1 &:= \frac{a_1(t,x)}{x} \to 1 \\
\nu_2 &:= \frac{a_2(t,x)}{x} \to 1.
\end{aligned}\right\}$$

Recall from Section 5.4.1 that the point (t, x) is the intersection of the C_1 and C_2 characteristics where the C_1-characteristics originate from the point $(0, a_1(t,x))$ and are increasing and the C_2-characteristics originate from the point $(0, a_2(t,x))$ and are decreasing. We also established therein that $a_1(t,x) \leq x \leq a_2(t,x)$. So for a fixed t and $0 \leq \ s \leq t$, we have that $a_1(t,x) \leq X_1(s) \leq X_2(s) \leq a_2(t,x)$. Hence,

$$\left.\begin{aligned} \nu_1 \leq \frac{X_1(s)}{x} \leq \nu_2; \\ \nu_1 \leq \nu := \frac{a_2(s, X_1(s))}{x} \leq \nu_2. \end{aligned}\right\} \tag{6.5}$$

Thus $X_1(s)/x \to 1$ and $\nu \to 1$ uniformly in $s \in [0,t]$ when $x \to \infty$.

Next we apply the Mean Value Theorem and conclude that there exists at least one value ξ, with $a_2(s, X_1(s)) < \xi$ or $x < \xi < a_2(s, X_1(s))$ such that

$$\begin{aligned} p_0(a_2(s, X_1(s))) - p_0(x) &= \big(a_2(s, X_1(s)) - x\big)\, p_0^{'}(\xi), \\ &= \left(\frac{a_2(s, X_1(s))}{x} - 1\right) x p_0^{'}(\xi), \\ &= [\nu - 1] \left(\frac{x}{\xi}\right) \xi p_0^{'}(\xi). \end{aligned} \tag{6.6}$$

Combination of (6.4) and (6.6) yields

$$\big|p_0(a_2(s, X_1(s)) - p_0(x)\big| \leq N\,|\nu - 1| \left(\frac{x}{\xi}\right). \tag{6.7}$$

From (6.5) either $\frac{x}{\xi} \leq \frac{1}{\nu_1}$ or $\frac{x}{\xi} < 1$. Hence (6.7) implies that $p_0(a_2(s, X_1)) \to p_0(x)$ uniformly as $x \to \infty$. Therefore the integral in (6.1) converges to $t[1 + p_0(x)]^{\frac{1}{2}}$ as $x \to \infty$.

The argument for the convergence of the integral in (6.2) proceeds analogously but with some minor modifications as follows: From expression (6.2) let $\theta = a_1(s, X_2(s))/x$. From the Mean Value Theorem, there exists at least one value λ with $a_1 < \lambda < x$ such that

$$\begin{aligned} p_0(x) - p(a_1(s, X_2(s))) &= (x - a_1(s, X_2(s)))p_0'(\lambda), \\ &= \left(1 - \frac{a_1(s, X_2(s))}{x}\right) x p_0'(\lambda), \\ &= \left(1 - \frac{a_1(s, X_2(s))}{x}\right) \left(\frac{x}{\lambda}\right) \lambda p_0'(\lambda), \\ &= (1 - \theta) \left(\frac{x}{\lambda}\right) (\lambda p_0'(\lambda)). \end{aligned}$$

But $a_1(s, X_2(s)) < \lambda < x$ implies that $1 < \dfrac{x}{\lambda} < \dfrac{x}{a_1(s, X_2(s))}$. It follows that,

$$p_0(x) - p_0(a_1) < (1 - \theta) \left(\frac{1}{\theta}\right) (\lambda p_0'(\lambda)).$$

Hence,

$$|p_0(a_1) - p_0(x)| \leq \left|\frac{1-\theta}{\theta}\right| |\lambda p_0'(\lambda)| \leq \left(\frac{1-\theta}{\theta}\right) N.$$

But (6.5) implies that $\theta \to 1$ as $x \to \infty$. Therefore for large $x, p_0(a_1(s, X_2(s))) \sim p_0(x)$. So the integrand (6.2) behaves like $[1 + p_0(x)]^{\frac{1}{2}}$ as $x \to \infty$. Thus

$$a_2(t, x) - x = [1 + p_0(x)]^{\frac{1}{2}} \int_0^t [1 + p_0(x)]^{\frac{1}{2}} ds \sim [1 + p_0(x)]t.$$

This establishes the first assertion of the theorem.

Case 2: $x \to -\infty$

This case can be dispensed with when we notice that it is equivalent to proving the result when $-x \to \infty$. Then we can effect a change of variable by reflection and let $x^* = -x$. From Theorem 5.6, the system of equations and constraint (5.1), (5.2) retain their form under this reflection. So we can apply the result we have just established in the preceding case and conclude that for fixed t

$$\left.\begin{aligned} a_1^*(t, x^*) &\sim x^* - [1 + p_0^*(x^*)]t \\ a_2^*(t, x^*) &\sim x^* + [1 + p_0^*(x^*)]t. \end{aligned}\right\}$$

as $x^* \to +\infty$. This completes the proof. □

Remark 6.2. Recall that d'Alembert's classical solution of the one-dimensional wave equation is $w(t, x) = F(x - ct) + G(x + ct)$ where F and G are arbitrary functions and c is the speed of propagation ([10, Chapter 12]). We see that under assumptions (6.3) and (6.4), wave-like behaviour is asymptotically exhibited with a_1 representing downstream motion and a_2 representing upstream motion with speed of propagation $1 + p_0(x)$.

6.3 Singularities

In Chapter 5 (see (5.5)) we intimated the possibility of singularities. In this section we expand on this intuition. The principal tools of investigation we will use are the ideas of *time-like* and *space-like* curves. These ideas are well-established in the literature, see for example Jeffrey ([13, Chapter 2]) and Courant-Friedrichs ([3, Chapter 3]). For our purposes, we take time-like curves to be the trajectories in the a_1a_2-plane of the solutions of the system of ordinary differential equations (5.29), (5.31) under the initial conditions (5.32) with x as a parameter. The trajectories describe flow lines followed by the various single 'particles'. Similarly, we take space-like curves to be the trajectories in the xa_1, xa_2-planes along which t is fixed. Space-like curves provide a means to identify and compute discontinuities in the functions a_1, a_2 and therefore in the solutions of the Cauchy problem (5.11), (5.12).

6.3.1 Time-like curves

We want to devise a method of computing time-like curves. To this end, we make the following (nonnegative) substitutions, by courtesy of (5.26) and (5.27):

$$\left.\begin{aligned} b_1(t,x) &= x - a_1(t,x), \\ b_2(t,x) &= a_2(t,x) - x. \end{aligned}\right\} \tag{6.8}$$

and by implication,

$$\left.\begin{aligned} -b_{1,t}(t,x) &= a_{1,t}(t,x), \\ b_{2,t}(t,x) &= a_{2,t}(t,x). \end{aligned}\right\} \tag{6.9}$$

We may now combine (6.8) and (6.9) with the differential equations (5.29) and (5.31) to obtain an expression that links b_1 and b_2:

$$\Big[1 + \frac{1}{2} b_1 q_0'(x - b_1)\Big] b_{1,t} = \Big[1 - \frac{1}{2} b_2 q_0'(x + b_2)\Big] b_{2,t} \tag{6.10}$$

The initial conditions derived from (5.32) become

$$b_1(0,x) = b_2(0,x) = 0. \tag{6.11}$$

At this stage we could attack problem (6.10) and (6.11) through suitable numerical methods. But this would be akin to intentionally making life difficult for ourselves and not a rational thing to do. So instead we integrate (6.10) over the interval $[0,t]$. The result is the following relation between b_1 and b_2:

$$b_1(t,x) + \frac{1}{2}\int_0^t b_1 q_0'(x - b_1) b_{1,s}\, ds = b_2(t,x) - \frac{1}{2}\int_0^t b_2 q_0'(x + b_2) b_{2,s}\, ds. \tag{6.12}$$

For convenience, let us denote the left hand side of (6.12) by $I_1(b_1, x)$ and the right hand side by $I_2(b_2, x)$, and in addition, also make the following substitutions

$$\begin{aligned} \sigma_1 &= b_1(s,x); \quad & \sigma_2 &= b_2(s,x) \\ d\sigma_1 &= b_{1,s}(s,x)ds; \quad & d\sigma_{2,s} &= b_2(s,x)ds. \end{aligned}$$

The result is:

$$\left.\begin{aligned} I_1(b_1, x) &:= \int_0^{b_1} \left[1 + \frac{1}{2}\sigma_1 q_0'(x - \sigma_1)\right] d\sigma_1 \\ &= \int_0^{b_2} \left[1 - \frac{1}{2}\sigma_2 q_0'(x + \sigma_2)\right] d\sigma_2 =: I_2(b_2, x). \end{aligned}\right\} \tag{6.13}$$

We can also integrate by parts to obtain

$$\left.\begin{aligned} I_1(b_1, x) &= b_1 - \frac{1}{2}b_1 q_0(x - b_1) + \frac{1}{2}\int_0^{b_1} q_0(x - \sigma_1)\, d\sigma_1 \\ I_2(b_2, x) &= b_2 - \frac{1}{2}b_2 q_0(x + b_2) + \frac{1}{2}\int_0^{b_2} q_0(x + \sigma_2)\, d\sigma_2 \end{aligned}\right\}$$

We thus have the following useful relation between b_1 and b_2:

$$b_1 - \frac{1}{2}b_1 q_0(x - b_1) + \frac{1}{2}\int_0^{b_1} q_0(x - \sigma_1)\, d\sigma_1 = b_2 - \frac{1}{2}b_2 q_0(x + b_2) + \frac{1}{2}\int_0^{b_2} q_0(x + \sigma_2)\, d\sigma_2. \tag{6.14}$$

This equation is amenable to elementary numerical methods for as long as the substitutions are valid. Indeed, to calculate the time-like curves associated with the parameter x, we may proceed algorithmically as follows : Specify a value of a_1, find $b_1 = x - a_1$, calculate the left of (6.14), solve for b_2 and then find $a_2 = b_2 + x$. The algorithm may not be a straight highway to paradise; some treacherous bends may exist. For example, suppose that $p_0 = \exp\{q_0(x)\} - 1$ is decreasing so that $q_0'(x) \leq 0$. Then the integrand in $I_2(b_2, x)$ is non-negative so that the right hand side of (6.13) is increasing. On the other hand $I_1(b_1, x)$ may be positive, negative or even multivalued. In such cases the procedure fails and may yield multiple solutions or no solutions at all. However, if b_1 is close to zero, this cannot happen. We will provide illustrative examples of the computations of time-like curves in Chapter 7.

As an interesting aside, we can examine the envelopes of trajectories to further explore the above possibilies. We defer this to Chapter 7 where we present worked examples.

