# Canard solutions in equations with backward bifurcations of the quasi-steady state manifold*

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**Abstract**

In this paper we consider a delayed exchange of stability for solutions of a singularly perturbed nonautonomous equation in the case when a backward bifurcation of its quasi-steady (critical) manifolds occurs. This result is applied to provide a precise descriptions of canard solutions to singularly perturbed predator-prey models of Rosenzweig–MacArthur and Leslie–Gowers type.

**Key words:** singularly perturbed dynamical systems, multiple time scales, Tikhonov theorem, delayed stability switch, predator-prey models, canard solutions, backward bifurcation.
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## 1 Introduction

Singular perturbation theory describes the behaviour of solutions to systems of differential equations of the form

$$\begin{aligned} \varepsilon u_t &= f(t,u,v,\varepsilon), \quad u(t_0)=u_0 \\ v_t &= g(t,u,v,\varepsilon), \quad v(t_0)=v_0 \end{aligned}, \tag{1.1}$$

where $(f,g): U \to \mathbb{R}^{n+m}$ is a sufficiently regular function on an open set $U \subset \mathbb{R}^{m+n+2}$, as the parameter $\varepsilon$ tends to 0. Problems of this type arise in almost all fields of science and engineering, where one attempts to model systems driven by mechanisms operating at widely different scales (whose ratio is represented as

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the parameter $\varepsilon$), see e.g. [3, 17, 25, 31], and the interest is to use the solution of the simplified equation obtained from (1.1) by putting $\varepsilon = 0$ to approximate true solutions to (1.1) for small, but nonzero, $\varepsilon$. Numerous methods to approach such problems have been developed, ranging from classical asymptotic expansions, through the Tikhonov-Vasilieva theory, [30], the geometric singular perturbation theory, [12], to the recent renormalization group approach, [8, 9, 20]. A typical, and somehow expected, result is that, under some technical assumption, the solutions to (1.1) can be approximated by the solutions on the *quasi-stationary manifold*; that is, the manifold $\Gamma$ determined by

$$0 = f(t, u, v, 0); \tag{1.2}$$

the solution $\bar{u}(t, v)$ of (1.2) is then used to find the approximation of $v$ given by the solution $\bar{v}$ to

$$v_t = g(t, v, \bar{u}(t, v), \varepsilon), \quad v(t_0) = v_0. \tag{1.3}$$

Whether $(\bar{u}, \bar{v}) \approx (u, v)$ for small $\varepsilon$ depend on two fundamental assumptions: the solution $\bar{u}$ must be isolated and $\Gamma$ must be attractive (hyperbolic) in the sense that $\Gamma$ must consist of uniformly asymptotically stable equilibria of the *layer equation*

$$u_\tau = f(t, u, v, 0), \tag{1.4}$$

where $(t, v)$ are treated as parameters. In many cases, however, the solution of (1.2) is a complicated self-intersecting geometrical structure with changes of stability, as defined above, occurring at the bifurcation points. In such cases an expectation is that the solutions to (1.1) will follow the part of $\Gamma$ consisting of its attracting parts. While this often happens, see e.g. [13, 14, 15, 17, 18, 19], there are cases when the solution, having passed close to the intersection, follows the repelling part of $\Gamma$ for some fixed time and only after that jumps to the closest attracting part of $\Gamma$. Such a phenomenon often is termed the *delayed stability switch* and the solutions having this property are referred to as *canards*, e.g. [4, 6, 10, 11, 17, 16, 24, 27]. The existence of canards have been investigated in those papers by various methods, such as nonstandard analysis, see e.g. [6], matched asymptotic expansions, [11, 23], or singularity blow-ups, [10]. An alternative method, based on upper and lower solutions, was proposed in [7], where the authors considered one-dimensional non-autonomous problems with $\Gamma$ exhibiting transcritical and pitchfork bifurcations. The approach of [7] was extended to a class higher dimensional monotone problems in [4, 5]. While not as general, this method has the advantage of being quite elementary; it also handles non-autonomous problems and provides an explicit formula for the time of the stability switch, that is not available for most of other methods.

The aim of this note is to extend the method of [7] to backward bifurcations of the quasi-steady manifold. The proof also covers the backward (or subcritical) transcritical bifurcation that typically occurs alongside the forward transcritical bifurcation, when two branches of $\Gamma$ intersect in a transversal but not orthogonal way. In this way are able to complete the proof of [7, Theorem 2.1] in the case of negative initial values as the transcritical bifurcation considered there, is backward in the fourth quadrant.

As an application, we consider two predator-prey models: the Rosenzweig–MacArthur model, [22, 28] and the Leslie–Gowers/Holling model, [2]. If the prey dynamics is very fast, the existence of slow-fast cycles and canard solutions were investigated in [26, 13] in the former case, while the latter was considered in [1]. In both these cases the quasi-steady manifolds intersect and a backward bifurcation occurs along their intersection. Applying the one dimensional theory developed in this paper, combined with the approach of [4, 5], we give an elementary proof of the existence of canards and provide an exact value of time at which the stability switch occurs.

## 2 Formulation of the problem and assumptions

As in [7, Theorem 2.1], we consider the initial value problem for the singularly perturbed differential equation

$$\varepsilon u_t = f(t,u,\varepsilon), \quad u(t_0,\varepsilon) = u_0, \tag{2.1}$$

in $D = D_0 \times I_{\varepsilon_0}$, where $D_0 = I_T \times I_N$, $I_N = (-N, N)$ with $N > 0$, $I_{\varepsilon_0} = \{\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ and $t \in I_T = \{t : t_0 < t < T\}$. We use the notation for the flows; that is, if it is necessary to indicate the initial time or condition, the solution $u$ to (2.1) will be denoted by $u(t, t_0, u_0, \varepsilon)$. This notation will be also extended to indicate further dependence of $u$ with respect to additional parameters that may arise in the problem.

As discussed in Introduction, we are interested in solutions to (2.1) for small $\varepsilon$ and that strongly depends on the roots of the degenerate equation

$$f(t,u,0) = 0; \tag{2.2}$$

each solution to (2.2) is an equilibrium of the layer equation

$$u_\tau = f(t,u,0), \quad \tau > 0,$$

where $t$ and $u$ are considered as a parameters (related by (2.2)). The following assumptions will we needed throughout the paper:

(A1) $f \in C^3(\overline{D}, \mathbb{R})$.

(A2) Let $t_0 \leq t_a < t_b < T$. In $\overline{I}_T \times \overline{I}_N$ the solution set of (2.2) consists of three roots: $\phi_0 = \{(t,u) \in D_0 : u = 0\}$, $\phi_1 = \{(t,u) \in D_0 : u = \varphi_1(t)\}$, $\phi_2 = \{(t,u) \in D_0 : u = \varphi_2(t)\}$ for $t \in [t_a, T]$ and $\varphi_1, \varphi_2$ are twice differentiable on $[t_a, T]$. Moreover, we have

$$\begin{aligned}
&\varphi_1(t_a) \leq \varphi_2(t_a),\\
&\varphi_1(t) < \varphi_2(t), \;\; t \in (t_a, T],\\
&\varphi_2(t) > 0, \;\; t \in [t_a, T],\\
&\varphi_1(t) > 0, \;\; t \in [t_a, t_b) \quad \text{and} \quad \varphi_1(t) < 0, \;\; t \in (t_b, T].
\end{aligned}$$

**Remark 2.1.** The geometry of the quasi-steady steady states introduced in assumption (A2) depends on the mutual relations between the parameters. In general, by a backward (transcritical) bifurcation we understand the configuration, when there are solutions to (2.2) in the first quadrant before the time they intersect; that is, for $t_0 \le t < t_b$ in the current notation. However, recently there has been an interest in more specific cases of backward bifurcation, such as the one shown on Fig. 1, when the backward branch of the quasi-steady state after some time folds again forward, creating the second attracting branch that coexists for some time with the attracting trivial steady state. Such backward bifurcations are of importance e.g. in epidemiology, see e.g. [21], as they describe situations in which locally stable disease free and endemic equilibria can coexist. In the notation of (A2), the latter case occurs if $t_a > t_0$ and $\varphi_1(t_a) = \varphi_2(t_a)$; that is, for $t \in [t_0, t_a)$ there is only the trivial attractive quasi-steady state and two new quasi-steady, one repelling and one attractive, branch out at $t_a$. The backward bifurcation occurs at $t_b > t_a$ (Fig. 1). If, however, $\varphi_1(t_a) < \varphi_2(t_a)$, then $\varphi_2$ is an isolated quasi-steady and $\phi_1$ and $\phi_0$ intersect at $t_b$ forming a backward transcritical (sometimes called subcritical) bifurcation. Such a case would occur if on Fig. 1 the turning point of the quasi-steady state occurred at some $t < t_0$ and we took $t_a = t_0$. As the main objective of the paper is to analyse the behaviour of the solutions close to $t_b$, distinguishing these cases does not make any difference in the proof, apart from a technical condition that $u_0 < \phi_1(t_0)$ if $t_a = t_0$. Finally, if the turning point of the quasi-steady state on Fig. 1 was in the fourth quadrant, then the upper branch of $\varphi_2$ would be a standard transcritical bifurcation and thus we will not consider this case here.

