Modified Newton's Method in the Leapfrog Method for Mobile Robot Path Planning

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Abstract. The problem of determining an optimal trajectory for an autonomous mobile robot in an environment with obstacles is considered. The Leapfrog approach is used to solve the ensuing system of equations derived from the first order optimality conditions of the Pontryagin Minimum principle. A comparison is made between a case in which the classical Newton Method and the Modified Newton method is used in the shooting method for solving the two point boundary value problem in the inner loop of the Leapfrog algorithm. It can be observed that with this modification of one observes gains in convergence rates of the Leap-frog algorithm in general.

Keywords: Trajectory planning, Obstacle avoidance, Leapfrog algorithm, Pontryagin Minimum Principle, Modified Newton algorithm

1 Introduction

Advances in optimal control provide the necessary tools to determine optimal trajectories for problems such as those arising from applications in robotics. An amount of research is done on optimal control with application to mobile robot path planning [1], [2] and [3]. Most of these problems are nonlinear and can today be solved using numerical methods. The general numerical approaches for solving optimal control problems can be classed into direct and indirect methods [4]. In the direct method the differential equations and their integrals are discretized. The drawback of direct methods is that they suffer from lower accuracy. Indirect methods on the other hand have outstanding precision and have a possibly of verifying necessary conditions. The necessary conditions of optimality are formulated through Pontryagin Minimum Principle (PMP) resulting to a system of boundary value problem. The disadvantage of using the indirect methods is that a good approximation of the initial guess for the costates is required.

The Leapfrog method proposed in [5] can be viewed as an indirect method for solving TPBVP. In the method, a feasible path given from the starting and the final states is subdivided into segments where local optimal paths are computed. Application of the Leapfrog algorithm to path planning of a two-wheeled mobile robot was introduced in [6] to determine optimal paths for kinematic model. Later on the work was extended to finding optimal trajectories for a mobile robot while avoiding obstacles [7]- [8]. The algorithm gave promising results for path planning in both obstructed and unobstructed workspace.

With the Leapfrog method one does not need a good approximation for initial guesses for the costates along a trajectory. To initialize the costates values affine approximation of the local problem in a segment is used to provide initial guesses needed in simple shooting. In [9] it was shown that the approximation approach provides a good guesses for at least some of the costates. The focus of their work was finding a way to choose initial guesses for the costates. Numerical and experimental solutions were done to validate the proposed approach.

Even though initial guess of the costates is not crucial in the Leapfrog method, the guesses are needed for the success of local shooting method. In addition, the algorithm is convergent to a critical trajectory provided that these local shooting methods produce optimal trajectories. As mentioned in [5], improvements to the simple shooting method may be necessary for more difficult optimal control problems. This may reduce the computation time that the Leapfrog takes to execute, especially in the first and last iterations. Moreover, improve the stability of the Leapfrog numerical solutions.

In this paper, the Modified Newton Method (MNM) implemented in [10] is incorporated in simple shooting [5] in an attempt to improve the convergence of the Leapfrog method for solving mobile robot path planning problem. A general discussion on the convergence of the MNM is given in [11]. For simulation purposes, the work done in [6] - [8] is revisited to evaluate the MNM with simple shooting. A comparison is made between a case in which the classical Newton Method used in our previous work and the Modified Newton method is used in the shooting method. The computational time along the path and cost for each example are tabled.

The paper organization is as follows: In Section 2 the Leapfrog algorithm is described. Section 3 presents the simulations with Modified Newton's Method and Section 4 provide discussion and conclusions, respectively.

2 Optimal Control

A general optimal control system can be modelled by

$$\min_{u \in U} \int_{t_0}^{t_f} L\big(x(t), u(t)\big)\, dt \tag{1}$$

subject to state equation

$$\dot{x} = f(x(t), u(t)), \tag{2}$$

$$x(t_0) = x_0, \; x(t_f) = x_f,$$

In the equations above, x is the state variable, u is a control input, t_0 and x_0 are initial time and state.

Following the Pontryagin Minimum Principle (PMP) necessary conditions of optimality are formulated as

$$\dot{x} = \partial H / \partial \lambda \tag{3}$$

$$\dot{\lambda} = -\partial H / \partial x \tag{4}$$

$$0 = \partial H / \partial u \tag{5}$$

with the Hamiltonian function

$$H(t, \lambda, x, u) = L + \lambda^T f(t, x, u) \tag{6}$$

where λ is for the costates variable. Solving for $u(t)$ in Equation (5) and substituting it to (3) and (4) reduces to a TPBVP which is solved using indirect numerical method.