6.3.2 Time-like curves as an alternative measure of time

It is convenient to measure time along an appropriate time-like curve. Such a curve must of necessity be unbroken. This means that for some x the equations

$$\left.\begin{aligned} 1 + \frac{1}{2}\sigma_1 q_0'(x - \sigma_1) = 0; \\ 1 - \frac{1}{2}\sigma_2 q_0'(x + \sigma_2) = 0, \end{aligned}\right\}$$

should not have positive solutions so that we can, by virtue of Theorem 5.6, take $x = 0$. Then points on such curves in $a_1 a_2$-space are of the form

$$T(t) = (a_1(t,0), a_2(t,0)) = (-b_1(t,0), b_2(t,0)),$$

by the substitutions (6.8). Now let $\tau_1(t) = b_1(t,0)$ and $\tau_2(t) = b_2(t,0)$ so that $T(t) = (-\tau_1(t), \tau_2(t))$. With the help of equations (6.13) we can establish the following relation between τ_1 and τ_2:

$$\int_0^{\tau_1} [1 + \frac{1}{2}\sigma_1 q_0'(-\sigma_1)\ d\sigma_1 = \int_0^{\tau_2} [1 - \frac{1}{2}\sigma_2 q_0'(\sigma_2)]\ d\sigma_2. \tag{6.15}$$

Next we consider the differential equations (5.29) and (5.31) at $x = 0$. There, from (6.8), $a_{1,t} = -b_{1,t} = -\tau_1(t)$, $a_{2,t} = b_{2,t} = \tau_2(t)$, $a_{1,t} = -\tau_1'(t)$ and $a_{2,t} = \tau_2'(t)$. The equations are expressed as follows:

$$\left.\begin{aligned} \left[1 + \frac{1}{2}\tau_1 q_0'(-\tau_1)\right]\tau_1'(t) = \exp\left\{\frac{1}{2}[q_0(-\tau_1) + q_0(\tau_2)]\right\} \\ \left[1 - \frac{1}{2}\tau_2 q_0'(\tau_2)\right]\tau_2'(t) = \exp\left\{\frac{1}{2}[q_0(-\tau_1) + q_0(\tau_2)]\right\} \end{aligned}\right\} \tag{6.16}$$

We also observe that (6.15) allows us to express τ_2 in terms of τ_1 so that the first equation in (6.16) can be integrated explicitly to obtain t in terms of τ_1 under the initial condition $\tau_1(0) = 0$ derived from (6.11). So what has just happened here? We have shown that the curve $T(t)$ in the $a_1 a_2$-plane is related to time in an explicit way. Further, the functions in brackets on the left of equation (6.16) are positive because at $\tau_{1,2} = 0$ they are positive and they are never zero, courtesy of a property of continuous functions. So if they are not bounded above, we can use the parameter $\tau = \tau_1$ instead of t, which is what we will do in the sections that follow. Theorem 6.3 below places this possibility on firm ground.

Theorem 6.3. *Suppose that the equations* $1 + \frac{1}{2}\sigma q_0'(-\sigma) = 0$ *and* $1 - \frac{1}{2}\sigma q_0'(\sigma) = 0$ *have no positive solutions. If for every* $\tau > 0$,

$$\frac{1}{\tau}\int_{-\tau}^0 q_0(\sigma)\ d\sigma \geq q_0(-\tau) \quad \textit{and} \quad \frac{1}{\tau}\int_0^\tau q_0(\sigma)\ d\sigma \geq q_0(\tau),$$

the equation (6.15) *represents a unique relation betweeen* τ_1 *and* τ_2*;* $\tau_2 \to \infty$ *as* $\tau_1 \to \infty$ *and vice-versa.*

If p_0 *is an even function,* $\tau_1 = \tau_2 = \tau$*.*

If it is assumed that $1 + p_0$ *is bounded as per* (6.3)*, then* $t \to \infty$ *as* $\tau = \tau_1 \to \infty$*. The same is true when* $\tau_2 \to \infty$*.*

Proof. Consider the equations (6.15). Integration by parts yields

$$\int_0^s \sigma q'(-\sigma)\ d\sigma = \int_0^s q_0(-\sigma)\ d\sigma - sq_0(-s) = \int_{-s}^0 q_0(u)\ du - sq_0(-s) \geq 0,$$

by hypothesis (after the substitution $\sigma = -u$).

Similarly,

$$\int_0^s \sigma q'(\sigma)\ d\sigma = sq_0(s) - \int_0^s q_0(\sigma)\ d\sigma \leq 0.$$

From previous remarks, the integrals in (6.15) are increasing functions of τ_1 and τ_2 and

$$\left.\begin{aligned} \int_0^{\tau_1} h_1(\sigma)\ d\sigma &= \int_0^{\tau_1} \left[1 + \frac{1}{2}\sigma q_0'(-\sigma)\right] d\sigma = \tau_1 + \frac{1}{2}\int_0^{\tau_1} \sigma q_0'(-\sigma)\ d\sigma \geq \tau_1 \\ \int_0^{\tau_2} h_2(\sigma)\ d\sigma &= \int_0^{\tau_2} \left[1 - \frac{1}{2}\sigma q_0'(\sigma)\right] d\sigma = \tau_2 - \frac{1}{2}\int_0^{\tau_2} \sigma q_0'(\sigma)\ d\sigma \geq \tau_2 \end{aligned}\right\} \tag{6.17}$$

so that the positive integrals in (6.15) are unbounded functions of their upper limits. Hence $\tau_2 \to \infty$ as $\tau_1 \to \infty$ and vice-versa.

Since $q_0 = \ln[1 + p_0]$, an even p_0 implies an even q_0 and therefore an odd q_0'. But then $h_2(-\sigma) = h_1(\sigma)$ and we conclude that $\tau_1 = \tau_2$.

Finally, suppose that $1 + p_0(x) \leq M$. It follows from the first equation in (6.16) that

$$\begin{aligned} \int_0^t h_1(\tau_1(s))\tau_1'\ ds &= \int_0^t \exp\left\{\frac{1}{2}[q_0(-\tau_1(s)) + q_0(\tau_2(s))]\right\} ds \\ &= \int_0^t \left[1 + p_0(-\tau_1(s))\right]^{\frac{1}{2}} \left[1 + p_0(\tau_2)(s)\right]^{\frac{1}{2}} ds \leq Mt. \end{aligned}$$

But, from (6.17) we see that

$$\int_0^t h_1(\tau_1(s))\tau_1'(s)\ ds = \int_0^{\sigma_1} h_1(\sigma_1)\ d\sigma_1 \geq \tau_1.$$

It follows that $\tau_1 \leq Mt$ and similarly $\tau_2 \leq Mt$. This establishes the last assertion.

□

Henceforth we shall assume all the hypotheses of Theorem 6.3.

6.3.3 Space-like curves

To complete the picture we need to study the curves in the $a_1 x, a_2 x$-planes along which t is fixed. We call these the space-like curves associated with our problem. Our point of departure is the first equation in (5.36) and we proceed to re-write it in the form

$$\frac{1}{a_{1,x}} = f_1(a_1; x) = 1 + \frac{1}{2}(x(a_1) - a_1)q_0'(a_1). \tag{6.18}$$

For small t, a_1 is near x and the derivative $a_{1,x}$ is positive so x can be expressed as a function of a_1. The expression (5.34) can now be rewritten in the form

$$\frac{d}{da_1}[x(a_1) - a_1] = \frac{1}{2}(x(a_1) - a_1)q_0'(a_1). \tag{6.19}$$

This is a first-order linear ordinary differential equation whose general solution is given implicitly by

$$\ln[x(a_1) - a_1] = \frac{1}{2}q_0(a_1) + \ln C.$$

For a particular solution of we need the value of x at some point. If the curve passes through a given point $a_1 = A_1$, $x = X_1$, that is, $x(A_1) = X_1$, then

$$\ln C = \ln[X_1 - A_1] - \frac{1}{2}q_0(A_1),$$

so that,

$$\ln[x(a_1) - a_1] = \frac{1}{2}q_0(a_1) + \ln[X_1 - A_1] - \frac{1}{2}q_0(A_1).$$

Or,

$$\ln\left\{\frac{x(a_1) - a_1}{X_1 - A_1}\right\} = \frac{1}{2}\Big\{q_0(a_1) - q_0(A_1)\Big\}.$$

Hence

$$x(a_1) - a_1 = [X_1 - A_1]\exp\left\{\frac{1}{2}[q_0(a_1) - q_0(A_1)]\right\}.$$

Thus a particular solution of (6.19) is

$$x(a_1) = a_1 + (X_1 - A_1)\exp\left\{\frac{1}{2}[q_0(a_1) - q_0(A_1)]\right\}. \tag{6.20}$$

We can elegantly express the solution (6.20) in terms of the initial pressure p_0 with the help of definition (5.4) and the notation

$$P_0 := \Big[1 + p_0\Big]^{\frac{1}{2}} = \exp\left\{\frac{1}{2}q_0\right\}. \tag{6.21}$$

The result is

$$x(a_1) = a_1 + \left[\frac{X_1 - A_1}{P_0(A_1)}\right]P_0(a_1). \tag{6.22}$$

For later use, let us note that from (6.21)

$$\frac{1}{2}q_0' = \frac{P_0'}{P_0}. \tag{6.23}$$

We now employ the alternative measure of time introduced in the previous section by assuming that the time-like curve corresponding to $x = 0$ exists and use $\tau = \tau_1$ instead of t. $a_1(t, 0) = -\tau_1(t)$ and $a_2(t, 0) = \tau_2(t)$. Then we are able to compute space-like curves associated with a_1 when we let τ assume different values. Every curve will be associated with a fixed τ which in turn can be associated with a fixed time t calculated from (6.16). In our case we have $A_1 = -\tau$ and $X_1 = 0$ so the expression (6.22) reduces to

$$x(a_1) = a_1 + \Big[\frac{\tau}{P_0(-\tau)}\Big] P_0(a_1). \tag{6.24}$$

For a_2 we follow the same procedure, this time starting from (5.35). Now the equation (6.19) is replaced by

$$\frac{d}{da_2}[a_2 - x(a_2)] = \frac{1}{2}(a_2 - x_2(a_2)q_0'(a_2). \tag{6.25}$$

The solution of (6.25) under the boundary condition $a_2(0, t) = \tau_2$ can be obtained in a manner analogous to what was done before. The result is

$$x(a_2) = a_2 - \Big[\frac{\tau_2}{P_0(\tau_2)}\Big] P_0(a_2). \tag{6.26}$$

An immediate consequence is that for $\tau_1 = 0$, (that is for $t = 0$), $a_1 = a_2 = x$. The expressions (6.24) and (6.26) also present the possibility of calculating a_1 and a_2 when x is given. We shall continue to write $a_1(\tau, x)$ instead of $a_1(t(\tau), x)$ and similarly for a_2. We shall also use the notation $x = x(a_1; \tau)$, $x = x(a_2; \tau_2)$ for the space-like curves we have found. This indicates dependence on the time-parameters τ, τ_2. Derivatives with respect a_1 and a_2 will be denoted by $x'(a_1; \tau)$ and $x'(a_2; \tau_2)$ respectively. When no confusion can arise, the time-pameters will be left out.

6.3.4 Discontinuities

Discontinuities of space-like curves are, under certain circumstances, to be expected. The expressions in (5.37) show that if either f_1 or f_2 has a zero, a_1 or a_2 will be discontinuous and at such a *point of singularity* (x, a_1) the relation

$$x = a_1 - \frac{2}{q_0'(a_1)}. \tag{6.27}$$

follows from (5.36). This relation describes a *curve of discontinuity* of a_1. The relation (6.27) can, by virtue of (6.23), be expressed in terms of the initial pressure:

$$x = S(a_1) := a_1 - \frac{P_0(a_1)}{P_0'(a_1)}. \tag{6.28}$$

We study discontinuities under the following assumptions:

I. The initial pressure p_0 is strictly decreasing over an interval $I = (A_1^\dagger, \infty)$ and $\lim_{a_1 \to A_1^\dagger} S(a_1) = \infty$.

II. The function S is strictly convex on the interval I.

III. On the interval I the function p_0 has a second derivative.

IV. For all $\tau > 0$, $P_0(-\tau) + \tau P_0'(-\tau) > 0$.

These assumptions have important consequences:

Proposition 6.4. $\lim_{a_1 \to \infty} S(a_1) = \infty$. *The curve* $x = S(a_1)$ *lies above the straight line* $x = a_1$.

Proof. From (6.28), since P_0 is decreasing, we have for $a_1 \in I$, that $S(a_1) > a_1$. □

Proposition 6.5. *The curve of discontinuity has a unique minimum point.*

Proof. Since the curve extends to infinity at both ends of I, there must, by Rolle's theorem, be at least one local minimum. The assumption of strict convexity ensures that there is only one such point. □

The convex region $H = \{(a_1, x) : S(a_1) < x\}$ enclosed by the curve of discontinuity will be called the *region of discontinuity.*

Proposition 6.6. *Let* $x = x(a_1)$ *be a space-like curve. If the point* $(a_1, x(a_1))$ *is on the curve of discontinuity,* $x'(a_1) = 0$. *If such a point is in* H, $x'(a_1) < 0$. *If the point is outside the closure of* H, $x'(a_1) > 0$.