For the stability of the quasi-steady states we assume

(A3) $f_u(t,0,0) < 0,\ t \in (t_0, t_b) \quad \text{and} \quad f_u(t,0,0) > 0,\ t \in (t_b, T],$
$f_u(t,\varphi_2(t),0) < 0,\ t \in (t_a, T],$
$f_u(t,\varphi_1(t),0) > 0,\ t \in (t_a, t_b) \quad \text{and} \quad f_u(t,\varphi_2(t),0) > 0,\ t \in (t_b, T].$

For $u \equiv 0$ we additionally assume

(A4) $f(t,0,\varepsilon) = 0$ for $(t,\varepsilon) \in \overline{I}_{\varepsilon_0} \times \overline{I}_T$.

**Remark 2.2.** Assumption (A4) ensures that $u \equiv 0$ is a solution to the problem (2.1) for all $\varepsilon \in \overline{I}_{\varepsilon_0}$ and $\overline{I}_T$. Therefore, if we let $I_N^+ = \{u \in I_N : u \ge 0\}$ and $I_N^- = \{u \in I_N : u \le 0\}$, then $I_N^\pm$ are invariant under the flow generated by (2.1).

Next, let us define the function

$$F(t,\varepsilon) = \int_{t_0}^{t} f_u(s,0,\varepsilon)ds. \tag{2.3}$$

By (A3), $F(t,0) = 0$ has at most one root in $(t_0, T)$. Further, we assume

(A5) The equation $F(t,0) = 0$ has a root $t^*$ in $(t_0, T)$.

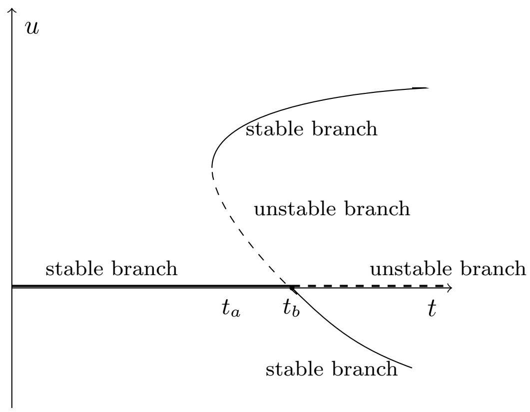


Figure 1: Backward bifurcation at the bifurcation time $t_b$

We observe that $F(t,0)$ attains global minimum on $[t_0,\infty)$ at $t = t_a$, hence $t_b < t^*$ and $F(t,0) < 0$ on $(t_0,t^*)$ with $\frac{dF}{dt}\big|_{t=t^*} > 0$.

The last assumption on $f$ is:

(A6) There is a positive number $c_0$ such that $c_0 \in I_N^+$ and

$$f(t,u,\varepsilon) \le f_u(t,0,\varepsilon)u + \frac{1}{2}f_{uu}(t,0,\varepsilon)u^2,$$

for $t_0 \le t \le t^*$, $\varepsilon \in \overline{I}_{\varepsilon_0}$ and $0 \le u \le c_0$.

**Remark 2.3.** Note that assumption (A6) holds if the third derivative of $f$ with respect to $u$ at $u = 0$ is negative.

**Remark 2.4.** As will be clear from the proof, most of the assumptions above, such as (A1) and (A6), must hold only in a neighbourhood of the bifurcation point. A more exhaustive discussion of possible relaxations of the assumptions for transcritical bifurcation can be found in [5, Appendix].

**Remark 2.5.** In this paper we restrict ourselves to positive solutions. The reason for this is that the problem in $I_N^-$ can be transformed, by replacing $u$ by $-u$, to a problem with forward transcritical bifurcation in $I_N^+$, [29]. That problem, under the assumption that $u = \varphi_1(t)$ is attractive for $t > t_b$, has been studied in [7, Theorem 2.1]. In the same theorem the authors also stated the result for negative solutions. However, the proof of this part is not correct as they used the same construction as for the positive solutions without recognizing that the bifurcation they considered is backward in the negative half-plane and, under the adopted assumption ([7, Theorem 2.1, assumption (A5)]) the exponential function is an upper solution for both positive and negative initial conditions. Thus our result can be considered as filling this gap.

## 3 Delayed exchange stabilities in the backward bifurcation

As in [7], our approach is based on the concept of upper and lower solutions. We recall that functions $\alpha$ and $\beta$ are, respectively, a lower and upper solution to (2.1), if they are continuous, piecewise differentiable with respect to $t$ and

$$\alpha(t,\varepsilon) \leq \beta(t,\varepsilon), \quad \alpha(t_0,\varepsilon) \leq u_0 \leq \beta(t_0,\varepsilon) \tag{3.1}$$

and

$$\varepsilon\alpha_t(t,\varepsilon) \leq f(t,\alpha,\varepsilon), \quad \varepsilon\beta_t(t,\varepsilon) \geq f(t,\beta,\varepsilon), \tag{3.2}$$

for $t \in \overline{I_T}$, $\varepsilon \in I_{\varepsilon_0}$. If there exit lower and upper solutions to (2.1), then there exists its unique solution that satisfies

$$\alpha(t,\varepsilon) \leq u(t,\varepsilon) \leq \beta(t,\varepsilon), \;\; t \in \overline{I_T}, \varepsilon \in I_{\varepsilon_0}.$$

We also recall that, for a sequence of function defined on a noncompact $I$, we say that it converges almost uniformly if it converges uniformly on any compact subset of $I$.

The following technical lemma is basic for the considerations.

**Lemma 3.1.** *Assume that $G$ is a continuously differentiable function on $[a,b]$ satisfying $G(a) = G(b) = 0$, $G(t) < 0$ on $(a,b)$ with $G_t(a) < 0$ and $G_t(b) > 0$. Further, let $\{\phi(\cdot,\varepsilon)\}_{\varepsilon>0}$ be a family of continuous functions on $[a,b]$ such that $\lim_{\varepsilon\to 0^+}\phi(t,\varepsilon) = \phi(t)$ uniformly for $t \in [a,b]$. Then for any $t \in (a,b)$, we have*

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_a^t e^{\frac{G(s)}{\varepsilon}}\phi(s,\varepsilon)ds &= -\frac{\phi(a)}{G_t(a)}, \\ \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_t^b e^{\frac{G(s)}{\varepsilon}}\phi(s,\varepsilon)ds &= \frac{\phi(b)}{G_t(b)} \end{aligned} \tag{3.3}$$

*and the convergence is almost uniform on $(a,b)$. Moreover, there is $M < \infty$ such that*

$$\sup_{0<\varepsilon<\varepsilon_0} \frac{1}{\varepsilon}\int_a^b e^{\frac{G(s)}{\varepsilon}}|\phi(s,\varepsilon)|ds \leq M. \tag{3.4}$$

*Proof.* First observe that

$$\lim_{\varepsilon\to 0^+} e^{\frac{G(t)}{\varepsilon}} = \begin{cases} 1, & t \in \{a,b\}, \\ 0, & t \in (a,b), \end{cases} \tag{3.5}$$

hence

$$\lim_{\varepsilon\to 0^+} \int_{a_1}^{b_1} e^{\frac{G(s)}{\varepsilon}}\phi(s,\varepsilon)ds = 0. \tag{3.6}$$

for any $a_1, b_1 \in [a,b]$. Moreover, for any $a_1, b_1 \in (a,b)$ there is $k > 0$ such that $e^{\frac{G(t)}{\varepsilon}} \le e^{-\frac{k}{\varepsilon}}$ and hence

$$\left| \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{a_1}^{b_1} e^{\frac{G(s)}{\varepsilon}} \phi(s,\varepsilon) ds \right| \le \lim_{\varepsilon\to 0^+} \frac{\max_{t\in[a,b]} |\phi(t,\varepsilon)| e^{-\frac{k}{\varepsilon}}}{\varepsilon} (b_1 - a_1) = 0. \quad (3.7)$$

We begin with the first equality. Since $G_t(a) < 0$, there is $a_1 > a$ such that $G_t(t) < 0$ on $[a, a_1]$. Therefore, for any $0 < t < a_1$, $G$ maps $(a,t)$ onto $(G(t), 0) =: (w, 0)$ in a one-to-one way. Then we can set $z = G(s)$ so that $dz = G_s(s)ds$ and $s = G^{-1}(z)$. Thus, for $a < t < a_1$,

$$\lim_{\varepsilon\to 0^+} \int_a^t \frac{e^{\frac{G(s)}{\varepsilon}}}{\varepsilon} \phi(s,\varepsilon) ds = -\lim_{\varepsilon\to 0^+} \int_w^0 \frac{e^{\frac{z}{\varepsilon}}}{\varepsilon} \frac{\phi(G^{-1}(z),\varepsilon)}{G_s(G^{-1}(z))} dz = -\lim_{\varepsilon\to 0^+} \int_w^0 \frac{e^{\frac{z}{\varepsilon}}}{\varepsilon} \psi(z,\varepsilon) dz$$
$$= -\lim_{\varepsilon\to 0^+} \int_w^0 \frac{e^{\frac{z}{\varepsilon}}}{\varepsilon} \psi(z) dz - \lim_{\varepsilon\to 0^+} \int_w^0 \frac{e^{\frac{z}{\varepsilon}}}{\varepsilon} (\psi(\varepsilon, z) - \psi(z)) dz,$$

where $\psi(z) = \frac{\phi(G^{-1}(z))}{G_s(G^{-1}(z))}$. Define

$$\delta_\varepsilon(z) = \begin{cases} \frac{1}{\varepsilon} e^{\frac{z}{\varepsilon}}, & w \le z \le 0, \\ 0, & z \notin [w, 0]. \end{cases}$$

