ON THE MULTI-DIMENSIONAL PORTFOLIO OPTIMIZATION WITH STOCHASTIC VOLATILITY.

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Abstract. In a recent paper by Mnif [17], a solution to the portfolio optimization with stochastic volatility and constraints problem has been proposed, in which most of the model parameters are time-homogeneous. However, there are cases where time-dependent parameters are needed, such as in the calibration of financial models. Therefore, the purpose of this paper is to generalize the work of Mnif [17] to the time-inhomogeneous case. We consider a time-dependent exponential utility function of which the objective is to maximize the expected utility from the investor's terminal wealth. The derived Hamilton-Jacobi-Bellman(HJB) equation, is highly nonlinear and is reduced to a semilinear partial differential equation (PDE) by a suitable transformation. The existence of a smooth solution is proved and a verification theorem presented. A multi-asset stochastic volatility model with jumps and endowed with time-dependent parameters is illustrated.

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1. Introduction

An optimal investment problem has occupied researchers for a long time especially after Merton's article appeared in 1971, [16]. The problem is about the investor who seeks to maximize his expected utility from the terminal wealth. Numerous extensions to this optimization problem have been proposed since then, (see e.g., [2], [9], [10], [13], [19], [17], and references there-in). For instance, Pham [19] considered a multi-dimensional problem with random volatilities which include the degenerate cases and correlation between assets and the stochastic factors. Motivated by Pham, Mnif [17] considered convex constraints on the amount of the portfolio, and a jump-diffusion process in the n-dimensional risky assets. These authors solved the optimal investment problem in a time-homogeneous framework. In this paper, we solve the same problem in a framework where the parameters are generalized to include time-dependent ones.

We consider a time-dependent exponential utility function, (see Karatzas [12] page 34, for this concept), of which the objective is to maximize the expected utility from the investor's terminal wealth. We derive an HJB equation and reduce this highly nonlinear equation to a semilinear PDE using an exponential transform. Various authors used different suitable transformations which express value functions in terms of a linear or semilinear PDE (see Benth and Kalsen [2], Kufakunesu [13], Hobson [10], and Heston [9], Zariphopoulou ([23],

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[24]), among others). One example is the logarithmic transform assuming some constant relative risk aversion utility function introduced by Fleming [4].

During the review process, we became aware of a recent paper by Aktar and Taflin [1] with the same philosophy as ours, of considering the time-inhomogeneous market model. Aktar and Taflin [1] considered smooth solutions to a multi-dimensional diffusion model with stochastic volatility but did not analyze a jump-diffusion risky-asset model as in our case.

In this paper, we make a theoretical contribution to the time-inhomogeneous case of the problem solved in [17]. The structure of the rest the paper is as follows. In Section 2, we discuss the problem formulation and present various conditions fulfilled by the model parameters. In Section 3, a highly nonlinear Hamilton-Jacobi-Bellman equation is transformed to a semilinear PDE. A verification theorem relating the value function to the solution of the semilinear PDE is stated and proved. In Section 4, the conditions for the existence of a smooth solution to the semilinear PDE are presented; lemmas and the main theorem are proved. Lastly, in Section 5, we discuss some important financial examples where this framework may be applied and present an example to illustrate the theory from the previous sections of the paper.

2. The Problem Formulation

Consider $(\Omega, \mathcal{F}, P)$ to be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,\mathbb{T}]}$ satisfying the usual conditions, where $\mathbb{T} < \infty$ is the time horizon (Protter [20]). Let a bond S_0 be given without loss of generality as

$$S_0 \equiv 1. \tag{2.1}$$

Let B be a d-dimensional and W be an m-dimensional standard Brownian motions, respectively, which may be correlated and are independent of $\bar{\mu}$, a Poisson random measure (see e.g., in Section 5). In our case, $\{\mathcal{F}_t\}^{B,W}_{t\in[0,\mathbb{T}]}$ represents the filtration generated by B and W. On the other hand, $\{\mathcal{F}_t\}^{\mu}_{t\in[0,\mathbb{T}]}$, is the filtration generated by μ. Hence, we have $\mathcal{F}_t = \mathcal{F}^{B,W}_t \vee \mathcal{F}^{\mu}_t$, for all $t \in [0, \mathbb{T}]$. The price of the n risky assets S are modelled as:

$$\begin{aligned} dS_t &= diag(S_t)[b(t,Y_t)dt + \sigma(t,Y_t)dB_t + \beta(t,Y_t)dW_t \\ &\quad + \int_{\mathbb{R}\backslash\{0\}} \gamma(t,Y_t,z)\bar{\mu}(dt,dz))\,], \quad S_0 > 0, \end{aligned} \tag{2.2}$$

where Y_t, valued in $\mathbb{R}^d$, is a stochastic volatility process which is given by the following:

$$dY_t = \eta(t,Y_t)dt + dB_t\,, \quad Y_0 > 0\,, \tag{2.3}$$

$\bar{\mu}(dt,dz) = \mu(dt,dz) - q(dz)dt$ is the compensated Poisson random measure, and $q(dz)$ is a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ called the Lévy measure with the property:

$$\int_{\mathbb{R}\backslash\{0\}} q(dz) \, < \infty. \tag{2.4}$$

The continuous time-dependent parameter functions which are assumed in Equations (2.2) and (2.3) are as follows. $b : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^{(n+1)\times d}$, $\eta : [0,\infty)\times\mathbb{R}^d \to$