Proof. The first assertion is a re-statement of the definition of the function S. Suppose that the point in question is in H. That is,

$$S(a_1) = a_1 - \frac{P_0(a_1)}{P_0'(a_1)} < x.$$

Since $P_0'(a_1) < 0$ (for $a_1 \in I$), it follows after a few manipulations that

$$(x - a_1)\frac{P_0'(a_1)}{P_0(a_1)} < -1.$$

From (6.18) and (6.23) however,

$$x'(a_1) = 1 + \frac{1}{2}(x - a_1)q_0'(a_1) = 1 + (x - a_1)\frac{P_0'(a_1)}{P_0(a_1)} < 0.$$

The remaining assertion is proved similarly. □

We can associate points on the curve of discontinuity with time. Suppose that the space-like curve $x = x(a_1;\tau)$ meets the curve of discontinuity. That is, $x(a_1;\tau) = S(a_1)$. By combining (6.24) and (6.28) one obtains

$$-P_0'(a_1) = \frac{P_0(-\tau)}{\tau} =: Q(\tau). \tag{6.29}$$

Proposition 6.7. *The function* $\tau \to \tau/P_0(-\tau) = 1/Q(\tau);\ \tau > 0$ *is increasing.*

Proof. From the definition (7.15),

$$Q'(\tau) = -\frac{P_0(-\tau) + \tau P_0'(-\tau)}{\tau^2}.$$

It follows from Assumption IV that $Q(\tau)$ is decreasing. Therefore $1/Q(\tau)$ is increasing. □

Proposition 6.8. *For fixed* $a_1 \in I$ *the function* $\tau \to x(a_1;\tau)$ *is increasing. Moreover,* $x(a_1;\tau) \to \infty$ *when* $\tau \to \infty$.

Proof. The first statement follows from (6.24) and Proposition 6.7. To complete the proof we note that from (6.24) and the boundedness of $1 + p_0$ (inequality (6.3)) that

$$x(a_1;\tau) = a_1 + \left[\frac{\tau}{P_0(-\tau)}\right] P_0(a_1) \geq a_1 + \left[\frac{\tau}{M^{1/2}}\right] P_0(a_1).$$

□

This means that if $\tau^* > \tau$ the space-like curve $x = x(a_1, \tau^*)$ lies above the curve $x = x(a_1;\tau)$ and there are infinitely many space-like curves stacked above each other.

Theorem 6.9. *Let (A_1^0, X^0) be the minimum point of the curve of discontinuity. This is a point of inflection of the curve $x = P_0(a_1)$. If the space-like curve $x = x(a_1, \tau)$ passes through this point, it also inflects there.*

There is a unique $\tau = \tau_c > 0$ such that the curve $x = x(a_1, \tau_c)$ passes through this point.

Proof. By Assumption III the differentiation of (6.28) is valid. Indeed,

$$S'(a_1) = \frac{P_0(a_1)P_0''(a_1)}{[P_0'(a_1)]^2}.$$

Since $S'(a_1) < 0$ for $a_1 \in (A_1^-, A_1^0)$ it follows that $P_0''(a_1) < 0$ for such a_1. To the right of A_1^0 it follows in a similar manner that $P_0''(a_1) > 0$. Thus P_0 is concave on (A_1^-, A_1^0) and convex on (A_1^0, ∞). Therefore (A_1^0, X^0) is a point of inflection of P_0.

If $X^0 = x(A_1^0; \tau)$ it follows from Proposition 6.6 that $x'(A_1^0, \tau) = 0$. From (6.24) we see that

$$x''(a_1; \tau) = \left[\frac{\tau}{P_0(-\tau)}\right] P_0''(a_1).$$

Thus $x(a_1; \tau)$ is also concave to the left and convex to the right of A_1^0.

By Proposition 6.8 $x(A_1^0; \tau)$ increases indefinitely from $A_1^0 = x(A_1^0; 0) < X^0$. For some $\tau = \tau_c$ it must reach the value X^0 and τ_c is unique. □

Let us interpret the results obtained from the assumptions above:

- For $\tau < \tau_c$ the curves $x = x(a_1; \tau)$; $a_1 \in I$ are increasing and do not enter the region of discontinuity H.
- For $\tau = \tau_c$ the curve touches the boundary of H at its minimum point where it has a point of inflection.
- For $\tau > \tau_c$ the curve $x = x(a_1; \tau)$ will increase until it reaches a point (A_1^-, X) on the curve of discontinuity. At that point, which depends on τ, the curve has a maximum. Upon entering the region H the curve will decrease downward and will exit the region at a point $(A_1^+, S(A_1^+))$ where it has a minumum. For $a_1 > A_1^+$ the curve will increase again. That is:
  - For $\tau < \tau_c$ the curve $x = x(a_1; \tau)$ increases and misses the region H entirely.
  - For $\tau = \tau_c$ the curve touches H at its lowest point.
  - For $\tau > \tau_c$: If $a_1 \in (A_1^\dagger, A_1^-)$, the curve increases; at $a_1 = A_1^-$ the curve has slope zero and enters H; if $a_1 \in (A_1^-, A_1^+)$ the curve traverses H with negative

slope; at $a_1 = A_1^+$ the curve exits H with zero slope; for $a_1 > A_1^+$ the curve is outside H and has positive slope.

We have noted in Section 6.3.3 that the curves $x = x(a_1)$ are inverses of the curves $a_1 = a_1(x)$ (for fixed t) if $a_{1,x} > 0$ and that in places where $x'(a_1) = 0$, a_1 jumps discontinuously. This jump discontinuity corresponds in the first place to the point $a_1 = A_1^-$ and goes along the the line $x = X = S(A_1^-)$ in the a_1x-plane. It will end when the line $x = X$ joins the up-going curve $x = x(a_1; \tau)$ at $a_1 = A_1$. Thus the curve has a jump discontinuity at $x = X$ and the size of the jump is $A_1 - A_1^- > 0$. The other space-like curves $x = x(a_2)$ do not exhibit this behaviour. Indeed, from (5.35), the fact that $a_2 \geq x$ and $q_0' \leq 0$, we see that $a_{2,x} > 0$. Thus the curves $x = x(a_2)$ are continuous, increasing and below the straight line $x = a_2$. This is illustrated in the figure below which also gives a visual guide to the notation.

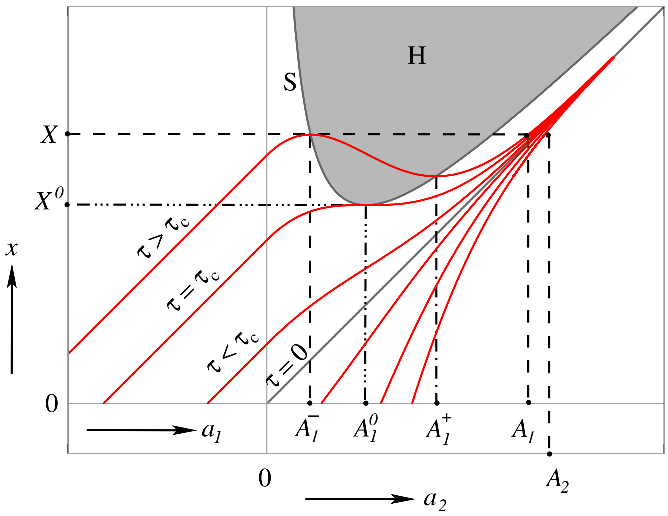

FIGURE 6.1: Space-like curves:
a_1x-plane above and a_2x-plane below diagonal.

Proposition 6.10. *$A_1^-(\tau)$ is decreasing and $A_1^+(\tau)$ is increasing.*

Proof. Consider the equation (6.29) at $a_1 = A_1(\tau) < A_1^0$ and at $a_1 = A_2(\tau) > A_1^0$. Then $-P_0'(A_1^\pm) = Q(\tau)$ and differentiation with respect to τ yields $-P_0''(A_1^\pm)\dfrac{d}{d\tau}\left[A_1^\pm(\tau)\right] = Q'(\tau)$. The result follows from Theorem 6.9 since P_0'' is negative to the left of A_1^0 and positive to the right of it. □

Theorem 6.11. *$A_1^-(\tau) \to A_1^\dagger$ and $X(\tau) \to \infty$ as $\tau \to \infty$. Additionally, $A_1^+(\tau) \to \infty$ as $\tau \to \infty$.*

Proof. By Proposition 6.10, $A_1^-(\tau)$ decreases. Thus the limit $a_1^* = \lim_{\tau\to\infty} A_1^-(\tau)$ exists and $a_1^* \geq A_1^\dagger$. Suppose that $a_1^* > A_1^\dagger$. Then it follows from (6.29) and the boundedness of

P_0 that $P'(a_1^*) = 0$. From (6.28) we see that $S(a_1^*)$ is undefined, which is a contradiction. It follows that $A_1^-(\tau) \to A_1^\dagger$.

Since $X(\tau) = S(A_1^-(\tau))$ it now follows (also from Assumption I) that $X(\tau) \to \infty$.

By Proposition 6.10, $A_1^+(\tau)$ increases. If it has a finite limit a contradiction follows from Proposition 6.4. □

From the assumptions of this section, it follows that the curve $x = x(a_2)$ is continuous and increasing. In fact, since q_0' is negative, it follows from (5.36) that $f_2 > 0$ and from (5.38) that $a_{2,x} > 0$. Let us denote by A_2 the point where this curve intersects the line $x = X$ (Figure 6.1). The following collection of results succintly explain the consequences.

Theorem 6.12. *$A_1(\tau) \to \infty$ and $A_2(\tau) \to \infty$.*

Proof. Since $A_2(\tau) > X(\tau)$ it follows from Theorem 6.11 that $A_2(\tau) \to \infty$ as $\tau \to \infty$. With respect to A_1 we note that since $x(a_1)$ increases for $a_1 > A_1^+$ from a value smaller than X, $A_1(\tau) > A_1^+$ so that by Theorem 6.11 $A_1(\tau) \to \infty$ as $\tau \to \infty$. □

6.4 So what happens when the constraint is challenged?

When one embarks on a journey to climb Mt Kilimanjaro one traverses through five distinct climate zones: rainforest, heath, moorland, alpine desert and arctic. The summit is naturally located in the arctic climate zone. Although trekking through the rainforest, heath, moorland and alpine desert is a beautiful experience on its own and a reward in itself, the goal of every climber is to reach the summit. In analogous terms, Sauer sparked the initial interest to climb our own "mathematical mountain" when he observed in [21] that under certain initial conditions, the solution to the system of equations (1.1) exhibits a strange behaviour. He speculatively attributed this to the effect of the initial pressure "pushing" the constraint (1.2). We took up the challenge and embarked on our own climb. Our equivalent summit goal was to provide definitive answers to Sauer's implied question "what happens when the constraint is challenged?". So if the work done up to this point has been about us trekking through our own "climate zones", learning some neat calculus and useful mathematics on the way, in this section, we fulfil the goal of every climber. We finally arrive at the summit (with some altitude sickness in no small measure!) and show that when the constraint is pushed, pressure and velocity shocks occur and that these develop over time in dramatically different ways. We continue with the assumptions made in Section 6.3.4.