Since

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(z) dz = \int_w^0 \frac{1}{\varepsilon} e^{\frac{z}{\varepsilon}} dz = 1 - e^{\frac{w}{\varepsilon}} \to 1 \quad (3.8)$$

when $\varepsilon \to 0^+$ and

$$\lim_{\varepsilon\to 0^+} \delta_\varepsilon(z) = \begin{cases} +\infty, & z = 0 \\ 0, & z \ne 0 \end{cases},$$

$\delta_\varepsilon$ is a delta sequence. Consequently, for any $a < t < a_1$ we get

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_w^0 e^{\frac{z}{\varepsilon}} \psi(z) dz = \frac{\phi(G^{-1}(0))}{G_t(G^{-1}(0))} = \frac{\phi(a)}{G_t(a)}.$$

Further

$$\left| \int_w^0 \frac{e^{\frac{z}{\varepsilon}}}{\varepsilon} (\psi(z,\varepsilon) - \psi(z)) dz \right| \le \sup_{z\in[w,0]} |\psi(z,\varepsilon) - \psi(z)| \int_{-\infty}^{+\infty} \delta_\varepsilon(z) dz$$
$$\le \sup_{s\in[a,a_1]} \frac{|\phi(s,\varepsilon) - \phi(s)|}{|G_s(s)|} \quad (3.9)$$

and hence

$$\lim_{\varepsilon\to 0^+} \int_w^0 \frac{e^{\frac{z}{\varepsilon}}}{\varepsilon} (\psi(z,\varepsilon) - \psi(z)) dz = 0.$$

Thus, for any $t \in (0, a_1)$ we have

$$\lim_{\varepsilon\to 0^+} \int_a^t \frac{e^{\frac{G(s)}{\varepsilon}}}{\varepsilon} \phi(s,\varepsilon) ds = -\frac{\phi(a)}{G_t(a)}$$

and then the first equation of (3.3) for arbitrary $t < b$ follows from (3.7).

To prove the almost uniform convergence, we observe that, by (3.7) and (3.9), it suffices to prove that for any $\delta > 0$ there is $\varepsilon_0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $w \in [G(a_1), w_0]$, where $w_0 < 0$ we have

$$\left| \frac{1}{\varepsilon} \int_w^0 e^{\frac{z}{\varepsilon}} \psi(z) dz - \psi(0) \right| < \delta.$$

By (3.8), we have

$$\frac{1}{\varepsilon} \int_w^0 e^{\frac{z}{\varepsilon}} \psi(z) dz - \psi(0) = \frac{1}{\varepsilon} \int_w^0 e^{\frac{z}{\varepsilon}} (\psi(z) - \psi(0)) dz - e^{\frac{w}{\varepsilon}} \psi(0)$$

and since it is clear that the last term converges to zero uniformly on any half-line $(-\infty, w_0]$, we only have to focus on the integral term. Then we split the integral and, using again (3.8), we obtain the following estimates

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} \int_w^0 e^{\frac{z}{\varepsilon}} (\psi(z) - \psi(0)) dz \right| \\
&\leq \left| \int_w^{-\sqrt{\varepsilon}} \frac{1}{\varepsilon} e^{\frac{z}{\varepsilon}} (\psi(z) - \psi(0)) dz \right| + \left| \int_{-\sqrt{\varepsilon}}^0 \frac{1}{\varepsilon} e^{\frac{z}{\varepsilon}} (\psi(z) - \psi(0)) dz \right| \\
&\leq 2 \max_{z \in [w,0]} |\psi(z)| \left| \int_w^{-\sqrt{\varepsilon}} \frac{1}{\varepsilon} e^{\frac{z}{\varepsilon}} dz \right| + \max_{z \in [-\sqrt{\varepsilon},0]} |\psi(z) - \psi(0)| \left| \int_w^0 \frac{1}{\varepsilon} e^{\frac{z}{\varepsilon}} dz \right| \\
&\leq 2 \max_{z \in [w,0]} |\psi(z)| \left| e^{-\frac{1}{\sqrt{\varepsilon}}} - e^{\frac{w}{\varepsilon}} \right| + \max_{z \in [-\sqrt{\varepsilon},0]} |\psi(z) - \psi(0)|.
\end{aligned}$$

We see that the first term uniformly converges to zero as long as $w \leq w_0 < 0$ and the second term also can be estimated independently of $w$ du to the continuity of $\psi$ at 0.

The proof of the statements for the second equation in (3.3) follows in the same way.

To prove (3.4), we split

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_a^b e^{\frac{G(s)}{\varepsilon}} |\phi(s,\varepsilon)| ds \\
&\leq \sup_{s \in [a,b], 0 < \varepsilon < \varepsilon_0} |\phi(s,\varepsilon)| \left( \frac{1}{\varepsilon} \int_a^{a_1} e^{\frac{G(s)}{\varepsilon}} ds + \frac{1}{\varepsilon} \int_{a_1}^{b_1} e^{\frac{G(s)}{\varepsilon}} ds + \frac{1}{\varepsilon} \int_{b_1}^{b} e^{\frac{G(s)}{\varepsilon}} ds \right) \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $b_1 < b$ is such that $G_t > 0$ on $[b_1, b]$. $I_2$ is clearly bounded by (3.7).

Further, as in the first part of the proof,

$$I_2 = \frac{1}{\varepsilon}\int\limits_a^{a_1} e^{\frac{G(s)}{\varepsilon}} ds = \frac{1}{\varepsilon}\int\limits_w^{0} e^{\frac{z}{\varepsilon}} \frac{1}{G_s(G^{-1}(z))} dz \leq \frac{1}{\inf_{t\in[a,a_1]}|G_t(t)|}(1-e^{\frac{w}{\varepsilon}})$$
$$\leq \frac{1}{\inf_{t\in[a,a_1]}|G_t(t)|} < \infty \tag{3.10}$$

and an analogous calculation is valid for $I_3$. □

**Theorem 3.1.** *Let assumptions (A1)–(A6) and the notations introduced therein hold and let $u_0 \in I_N^+$. Then for sufficiently small $\varepsilon$ there is a unique solution $u(t,\varepsilon)$ to the problem* (2.1) *which satisfies the following conditions*

$$\lim_{\varepsilon\to 0^+} u(t,\varepsilon) = 0, \quad \textit{for} \quad t \in (t_0, t^*), \tag{3.11}$$

$$\lim_{\varepsilon\to 0^+} u(t,\varepsilon) = \varphi_2(t), \quad \textit{for} \quad t \in (t^*, T], \tag{3.12}$$

*and the convergence is almost uniform on the respective intervals.*

*Proof.* Since the proof of this theorem is long and rather technical, first we provide a sketch of it. The idea of the proof is based on the observation that, by the Tikhonov theorem, after the initial transition time the solution $u$ of (2.1) is close to zero and, as long as it stays there, the right hand side $f$ of (2.1) can be approximated by its Taylor expansion around $u = 0$. While for the transcritical bifurcation, as in [7], the linearization works, here the quadratic approximation turns out to be necessary. The resulting equation is the Bernoulli equation that can be explicitly solved and its solution, under the adopted assumptions, is an upper solution for (2.1). Using Lemma 3.3 we find that this upper solution converges to zero on an interval $(t_0, t^*)$ with $t^* > t_a$ and that this is the largest interval on which the convergence is almost uniform. Since the trivial solution is an obvious lower solution to (2.1), we obtain (3.11). To prove (3.12), we construct lower solutions to (2.1) that detach themselves from the trivial solution soon after $t^*$ – it is sufficient to construct lower solutions that become positive uniformly in $\varepsilon$ for $t > t^*$ as then they enter into the domain of attraction of $\varphi_2$ and, by a standard Lyapunov type argument, become attracted by it, dragging with them the solution to (2.1).

Let us now make the above ideas mathematically rigorous. First, observe that, by assumption (A1), applying the Mean Value Theorem and using e.g. the Taylor theorem with the remainder in the integral form, there are (uniformly) continuous functions $\Psi$ and $\psi$ on $[t_0, t^*] \times [0, \varepsilon_0]$ such that

$$F(t,\varepsilon) = F(t,0) + \Psi(t,\varepsilon)\varepsilon. \tag{3.13}$$

In particular,

$$\Psi(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^{\varepsilon} F_s(t,s)ds = \int_0^1 D_2F(t,\varepsilon z)dz \tag{3.14}$$

where $D_2$ denotes differentiation with respect to the second variable.