$\mathbb{R}^d$, and $\beta : [0,\infty) \times \mathbb{R}^{n\times d} \to \mathbb{R}^{(n+1)\times m}$. Here, $diag(S_t)$ is the diagonal $n \times n$ matrix with diagonal elements S_t^i, where S_t^i is the price of asset number i, with $i = 1, \cdots, n$ and the $[0,\infty) \times \mathbb{R}^n$-valued functions γ are assumed to be continuous. We suppose that μ is right-continuous and assume the following Lipschitz condition on the $\mathbb{R}^d$-valued function η:

$$(2.5) \qquad (\mathbf{A1})\ \exists\, C > 0\ :\ |\eta(y^*) - \eta(z^*)| \le C|y - z|, \qquad \forall\, y, z \in [0,\infty) \times \mathbb{R}^d,$$

where $y^* = (t, y)$ and $z^* = (t, z)$. In (2.5), the linear growth condition is fulfilled, therefore, by Gronwall's lemma there exists a constant C [19] satisfying

$$|Y_t| \le C\left(1 + \int_0^t |W_u| du + |W_t|\right), \quad t \in [0, \mathbb{T}]\,.$$

Hence, there exists some $\epsilon > 0$ such that

$$(2.6) \qquad \sup_{t\in[0,\mathbb{T}]} \mathbb{E}[\exp(\epsilon|Y_t|^2)] < \infty.$$

Since the process Y is continuous, the price process S is well defined[17]. Put

$$\Sigma(t, y) = (\sigma(t, y),\ \beta(t, y)),$$

as the $(n + 1) \times (d + m)$ matrix-valued volatility function of the risky assets. We assume that for a.e $y^* \in [0,\mathbb{T}] \times \mathbb{R}^d$, $\Sigma(t, y)$ is the full rank equal to $n + 1$ so that $\Sigma(t, y)\Sigma(t, y)^T$ is a nonsingular $(n + 1) \times (n + 1)$ matrix. Here, the symbol T represents the transposition operator. Therefore:

$$(2.7) \qquad \alpha(t, y) = \inf_{\pi\in\mathbb{R}^n,\pi\neq 0} \frac{|\Sigma(y^*)^T\pi|^2}{|\pi|^2}, \quad y^* \in [0,\mathbb{T}] \times \mathbb{R}^d,$$

is the smallest eigenvalue of $\Sigma(t, y)\Sigma(t, y)^T$, which is then strictly positive a.e., $y^* \in [0,\mathbb{T}] \times \mathbb{R}^d$. Another assumption is the following (see [19]): there exists some positive constant C such that for a.e., $y^* \in [0,\mathbb{T}] \times \mathbb{R}^d$:

A2 (i) $|\mu(t, y)/\sqrt{\alpha(t, y)} \le C(1 + |y^*|)$.

A2(ii) $||\sigma(t, y)||/\sqrt{\alpha(t, y)} \le C$.

The two conditions above are referred to as **A2**. If $n = 1$, we have

$$\alpha(t, y) = ||\Sigma(t, y)||^2 = ||\sigma(t, y)||^2 + ||\beta(t, y)||^2\,,$$

and its clear that Assumption **A2(ii)** is fulfilled. For a general n, $\alpha(t, y)$ is larger than the smallest eigenvalue of $\sigma\sigma^T(t, y)$, that is:

$$\alpha(t, y) \ge \inf_{\pi\in\mathbb{R}^n,\pi\neq 0} \frac{|\Sigma(y^*)^T\pi|^2}{|\pi|^2}, \quad y^* \in [0,\mathbb{T}] \times \mathbb{R}^d$$

and $||\sigma(t, y)||^2$ is bounded by the largest eigenvalue of $\sigma\sigma^T(t, y)$. Assumption **A2(i)** can be seen as a condition on the growth of $\mu(t, y)$. As in [19], we consider an investor allocating at any time $t \in [0,\mathbb{T}]$ the amount $\pi_t = (\pi_t^1, \cdots, \pi_t^n)^T$ of the wealth in the risky asset S and

$\mathbf{1} - \pi_t^T \mathbf{e}_n$ in the bond. Due to the self-financing hypothesis the wealth process is given by the following stochastic differential equation :

$$
\begin{aligned}
X_t &= x + \int_0^t \pi_s^T diag(S_{s^-})^{-1} dS_s \\
&= x + \int_0^t \pi_s^T b(s, Y_s) ds + \pi_s^T \sigma(s, Y_s) dB_s + \pi_s^T \beta(s, Y_s) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}\backslash\{0\}} \pi_s^T \gamma(s, Y_s, z) \bar{\mu}(ds, dz)) \,.
\end{aligned} \tag{2.8}
$$