2.1 Leapfrog Method

Given a feasible path μ_z^k, from the initial state x_0 and the final state x_f , the Leapfrog method starts by dividing the path into q partitions with

$$z_0^{(k)}, z_1^{(k)}, \dots, z_{q-1}^{(k)}, z_q^{(k)}$$

as partition points. On each iteration, k a sub-problem:

$$\min_{u \in U} \int_{t_{i-1}}^{t_{i+1}} f\big(x(t), u(t)\big)\, dt$$

$$\dot{x}(t) = f\big(x(t), u(t)\big) \tag{7}$$

$$x(t_{i-1}) = z_{i-1}, \; x(t_{i+1}) = z_{i+1}$$

is solved updating the initial feasible path towards the optimal path. This is achieved by midpoint mapping scheme where a point z_i^{k+1} is selected as the point which is reached roughly half the time along the optimal trajectory between z_{i-1}^k and z_{i+1}^k. This continues for all the partitions until the maximum number of iterations is attained. It is noted that for every update on z_i^{k+1} and t_i^{k+1} the states and costates are also updated. On each iteration of the Leapfrog algorithm the cost decreases. Hence the number of partition points decreased as the algorithm is executing. If the partitions are reduced accordingly, the convergence towards optimal trajectory proceeds efficiently. However, when $q = 2$, a simple shooting method is used to determine the optimal solution between the starting and final states z_0 and z_f. This causes a slow convergence for the algorithm on the final iteration.

2.2 Modified Newton's Method

The Newton's Method (NM) is a standard root-finding method which uses the first few terms of Taylor series of a function $f(x)$. With the assumption that $f(x)$ is continuous and is a real valued, the method finds numerical solution of $f(x) = 0$.
An iterative NM is given by

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} \tag{8}$$

Given the initial guess x_j, one can find the next approximation of x_{j+1}. In [11] it is shown that the NM converges provided the initial guess is close enough to the estimated point. It is stated that the Newton's method may however diverge. Hence the Modified Newton's Method was introduced to solve a large class of function f in which global convergence can be proven. The Modified Newton's Method (MNM) introduces an extra term, λ, and Equation (8) becomes

$$x_{j+1} = x_j - \lambda_j s_j \tag{9}$$

where $s_j = d_j \equiv \frac{f(x_j)}{f'(x_j)}$ is a search direction.

The steps for the MNM [11] are as follows:

1. Choose a starting point x_0
2. For each $j = 0,1, \ldots$ define x_{j+1} from x_j as follows
   a) Set

$$d_j = DF(x_j)^{-1}F(x_j),$$

$$\gamma_j = \frac{1}{cond(DF(x_j))},$$

and let $h_j(\tau) = h(x_j - \tau\, d_j)$, where $h(s) = F(s)^T F(s)$.

Determine the smallest integer $m \geq 0$ satisfying

$$h_j(2^{-m}) \leq h_j(0) - 2^{-m}\frac{\gamma_j}{4}\, \|d_j\|\, \|Dh(x_j)\|\,.$$

   b) Determine λ_j so that $h(x_{j+1}) = \min_{0 \leq \kappa \leq m} h_j(2^{-\kappa})$ and let $x_{j+1} = x_j - \lambda_j d_j$.

A theoretical analysis for Modified Newton's Method can be found in [11]. The method was used in [12] and [13], and the implementation can be found in [14].

3 Simulation Results

In this section a comparison is made between the case in which the classical Newton's Method (NM) and Modified Newton's Method (MNM) is used in Leapfrog algorithm to find optimal solutions of mobile robot path planning. The simulations presented utilize the set of examples adapted from our previous work. The arrows in the figures indicate the orientations of the robot at key positions.

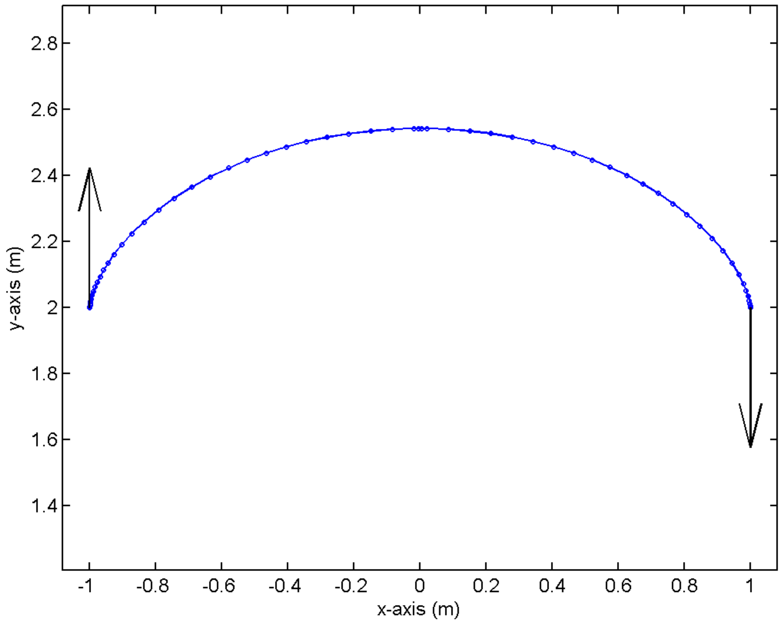

Fig. 1. Optimal path produced by Leapfrog where the robot is considered to be moving from the initial state $[-1\ 2\ {}^{\pi}/_{2}]$ to the final state $[1\ 2-{}^{\pi}/_{2}]$.