Observe that $u \equiv 0$ is a solution for any $\varepsilon \in I_{\varepsilon_0}$ and also it is an isolated attracting quasi-steady state in the domain $[t_0, \bar{t}] \times [0, \bar{u}]$, with $\bar{t} < t_b$ and sufficiently small $\bar{u} > 0$. Hence, the solutions to (2.1) starting from positive initial conditions are nonnegative and are attracted to the quasi-steady state $u \equiv 0$ on $(t_0, \bar{t}]$. In other words, if we fix $0 < u_0 < N$, then for any $\eta > 0$ and any $\bar{t} < t_b$ there is $\varepsilon_0$ such that $0 < u(\bar{t}, t_0, u_0, \varepsilon) < \eta$. Thus, using the fact that the solutions cannot cross each other, we see that for $t > \bar{t}$ we have $0 < u(t, t_0, u_0, \varepsilon) \leq u(t, \bar{t}, \eta, \varepsilon)$. Thus, without loosing generality, we can assume that $t_0 = \bar{t}$ and $u_0 = \eta$ for the estimates from above of the solution close to $t_b$.

Let us consider the quadratic approximation to (2.1),

$$\varepsilon\beta_t(t, \varepsilon) = f_u(t, 0, \varepsilon)\beta(t, \varepsilon) + \frac{1}{2}f_{uu}(t, 0, \varepsilon)\beta^2(t, \varepsilon), \quad \beta(t_0, \varepsilon) = u_0. \tag{3.15}$$

Using (A6), we see that $\beta$ is an upper solution to (2.1) as long as it remains small. Since (3.15) is a Bernoulli type equation, it can be easily solved, giving

$$\beta(t, \varepsilon) = \frac{u_0 e^{\frac{F(t,\varepsilon)}{\varepsilon}}}{1 - \frac{u_0}{2\varepsilon}\int_{t_0}^{t} e^{\frac{F(t,\varepsilon)}{\varepsilon}} f_{uu}(s, 0, \varepsilon)ds}. \tag{3.16}$$

Now, observe that, by (3.13), one gets

$$e^{\frac{F(t,\varepsilon)}{\varepsilon}} f_{uu}(t, 0, \varepsilon) = e^{\frac{F(t,0)}{\varepsilon}} e^{\Psi(t,\varepsilon)} f_{uu}(t, 0, \varepsilon) = e^{\frac{F(t,0)}{\varepsilon}} \phi(t, \varepsilon) \tag{3.17}$$

and, by (3.14),

$$\lim_{\varepsilon \to 0^+} \phi(t, \varepsilon) = e^{F_\varepsilon(t,0)} f_{uu}(t, 0, 0).$$

Thus, using Lemma 3.1,

$$\lim_{\varepsilon \to 0^+} \frac{u_0}{2\varepsilon} \int_{t_0}^{t} e^{\frac{F(t,\varepsilon)}{\varepsilon}} f_{uu}(s, 0, \varepsilon)ds = -\frac{u_0 e^{F_\varepsilon(t_0,0)} f_{uu}(t_0, 0, 0)}{2f_u(t_0, 0, 0)} \tag{3.18}$$

almost uniformly on $(t_0, t^*)$. By the comments at the beginning of the proof, we can select $u_0$ small enough for $\left|\frac{u_0 e^{F_\varepsilon(t_0,0)} f_{uu}(t_0,0,0)}{2f_u(t_0,0,0)}\right| < 1/2$ and hence

$$\lim_{\varepsilon \to 0^+} \beta(t, \varepsilon) = 0$$

almost uniformly on $(t_0, t^*)$. Moreover, again by Lemma 3.1,

$$\left|\frac{1}{\varepsilon} \int_{t_0}^{t} e^{\frac{F(t,\varepsilon)}{\varepsilon}} f_{uu}(s, 0, \varepsilon)ds\right| \leq M$$

for all $\varepsilon \in I_{\varepsilon_0}$ and $t \in [t_0, t^*]$ and thus, by selecting sufficiently small $u_0$ we can make $0 < \beta(t, \varepsilon) < c_0$ (see assumption (A6)), hence $\beta$ is an upper solution and (3.11) holds.

Let $\nu > 0$ be such that $t^* - \nu > t_b$. We can also estimate an upper bound of $u(t,\varepsilon)$ on $[t^* - \nu, T]$. From assumption (A2), there exists $\varepsilon_1$ such that for $\varepsilon < \varepsilon_1$ we have $u(t^* - \nu, \varepsilon) < \varphi_2(t^* - \nu)$. Since $\phi_2$ is an attracting quasi-steady state on $[t_b, T]$, we can fix sufficiently small $\omega$ and choose $\varepsilon < \varepsilon_2$ such that, by a classical Lyapunov type argument, e.g. [3, p. 90], if a solution $u(t,\varepsilon)$ enters the strip

$$\{(t,u) : t \in [t^* - \nu, T],\ \varphi_2(t) - \omega < u < \varphi_2(t) + \omega\},$$

then it stays there. Consequently, one has

$$u(t,\varepsilon) \leq \varphi_2(t) + \omega,$$

for every $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$.

Now, we shall construct a nontrivial lower solution to (2.1). Let us define

$$\alpha(t,\varepsilon,\gamma) = \frac{\nu e^{\frac{F(t,\varepsilon)-\gamma(t-t_0)}{\varepsilon}}}{1 - \frac{\nu}{2\varepsilon}\int_{t_0}^{t} e^{\frac{F(t,\varepsilon)-\gamma(t-t_0)}{\varepsilon}} f_{uu}(s,0,\varepsilon)ds}, \tag{3.19}$$

where $\gamma > 0$ and $\nu > 0$ are sufficiently small constants to be determined later. Notice that $\alpha(t,\varepsilon,\gamma)$ satisfies the following initial value problem

$$\varepsilon\alpha_t(t,\varepsilon,\gamma) = (f_u(t,0,\varepsilon) - \gamma)\,\alpha(t,\varepsilon,\gamma) + \frac{1}{2} f_{uu}(t,0,\varepsilon)\alpha^2(t,\varepsilon,\gamma), \quad \alpha(t_0,\varepsilon) = \nu.$$

Moreover, observe that $\alpha(t,\varepsilon,\gamma) \leq \beta(t,\varepsilon)$ for every $\gamma$, $t \in \overline{I}_T$ and $\varepsilon \in I_{\varepsilon_0}$. Set

$$G(t,\gamma) = F(t,0) - \gamma(t - t_0).$$

Using the Implicit Function Theorem and the properties of $F$, for sufficiently small $\gamma$ there is a unique $t_1(\gamma)$ for which $G$ reaches its minimum and there is a unique exists $t_2(\gamma) > t^*$ such that $G(t_2(\gamma),\gamma) = 0$ and $G_t(t_2(\gamma),\gamma) > 0$. Furthermore, $t_1(\gamma) \to t_b$ and $t_2(\gamma) \to t^*$ as $\gamma \to 0^+$. Now, we have as in (3.10)

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_{t_0}^{t_2(\gamma)} e^{\frac{F(s,\varepsilon)-\gamma(s-t_0)}{\varepsilon}} f_{uu}(s,0,\varepsilon)ds\right| \leq \frac{1}{\varepsilon}\int_{t_0}^{t_2(\gamma)} e^{\frac{G(s,\gamma)}{\varepsilon}} e^{\Psi(s,\varepsilon)} |f_{uu}(s,0,\varepsilon)|ds \\
&\leq \max_{s\in[0,T], 0<\varepsilon<\varepsilon_0} e^{\Psi(s,\varepsilon)}|f_{uu}(s,0,\varepsilon)| \left(M_1 + \frac{1}{\varepsilon}\int_{b_1}^{t_2(\gamma)} e^{\frac{G(s,\gamma)}{\varepsilon}} ds\right) \\
&\leq M_2\left(M_1 + \frac{1}{\inf_{t\in[b_1,t_2(\gamma)]}|f_u(t,0,0)-\gamma|}\right) \leq M_3 < \infty,
\end{aligned}$$

where $b_1$ is an arbitrary number such that $f_u(t,0,0) - \gamma > 0$ on $[b_1, t_2(\gamma)]$ and hence $M_3$ can be made independent of $\gamma$ for sufficiently small $\gamma$. Thus, we can select $\nu$ small enough for $\nu M_3/2 < 1/2$ and thus, for sufficiently small $\varepsilon, \nu$ and $t \in [t_0, t_2(\gamma)]$ we obtain

$$0 < \alpha(t,\varepsilon,\gamma) \leq M_4\nu \tag{3.20}$$

where $M_4$ is independent of $\gamma$. Further, let us define

$$\alpha(t_2(\gamma), \varepsilon, \gamma) = \frac{\nu e^{\Psi(t_2(\gamma),\varepsilon)}}{H(\gamma, \varepsilon)}, \tag{3.21}$$

where, as in (3.17),

$$H(\gamma, \varepsilon) := 1 - \frac{\nu}{2\varepsilon} \int_{t_0}^{t_2(\gamma)} e^{\frac{G_\gamma(s)}{\varepsilon}} e^{\Psi(s,\varepsilon)} \, f_{uu}(s, 0, \varepsilon) ds. \tag{3.22}$$