Let $K \in \mathbb{R}^d$ be a fixed closed convex set which contains the origin. The portfolio control $\pi = (\pi_t)$ is called *admissible* if it is an $\mathcal{F}_t$ - adapted stochastic process satisfying the following condition:

$$
\sup_{t \in [0,\mathbb{T}]} \mathbb{E}[\exp(\epsilon |\Sigma(t, Y_t)^T \pi_t|)] < \infty \,, \tag{2.9}
$$

for some $\epsilon > 0$, and the constraint $\pi_t \in K, \ t \in [0, \mathbb{T}]$ a.s. The set of admissible controls is donated by $\mathcal{A}(K)$. In our case, we consider the Markov controls, and this means that the investor will allocate the amount $\pi \equiv \pi(t, x, y) \in \mathcal{A}(K)$, for $y \in \mathbb{R}^d$, into the risky asset when the wealth $X_t = x$ and volatility $Y_t = y$ (see e.g., [2],[13], [19]). We consider an investor with a time-dependent exponential utility function $U : [0, \mathbb{T}] \times \mathbb{R}_+ \to \mathbb{R}$

$$
U(t, x) = -\exp(-\delta(t) x), \ x \in \mathbb{R} \,,
$$

for $\delta(t) > 0$. The expected utility of terminal wealth, for the stochastic volatility Y and X controlled by π, is given by:

$$
J(t, x, y, \pi) = \mathbb{E}\left[U(X_{\mathbb{T}}) \,|\, X_t = x \,, Y_t = y\right] \,.
$$

The objective of the investor is to find the value which is defined as

$$
v(t, x, y) = \sup_{\pi \in \mathcal{A}(K)} J(t, x, y, \pi), \quad (t, x, y) \in [0, \mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d \,. \tag{2.10}
$$

In this paper, we need to characterize the value function v as a classical solution of a semilinear partial differential equation taking into consideration the time-inhomogeneous case of some parameters.

3. Derivation of the HJB Equation

The HJB equation associated with the control problem (2.10) is the following [17]:

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + rx \frac{\partial v}{\partial x} + \eta(t, y)^T D_y v + \frac{1}{2} \Delta_y v \\
&+ \max_{\pi \in K} \{ \pi^T \mu(t, y) x \frac{\partial v}{\partial x} + \frac{1}{2} |\Sigma(t, y)^T \pi|^2 x^2 \frac{\partial^2 v}{\partial x^2} + \pi^T \sigma(t, y) x D_{xy}^2 v \\
&+ \int_{\mathbb{R}\backslash\{0\}} \left(v(t, x + \pi^T \gamma(t, y, z), y) - v(t, x, y) - \pi^T \gamma(t, y, z) \frac{\partial v}{\partial x} \right) q(dz) \} = 0 \,.
\end{aligned} \tag{3.1}
$$

for $(t, x, y) \in [0, \mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d$, where $D_y v$ represents the gradient vector of v with respect to y, $\Delta_y v$ the Laplacian of v with respect to y and $D^2_{xy} v$ as the second order derivative vector v with respect to (t, x, y) [19]. Furthermore, the terminal data is given as follows:

$$v(\mathbb{T}, x, y) = -\exp(-\delta(t)x)\,, \tag{3.2}$$

where $(t, x, y) \in [0, \mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d$. To solve (3.1), we introduce the following candidate of the HJB of the form

$$v(t, x, y) = \exp(-\delta(t)x) \exp(-\phi(t, y))\,. \tag{3.3}$$

Differentiating (3.7), we get:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \left[\frac{\partial \phi}{\partial t} - x\delta'(t)\right] v, & \frac{\partial v}{\partial x} &= -\delta(t)v, \\ \frac{\partial^2 v}{\partial x^2} &= \delta^2(t)v, & D_y v &= vD\phi, \\ \Delta_y v &= [\Delta\phi + D\phi^T D\phi]v, & D^2_{xy} v &= -\delta(t)vD\phi. \end{aligned}$$

Substituting these in (3.1)-(3.2) and obtain the following semilinear PDE:

$$-\phi_t - \frac{1}{2}\Delta\phi_k + F(t, y, D\phi_k) = 0, \quad (t, y) \in [0, \mathbb{T}] \times \mathbb{R}^d\,, \tag{3.4}$$

with terminal condition:

$$\phi(\mathbb{T}, y) = 0, \quad y \in \mathbb{R}^d\,, \tag{3.5}$$

and the nonlinear Hamiltonian function F is defined on $[0, \mathbb{T}) \times \mathbb{R}^d \times \mathbb{R}^d$, is defined as

$$\begin{aligned} F(t, y, p) \quad = \quad & -\frac{1}{2}|p|^2 - p^T\eta(t, y) - x\delta'(t) \\ & + \max_{\pi \in K}\{\delta(t)\pi^T(b(t, y) + \sigma(t, y)p) - \frac{\delta(t)^2}{2}|\Sigma(t, y)^T\pi|^2 \\ & - \int_{\mathbb{R}\setminus\{0\}} \big(\exp(-\delta(t)\pi^T\gamma(t, y, z)) - 1 + \delta(t)\pi^T\gamma(t, y, z)\big)\, q(dz)\}\,. \end{aligned} \tag{3.6}$$

For $\pi \in \mathcal{A}(K)$, we introduce the probability measure Q^π whose density process is given by:

$$\begin{aligned} Z^\pi_{1t} \quad = \quad & \exp\Big(-\int_0^t \delta(t)\pi_s^T\sigma(t, Y_s)dB_s - \int_0^t \delta(t)\pi_s^T\beta(t, Y_s)dW_s \\ & -\frac{1}{2}\int_0^t |\delta(t)\Sigma(t, Y_s)^T\pi_s|^2\Big),\ t \in [0, \mathbb{T}]. \end{aligned}$$

This is well defined (see [17], [18]). The following verification result is somewhat similar to Theorem 3.1 in [17] and relates a solution of the semilinear PDE equation to the stochastic control problem (2.10) .