6.4.1 Shock discontinuities

The results of Section 6.3.4 provide crucial stepping stones to the summit. From (5.4), (5.33) and (6.21) we find that,

$$\begin{aligned} 1 + p(t(\tau), x) &= \exp\{q(t(\tau), x\}, \\ &= \exp\frac{1}{2}\Big\{q_0(a_1)\Big\} \exp\frac{1}{2}\Big\{q_0(a_2)\Big\}, \\ &= P_0(a_1)P_0(a_2). \end{aligned} \tag{6.30}$$

In the same vein, by (5.33) and (6.21),

$$\begin{aligned} v(t, x) &= \frac{1}{2}\left[q_0(a_1) - q_0(a_2)\right] \\ &= \ln\left\{\frac{P_0(a_1)}{P_0(a_2)}\right\} \end{aligned} \tag{6.31}$$

For fixed $\tau > \tau_c$, let us consider the line $x = X(\tau)$. There are two distinct values of a_1 for which on this line, namely, $a_1 = A_1^-(\tau)$ and $a_1 = A_1(\tau) > A_1^-(\tau)$ (see Figure 6.1). Thus, the inverse function $x \to a_1(t(\tau), x)$ is multi-valued on $x = X(\tau)$ having the two values just described. On the other hand, the function $x \to a_2(t(\tau), x)$ is single-valued. In fact, $a_2(t(\tau), X(\tau)) = A_2(\tau)$. Thus, from (6.30), $p(t, x) = p(t(\tau), X(\tau))$ has two distinct values, $P_0(A_1^-(\tau))P_0(A_2(\tau)) - 1$ and $P_0(A_1(\tau))P_0(A_2(\tau)) - 1$. We define the "jump" (the size of the discontinuity) in p as the difference of the two values:

$$\begin{aligned} [p](\tau) &= P_0(A_1^-(\tau))P_0(A_2(\tau)) - 1 - \{P_0(A_1(\tau))P_0(A_2(\tau)) - 1\} \\ &= \{P_0(A_1^-(\tau) - P_0(A_1(\tau))\}P_0(A_2(\tau)). \end{aligned} \tag{6.32}$$

Similarly, according to (6.31), the velocity $v(t, x)$ has two distinct values that define the jump in v as their difference:

$$[v](\tau) = \ln\left\{\frac{P_0(A_1^-(\tau))}{P_0(A_1(\tau))}\right\}. \tag{6.33}$$

Finally what exactly is meant by the term "challenging the constraint"? It turns out (quite intuitively) that this is a limiting notion. For if we let $m = \lim_{x\to\infty} P_0(x)$ then we can say that the constraint is *challenged* if $m = 0$. If $m > 0$ the constraint is *not challenged.*

Theorem 6.13. *If the constraint is not challenged,* $\lim_{\tau\to\infty} [p](\tau) = \{m - P_0(A_1^\dagger)\}m$ *and* $\lim_{\tau\to\infty} [v](\tau) = \frac{1}{2}[q_0(A_1^\dagger) - \ln(m)]$*. If the constraint is challenged* $[p](\tau) \to 0$ *and* $[v](\tau) \to \infty$ *when* $\tau \to \infty$.

Proof. The result follows from (5.4), Theorem 6.11, (7.8) and (7.9). □

The conclusion is that when the constraint is challenged, the "pressure shock" decays to zero while the "velocity shock" runs riot. We can understand this with reference to the physical model in the following way: If the initial pressure is low far away, it will tend to distribute evenly while the gas, initially at rest, will rush towards places where the pressure is low. But not only that. For under the hypotheses of Theorem 6.1 we see that from (6.30) and (6.31) that for fixed $\tau > \tau_c$ and $x \gg X(\tau)$, $1 + p(t,x) \sim P_0^2(x)$ and $v(t,x) \sim 0$. Thus, beyond the shock everything is quite calm. The confluence of the space-like curves in Figure 6.1 vividly illustrates this observation and Theorem 6.12. We note however, that the "shock point" $X(\tau)$ runs away when $\tau \to \infty$ (Theorem 6.11). Mathematically, to challenge the constraint means to challenge the hyperbolicity of the system as can be seen from Section 5.2.

The observation that velocity shock runs riot leaves a lingering feeling of incompleteness. To resolve this, we may refer to the physical model again, and consider the power (Watt = force × velocity) experienced by x (reference configuration) at time t defined by

$$W(t,x) := (1+p)v = P_0(a_2) \cdot P_0(a_1) \ln P_0(a_1) - P_0(a_1) \cdot P_0(a_2) \ln P_0(a_2). \tag{6.34}$$

Notice that the above definition has only pressure, not force. This is because we have left out the constant cross-sectional area. The unit is actually Watt per square meter. When the constraint is challenged $\lim_{x\to\infty} P_0(x) = 0$. So we consider the limit of (6.34) as $P_0(a_{1,2}) \to 0^+$. We have

$$\begin{aligned}
\lim P_0(a_{1,2}) \ln(P_0(a_{1,2})) &= \lim \frac{\ln(P_0(a_{1,2}))}{1/P_0(a_{1,2})} \\
&= \lim \frac{1/P_0(a_{1,2})}{-1/(P_0(a_{1,2}))^2} \\
&= -\lim P_0(a_{1,2}) \\
&= 0.
\end{aligned}$$

Since at $x = X$, $a_2 = A_2(\tau)$ it follows that $W(t(\tau), X) \to 0$ as $\tau \to \infty$. Thus the logarithmic growth of the velocity shock is countered by the rapid drop of pressure. We will have more to say on this in the next section.

6.4.2 Decay of pressure and power

In the previous section we demonstrated the decay of pressure at the shock front when the constraint is challenged. This may also be true for "particles" not in shock. We need to consider the dependence of $a_2(t,x)$ on t when x is fixed. In view of Theorem 6.3, we only need to consider the dependence of a_2 on τ_2 as given by (6.26) in the form

$$a_2 - \left[\frac{\tau_2}{P_0(\tau_2)}\right] P_0(a_2) = x.$$

Let $a_2(\tau_2;x)$ denote the solution of this equation.

Proposition 6.14. *If $x > A_1^\dagger$ the function $\tau_2 \to a_2(\tau_2;x)$; $\tau_2 > A_1^\dagger$, is an increasing function of τ_2.*

If the initial pressure p_0 is non-increasing, this is true for arbitrary (fixed) x and $\tau_2 > 0$.

Proof. By (6.26), $a_2(\tau_2;x) > x > A_1^\dagger$ and hence $P_0'(a_2(\tau;x)) \leq 0$. Also, since $\tau_2 \in I$, $P_0(\tau_2)$ cannot increase. If $p_0' \leq 0$ everywhere, the same is true for P_0' and then the first conclusion remains valid.

Let $Q_2(\tau_2) := \tau_2/P_0(\tau_2)$. Then (6.26) can be written as $x = a_2 - Q_2(\tau_2)P_0(a_2)$. Differentiation with respect to τ_2 (x fixed) yields after some re-arrangement

$$[1 - Q_2(\tau_2)P_0'(a_2;x)]\frac{da_2}{d\tau_2} = Q_2'(\tau_2)P_0(a_2).$$

Since P_0 is decreasing, Q_2 is increasing. The required assertion now follows under any one of the two hypotheses. □

Theorem 6.15. *Suppose that the constraint is challenged. For fixed $x > A_1^\dagger$, $1 + p(t(\tau),x) \to 0$ when $\tau \to \infty$. If p_0 is non-decreasing, this is true for arbitrary x. In either case the power $W(t(\tau),x) \to 0$ as $\tau \to \infty$.*

Proof. We first show that $a_2 \to \infty$ when $\tau_2 \to \infty$. For this we re-write (6.14) as follows:

$$\frac{x}{Q_2(\tau_2)} = \frac{a_2}{Q_2(\tau_2)} - P_0(a_2), \tag{6.35}$$

Suppose the increasing function $a_2(\tau_2)$ has a finite limit a_2^* then, since $Q_2(\tau_2) \to \infty$, the left and first term on the right of (6.35) tend to zero as $\tau_2 \to \infty$. Hence $P_0(a_2^*) = 0$ and that cannot be. It follows from (6.29) and (6.34) that $1 + p(t(\tau),x)$ and $W(t(\tau),x)$ tend to zero as $\tau \to \infty$. □

6.5 Summary

This chapter sought to flesh out Sauer's speculation "what happens when the constraint is challenged?". We showed that when the constaint is challenged (as suitably defined) shocks in both pressure and velocity occur. Further, the shock fronts behave differently: the pressure shock front decays to zero and the velocity shock runs riot, uncontrollably. We conclude with Chapter 7 which contains some examples to illustrate our results.

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Chapter 7

Examples

7.1 Introduction

In this final chapter we present some illustrative examples. In the first example the initial pressure distribution is modelled by

$$q_0(x) = -x^2, \quad -\infty < x < \infty,$$

which means that $1+p_0(x) = \exp\{-x^2\}$. This example was first reported on in Sauer[21] and describes a situation where the constraint is challenged from both sides ($\pm\infty$). Quite naturally we call this example the "two-sided challenge". In the second example the initial pressure distribution is modelled by

$$q_0(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ -x^2 & \text{otherwise.} \end{cases}$$

It is a situation where the constraint is challenged from one side ($+\infty$). We call this example the "one-sided challenge".

Before proceeding with the examples, we would like to confirm that the examples are valid in the sense that they conform to Assumptions I – IV of Section 6.3.4.

7.1.1 Two-sided challenge and Assumptions

The initial pressure is $p_0 = \exp\{-x^2\} - 1$. Hence $P_0 := [1 + p_0]^{\frac{1}{2}} = \exp\{-\frac{1}{2}x^2\}$ for $-\infty < x < \infty$, (see (6.21)). It follows that the curve of discontinuity is given by

$$\begin{aligned} x = S(a_1) &= a_1 - \frac{P_0(a_1)}{P_0'(a_1)}, \\ &= a_1 + \frac{1}{a_1}. \end{aligned} \tag{7.1}$$

Assumption I. $p_0' = -2x\exp\{-x^2\} < 0$. We need to discount the possibility that $x \leq 0$. This follows since p_0 is strictly decreasing only on the interval $(0, \infty)$ and so we can choose $A^\dagger$ as zero. Therefore $x > 0$ necessarily on the curve.
For the second part, we note that its consequence is Proposition 6.4. From (7.1), this proposition is satisfied. Thus Assumption I is satisfied.

Assumption II. Since $S''(a_1) = 2/a_1^3 > 0$ and exists for $a_1 > 0$, S is convex on the interval I. This is a consequence of the mean-value theorem [14, Theorem 2.6, p.68].

Assumption III. The function p_0 is infinitely differentiable.

Assumption IV. This is satified because for all $\tau > 0$,

$$P_0(-\tau) + \tau P_0'(-\tau) = (1 + \tau^2)\exp\{\tfrac{1}{2}\tau^2\} > 0.$$

7.1.2 One-sided challenge and Assumptions

Assumptions I –IV are satisfied. This follows from the aforegoing assumption-analysis for the two-sided challenge because the one sided challenge is a restriction of the two-sided challenge to the domain $0 < x < \infty$.