Using Lemma 3.1, we get

$$\lim_{\varepsilon \to 0^+} \alpha(t_2(\gamma), \varepsilon, \gamma) = \frac{\nu e^{F_\varepsilon(t_2(\gamma),0)}}{H(\gamma, 0)}, \tag{3.23}$$

where

$$H(\gamma, 0) = 1 + \frac{\nu}{2} e^{F_\varepsilon(t_0,0)} \, \frac{f_{uu}(t_0, 0, 0)}{f_u(t_0, 0, 0)} - \frac{\nu}{2} e^{F_\varepsilon(t_2(\gamma),0)} \, \frac{f_{uu}(t_2(\gamma), 0, 0)}{f_u(t_2(\gamma), 0, 0)},$$

where, by (A3), $f_u(t_2(\gamma), 0, 0) > 0$. Let us recall that $t_2(\gamma) > t^*$ and $t_2(\gamma) \to t^*$, as $\gamma \to 0^+$. Hence

$$\lim_{\gamma \to 0^+} H(\gamma, 0) = H(0, 0) = 1 + \frac{\nu}{2} e^{F_\varepsilon(t_0,0)} \frac{f_{uu}(t_0, 0, 0)}{f_u(t_0, 0, 0)} - \frac{\nu}{2} e^{F_\varepsilon(t^*,0)} \frac{f_{uu}(t^*, 0, 0)}{f_u(t^*, 0, 0)}.$$

Next, by (3.20), for sufficiently small $\varepsilon, \nu$, we have $\alpha(t, \varepsilon, \gamma) \leq c_0$ for every $t \in [t_0, t_2(\gamma)]$. By assumptions (A1) and (A4) we can write

$$f(t, u, \varepsilon) = f_u(t, 0, \varepsilon) u + \frac{1}{2} f_{uu}(t, 0, \varepsilon) u^2 + \frac{1}{6} f_{uuu}(t, \varsigma_u, \varepsilon) u^3,$$

where $\varsigma_u \in I_N$, hence assumption (A6) yields

$$\begin{aligned} \varepsilon \alpha_t(t, \varepsilon, \gamma) &= \left(f_u(t, 0, \varepsilon) - \gamma\right) \alpha_\gamma + \frac{1}{2} f_{uu}(t, 0, \varepsilon) \alpha_\gamma^2 \\ &= f(t, \alpha_\gamma, \varepsilon) - \gamma \alpha_\gamma - \frac{1}{6} f_{uuu}(t, \varsigma_u, \varepsilon) \alpha_\gamma^3 \\ &\leq f(t, \alpha_\gamma, \varepsilon) - \gamma \alpha_\gamma + k_4 \alpha_\gamma^3 \\ &\leq f(t, \alpha_\gamma, \varepsilon), \end{aligned}$$

assuming additionally that $\nu \leq \frac{1}{M_4} \sqrt{\frac{\gamma}{k_4}}$, with $k_4 := \max_{\overline{D}} |\frac{1}{6} f_{uuu}(t, u, \varepsilon)|$.

Hence, imposing a further restriction that $\nu \leq u_0$, $\alpha(t, \varepsilon, \gamma)$ is a lower solution to (2.1) for $t \in [t_0, t_2(\gamma)]$; that is,

$$u(t, \varepsilon) \geq \alpha(t, \varepsilon, \gamma).$$

Furthermore, by (3.23), there is $0 < \theta < \varphi_2(t_2(\gamma))$ such that for all sufficiently small $\varepsilon$, $\theta \leq \alpha(t_2(\gamma), \varepsilon, \gamma) \leq u(t, \varepsilon)$. Since the point $(t_2(\gamma), \theta)$ belongs to the domain of attraction of $\phi_2$, $u(t, t_2(\gamma), \theta, \varepsilon)$ converges almost uniformly to $\varphi_2(t)$ on $(t_2(\gamma, T]$. Since solutions cannot intersect, we have

$$\lim_{\varepsilon \to 0^+} u(t, \varepsilon) = \varphi_2(t)$$

almost uniformly on $(t_2(\gamma), T]$. Since $t_2(\gamma) \to t^*$ as $\gamma \to 0^+$, equation (3.12) is satisfied. □

**Example 3.1.** As an illustration we consider the

$$\varepsilon u_t = -u((u-1)^2 + 1 - t),$$

whose quasi-steady states and delay in the stability switches are presented in Figure 2.

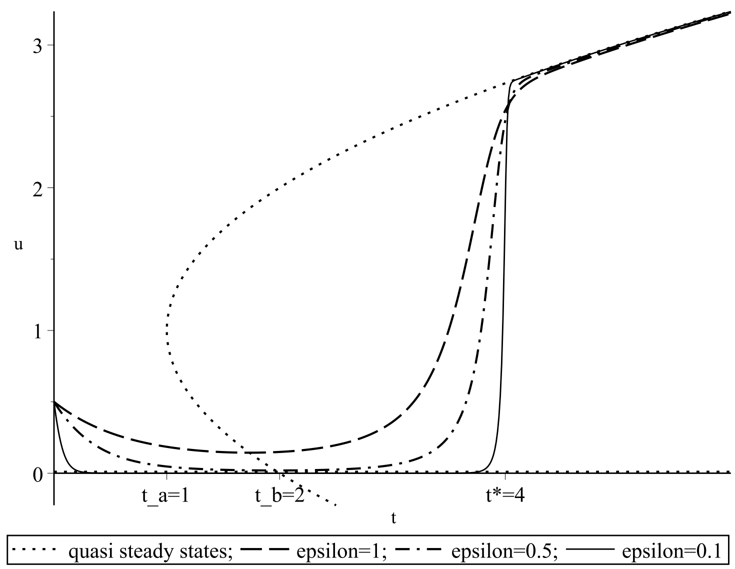


Figure 2: Delay in the stability switch in the case of backward bifurcation of quasi-steady states in one dimension.

# 4 Applications

**The Rosenzweig–MacArthur model.** Let us consider the following predator-prey model, [13, 26],

$$\begin{aligned} \varepsilon u_t &= u\left(r\left(1 - \frac{u}{k}\right) - \frac{av}{b+u}\right), \quad u(0,\varepsilon) = u_0 \\ v_t &= v\left(\frac{cu}{b+u} - d\right), \quad v(0,\varepsilon) = v_0 \end{aligned}, \tag{4.1}$$

which is a scaled version of the Rosenzweig–MacArthur model, see [22, 28]. In the model the prey $u$ has logistic, while the predators $v$ have Holling type II, functional responses. The coefficients are assumed to be positive and the small parameter $\varepsilon$ accounts for fast demography of the prey and a high aggressiveness of the predator.

It is easy to observe that the set $\{(u,v) : u, v \geq 0\}$ is invariant. Consequently, for any positive initial conditions the are unique local solutions to (4.1) that are positive on the interval of their existence. Furthermore, as long as the solution is positive, (4.1) implies that $u$ is bounded by a solution of the logistic equation, while $v$ is bounded by a solution to a linear equation (as $u/(b+u) \leq 1$) and hence the solutions are global.

Let us consider the geometry of the quasi-steady states. By $\Phi_0$ we denote the trivial quasi-steady state $\{u = 0\}$. The nontrivial quasi-steady state $\Phi$ of (4.1) is given by

$$v = \frac{r(bk + u(k-b) - u^2)}{ak}$$

that is a paraboloid in the $(t, v, u)$ coordinate system. The maximum value of $v$, that is

$$v_{\max} = \frac{r(b+k)^2}{4ak}, \tag{4.2}$$

is attained at $u = \frac{k-b}{2}$. Moreover, the $\Phi$ intersects $\Phi_0$ at $v = \frac{br}{a}$ and $\Phi_0$ is attracting for $v > \frac{br}{a}$ and repelling for $0 < v < \frac{br}{a}$. The turning point of the quasi-steady state is in the positive octant provided $k - b > 0$ and if $\frac{br}{a} < v_0 \leq \frac{r(b+k)^2}{4ak}$; that is, when $v_0$ is situated under the overhang of $\Phi$ (which places addition restriction on $u_0$ if we want $(u_0, v_0)$ to be in the basin of attraction of $\Phi_0$). On the other hand, if $v_0 > \frac{r(b+k)^2}{4ak}$, then any $(u_0, v_0)$ is in the basin of attraction of $u \equiv 0$.