Proposition 3.1. *Assume that Equations* (2.5), (2.6) *given by Assumption* **A1** *and Assumption* **A2** *respectively hold. Suppose there exists a solution* $\phi \in C^{1,2}([0,\mathbb{T}],\mathbb{R}^d) \cap ([0,\mathbb{T}],\mathbb{R}^d)$ *to the semilinear PDE* (3.4) *with terminal condition* (3.5). *Then the value function of* (2.10) *is given by*

$$v(t,x,y) = -\exp(-x\delta(t))\exp(-\phi(t,y)) \qquad (t,x,y) \in [0,\mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d \,. \tag{3.7}$$

Moreover, the Markov optimal control π^* *is given by*

$$\begin{aligned} \pi^*(t,y) \;\; &\in \;\; \arg\min_{\pi\in K}\Big[\frac{\delta(t)^2}{2}|\Sigma(t,y)^T\pi|^2 - \delta(t)\pi^T\left(b(t,y) - \sigma(t,y)D\phi(t,y)\right) \\ &\qquad + \int_{\mathbb{R}\setminus\{0\}} \left(\exp(-\delta(t)\pi^T\gamma(t,y,z)) - 1 + \delta(t)\pi^T\gamma(t,y,z)\right) q(dz)\Big]\,, \end{aligned}$$

for almost every $(t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d$.

Proof. The proof goes along the lines of (Mnif [17], Theorem 3.1). Let the following local martingale be defined:

$$Z^\pi_{2t} = \varepsilon\left(\int_0^t \int_{\mathbb{R}\setminus\{0\}} \left(\exp(-\delta(t)\pi^T\gamma(t,y,z)) - 1\right)\bar{\mu}(ds,dz)\right).$$

Using the generalized Itô formula, we obtain:

$$\begin{aligned} & U(X_\mathbb{T}) \\ = \;\; & U(X_t) - \int_t^\mathbb{T} \delta(s)U(X_{s^-})\pi_s^T b(s,Y_s)ds \\ & - \int_t^\mathbb{T} \delta(s)U(X_{s^-})\pi_s^T\sigma(s,Y_s)dB_s + \pi_s^T\beta(s,Y_s)d\bar{W}_s \\ & + \int_t^\mathbb{T}\int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,Y_s,z)) - 1 + \delta(s)\pi^T\gamma(s,Y_s,z)\right) q(dz)dt \\ & + \frac{\delta(t)^2}{2}\int_t^\mathbb{T} U(X_{s^-})\,|\sum(s,Y_s)^T\pi_s|^2 ds \\ & + \int_t^\mathbb{T}\int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,y,z)) - 1\right)\bar{\mu}(ds,dz)\,. \end{aligned} \tag{3.8}$$

The solution of (3.8) is given by the Doléans-Dade Exponential Formula

$$U(X_T) = U(X_t)\frac{Z^\pi_{1\mathbb{T}}}{Z^\pi_{1t}}\frac{Z^\pi_{2\mathbb{T}}}{Z^\pi_{2t}}\exp\left(\int_t^\mathbb{T} h(s,Y_s,\pi_s)ds\right),$$

where

$$\begin{aligned} h(t,y,\pi) \;\; &= \;\; -\delta(t)\pi^T b(t,y) - \frac{\delta(t)^2}{2}\,|\sum(t,y)^T\pi|^2 \\ &\qquad + \int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,y,z)) - 1 + \delta(s)\pi^T\gamma(s,y,z)\right) q(dz)\,. \end{aligned}$$

We obtain the following :

$$J(t,x,y,\pi) = U(x)\mathbb{E}^{\pi}\left[\frac{Z^{\pi}_{2\mathbb{T}}}{Z^{\pi}_{2t}}\exp\left(\int_t^{\mathbb{T}} h(s,Y_s,\pi_s)ds\right) \mid X_t = x\,, Y_t = y\right].$$

Applying the Itô's formula to $\phi(t,Y_t)$ under Q^{π}, we obtain

$$\begin{aligned}
\phi(t,Y_{\mathbb{T}}) &= \phi(t,Y_t) + \int_t^{\mathbb{T}}\left(\frac{\partial\phi}{\partial u} + \eta^T D\phi\right)(u,Y_u)du \\
&\quad + \int_t^{\mathbb{T}}(-\delta(u)\pi^T\sigma D\phi + \frac{1}{2}\Delta\phi)(u,Y_u)du + \int_t^{\mathbb{T}} D\phi(u,Y_u)^T dB^{\pi}_u \\
&\geq \phi(t,Y_t) - \int_t^{\mathbb{T}} h(u,Y_u,\pi_u)du + \int_t^{\mathbb{T}} D\phi(u,Y_u)^T dB^{\pi}_u \\
&\quad -\frac{1}{2}\int_t^{\mathbb{T}} |D\phi(u,Y_u)|^2 du\,.
\end{aligned}$$

The rest of the proof follows that of Theorem 3.1 in ([17]), by adjusting certain parameters into time-dependent ones. □