7.2 Time-like curves

7.2.1 The two-sided challenge

The starting point is the relation (6.14), $I_1(b_1, x) = I_2(b_2, x)$ where

$$\left.\begin{aligned} I_1(b_1, x) &= b_1 - \frac{1}{2}b_1 q_0(x - b_1) + \frac{1}{2}\int_0^{b_1} q_0(x - \sigma_1)d\sigma_1 \\ I_2(b_2, x) &= b_2 - \frac{1}{2}b_2 q_0(x + b_2) + \frac{1}{2}\int_0^{b_2} q_0(x + \sigma_2)\, d\sigma_2 \end{aligned}\right\} \tag{7.2}$$

With q_0 as given, we may compute the integrals I_1, I_2 explicitly:

$$
\begin{aligned}
I_1(b_1, x) &= b_1 - \tfrac{1}{2}b_1(-(x-b_1)^2) + \tfrac{1}{2}\int_0^{b_1} -(x-\sigma_1)^2\, d\sigma_1 \\
&= b_1 + \tfrac{1}{2}b_1 x^2 - xb_1^2 + \tfrac{1}{2}b_1^3 - \tfrac{1}{2}(x^2\sigma_1 - x\sigma_1^2 + \tfrac{1}{3}\sigma_1^3)\Big|_0^{b_1} \\
&= b_1 - \tfrac{1}{2}xb_1^2 + \tfrac{1}{3}b_1^3 \\
&= b_1\left[1 - \tfrac{1}{2}xb_1 + \tfrac{1}{3}b_1^2\right], \qquad (7.3)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(b_2, x) &= b_2 - \tfrac{1}{2}b_2(-(x+b_2)^2) + \tfrac{1}{2}\int_0^{b_1} -(x+\sigma_2)^2\, d\sigma_2 \\
&= b_2 + \tfrac{1}{2}b_2 x^2 + xb_2^2 + \tfrac{1}{2}b_2^3 - \tfrac{1}{2}(x^2\sigma_2 + x\sigma_2^2 + \tfrac{1}{3}\sigma_2^3)\Big|_0^{b_2} \\
&= b_2 + \tfrac{1}{2}xb_2^2 + \tfrac{1}{3}b_2^3 \\
&= b_2\left[1 + \tfrac{1}{2}xb_2 + \tfrac{1}{3}b_2^2\right]. \qquad (7.4)
\end{aligned}
$$

The critical step in the practical implementation of the relation is to choose b_1 and then calculate b_2. This involves the study of cubic polynomials.

Let us, for the time being, consider only cases for which $x \geq 0$. From (7.4), we see that I_2 is strictly increasing, with the only zero at $b_2 = 0$. On the other hand, I_1 is different. The zeros of I_1 are at

$$
b_1 = \begin{cases} 0, \\ \eta_1 = \frac{3}{4}\left[x - \sqrt{x^2 - x_c^2}\right], & \text{if } x \geq x_c = \sqrt{48}/3 = 2.309\ldots; \\ \eta_2 = \frac{3}{4}\left[x + \sqrt{x^2 - x_c^2}\right], & \text{if } x \geq x_c. \end{cases}
$$

The local extrema of I_1 occur at the zeros of $1 - xb_1 + b_1^2$ and they exist only if $x \geq 2$. These are:

$$
b_1 = \begin{cases} H_1 = \frac{1}{2}[x - \sqrt{x^2-4}], & \text{a local maximum;} \\ H_2 = \frac{1}{2}[x + \sqrt{x^2-4}], & \text{a local minimum.} \end{cases}
$$

Thus we are able to distinguish three cases: $0 \leq x \leq 2$, $2 < x < x_c$ and $x \geq x_c$.

Case 1: $0 \leq x \leq 2$. Figure 7.1 illustrates Case 1. It is clear from this sketch that a value of b_1 may be chosen and that the equation $I_2(b_2, x) = I_1(b_1, x)$ has a unique solution b_2.

Case 2: $2 < x < x_c$. This is depicted in Figure 7.2. Here it is apparent that three different branches of the trajectory have to be calculated. Branch 1 corresponds to the

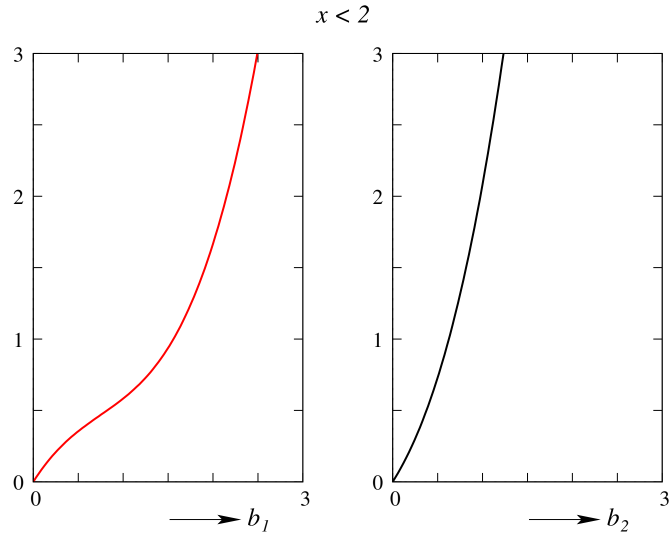

FIGURE 7.1: Case 1: I_1 (left), I_2 (right); $0 \leq x \leq 2$

choice $0 \leq b_1 \leq H_1$. Branch 2 is associated with $H_1 < b_1 < H_2$ while Branch 3 is obtained from $b_1 \geq H_2$.

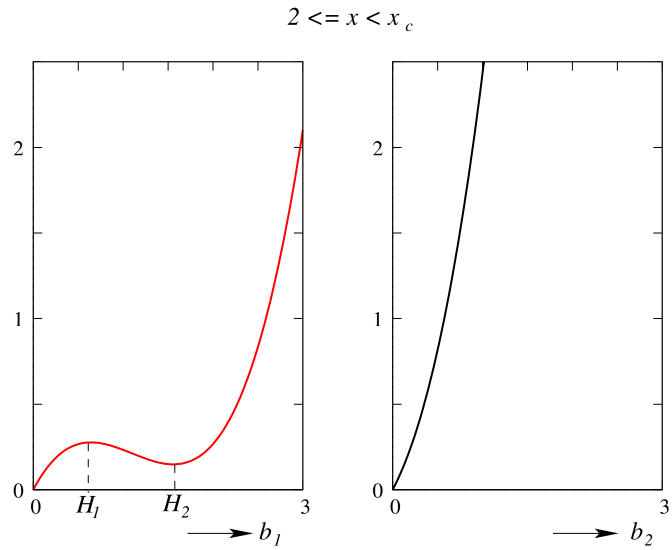

FIGURE 7.2: Case 2: I_1 (left), I_2 (right); $2 \leq x \leq x_c$

Case 3: $x \geq x_c$. This is depicted in Figure 7.3. This case is tricky and it is one of those "treacherously bended" situations (alluded to in Section 6.3.1). For $\eta_1 < b_1 < \eta_2$ the equation cannot be solved. The branches of the solution are defined by $0 \leq b_1 \leq H_1$ (Branch 1), $H_1 < b_1 \leq \eta_2$ (Branch 2) and $b_1 \geq \eta_2$. It should be noted that if $x = x_c$, the zeros η_1, η_2 and the minimum point H_2 coincide ($= \sqrt{3}$).

Once we know how to calculate b_2 when b_1 is given, the next step is to construct points on the time-like curves $(a_1(t, x), a_2(t, x))$ for fixed x and $t \geq 0$. For this the transformation (6.8) has to be be inverted. In fact, $a_1 = b_1 - x$ and $a_2 = b_2 + x$.

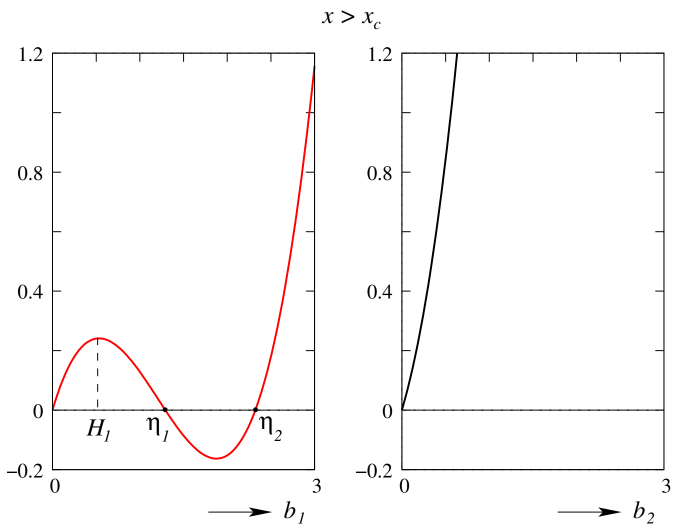

FIGURE 7.3: Case 3: I_1 (left), I_2 (right); $x \geq x_c$

For Case 1, there is a single curve in $a_1 a_2$-space. This is for $0 \leq x \leq 2$. For Case 2 ($2 < x \leq x_c$) the curve consists of three branches. Branch 1 results from choices of $b_1 \in [0, H_1]$; for Branch 2, $b_1 \in (H_1, H_2)$ and Branch 3 is obtained when $b_1 \geq H_2$. Case 3 ($x > x_c$) yields the three brances corresponding to b_1 in $[0, H_1]$, $(H_1, \eta_1]$ and $[\eta_2, \infty)$. The interval (η_1, η_2) is excluded. In Figure 7.4 a "broken" trajectory from Case 3 is shown.

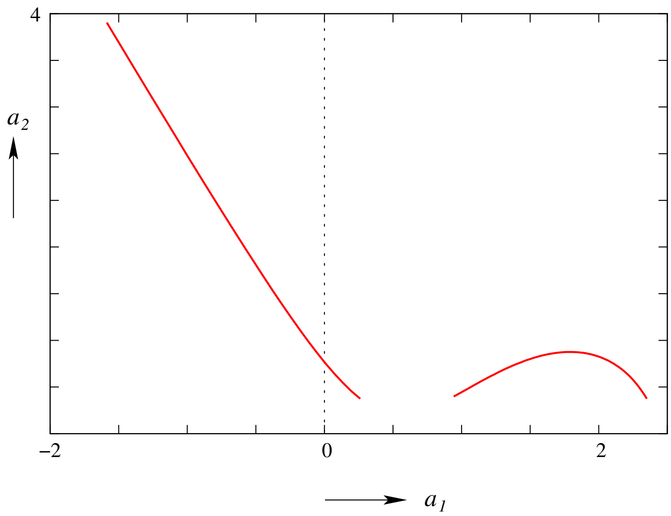

FIGURE 7.4: Typical trajectory for Case 3

A family of the trajectories such as shown in Figure 7.4 where the different curves correspond to different values of the parameter x is called a phase portrait. The phase portrait is precisely the time-like curves that we theoretically discussed in section 6.3.1. For different values of x a phase portrait is shown in Figure 7.5

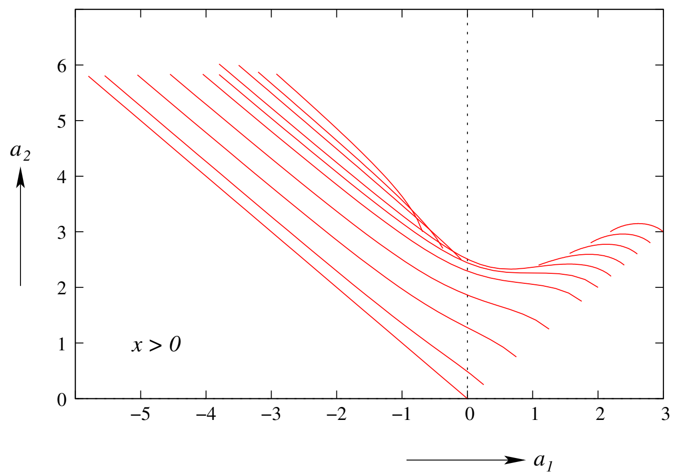

FIGURE 7.5: Phase portrait: $x \geq 0$

The analysis above is only for the case $x \geq 0$. Let us suppose that x is negative ($-x > 0$). It now follows from Theorem 5.6 that the expressions (7.3), (7.4) take the form $I_1(b_1, -x) = I_2(b_1 .x)$ and $I_2(b_2, -x) = I_1(b_2, x)$. This means that for negative x the roles of a_1 and a_2 interchange from what they are when $x > 0$.