Our aim is to prove that, for small $\varepsilon$, the solution $(u(t,\varepsilon), v(t,\varepsilon))$ stays close to $\Phi_0$ for some time that is independent of $\varepsilon$ even after $v(t,\varepsilon)$ crosses $v = br/a$ or, in other words, that $(u(t,\varepsilon), v(t,\varepsilon))$ stays close to the repelling branch of $\Phi_0$. Precisely, we prove

**Proposition 4.1.** *Let*

$$k - b > 0 \tag{4.3}$$

*and*

$$v_0 > \frac{r(k+b)^2}{4ak}, \quad u_0 \geq 0. \tag{4.4}$$

*Then there exists $t^{\beta,*}$, a unique positive solution to*

$$rt + \frac{av_0}{bd}e^{-dt} - \frac{av_0}{bd} = 0, \tag{4.5}$$

*such that $v(t^{\beta,*}, \varepsilon) < br/a$ uniformly for sufficiently small $\varepsilon$,*

$$\lim_{\varepsilon \to 0} u(t,\varepsilon) = 0 \quad \textit{for} \quad t \in (0, t^{\beta,*}) \tag{4.6}$$

*and* $(0, t^{\beta,*})$ *is the largest time interval on which the convergence is almost uniform. Moreover*

$$\lim_{\varepsilon\to 0} v(t,\varepsilon) = v_0 e^{-dt} \quad \textit{for} \quad t \in [0, t^{\beta,*}), \tag{4.7}$$

*also in an almost uniform way.*

*Proof.* The proof is an application of Theorem 3.1. First we construct an appropriate upper solution so that its one-dimensional theory can be applied. From (4.1) we see that

$$v_t = v\left(\frac{cu}{b+u} - d\right) \geq -dv,$$

hence $\alpha(t) := v_0 e^{-dt} \leq v(t,\varepsilon)$ and thus

$$u_t = \frac{1}{\varepsilon}u\left(r\left(1-\frac{u}{k}\right) - \frac{av}{b+u}\right) \leq \frac{1}{\varepsilon}u\left(r\left(1-\frac{u}{k}\right) - \frac{av_0e^{-dt}}{b+u}\right).$$

Hence, the solution $\beta(t,\varepsilon)$ to

$$\varepsilon\beta_t = \beta\left[r\left(1-\frac{\beta}{k}\right) - \frac{av_0e^{-dt}}{b+\beta}\right]. \tag{4.8}$$

is an upper solution to $u$, i.e., $\beta(t,\varepsilon) \geq u(t,\varepsilon)$ for $t \geq 0$.

Let

$$f(t,u) := u\left(r\left(1-\frac{u}{k}\right) - \frac{av_0e^{-dt}}{b+u}\right) \tag{4.9}$$

and, as in (2.3),

$$F(t) = \int_0^t f_u(s,0)ds = rt + \frac{av_0}{bd}e^{-dt} - \frac{av_0}{bd},$$

so that (4.5) is the equation $F(t) = 0$. The zeroes of $f$ are given by $\phi_0 = \{u = 0\}$ and the curve $\phi$ determined by the equations

$$\varphi^\beta_\pm(t) = \frac{1}{2}\left((k-b) \pm \sqrt{(k-b)^2 + 4kb - \frac{4av_0ke^{-dt}}{r}}\right).$$

We observe that the square root is real; that is, we have two real branches of $\phi$, if $t > t^\beta_a$, where

$$t^\beta_a := -\frac{1}{d}\ln\frac{r(k+b)^2}{4akv_0} > 0,$$

where the positivity is ensured by the first condition of (4.4). The position of the quasi-steady manifold determined by $\varphi_\pm$ depends on the parameters of the problem. By (4.3), the turning point of $\phi$ is in the upper half plane (otherwise only one branch would be there and we would have at most a transcritical bifurcation that has been comprehensively analysed in [5]). The other important parameter is the time, when the lower branch cuts the $t$-axis, given by

$$t_b^\beta := -\frac{1}{d} \ln \frac{br}{av_0}. \tag{4.10}$$

It is clear that $t_b^\beta > t_a^\beta > 0$ and hence we have a bona fide backward bifurcation satisfying (A2) with the turning point of $\phi$ occurring at $t_a^\beta > 0$ and thus any $u_0$ is in the basin of attraction of $\phi_0$, see Fig. 3.

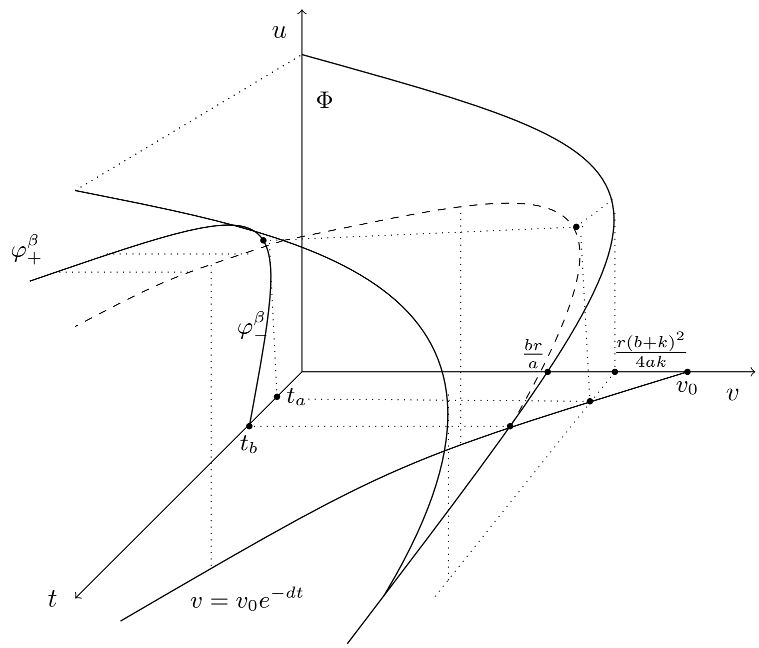


Figure 3: Interplay of one and two dimensional structures in the model.

Next, we observe that $f_u(t,0) < 0$ for $t < t_b^\beta$ and $f_u(t,0) > 0$ for $t > t_b^\beta$. Then a continuity argument shows that $f_u(t,\phi_-^\beta(t)) > 0$ for $t_a^\beta < t < t_b^\beta$ (with $f_u(t,\phi_-^\beta(t)) < 0$ for $t > t_b^\beta$) and $f_u(t,\phi_+^\beta(t)) > 0$ for $t_a^\beta < t$.

Let us consider now equation (4.5). Since $F(0) = 0$, $F(+\infty) = +\infty$ and $F_t(t) = r - \frac{av_0}{b}e^{-dt}$, by (A3) $F(t)$ reaches a single absolute minimum at $t_b^\beta$. Consequently, there is a unique $t^{\beta,*} > t_b$ such that $F(t^{\beta,*}) = 0$. Moreover, since $f_{uuu}(t,0) = -6av_0e^{-dt}/b^3 < 0$, assumption (A6) holds as well. Hence, Theorem 3.1 implies that the solution to (4.8) satisfies

$$\lim_{\varepsilon\to 0} \beta(t,\varepsilon) = 0 \quad \text{for} \quad t \in (0, t^{\beta,*})$$

and the convergence is almost uniform which yields the convergence part of (4.6). To prove that this is the maximal interval of convergence, we assume, to the contrary, that there is $t_1 > t^{\beta,*}$ such that

$$\lim_{\varepsilon\to 0} u(t,\varepsilon) = 0 \quad \text{for} \quad t \in (0,t_1) \tag{4.11}$$

almost uniformly. This means that for any $\eta > 0$ and $\delta > 0$ there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have

$$0 \le u(t,\varepsilon) \le \delta \quad \text{for} \quad t \in [\eta, t_1 - \eta]. \tag{4.12}$$

We can choose $\delta < \phi_+(t)$ in the neighbourhood of $t^{\beta,*}$. From (4.1), one gets

$$v_t = v\left(\frac{cu}{b+u} - d\right) \le v\left(\frac{c\delta}{b+\delta} - d\right).$$

Let us choose $\delta$ small enough such that $d(\delta) := d - \frac{c\delta}{b+\delta} > 0$. Then

$$0 \le v(t,\varepsilon) \le \beta(t,\delta,\eta) := v(\delta,\eta)e^{-d(\delta)t}, \quad t \in [\eta, t_1-\eta], \tag{4.13}$$

as long as $v(\delta,\eta)e^{-d(\delta)\eta} \ge v(\eta,\varepsilon)$. Moreover, from (4.1), we obtain

$$v_t = v\left(\frac{cu}{b+u} - d\right) \le v(c-d), \quad t \ge 0.$$

Therefore, $\beta(t) = v_0 e^{(c-d)t}$ satisfies

$$0 \le v(t,\varepsilon) \le \beta(t), \quad [0,\eta],$$

hence (4.13) will be satisfied if we select $v(\delta,\eta) = v_0 e^{\frac{c\eta b}{b+\delta}}$. Since $\frac{c\delta}{b+\delta} - d < c - d$, we see that $\beta(t,\delta,\eta)$ is also an upper solution to $v(t,\varepsilon)$. Therefore

$$u_t = \frac{1}{\varepsilon}u\left(r\left(1-\frac{u}{k}\right) - \frac{av}{b+u}\right) \ge \frac{1}{\varepsilon}u\left(r\left(1-\frac{u}{k}\right) - \frac{a\beta}{b+u}\right)$$

and thus $u(t,\varepsilon) \ge \alpha(t,\varepsilon,\delta,\eta), t \in [t_0, t_1-\eta]$, where $\alpha$ is the solution to

$$\varepsilon\alpha_t = \frac{1}{\varepsilon}\alpha\left(r\left(1-\frac{u}{k}\right) - \frac{a\beta}{b+\alpha}\right), \quad u(0,\varepsilon) = u_0. \tag{4.14}$$

As before, for (4.14) we obtain three potential quasi-steady states: $u \equiv 0$ and

$$\varphi^\alpha_\pm(t) = \frac{1}{2}\left[(k-b) \pm \sqrt{(k+b)^2 - \frac{4ak\beta(t)}{r}}\,\right],$$

where

$$t^\alpha_a(\delta,\eta) := -\frac{1}{d(\delta)}\ln\frac{r(k+b)^2}{4akv(\delta,\eta)} \quad \text{and} \quad t^\alpha_b(\delta,\eta) := -\frac{1}{d(\delta)}\ln\frac{br}{av(\delta,\eta)}.$$

Observe that $t_a^\alpha(\delta,\eta) \to t_a^\beta$ and $t_b^\alpha(\delta,\eta) \to t_b^\beta$ when $\delta,\eta \to 0$.