4. Regularity of the Value function

In this section, we discuss the existence of a classical solution to the semilinear PDE (3.4). We follow the idea in [19] and [17]. Firstly, we establish the smoothness of the value function under at least one of the following general conditions

(G3):

(i) $\eta,\ \sigma^T(\Sigma\Sigma^T)^{-1}\sigma,\ \sigma^T(\Sigma\Sigma^T)^{-1}\mu$ are C^1 and Lipschitz,
(ii) $\sigma^T(\Sigma\Sigma^T)^{-1}\mu$ is C^1, bounded and Lipschitz,
(iii) $\gamma(t,y,z) = \gamma(t,z) \leq M$ for all $(t,y,z) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^n$ where $M > 0$,
(iv) $K = \mathbb{R}^n_+$.

In our case, the Hamiltonian F is not Lipschitz in p. We prove a smooth solution for the semilinear PDE (3.1). Since

$$-\frac{1}{2}|p|^2 - p^T\eta(t,y) = \min_{\omega\in\mathbb{R}^d}[-\omega^T p + \frac{1}{2}|\omega - \eta(t,y)|^2]\,,$$

for all $(t,y,p) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, as in [17], we make the following transformation on the Hamiltonian F:

$$F(t,y,p) = \min_{\omega\in\mathbb{R}^d}[-\omega^T p + \bar{L}(t,y,\omega)]\,,$$

where

$$\bar{L}(t,y,\omega) = \max_{\pi\in K}[\bar{L}(t,y,\omega,\pi)]\,,$$

and

$$\bar{L}(t,y,\omega,\pi) = \frac{1}{2}|\omega - \eta(t,y)|^2 + \delta(t)\pi^T b(t,y) - \frac{\delta(t)^2}{2}\mid\sum(t,y)^T\pi|^2$$

$$-\int_{\mathbb{R}\setminus\{0\}} \left(\exp(-\delta(s)\pi^T\gamma(s,z)) - 1 + \delta(s)\pi^T\gamma(s,z)\right) q(dz)\,.$$

Consider compact sets $\mathcal{B}_k := \{\omega \in \mathbb{R}^d : |\omega| \leq k\},\ k > 0$ and $\mathcal{A}_k(K) := \{\pi \in K : |\pi| \leq k\},\ k > 0$ where $\mathcal{A}_k(K),\ \mathcal{B}_k \subset \mathbb{R}^d$. We have the following truncated Hamiltonian functions:

$$F_k(t,y,p) = \max_{\omega\in\mathcal{B}_k}\{-\omega^T p - \bar{L}(t,y,\omega)\}\,. \tag{4.1}$$

We consider the following function $(t,\pi,y) \mapsto f(t,\pi,y)$ as in[17]:

$$\begin{aligned} f(t,\pi,y) &= -\delta(t)\pi^T b(t,y) + \frac{\delta(t)^2}{2}\,|\sum(t,y)^T\pi|^2 \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \left(\exp(-\delta(s)\pi^T\gamma(s,z)) - 1 + \delta(s)\pi^T\gamma(s,z)\right) q(dz) \end{aligned}$$

and the following constraints $\pi_i \geq 0,\ i = 1,\cdots,n$ and $\pi_i - k \leq 0,\ i = 1,\cdots,n$. It is clear that the function f is $C^{1,2}$ in π_i, convex in π_i and coercive. Moreover, $\mathcal{A}_k(K)$ is a closed and convex, and there exists a unique solution $\pi^*(t,y)$ to the problem $\min_{\pi\in\mathcal{A}_k(K)} f(t,\pi,y)$ which implies that

$$\bar{L}_k(t,y,\omega) = \bar{L}(t,y,\omega,\pi^*(t,y))\,.$$

The Assumptions (**G3**) which satisfy a polynomial growth condition imply that the function L is $C^{1,1}$ in (t,y) and that L, D_yL are also $C^{1,1}$ on $[0,\mathbb{T}]\times\mathbb{R}^d\times\mathcal{B}_k\times\mathcal{A}_k(K)$. Similarly, the functions $\bar{L}_k,\ D_y\bar{L}_k$ are $C^{1,1}$ in (t,y) and satisfy a polynomial growth condition in (t,y) on $[0,\mathbb{T}]\times\mathbb{R}^d\times\mathcal{B}_k$. Hence (from Mnif[17], page 254 and references therein) there exists a unique $\phi_k \in C^{1,2}([0,\mathbb{T}),\mathbb{R}^d)\cap C^0([0,\mathbb{T}],\mathbb{R}^d)$ with a polynomial growth condition, to the following partial differential equation:

$$-\frac{\partial\phi_k}{\partial t} - \frac{1}{2}\Delta\phi_k + F_k(t,y,D\phi_k) = 0, \quad (t,y)\in[0,\mathbb{T}]\times\mathbb{R}^d\,, \tag{4.2}$$

with Cauchy data $\phi_k(\mathbb{T},y) = 0$. The following lemma gives a linear growth condition on the derivative $D\phi_k$.