7.2.2 The one-sided challenge

The integrals in (7.2) need to be considered in the light of the particular construction of q_0 and the translations in the integrands. The following is obtained after a careful examination:

$$J_1(b_1, x) = \int_0^{b_1} q_0(x-\nu)d\nu = \begin{cases} 0 & \text{if } x \leq 0; \\ \int_0^{b_1} q_0^+(x-\nu)d\nu & \text{if } x \geq b_1; \\ \int_0^{x} q_0^+(x-\nu)d\nu & \text{elsewhere,} \end{cases} \tag{7.5}$$

Thus discontinuities in a_1 and a_2 transfer to discontinuities in p and v and these occur at $x = X(\tau)$ and for $\tau > \tau_c$ as illustrated in Figure 6.1. We can express these discontinuities in equations called *shock conditions*. These are:

$$[p](\tau) = p(t(\tau), A_1(\tau)) - p(t(\tau)), A_1^-(\tau)), \tag{7.6}$$

$$[v](\tau) = v(t(\tau), A_1(\tau)) - v(t(\tau)), A_1^-(\tau)), \tag{7.7}$$

where [ ] means "the jump of". At $x = X$, the curve $x = x(a_1)$ is discontinuous at $a_1 = A_1^-$ and this discontinuity ends at $a_1 = A_1$. The curve $x = x(a_2)$ is continuous and intersects the line $x = X$ at the point $a_2 = A_2$ (see Figure 6.1). The expressions

(7.6),(7.7) explicitly are as follows:

$$[p](\tau) = P_0(A_1(\tau))P_0(A_2(\tau)) - 1 - P_0(A_1^-(\tau))P_0(A_2(\tau)) + 1,$$
$$= \Big[P_0(A_1(\tau)) - P_0(A_1^-(\tau))\Big] P_0(A_2(\tau)), \tag{7.8}$$
$$[v](\tau) = \frac{1}{2}q_0(A_1(\tau)) - \frac{1}{2}q_0(A_2(\tau)) - \frac{1}{2}q_0(A_1^-(\tau)) + \frac{1}{2}q_0(A_2(\tau)),$$
$$= \frac{1}{2}\Big[q_0(A_1(\tau)) - q_0(A_1^-(\tau))\Big]. \tag{7.9}$$

and

$$J_2(b_2, x) = \int_0^{b_2} q_0(x+\nu)d\nu = \begin{cases} 0 & \text{if } x \le -b_2; \\ \int_0^{b_2} q_0^+(x+\nu)d\nu & \text{if } x \ge 0; \\ \int_{-x}^{b_2} q_0^+(x+\nu)d\nu & \text{elsewhere,} \end{cases} \tag{7.10}$$

We can calculate the expressions (7.5), (7.10) for the case $q_0 = -x^2$. The outcomes are:

$$J_1(b_1, x) = \begin{cases} 0 & \text{if } x \le 0; \\ \frac{1}{3}[(x-b_1)^3 - x^3] & \text{if } x \ge b_1; \\ -\frac{1}{3}x^3 & \text{elsewhere,} \end{cases} \tag{7.11}$$

and

$$J_2(b_2, x) = \begin{cases} 0 & \text{if } x \le -b_2; \\ \frac{1}{3}[x^3 - (x+b_2)^3] & \text{if } x \ge 0; \\ -\frac{1}{3}(x+b_2)^3 & \text{elsewhere,} \end{cases} \tag{7.12}$$

A phase portrait based on (7.2), (7.11) and (7.12) may now be calculated. It is represented in Figure 7.6

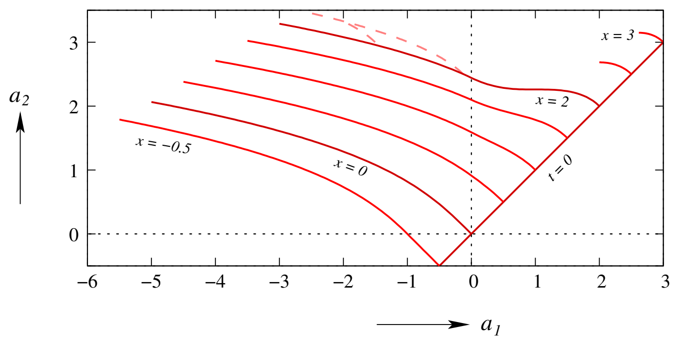

FIGURE 7.6: One-sided phase portrait:$1 \le x \le 3$

Remark 7.1. Figures 7.5 and 7.6 suggest that the respective families of curves have an envelope at the upper end that seems to serve as a demarcation of the "permissible regions" and "forbidden regions" to the solution space of the Cauchy Problem (5.1) - (5.3). We investigate this possibility in Appendix A.

7.3 Space-like curves

This section is best understood with Section 6.3.4 and Fig 6.1 in mind. The computation of space curves rely on $P_0 = \exp\{-\frac{1}{2}x^2\}$. For both the one-sided and the two-sided examples, q_0 is the same and we have seen from the assumption-analysis that x is automatically restricted to positive values only, otherwise we will have the absurd result that $a_{1,2}^2 \leq -1$. Thus for purposes of this section, which is an illustration of the underlying theory of space-like curves using the two examples, it matters not which example we use. In what follows, $q_0 = -x^2$. From (6.24) and (6.26)

$$x(a_1) = a_1 + \Big[\frac{\tau_1}{\exp\{-\frac{1}{2}\tau_1^2\}}\Big] \exp\Big\{-\frac{1}{2}a_1^2\Big\}. \tag{7.13}$$

and

$$x(a_2) = a_2 - \Big[\frac{\tau_2}{\exp\{-\frac{1}{2}\tau_2^2\}}\Big] \exp\Big\{-\frac{1}{2}a_2^2\Big\}. \tag{7.14}$$

The space-like curves associated with a_1, a_2 are then computed by letting $\tau_{1,2}$ run through different values.

7.3.1 Discontinuities

Given a particular a_1, from (6.28) and (7.1) a curve of discontinuity of a_1 is given by $x = S(a_1) := a_1 + 1/a_1$. For a_2 an similar expression can be obtained. For this curve, $a_1 = 0$ is an asymptote and it can be easily established that the point $a_1 = 1$ is a local minimum point and this minimum is equal to 2. This quick mental sketch conforms to Fig 6.1. Thus the point $(1, 2)$ is the point (A_1^0, X^0) referenced in Theorem 6.9. The region H in Fig 6.1, which is the region enclosed by the curve of discontinuity, is the region that satifies $\{(a_1, x) : (1 + a_1^2)/a_1 < x\}$.

Still with Theorem 6.9, we see that the point $\tau = \tau_c$ which is the point (a_1, τ_c) at which the space-like curve $x = x(a_1, \tau)$ passes through (A_1^0, X^0), the minimum point of the curve of discontinuity. Within the context of our specific examples, we have established that $(A_1^0, X^0) = (1, 2)$. So $a_1 = 1$. Hence we can use (7.15) to compute τ_c. Therefore

$$\exp\{-0.5\}\tau_c = \exp\{-0.5\tau_c^2\}. \tag{7.15}$$

It follows that $\tau_c = 1$.

7.4 What happens when the constraint is challenged?

From the theoretical discussion in Section 6.4 we know that when the constraint is challenged, shocks in both pressure and velocity occur. For fixed time (represented by τ) over a range of values of x, expressions (7.13) and (7.14) give the inverse characteristics $a_1(\tau, x), a_2(\tau, x)$. From there the velocity $v(t, x)$ and $p(x, t)$ can be evaluated. The use of (5.8), (5.22) and (5.24) results in evaluations of $u_1(t, x)$ and $u_2(t, x)$ and the identities

$$v = u_1 + u_2 = \tfrac{1}{2}[q_0(a_1) - q_0(a_2)] = \frac{1}{2}\left[a_2^2 - a_1^2\right], \tag{7.16}$$

and

$$q = u_1 - u_2 = \tfrac{1}{2}[q_0(a_1) + q_0(a_2)] = -\frac{1}{2}\left[a_1^2 + a_2^2\right]. \tag{7.17}$$

Further use of the transformation (5.4), results in an expression of pressure p in terms of the inverse characteristics. The result is

$$p = \exp\{q\} - 1 = \exp\left\{-\frac{1}{2}(a_1^2 + a_2^2)\right\}.$$

With (7.16) and (7.17), one is in a position to compute velocity and pressure profiles for the example $q(x) = -x^2$ for $x > 0$. The results are shown in Figure 7.7.

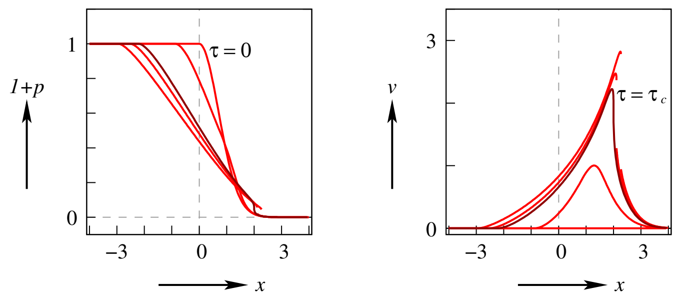

FIGURE 7.7: Evolution of pressure (left) and velocity (right)

The development of discontinuities is evident. Evident as well is the decline of the initial pressure and the apparent confinement of both velocity and pressure in the downstream direction, a result of onset of shocks. The shock occurs at $x = X(\tau)$, a point that keeps going forward in time. The *shock curve* $x = X(\tau)$ increases quite rapidly. The shock curve or shock front for a_1, $x = X(\tau)$ is actually a curve of discontinuities discussed in Section 6.3.4. From (6.28) and (7.15)

$$x(a_1; \tau) = a_1 + \left[\frac{\tau}{P_0(-\tau)}\right] P_0(a_1),$$

so that in particular,

$$X(\tau) = a_1 + \Big[\frac{\tau_1}{\exp\{-\frac{1}{2}\tau_1^2\}}\Big] \exp\Big\{-\frac{1}{2}a_1^2\Big\}.$$

By Theorem 6.11, in the limit as $\tau \to \infty$,

$$X(\tau) \to A_1^\dagger + P_0(A_1^\dagger)\frac{\tau}{P_0(-\tau)} \text{ as } \tau \to \infty.$$

For our example with $q = -x^2$, we established in the discussion on space-like curves that the asymptote of $S(a_1) = a_1 + 1/a_1$ is $a_1 = 0$. Hence $A_1^\dagger = 0$. Thus in the limit

$$X(\tau) \to \tau \exp\{-\tau^2/2\} = \tau \exp\{\tau^2/2\} \text{ as } \tau \to \infty.$$

For the two-sided challenge the computational procedure is similar, and instead of Figure 7.7, we obtain Figure 7.8 below. This should be compared to Figure 1.1.

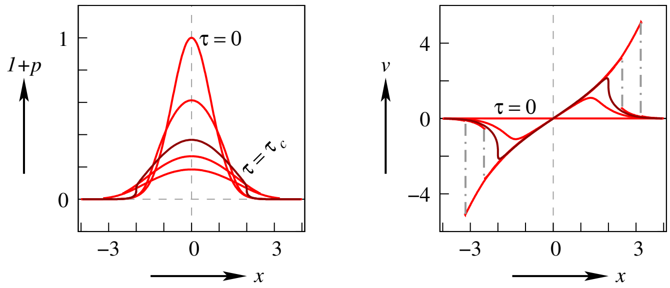

FIGURE 7.8: Evolution of pressure (left) and velocity (right)

7.5 Concluding remarks

We have seen that when the input pressure decreases, shock discontinuities in both pressure and velocity will develop and pressure will decay. If the constraint is challenged, pressure will decay towards the allowed lower limit and velocity will increase unboundedly with time at the shock. This seems to be un-physical. The mathematical constraint we impose may be at the core. Physically the constraint should be stricter in the sense that a positive lower bound would be more appropriate. In that case, we found, that velocity will indeed not get out of hand.

The sudden drop of velocity ahead of the shock may be seen in the pressure- dependent propagation speed. When this quantity is near zero, disturbances will find it hard to propagate, which heuristically accounts for the violent shock. Challenges to the constraint as studied here, also have a mathematical side. It is the constraint that makes the system of differential equations hyperbolic. When the constraint is challenged, hyperbolicity is challenged in the sense that two crucial eigenvectors become almost linearly dependent.