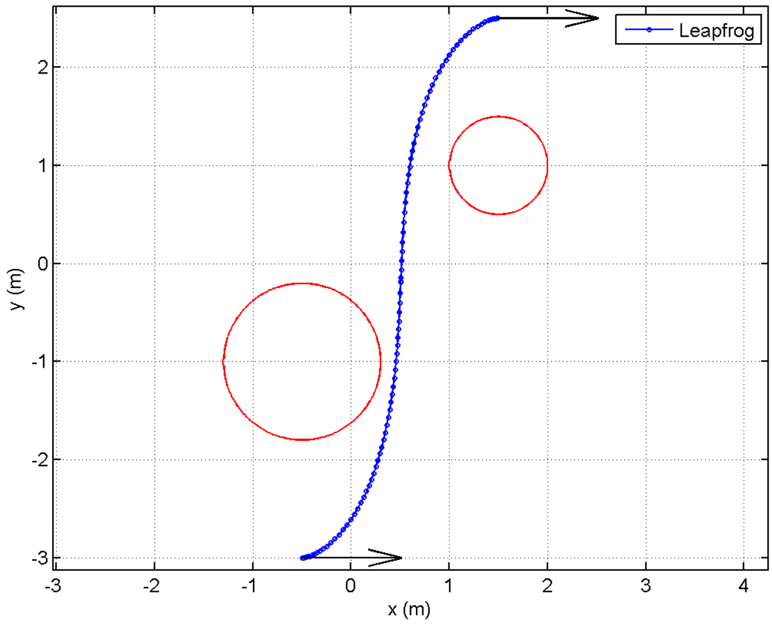

Fig. 2. Optimal path returned by Leapfrog in the presence of two obstacles. The initial and final states of the robot are [0 0 0] and [3.5 3.5 0], respectively.

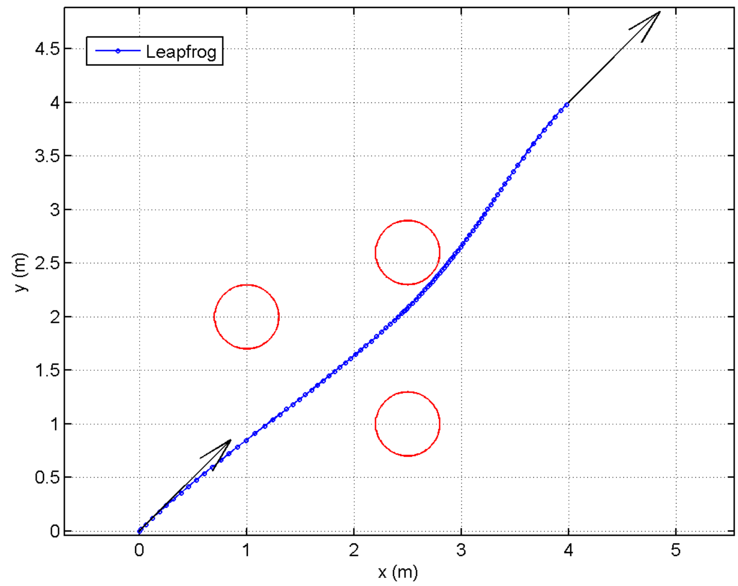

Fig. 3. Shows the optimal path obtained from Leapfrog in the presence of three obstacles with the robot initial state $[0\ 0\ {}^{\pi}/_{4}]$ and final state $[1\ \ 1\ \ {}^{\pi}/_{4}]$.