Also, in the same way,

$$F(t,\varepsilon,\delta,\eta) = rt - \frac{av(\delta,\eta)}{bd(\delta)} e^{-d(\delta)t} - \frac{av(\delta,\eta)}{bd(\delta)}$$

Proceeding in the same way as before, we get that there is $t^{\alpha,*}(\delta,\eta) > t_b^\alpha(\delta,\eta)$ such that $F(t^{\alpha,*}(\delta,\eta),0,\delta,\eta) = 0$. Moreover

$$\lim_{\delta,\eta\to 0} t^{\alpha,*}(\delta,\eta) = t^{\beta,*}. \tag{4.15}$$

Observe that assumptions (A1)–(A6) are satisfied for (4.14) and hence, from Theorem 3.1,

$$\lim_{\varepsilon\to 0^+} \alpha(t,\varepsilon,\delta,\eta) = 0, \quad \text{for} \quad t \in (0, t^{\alpha,*}(\delta,\eta))$$

and

$$\lim_{\varepsilon\to 0^+} \alpha(t,\varepsilon,\delta,\eta) = \varphi_+^\alpha(t), \quad \text{for} \quad t > t^{\alpha,*}(\delta,\eta).$$

Since $\alpha(t,\varepsilon,\delta,\eta)$ is a lower solution for $u$, by (4.12) and (4.15), $u(t,\varepsilon)$ cannot be bounded from above by $\delta$ on any interval $(t^{\beta,*}, t_1]$. This contradiction shows that $(0, t^{\beta,*})$ is the largest interval on which $u(t,\varepsilon)$ almost uniformly converges to 0 as $\varepsilon \to 0$.

As far as the convergence of $v(t,\varepsilon)$ is concerned, on $[0,\bar{t}]$, $0 < \bar{t} < t_b^\beta$, the convergence to $\alpha(t) = v_0 e^{-dt}$ is ensured by the Tikhonov theorem, whereas on $[\bar{t}, t^{\beta,*}]$ it follows by writing

$$v_t = v\left(\frac{cu(t,\varepsilon)}{b+u(t,\varepsilon)} - d\right),$$

with the initial condition at $\bar{t}$ taken as $v(\bar{t},\varepsilon)$, using the fact that $u(t,\varepsilon) \to 0$ uniformly on $[\bar{t}, t_2]$ for any $t_2 < t^{\beta,*}$ and the regular perturbation theory. □

**Remark 4.1.** As pointed out in [26], the importance of recognizing possible canard solutions is that a 'naive' asymptotic analysis can lead to a serious over (or under) estimate of the minimum/maximum of the population. For instance, in (4.1) an expectation is that the minimum population $v_{\min}$ is attained at the intersection of quasi-steady manifolds; that is, it should be $br/a$ and henceforth it should increase as the prey population should have jumped from $u \equiv 0$ to the upper branch of the other quasi-steady manifold. However, as the prey solution continues close to $u \equiv 0$ for some time after $v$ reached $\bar{v} = br/a$, the predator population will still decrease. The formula derived in [26, Eq. (11)] for $v_{\min}$ is

$$\frac{v_{\min}}{\bar{v}} - \ln\frac{v_{\min}}{\bar{v}} = \frac{v_{\max}}{\bar{v}} - \ln\frac{v_{\max}}{\bar{v}}, \tag{4.16}$$

where $v_{\max}$ is given by (4.2) and is just a reference value used in [26] as the only explicitly calculable value attained by $v$ in the oscillatory dynamics considered there. Our method not only allows for a precise estimate of $v_{\min}$ but also of the time at which it is attained and explicitly gives its dependence on the initial condition. Indeed, from the previous considerations, for small $\varepsilon$, $v_{min}$ is approximately attained at $t^{\beta,*}$ and $v_{\min} \approx v_0 e^{-dt^{\beta,*}}$. By (4.5), $t^{\beta,*}$ is the unique positive solution to

$$rt^{\beta,*} + \frac{av_0}{bd}\exp(-dt^{\beta,*}) - \frac{av_0}{bd} = 0 \tag{4.17}$$

that can be re-written as

$$-\ln\frac{v_{\min}}{v_0} + \frac{v_{\min}}{\bar{v}} - \frac{v_0}{\bar{v}} = 0.$$

This equation is the same as (4.16) if we take $v_{\max}$ as the initial condition for $v$ in (4.1).

The above considerations are illustrated on Fig. 4. We observe the qualitative change of the behaviour of the solutions that depends on the parameters $c$ and $d$. In the top row $c-d<0$ and the field in the first quadrant is directed to the left, resulting in the trajectories converging to the equilibrium $(k,0)=(3,0)$. On the other hand, in the bottom row $c-d>0$ and the field changes the direction from the left to the right at the isocline $u = bd/(c-d) > 0$. Then the trajectories take the more familiar form, known from [13, 26], leading to a slow-fast cycle.

**The Leslie-Gowers/Holling II model.** In a recent paper, [1], the authors considered the following predator-prey

$$\begin{array}{rl} \varepsilon u_t = u\left(1-u-\frac{av}{b+u}\right), & u(0,\varepsilon)=u_0,\\ v_t = v\left(1-\frac{v}{c+u}\right), & v(0,\varepsilon)=v_0, \end{array}, \tag{4.18}$$

which is a scaled version of the model introduced in [2]. In [1] the authors proved the existence of a unique attractive limit cycle showing also that near one of the fold points there is a delayed exchange of stabilities and hence there exists a canard solution. Here we shall show that our approach also shows the existence of a canard solution and, as in the previous paragraph, we will be able to explicitly estimate the time of the exchange of stabilities.

Most of the calculations for (4.18) are similar to that for (4.1) and thus we shall not go into details. First, it is clear that positive initial conditions produce globally defined positive solutions and hence in future we only will deal with such solutions. Since the first equation is the same as in (4.1), the structure of the quasi-steady manifolds is the same. Thus, the nontrivial one, say $\Phi$, is the paraboloid $v(u)$ with the maximum of $v$ attained at $u=(1-b)/2$ and equal to $v_{\max} = (b+1)^2/4a$, while the trivial is given by $\Phi_0 = \{u=0\}$. If we are interested in the backward bifurcation occurring in the positive octant, we must assume

$$1-b>0. \tag{4.19}$$

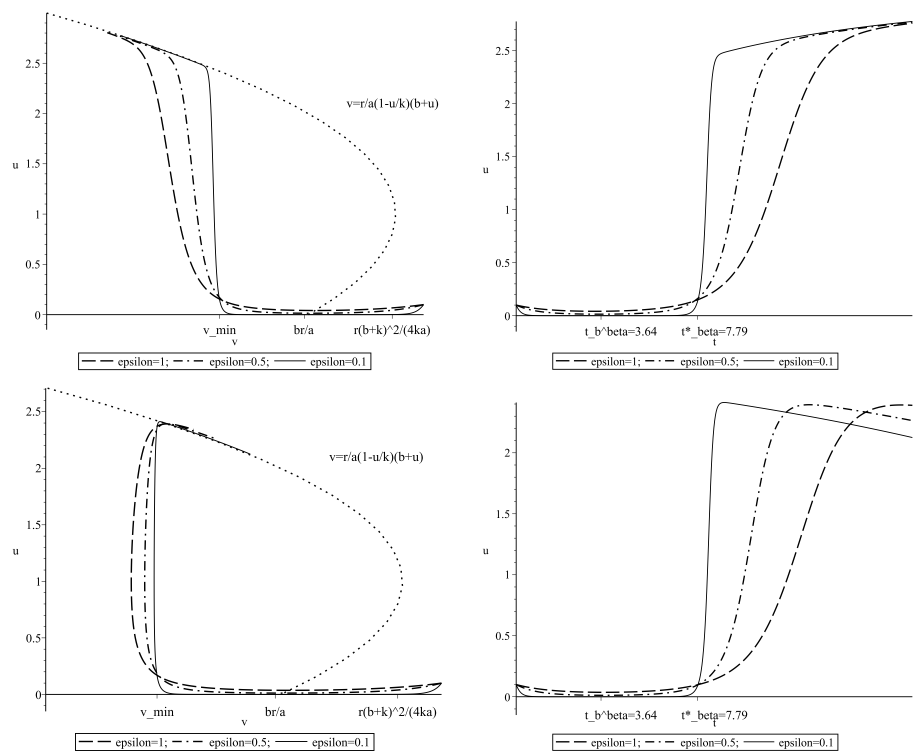


Figure 4: The phase portraits of solutions to (4.1) (left) and the solutions $u(t, \varepsilon)$ (right) for the parameters $k = 3, b = 1, r = 1.5, a = 0.18, d = 0.1$, $c = 0.01$ (top) and $c = 0.2$ (bottom), initial conditions $u(0) = 0.1, v(0) = 12$ and $\varepsilon = 1, 0.5, 0.1$.