Lemma 4.1. *Under the Assumptions of* **G3** *with time-dependent parameters there exists a positive constant C independent of k such that*

$$|D\phi_k(t,y)| \leq C(1+|y|), \quad \forall(t,y)\in[0,\mathbb{T}]\times\mathbb{R}^d\,.$$

Proof. This follows as in (Pham [19], Lemma 4.2). Here, we clarify some time-dependent parameters in the proof. From the standard verification theorem (see [19],[8]), ϕ_k can be represented as the solution of the stochastic control problem:

$$\phi_k(t,y) = \inf_{\omega\in\mathcal{B}_k,\pi\in\mathcal{A}_k(K)} \mathbb{E}^Q\left[\int_t^{\mathbb{T}} \bar{L}(u,Y_u,\omega_u)\,du \,|\, Y_t = y\right], \tag{4.3}$$

where $\mathcal{B}_k = \{\omega \in \mathbb{R}^d : |\omega| \leq k\}$ and $\mathcal{A}_k(K) = \{\pi \in K : |\pi| \leq k\}$ are compact sets and the controlled dynamics of Y under Q is given as

$$dY_t = \omega_t dt + dB_t\,. \tag{4.4}$$

Moreover, an optimal control for (4.3) is Markov with the policy given in feedback form by:

$$(\omega^*(t,y), \pi^*(t,y)) \in \arg\min_{|\omega|\leq k\,,|\pi|\leq k} \{\omega^T D\phi_k(t,y) + \bar{L}(t,y,\omega,\pi)\}\,. \tag{4.5}$$

Hence we obtain:

$$\psi_k(t,y) = \mathbb{E}^Q \left[\int_t^{\mathbb{T}} \bar{L}(u, Y_u^*, \omega^*(u, Y_u^*))\, du \,|\, Y_t^* = y \right], \tag{4.6}$$

where Y_t^* solves (4.4) with the control $\omega_t^* = \omega_k^*(t, Y_t^*)\,, \pi_t = \pi^*(t, Y_t^*)$. From [19] and standard theory (Fleming and Soner [8], Lemma 11.4 on page 209), we have:

$$D\psi_k(t,y) = \mathbb{E}^Q \left[\int_t^{\mathbb{T}} D_y\bar{L}(u, Y_u^*, \omega_k^*(u, Y_u^*))\, du \,|\, Y_t^* = y \right]. \tag{4.7}$$

From Equation (4.7) and assumption **G3**, we have that, $\forall (t,y,\omega,\pi) \in [0,\mathbb{T}] \times \mathbb{R}^d \times \mathbb{R}^d \times K$,

$$\begin{aligned}
&|D_y\bar{L}(t,y,\omega,\pi)| \\
\leq\ & |D\eta(t,y)||\omega - \eta(t,y)| + \frac{\delta(t)^2}{2}||D(\Sigma(t,y)\Sigma(t,y)^T)(t,y)|||\pi(t,y)|^2 \\
& +\delta(t)|Db||\pi(t,y)| - \int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,y,z)) - 1 + \delta(s)\pi^T\gamma(s,y,z)\right) q(dz) \\
\leq\ & C[|1 + |\omega - \eta(t,y)|^2 - \frac{\delta(t)^2}{2}||D(\Sigma(t,y)\Sigma(t,y)^T)(t,y)|||\pi(t,y)|^2 \\
& +\delta(t)|Db||\pi(t,y)| - \int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,y,z)) - 1 + \delta(s)\pi^T\gamma(s,y,z)\right)]q(dz) \\
\leq\ & C\left[|1 + \bar{L}(t,y,\omega,\pi)|\right],
\end{aligned}$$

where the generic positive constant C changes from line to line in the above estimation process. As in Mnif [17], the ellipticity property of $\Sigma(t,y)\Sigma(t,y)^T$ implies that $\pi^*(t,y)$ satisfies:

$$\begin{aligned}
& \epsilon|\pi^*(t,y)|^2 - \delta(t)|\pi^*(t,y)||b(t,y)| \\
& + \int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,z)) - 1 + \delta(s)\pi^T\gamma(s,z)\right) q(dz) \\
\leq\ & \frac{\delta(t)^2}{2}|D(\Sigma(t,y)^T\pi^*(t,y)|^2 - \delta(t)\pi^*(t,y)^T b(t,y) \\
& + \int_{\mathbb{R}\setminus\{0\}} U(X_{s^-})\left(\exp(-\delta(s)\pi^T\gamma(s,z)) - 1 + \delta(s)\pi^T\gamma(s,z)\right) q(dz) \\
\leq\ & 0,
\end{aligned}$$

for $\epsilon > 0$. This shows that $|\pi^*(t,y)| \le \frac{\delta(t)}{\epsilon}|b(t,y)|$. Hence, from (4.7) and the Jensen's inequality, we obtain:

$$\begin{aligned}
|D_y\phi_k(t,y)|^2 &\le C\left(1+\mathbb{E}^Q\Big[\int_t^{\mathbb{T}} D_y\bar{L}(u,Y_u^*,\omega_k^*(u,Y_u^*))\,du\,|\,Y_t^*=y\Big]\right)\\
&= C\left(1+\phi_k(t,y)\right)\\
&\le C\left(1+\mathbb{E}^Q\Big[\int_t^{\mathbb{T}} D_y\bar{L}(u,Y_u^*,0)\,du\,|\,Y_t=y\Big]\right)\\
&= C\left(1+\mathbb{E}^Q\Big[\int_t^{\mathbb{T}} D_y\bar{L}(u,Y_u^*,0,\pi^*(u,y))\,du\,|\,Y_t=y\Big]\right),
\end{aligned}$$

where $Y_t = y + B_t^Q$. Hence, the result follows. □

The following theorem states the existence of the smooth solution of the semilinear PDE with time-dependent parameters:

Theorem 4.2. *Under the Assumptions* **G3***, there exists a solution* $\phi \in C^{1,2}([0,\mathbb{T}),\mathbb{R}^d) \cap C^0([0,\mathbb{T}],\mathbb{R}^d)$ *, with linear growth condition in* y *on the derivative* $D\phi$*, to the semilinear equation* (3.4) *with terminal condition* (3.5)*.*

Proof. As in Pham [19], [17] the function $(\omega,\pi) \to -\omega^T D\phi_k(t,y) + \frac{1}{2}|\omega - \eta(t,y)|^2$ attains its maximum on $[0,\mathbb{T}] \times \mathbb{R}^d \times K$ for

$$\omega_k^*(t,y) = D\phi_k(t,y) + \eta(t,y)\,.$$

Furthermore, by Lemma 4.1, $D\phi_k$ satisfies a linear growth condition in y uniformly in k and $\eta(t,y)$ is Liptschiz. Therefore, for $M > 0$ arbitrarily large, there exists a $C > 0$ independent of k such that

$$|\pi^*(t,y)|,\ |\omega_k^*(t,y)| \le C, \quad \forall t \in [0,\mathbb{T}], \quad |(t,y)| \le M.$$

Hence for $K \ge C$, we obtain

$$\begin{aligned}
F_k(t,y,D\phi_k(t,y)) &= \max_{\omega\in\mathcal{B}_k}\left[-\omega^T D\phi_k(t,y) + L(t,y,\omega)\right]\,,\\
&= \max_{\omega\in\mathbb{R}^d}\left[-\omega^T D\phi_k(t,y) + \max_{\pi\in K} L(t,y,\pi,\omega)\right]\,,\\
&= F(t,y,D\phi_k(t,y)),
\end{aligned}$$

for all $(t,y) \in [0,\mathbb{T}] \times \{|y| \le M\}$. Letting M go to infinity we have that ϕ_k is a smooth solution with linear growth. □

5. Example

In this section, we illustrate the theoretical results in this paper in the time-inhomogeneous framework and discuss some other financial cases where time-dependency of parameters in models are important.

5.1. **A Time-Dependent Multi-Assets Stochastic Volatility Model with Jumps.** Consider a model with n-risky assets, $n \in \mathbb{N}$ adopted from [17]. We assume that these assets have nondegenerate volatility. The model is formalized as follows:

$$\frac{dS_t^i}{S_{t^-}^i} = b_i(t)dt + \nu_i(t, Y_{it}) \sum_{j=1}^{i} \rho_{ij} d\bar{W}_t^j + \int_{\mathbb{R}\setminus\{0\}} \gamma_i(t,z)\bar{\mu}(dt,dz), \tag{5.1}$$

$$dY_{it} = (a_i(t) - \theta_i(t)Y_{it})dt + dW_t^i, \text{ for all } i = \{1, 2, \cdots, n\}, \tag{5.2}$$

where $\nu_i(t,y)$ are bounded $C^{1,1}$ functions with bounded derivatives and lower bounded by positive constants $\varepsilon_i > 0$ for all $i = \{1, 2, \cdots, n\}$ and where $a_i(t)$, $b_i(t)$ and $\theta_i(t)$ are time-dependent functions and ρ_{ij} is the constant correlation between two Brownian motions $\bar{W}_t^i$ and $\bar{W}_t^j$, $(i,j) \in \{1, 2, \cdots, n\}^2$. Here we take $\nu_i(t,y) = \varepsilon_i + 1/\sqrt{\zeta_i^2(t)|y|^2 + \varepsilon_i}$ where ζ_i is a time-dependent function.

Remark. The volatility is modelled by an Ornstein-Uhlenbeck process $\theta_i(t)$, $i = \{1, 2, \cdots, n\}$ is the rate of mean reversion ($\theta_i(t) = 0$ in the Hull-White model [11] and $\theta_i(t) \neq 0$ in the Scott model [21] and in the Stein-Stein model [22]). We assume that there is no correlation between assets S^i , $i = \{1, 2, \cdots, n\}$ and its volatility.

With reference to the notations in Section 2, in this case we have $y = (y_1, \cdots y_n)$, $\eta(t,y) = (a_1(t) - \theta_1(t)y_1, \cdots, a_n(t) - \theta_n(t)y_n)^T$, $\sigma(t) = 0$, $\gamma(t,z) = (\gamma_1(t,z), \cdots, \gamma_n(t,z))^T$ and $\Sigma\Sigma^T(t,y) = \Big(\nu_i(t,y_i))(\nu_j(t,y_j)) \sum_{k=1}^{\inf(i,j)} \rho_{ik}\rho_{jk}\Big)_{1\le i,j\le n}$.