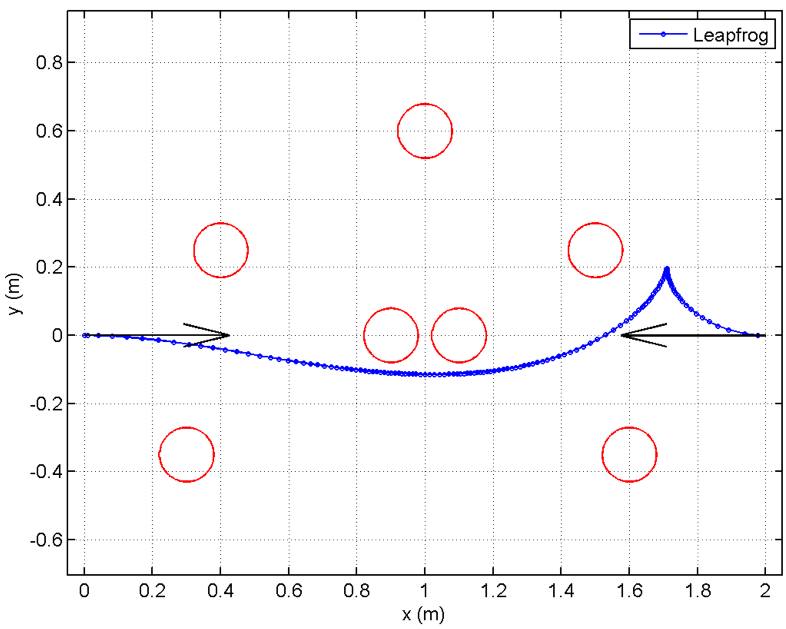

Fig. 4. Optimal path generated by Leapfrog in the presence of seven obstacles. The robot is considered to move from the initial state [0 0 0] to the final state [2 0 0] .

From the simulations shown above, the computational time and cost for the optimal solutions is recorded, see Table 1.

Table 1. Simulation Results for Simple Shooting and Modified Newton's Method

Case	No.of Obstacles	Cost	Computational Time	
			NM (s)	MNM (s)
1	0	8.64	34.07	32.61
2	2	5.98	78.91	77.80
3	3	3.15	95.66	86.49
4	7	2.79	113.36	106.75

4 Conclusion and Future Work

The numerical simulations showed that the Leapfrog method with MNM is capable of finding optimal paths. It was also noted that the final cost was the same for both implementation of the classical NM and MNM. On the last iteration of the Leapfrog algorithm the convergence is normally slower due to the simple shooting method, but after implementing the MNM, a faster convergence was noted. In the majority of the test cases in this study, the MNM took less time to compute the optimal path, as compared to the time required when simple shooting with NM was used. This indicates that an advantage is gained when using the MNM with the simple shooting. For future work, a more sophisticated simple shooting method with the MNM incorporated will be considered to improve the Leapfrog algorithm.

Bibliography

[1] H. Sussmann and G. Tang, "Shortest paths for the Reeds-Shepp car: A worked out example of the use of geometric techniques in nonlinear optimal control," Piscataway, NJ, 1991.

[2] J.-D. Boissonnat, A. C´er´ezo and J. Leblond, "Shortest paths of bounded curvature in the plane," *Journal of Intelligent and Robotic Systems,* vol. 11, pp. 5-20, 1994.

[3] F. E. Bertolazzi and M. D. Lio, "Symbolic-numeric indirect method for solving optimal control problems for large multibody systems," *Multibody System Dynamics,* vol. 13, pp. 233-252, 2005.

[4] A. Rao, "A survey of numerical methods for optimal control," in *AAS/AIAA Astrodynamics Specialist Conference*, 2009.

[5] C. Kaya and J. Noakes, "Leapfrog algorithm for optimal control," *SIAM Journal on Numerical Analysis,* vol. 46, pp. 2795-2817, 2008.

[6] B. T. Matebese, D. J. Withey and M. K. Banda, "Application of the Leapfrog method to robot path planning," in *IEEE International Conference on Information and Automation,* 2014.

[7] B. Matebese, D. Withey and K. B. Mapundi, "Path Planning with the Leapfrog Method in the Presence of Obstacles," in *IEEE International Conference on Robotics and Biomimetics*, 2016.

[8] B. Matebese, D. Withey and M. K. Banda, "Initialization of the leapfrog algorithm for mobile robot path planning," in *Pattern Recognition Association of South Africa and Robotics and Mechatronics International Conference,* 2016.

[9] A. Vamsikrishna, A. D. Mahindrakar and S. Tiwari, "Numerical and Experimental Implementation of Leapfrog Algorithm for Optimal Control of a Mobile Robot," in *Indian Control Conference,* 2017.

[10] R. Holsapple, R. Venkataraman and D. Doman, "A Modified Simple Shooting Method for Solving Two-Point Boundary-Value Problems," in *IEEE Aerospace Conference*, 2003.

[11] J. Stoer and R. Burlirsch, Introduction to Numerical Analysis, 2nd edition ed., Springer-Verlag, New York, 1993, pp. 272-286.

[12] R. Holsapple, R. Venkataraman and D. Doman, "A new, fast numerical method for solving two-point boundary value problems," *Journal of Guidance, Control and Dynamics,* vol. 26(3), pp. 505-508, 2004.

[13] A. Trent, R. Venkataraman and D. Damon, "Trajectory Generation Using A Modified Simple Shooting Method," in *IEEE Aerospace Conference,* 2004.

[14] A. Trent, "Trajectory generation using a modified simple shooting method," in *Thesis, Master of Science, Texas Tech University,* 2004.