Then the lower branch of the paraboloid intersects the plane $u = 0$ at $v = b/a$. Further, by

$$v_t = v\left(1 - \frac{v}{c+u}\right) \geq v\left(1 - \frac{v}{c}\right),$$

the function

$$\alpha(t) := \frac{cv_0e^t}{c - v_0 + v_0e^t} \tag{4.20}$$

is an upper solution for $v$. Thus, from the first equation of (4.18), the solution $\beta$ to

$$\varepsilon\beta_t = f(t, \beta) := \beta\left(1 - \beta - \frac{a\alpha(t)}{b+\beta}\right), \quad \beta(0, \varepsilon) = u_0 \tag{4.21}$$

is an upper solution for $u$. As before, potentially there are three quasi-steady states, $u \equiv 0$ and

$$\varphi^{\beta}_{\mp}(t) = \frac{b - 1 \mp \sqrt{(b+1)^2 - 4a\alpha(t)}}{2}.$$

Here the situation differs from that of the previous paragraph as $\alpha$ is a logistic function with a unique attractive equilibrium at $c$. Furthermore, $\alpha$ is increasing if $0 < v_0 < c$ and decreasing if $v_0 > c$. Hence, according to the geometry describe above, for the solution to pass close to the intersection of $\Phi$ and $\Phi_0$ we have to assume that

$$v_0 > \frac{b}{a} > c. \tag{4.22}$$

We observe that (4.22) is consistent with the assumptions in Section 2 of [1]. Under these assumptions there is exactly one $t^{\beta}_{b}$ such that

$$\alpha(t^{\beta}_{b}) = \frac{b}{a},$$

given by

$$t^{\beta}_{b} = \ln \frac{b(v_0 - c)}{v_0(b - ac)}. \tag{4.23}$$

This shows that assumption (A2) is satisfied and, as in the previous paragraph, clearly (A3) is also satisfied. Observe that

$$F(t) = t - \frac{ac}{b} \ln \frac{c - v_0 + v_0 e^t}{c}. \tag{4.24}$$

We have $F(0) = 0$ and, thanks to (4.22), $F(+\infty, 0) = +\infty$. Further, $F_t(t) = f_u(t, 0, 0)$ implies that $F(t)$ is strictly increasing for $t > t^{\beta}_{b}$. Hence, there is a unique $t^{\beta,*} > t_b$ such that $F(t^{*}_{\beta}) = 0$. Moreover, since $f_{uuu}(t, 0) = -\frac{6cav_0 \exp(t)}{b^3(c - v_0 + v_0 \exp(t))} < 0$, assumption (A6) holds as well. Thus we can state that $u(t, \varepsilon)$ is attracted to $u = 0$ up to $t = t^{\beta,*}$ and we have a canard solution.

# References


[1] B. Ambrosio, M. Aziz-Alaoui, and R. Yafia. Canard phenomenon in a slow-fast modified Leslie–Gower model. *Mathematical Biosciences*, 295:48–54, 2018.

[2] M. Aziz-Alaoui and M. D. Okiye. Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes. *Applied Mathematics Letters*, 16(7):1069–1075, 2003.

[3] J. Banasiak and M. Lachowicz. *Methods of small parameter in mathematical biology.* Springer, Heidelberg/New York, 2014.

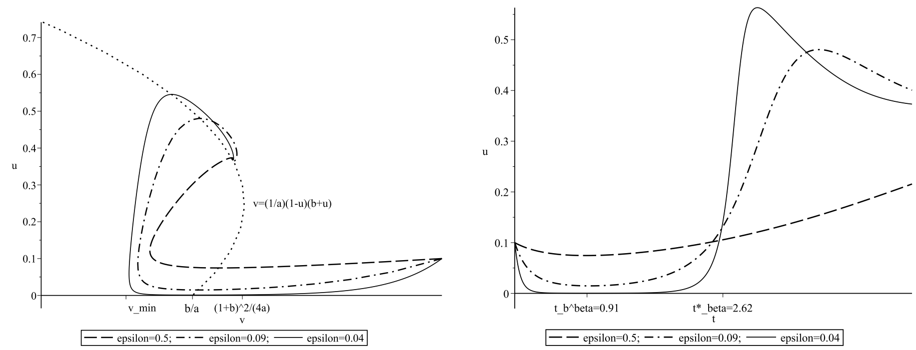


Figure 5: The phase portraits of solutions to (4.18) (left) and the solutions $u(t,\varepsilon)$ (right) for the parameters $b = 0.5, a = 0.4, c = 1$, initial conditions $u(0) = 0.1, v(0) = 2$ and $\varepsilon = 0.5, 0.09, 0.04$.


[4] J. Banasiak and E. Phongi. Canard-type solution in epidemiological models. *Dynamical Systems, Differential Equations and Applications, AIMS Proceedings*, 10:85–93, 2015.

[5] J. Banasiak and M. S. Tchamga. Delayed stability switches in singularly perturbed predator–prey models. *Nonlinear Analysis: Real World Applications*, 35:312–335, 2017.

[6] E. Benoît, J. Callot, F. Diener, and M. Diener. Chasse au canards. *Collectanea Mathematica*, 32:37–119, 1981.

[7] V. Butuzov, N. Nefedov, and K. Schneider. Singularly perturbed reaction-diffusion systems in cases of exchange of stabilities. *Natural Resource Modelling*, 13:247–269, 2000.

[8] L.-Y. Chen, N. Goldenfeld, and Y. Oono. Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory. *Physical Review E*, 54(1):376, 1996.

[9] H. Chiba. Extension and unification of singular perturbation methods for ODEs based on the renormalization group method. *SIAM Journal on Applied Dynamical Systems*, 8(3):1066–1115, 2009.

[10] F. Dumortier and R. Roussarie. *Canard cycles and center manifolds*. American Mathematical Society, Providence, 1996.

[11] W. Eckhaus. Relaxation oscillations including a standard chase on French ducks. In *Asymptotic analysis, II*, volume 985 of *Lecture Notes in Math.*, pages 449–494. Springer, Berlin, 1983.

[12] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations*, 31:53–98, 1979.

[13] G. Hek. Geometric singular perturbation theory in biological practice. *Mathematical Biology*, 60:347–386, 2010.

[14] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems (Montecatini Terme, 1994)*, volume 1609 of *Lecture Notes in Math.*, pages 44–118. Springer, Berlin, 1995.

[15] M. Krupa and P. Szmolyan. Extending slow manifolds near transcritical and pitchfork singularities. *Nonlinearity*, 14(6):1473–1491, 2000.

[16] M. Krupa and P. Szmolyan. Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions. *SIAM journal on mathematical analysis*, 33(2):286–314, 2001.

[17] C. Kuehn. *Multiple time scale dynamics*, volume 191 of *Applied Mathematical Sciences*. Springer, Cham, 2015.

[18] N. Lebovitz and R. Schaar. Exchange of stabilities in autonomous systems. *Studies in Applied Mathematics*, 54:229–260, 1975.

[19] N. Lebovitz and R. Schaar. Exchange of stabilities in autonomous systems II., vertical bifurcation. *Studies in Applied Mathematics*, 56:1–50, 1977.

[20] A. Marciniak-Czochra, A. Mikelić, and T. Stiehl. Renormalization group second order approximation for singularly perturbed nonlinear ordinary differential equations. *Mathematical Methods in the Applied Sciences*, 2018, doi.org/10.1002/mma.5107, in print.

[21] M. Martcheva. *An introduction to mathematical epidemiology*, volume 61 of *Texts in Applied Mathematics*. Springer, New York, 2015.

[22] R. M. May. Limit cycles in predator-prey communities. *Science*, 177(4052):900–902, 1972.

[23] E. F. Mishchenko, Y. S. Kolesov, A. Y. Kolesov, and N. K. Rozov. *Asymptotic methods in singularly perturbed systems*. Monographs in Contemporary Mathematics. Consultants Bureau, New York, 1994. Translated from the Russian by Irene Aleksanova.

[24] A. Neishtadt. Persistence of stability loss for dynamical bifurcations I. *Differentsial'nye Uravneniya*, 23:2060–2067, 1987.

[25] R. E. O'Malley, Jr. *Singular perturbation methods for ordinary differential equations*, volume 89 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1991.

[26] S. Rinaldi and S. Muratori. Slow-fast limit cycles in predator-prey models. *Ecological Modelling*, 61:287–308, 1992.

[27] J. Shen. Canard limit cycles and global dynamics in a singularly perturbed predator-prey system with non-monotonic functional response. *Nonlinear Analysis: Real World Applications*, 31:146–165, 2016.

[28] Y. Shimazu, K. Sugiyama, T. Kojima, and E. Tomida. Some problems in ecology oriented environmentology, terrestrial environmentology II. *The Journal of Earth Sciences, Nagoya University*, 20:31–89, 1972.

[29] M. S. Tchamga. *Study of Singularly Perturbed Models and its Applications in Ecology and Epidemiology.* PHD thesis, UKZN, 2017.

[30] A. Tikhonov, A. Vasileva, and A. Sveshnikov. *Differential equations.* Springer, Berlin, 1985.

[31] F. Verhulst. *Methods and applications of singular perturbations*, volume 50 of *Texts in Applied Mathematics.* Springer, New York, 2005.