We can interpret $\Sigma\Sigma^T(t,y)$ as YRY, where

$$Y_{ij} = \begin{cases} \nu_i(t,y_i) & \text{if } 1 \le i = j \le n, \\ 0 & \text{if not}, \end{cases}$$

and

$$R = \wedge\wedge^T = \left(\sum_{k=1}^{\inf(i,j)} \rho_{ik}\rho_{jk} \right) ,$$

with

$$\wedge_{ij} = \begin{cases} \rho_{ij} & \text{if } i \ge j, \\ 0 & \text{if } i < j\,. \end{cases}$$

Note, as in Mnif [17], that the matrix $\wedge$ is a lower triangle and invertible. Invertibility is achieved if and only if $\rho_{ii} \neq 0$ for all $i = \{1, 2, \cdots, n\}$. Under this property, R is symmetric positive definite and $\Sigma\Sigma^T(t,y)$ is a nonsingular $n \times n$ matrix. We therefore have that $\big(\Sigma\Sigma^T(t,y)\big)^{-1} = Y^{-1}R^{-1}Y^{-1}$ and

$$\big(\Sigma\Sigma^T(t,y)\big)_{ij}^{-1} = \frac{1}{(\nu_i(t,y_i))(\nu_j(t,y_j))} R_{ij}^{-1} \,, \text{ for all } 1 \le i,j \le n\,. \tag{5.3}$$

Denote by $\rho(\Sigma\Sigma^T(t,y))^{-1}$ the spectral radius of $(\Sigma\Sigma^T(t,y))^{-1}$. But

$$\rho(\Sigma\Sigma^T(t,y))^{-1} \le ||(\Sigma\Sigma^T(t,y))^{-1}||_1\,,$$

and since

$$||(\Sigma\Sigma^T(t,y))^{-1}||_1 = \max_{1\le j\le n}\sum_{i=1}^n |\left(\Sigma\Sigma^T(t,y)\right)^{-1}_{ij}|,$$

there exists a constant C independent of y such that

$$\begin{aligned}\rho(\Sigma\Sigma^T(t,y))^{-1} &\le C\max_{1\le j\le n}\sum_{i=1}^n \frac{1}{(\nu_i(t,y_i))(\nu_j(t,y_j))}\\ &\le C\max_{1\le j\le n}\sum_{i=1}^n \frac{1}{\varepsilon_i\varepsilon_j}\\ &= \frac{1}{\alpha},\end{aligned} \tag{5.4}$$

where α is a positive constant independent of y. The above inequality (5.4) implies the smallest eigenvalue of $\Sigma\Sigma^T(t,y)$, which is denoted by $\alpha(t,y)$, satisfies:

$$\alpha(t,y) = \frac{1}{\rho(\Sigma\Sigma^T(t,y))^{-1}} \text{ for all } (t,y)\in[0,T]\times\mathbb{R}^n.$$

The semilinear PDE is given by

$$\begin{aligned}&-\frac{\partial\phi}{\partial t} - \frac{1}{2}\sum_{i=1}^n\frac{\partial^2\phi}{\partial y_i^2} - \frac{1}{2}\sum_{i=1}^n\left(\frac{\partial\phi}{\partial y_i}\right)^2 - \sum_{i=1}^n(a_i-\theta_i y_i)\frac{\partial\phi}{\partial y_i}\\ &+\max_{\pi\in K}\{\delta(t)\sum_{i=1}^n\pi_i b_i - \frac{\delta^2(t)}{2}\sum_{i,j=1}^n\pi_i\pi_j\left(\nu_i(t,y_i))(\nu_j(t,y_j))\sum_{k=1}^{\inf(i,j)}\rho_{ik}\rho_{jk}\right)\\ &+\int_{\mathbb{R}\setminus\{0\}}\left(\exp(-\delta(t)\sum_{i=1}^n\pi_i\gamma_i(z)) - 1 + \delta(t)\sum_{i=1}^n\pi_i\gamma_i(z)\right)q(dz)\} = 0,\end{aligned} \tag{5.5}$$

with the terminal condition $\phi(\mathbb{T},y)=0$. The value function of the optimization problem is given by $v(t,x,y) = -\exp(-\delta(t)x\exp(\phi(t,y))$. The optimal solution is given by:

$$\begin{aligned}\pi^*(t,Y_t)\in\arg\min_{\pi\in K}\{&-\delta(t)\sum_{i=1}^n\pi_i b_i + \frac{\delta^2(t)}{2}\sum_{i,j=1}^n\pi_i\pi_j\left(\nu_i(t,y_i))(\nu_j(t,y_j))\sum_{k=1}^{\inf(i,j)}\rho_{ik}\rho_{jk}\right)\\ &-\int_{\mathbb{R}\setminus\{0\}}\left(\exp(-\delta(t)\sum_{i=1}^n\pi_i\gamma_i(z)) - 1 + \delta(t)\sum_{i=1}^n\pi_i\gamma_i(z)\right)q(dz)\},\end{aligned}$$

$a.s.$, $0\le t\le T$.

5.2. **Other financial Cases.** The time-inhomogeneous models are of extreme importance to fund managers and interest-rate-options traders. For instance, trading on the yield-curve, the trader takes quoted data as model input. To fit the observed yield-curve perfectly, time-dependent parameters need to be considered so that all risks on the curve are hedged away instantly (see e.g., [5], [6], [15], for other calibration purposes). In [3], a thorough empirical study of the time-dependent diffusion models has been done.

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