Appendix A

Envelopes and their equations

A.1 Envelopes

As alluded to in Chapter 7, Figures 7.5 and 7.6 seem to suggest strongly that the respective families of curves have an envelope at the upper end, and therefore a demarcation of sorts of "permissible regions" and "forbidden regions" to the solution space of the Cauchy Problem (5.1) - (5.3). We investigate this possibility.

A.1.1 General envelope equations

An envelope is a curve that is at a common tangent to a family of curves. The idea behind the determination of its analytic equation is to find a way of generating and solving two simultaneous equations: one that relates to the equation of the family of curves and the other to the equation of the tangent to the family. Since a family of curves is generated by varying a suitable parameter, it seems that the natural equations we seek are the parametric equation of the family of curves together with the equation that results when these are differentiated with respect to the parameter. This is essentially the gist of the theory. For more details see the excellent calculus text [9], pages 466 to 467.

As before the starting point is the relation $I_1(b_1, x) = I_2(b_2, x)$ with I_1, I_2 as per (7.2). Since the $a_1 a_2$ phase space is at stake, our first step is to rewrite this relation in terms of the variables a_1, a_2. Since $b_1 = x - a_1$ and $b_2 = a_2 - x$ we have

$$\begin{aligned}
I_1(b_1, x) &= J_1(a_1, x)\\
&= (x - a_1) - \frac{1}{2}(x - a_1)q_0(a_1) + \frac{1}{2}\int_0^{x-a_1} q_0(x - \sigma)d\sigma\\
&= (x - a_1) - \frac{1}{2}(x - a_1)q_0(a_1) + \frac{1}{2}\int_{a_1}^{x} q_0(\sigma)d\sigma\\
&= -(a_1 - x) + \frac{1}{2}(a_1 - x)q_0(a_1) - \frac{1}{2}\int_{x}^{a_1} q_0(\sigma)d\sigma\\
&= -(a_1 - x)[1 - \frac{1}{2}q_0(a_1)] - \frac{1}{2}\int_{x}^{a_1} q_0(\sigma)d\sigma,
\end{aligned}$$

and

$$\begin{aligned}
I_2(b_2, x) &= J_2(a_2, x)\\
&= (a_2 - x) - \frac{1}{2}(a_2 - x)q_0(a_2) + \frac{1}{2}\int_0^{a_2-x} q_0(x + \sigma)d\sigma\\
&= (a_2 - x) - \frac{1}{2}(a_2 - x)q_0(a_2) + \frac{1}{2}\int_{x}^{a_2} q_0(\sigma)d\sigma\\
&= (a_2 - x)[1 - \frac{1}{2}q_0(a_2)] + \frac{1}{2}\int_{x}^{a_2} q_0(\sigma)d\sigma.
\end{aligned}$$

We can now, through the relation $I_1(b_1, x) = I_2(b_2, x)$, define the expression

$$\begin{aligned}
&J(a_1, a_2, x) := J_2(a_2, x) - J_1(a_1, x)\\
&= (a_1 - x)[1 - \frac{1}{2}q_0(a_1)] + (a_2 - x)[1 - \frac{1}{2}q_0(a_2)] + \frac{1}{2}\int_{x}^{a_1} q_0(\sigma)d\sigma + \frac{1}{2}\int_{x}^{a_2} q_0(\sigma)d\sigma\\
&= 0.
\end{aligned}$$

For the tangent equation, the derivative of J with respect to x is zero. That is

$$J_x(a_1, a_2, x) = -2 + \tfrac{1}{2}q_0(a_1) + \tfrac{1}{2}q_0(a_2) - q_0(x) = 0.$$

Thus the general envelope equations are:

$$\left.\begin{aligned}
(a_1 - x)[1 - \frac{1}{2}q_0(a_1)] + (a_2 - x)[1 - \frac{1}{2}q_0(a_2)]&\\
+ \frac{1}{2}\int_{x}^{a_1} q_0(\sigma)d\sigma + \frac{1}{2}\int_{x}^{a_2} q_0(\sigma)d\sigma = 0;&\\
\tfrac{1}{2}[q_0(a_1) + q_0(a_2)] - q_0(x) - 2 = 0.&
\end{aligned}\right\} \tag{A.1}$$

A.1.2 The two-sided challenge

In this instance the general envelope equations (A.1) take the particular form:

$$\left.\begin{array}{c}(a_1 + a_2) - \frac{1}{2}x(a_1^2 + a_2^2) + \frac{1}{3}(a_1^3 + a_2^3) + \frac{1}{3}x^3 - 2x = 0 \\ (a_1^2 + a_2^2) = 2(x^2 - 2)\end{array}\right\} \tag{A.2}$$

We notice that these only make sense if $x^2 \geq 2$. Under this restriction, we introduce the "radius" $r(x)$ by setting $r^2(x) = a_1^2 + a_2^2$. The second of the equations (A.2) may then be expressed in the form $r^2(x) = 2(x^2 - 2)$. When this is substituted into the first equation we find

$$\begin{aligned}&(a_1 + a_2) - \tfrac{1}{2}x(a_1^2 + a_2^2) + \tfrac{1}{3}(a_1^3 + a_2^3) + \tfrac{1}{3}x^3 - 2x \\ &= (a_1 + a_2) - \tfrac{1}{2}xr^2(x) + \tfrac{1}{3}(a_1^3 + a_2^3) + \tfrac{1}{3}x^3 - 2x \\ &= (a_1 + a_2) - x(x^2 - 2) + \tfrac{1}{3}(a_1^3 + a_2^3) + \tfrac{1}{3}x^3 - 2x \\ &= (a_1 + a_2) + \tfrac{1}{3}(a_1^3 + a_2^3) - \tfrac{2}{3}x^2 = 0. \end{aligned} \tag{A.3}$$

Now let $y = a_1 + a_2$. We observe that

$$a_1^3 + a_2^3 = (a_1 + a_2)^3 - 3a_1a_2(a_1 + a_2) = y^3 - 3a_1a_2y$$

and

$$r^2 = a_1^2 + a_2^2 = (a_1 + a_2)^2 - 2a_1a_2 = y^2 - 2a_1a_2y. \tag{A.4}$$

Hence (A.3) can be transformed to a cubic equation in y:

$$\begin{aligned}&(a_1 + a_2) + \tfrac{1}{3}(a_1^3 + a_2^3) - \tfrac{2}{3}x^2 \\ &= y + \tfrac{1}{3}y^3 + \tfrac{1}{2}yr^2 - \tfrac{1}{2}y^3 - \tfrac{2}{3}x^3 \\ &= \left[1 + \tfrac{1}{2}r^2(x)\right] y - \tfrac{1}{6}y^3 - \tfrac{2}{3}x^3 \\ &= -\tfrac{1}{6}y^3 + (x^2 - 1)y - \tfrac{2}{3}x^3 = 0. \end{aligned} \tag{A.5}$$

We are now faced with the study of the roots of the cubic function F defined by

$$F(y) := -\tfrac{1}{6}y^3 + (x^2 - 1)y - \tfrac{2}{3}x^3.$$

This function has a local maximum at $y = y_+ = [2(x^2-1)]^{1/2} > 0$ and a local minimum at $y = y_- = -[2(x^2-1)]^{1/2} < 0$ (recall from (A.2) that $x \geq \sqrt{2} > 1$). Also, $F(y_-) = -\frac{2}{3}\{2^{1/2}(x^2-1)^{3/2} + x^3\} < 0$. Since $F(y) \to \infty$ as $y \to -\infty$, it has a zero at $y = y_0 < y_-$. Let us investigate the possibility of other zeros. To assist us, consider the value $F_+ = F(y_+)$. We have, after some arithmetic,

$$F_+ = \tfrac{2}{3}\{2^{1/2}(x^2-1)^{3/2} - x^3\}.$$

Since $F(y) \to -\infty$ when $y \to \infty$, the cubic polynomial F has, besides y_0, no other zeros if $F_+ < 0$, and this can happen if x is small. For those values of x for which $F_+ \geq 0$, there must be two additional zeros. The smallest x for which this can occur, is when $F_+ = 0$. That is, when $2^{1/3}(x^2-1) = x^2$, or when

$$x = x_e = \left[\frac{2^{1/3}}{2^{1/3}-1}\right]^{1/2} = 2.20166\ldots$$

Thus for $x = x_e$ there are two coincident zeros of F, namely $y = y_+$. For $x > x_e$, there are two distinct zeros, $y_2 < y_+ < y_1$. Also, since $F(0) = -\frac{2}{3}x^3 < 0$ in this range of x-values, it follows that the smaller of the two roots is positive.

If y is root of F we can find a_1 and a_2 in the following way: We have by (A.4) that

$$\left.\begin{aligned} y &= a_1 + a_2, \\ r^2 &= y^2 - 2a_1a_2. \end{aligned}\right\} \tag{A.6}$$

So that on upon the substitution $a_2 = y - a_1$, the following quadratic equation is satisfied:

$$a_1^2 - ya_1 + \tfrac{1}{2}(y^2 - r^2) = 0. \tag{A.7}$$

By symmetry of the equations in (A.6), a_2 should satisfy the same equation. The roots of (A.7) are $\frac{1}{2}[y \pm (2r^2 - y^2)^{1/2}]$. Since $a_1 \leq x \leq a_2$, we must have

$$\left.\begin{aligned} a_1 &= \tfrac{1}{2}[y - \sqrt{(2r^2 - y^2)}]; \\ a_2 &= \tfrac{1}{2}[y + \sqrt{(2r^2 - y^2)}]. \end{aligned}\right\} \tag{A.8}$$

The question now is, for which of the three roots y_0, y_1 and y_2 the expressions in (A.8) apply? To begin, we evaluate the function F at $y = \sqrt{(2)}y_- = -2(x^2-1)$. The result is $-\frac{2}{3}[(x^2-1)^{3/2} + x^3] < 0$ for $x \geq 1$. Thus $y_0 < y_\ddagger = -\sqrt{(2)}y_-$. Now, $2r^2 - y_0^2 < 2r^2 - y_\ddagger^2 = 4(x^2-2) - 4(x^2-1) = -4 < 0$. Thus for $y = y_0$ the square root in (A.8) is not defined and we are left with the two positive roots.

The hope is that the expressions in (A.8) should be valid for y_1 and y_2, which means that $2r^2 \geq y_1$ which only makes sense if $x \geq x_e > 2$. Since $y_1 \geq y_2$, we need not consider more. Let $y_\ddagger$ be defined $y_\ddagger = 2r^2 = 4(x^2 - 2)$. A direct calculation will show that $y_\ddagger > y_+$ for $x \geq x_e$. The next step requires some arithmetic finesse. We calculate $F(y_\ddagger)$. After some manipulations the result is

$$F(y_\ddagger) = \tfrac{2}{3}[(x^2 - 2)^{1/2}(x^2 + 1) - x^3].$$

Expansion of the term in brackets yields

$$(x^2 - 2)^{1/2}(x^2 + 1) - x^3 = (x^6 - 3x^2 - 2)^{1/2} - x^3 < x^3 - x^3 = 0.$$

Thus, $F(y_\ddagger) < 0$ and it follows that $y_\ddagger > y_1$. Indeed, the expressions in (A.8) are valid for $y = y_1$ and $y = y_2$. Figure A.1 illustrates the line of argument followed above.

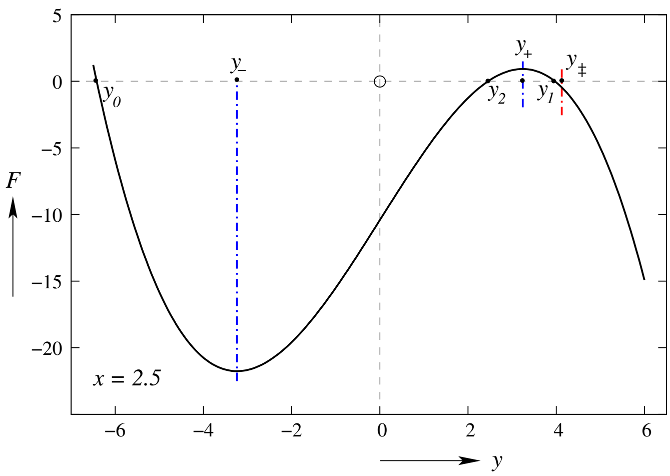

FIGURE A.1

The envelope has two additional branches. Branch 1 corresponds to the choice $y = y_1$ in (A.8) with $x \geq x_e$. Branch 2 corresponds to the choice $y = y_2$. Since y_1 increases with x, we will have $a_2 > x > a_1 > 0$ on Branch 2. On the other hand, y_2 decreases with x and therefore a_1 may become negative. This is shown in Figure A.2. Clearly, no time-like curve can move into the region above the envelope.

The analysis above is only for the case $x \geq 0$. As noted earlier, for the situation $x < 0$ the roles of a_1 and a_2 interchange from what they are when $x > 0$. The full phase portrait with positive and negative values of x is shown in Figure A.3. It leads to additional insights. We notice that the trajectories all emanate from the line $a_2 = a_1 = x$. Also, the line corresponding to $x = 0$ is $a_2 = -a_1$. This follows from (A.5) and from the fact that $y = a_1 + a_2$. The trajectories are all situated between the two envelopes. Outside of this region no points in the $a_1 a_2$ phase space are ever visited. The region between the trajectories corresponding to $x = -2$ and $x = 2$ is fully covered by trajectories while

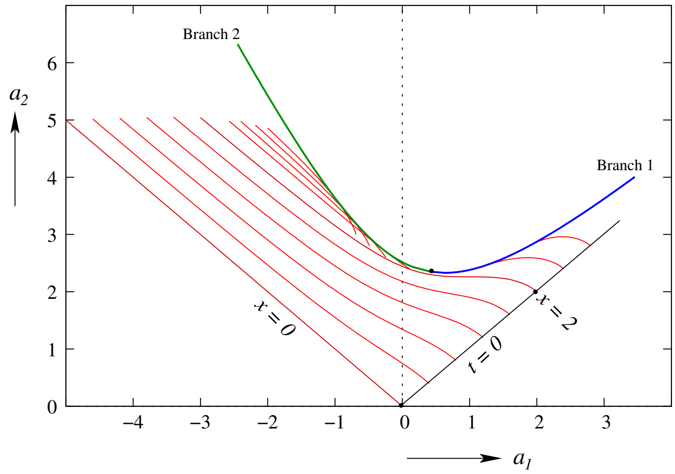

FIGURE A.2: Phase portrait with envelope: $0 \leq x \leq 3$

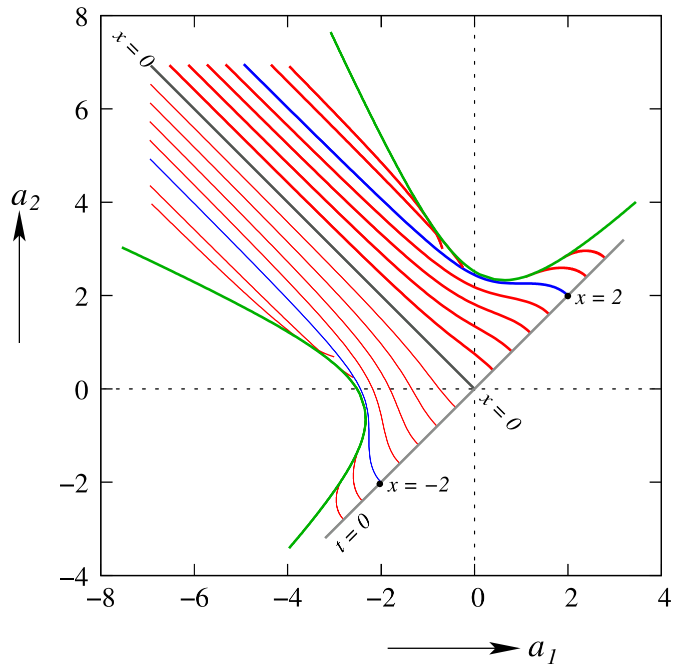

FIGURE A.3: Full phase portrait with envelopes: $-3 \leq x \leq 3$

in the regions bounded by the lower envelope and $x = -2$ or $x = 2$ and the upper envelope, trajectories may intersect or break up which could indicate the formation of discontinuities in the solution of the initial value problem (5.11), (5.10).

The picture shown in Figure A.3 shows time-like curves since each trajectory corresponds to a fixed value of x in tx-space. For the complete scene we need to construct space-like curves.

A.1.3 The one-sided challenge

This situation is somewhat similar to the two-sided challenge. Cognisance must, however, be taken of negative arguments of q_0. In both cases we have $x \geq \sqrt{2} > 0$

Since we always have $a_1(t,x) \leq x \leq a_2(t,x)$, we can distinguish two cases:

A. $a_1 > 0$. In this case, $a_2 \geq 0$ and the equations (A.1) are the same as for the two sided case:

$$(a_1 + a_2) - \tfrac{1}{2}x(a_1^2 + a_2^2) + \tfrac{1}{3}(a_1^3 + a_2^3) + \tfrac{1}{3}x^3 - 2x = 0; \qquad (a_1^2 + a_2^2) = 2(x^2 - 2).$$

to which we must add the restriction that $a_1 > 0$, which means that not all of the two-sided envelope is valid.

B. $a_1 \leq 0$. The envelope equations are now different. Since $a_2 \geq x \geq \sqrt{2}$, the first equation of (A.1) becomes

$$(a_1 - x) + (a_2 - x)[1 + \tfrac{1}{2}a_2^2] - \frac{1}{2}\int_x^0 \sigma^2 \, d\sigma - \frac{1}{2}\int_x^{a_2} \sigma^2 \, d\sigma = 0.$$

In expanded form this can be expressed as follows:

$$(a_1 + a_2) - \tfrac{1}{2}a_2^2 x + \tfrac{1}{3}a_2^3 + \tfrac{1}{3}x^3 - 2x = 0. \tag{A.9}$$

The second equation of (A.1) in this case is

$$a_2^2 = 2(x_2 - 2). \tag{A.10}$$

This branch (Branch 3) of the envelope can be calculated quite simply. For given x, calculate a_2 from (A.10) and then use (A.9) to calculate a_1, letting x run through values larger than $\sqrt{2}$. The useful expression here is

$$a_1 = \frac{2x^3 + [1 - 2x^2]a_2}{3}. \tag{A.11}$$

However one must take caution that the calculated a_1 is not positive.

Next is to find those x-es for which $a_1 \leq 0$. So we need to find $x \geq \sqrt{2}$ for which $a_1 = 0$. It is sufficient to consider the numerator in (A.9) in combination with (A.10). After some algebraic manipulations the equation to be solved turns out to be

$$x^6 - 6x^4 + \tfrac{9}{2}x^2 - 1 = 0. \tag{A.12}$$

The substitution $z = x^2$ in (A.12) leads to the cubic equation

$$z^3 - 6z^2 + \tfrac{9}{2}z - 1 = 0.$$

This can be solved numerically for the positive root and reverted back to obtain $x = 2.27299\ldots$. Now the envelope branches can be calculated. Figure A.4 below is the result. Branch 1 is the same as for the two-sided challenge. Branch 2 is part of the same for the two-sided challenge, while Branch 3 corresponds to the case $a_1 \leq 0$ discussed above.

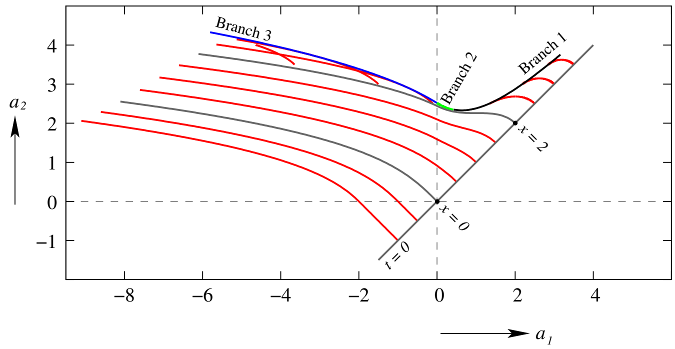

Figure A.4: Envelope for one-sided challenge

Bibliography

[1] D.T. Blackstock. History of nonlinear acoustics: 1750s-1930s. In Mark F. Hamilton and David T. Blackstock, editors, *Nonlinear Acoustics*, chapter 1, pages 1–23. The Acoustical Society of America, Melville, New York, USA, 2008.

[2] J. Challis. Lxv. on the velocity of sound, in reply to the remarks of the astronomer royal. *Philosophical Magazine.*, 32(218):494–499, 1848.

[3] R. Courant and K.O. Friedrichs. *Supersonic Flow and Shock Waves.* Springer-Verlag, Berlin, 1976.

[4] R. Courant and D. Hilbert. *Methods of Mathematical Physics Volume 2.* Interscience Publishers, New York, NY, USA, 1962.

[5] S. Earnshaw. On the mathematical theory of sound. *Philosophical Transactions of the Royal Society of London*, 150:133–148, 1860.

[6] L. Euler. Principes généraux du mouvement des fluides. *Mémoires de l'académie des sciences de Berlin*, 11:274–315, 1757.

[7] L. Euler. De la propagation du son. *Mémoires de l'académie des sciences de Berlin*, 15:185–209, 1759.

[8] A.C. Fowler. *Mathematical Models in the Applied Sciences.* Cambridge University Press, Cambridge, UK, 1997.

[9] W.A. Granville, P.F. Smith, and W.R. Longley. *Elements of the Differential and Integral Calculus.* Ginn & Company, Boston, MA, 1957.

[10] R. Haberman. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems.* Pearson Education Inc, Upper Saddle River, NJ, USA, fourth edition, 2004.

[11] S. Howison. *Practical Applied Mathematics: Modelling, Analysis, Approximation.* Cambridge University Press, Cambridge, UK, 2005.

[12] H. Hugoniot. Mémoire sur la propagation du mouvement dans les corps et spécialement dans les gaz parfaits. *Journal de l'École polytechnique.*, 57:3–97, 1887.

[13] A. Jeffrey. *Quasilinear hyperbolic systems and waves.* Research Notes in Mathematics. Pitman Publishing Ltd, London, 1976.

[14] S. Lang. *Undergraduate Analysis.* Springer-Verlag New York Inc, New York, NY, 1983.

[15] C.C. Lin and L.A. Siegel. *Mathematics Applied to Deterministic Problems in the Natural Sciences.* Macmillan Publishing Co., New York, NY, 1974.

[16] J.D. Logan. *Applied Mathematics.* Wiley-Interscience, Hoboken, NJ, 3rd edition, 2006.

[17] S.D. Poisson. Mémoire sur la théorie du son. *Journal de l'École polytechnique*, 7:319–392, 1808.

[18] W.J.M. Rankine. On the thermodynamic theory of waves of finite longitudinal disturbance. *Philosophical Transactions of the Royal Society of London*, 160:277–288, 1870.

[19] Lord Rayleigh. Aerial plane waves of finite amplitude. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 84(570):247–284, 1910.

[20] B. Riemann. Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. *Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, 8:43–65, 1860.

[21] N. Sauer. An excursion into classical nonlinear acoustics. *International Journal of Engineering Science*, 48:670–684, 2010.

[22] N. Sauer. The dynamic piston problem in classical nonlinear acoustics. *Math. Models and Meth. Appl. Sci.*, 21:149–167, 2011.

[23] G.G. Stokes. Liv. on a difficulty in the theory of sound. *Philosophical Magazine*, 33(223):349–356, 1848.

[24] G.I. Taylor. The conditions necessary for discontinuous motion in gases. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 84(571):371–377, 1910.

[25] E.F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction.* Springer-Verlag Berlin Heidelberg, Berlin, 2009.