Some theoretical comments regarding the run-length properties of the synthetic and runs-rules $\bar{X}$ monitoring schemes – Part 2: Steady-state

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Abstract

In this paper, the long term (also known as the steady-state mode) run-length theoretical properties of the four different types of synthetic and runs-rules $\bar{X}$ monitoring schemes that were empirically analyzed in another paper are discussed. That is, using the Markov chain imbedding technique (thoroughly discussed in Part 1 of this work), the closed-form expressions of the steady-state initial probabilities and average run-length (*ARL*) vectors are derived; so that the corresponding steady-state *ARL* and overall performance expressions, of each of the four different types of the synthetic and runs-rules $\bar{X}$ monitoring schemes, may be formulated. Since there is very little literature on steady-state analysis of the synthetic and runs-rules charts, the closed-form expressions derived here will ease the understanding and implementation of the different types synthetic and runs-rules $\bar{X}$ schemes in practice and in further academic research.

Keywords: Average run-length (*ARL*), Overall performance, Runs-rules charts, Steady-state, Synthetic charts, Transition probability matrix (TPM).

1. Introduction

This paper is primarily dependent on Shongwe and Graham (2018b) (henceforth referred to as Part 1 of this work) where all the important background information is thoroughly discussed. Hence, as directed by the referees, we will not re-define or re-introduce anything already discussed in Part 1, but instead we implore the reader to consult Part 1 of this work before reading this Part 2.

The steady-state mode is used to characterize long term run-length properties of a monitoring scheme. Hence, the steady-state run-length is the number of sampling points at which the chart first signals given that the process begins and stays IC for a long time, then at some random time, an OOC signal is observed, see Zhang and Wu (2005). Thus, the steady-state *ARL* (*SSARL*) is given by

$$SSARL(\delta) = \mathbf{s}_{(1\times M)} \cdot \boldsymbol{ARL}_{(M\times 1)}(\delta) = \mathbf{s}_{(1\times M)} \cdot (\mathbf{I}_{(M\times M)} - \mathbf{Q}(\delta)_{(M\times M)})^{-1} \cdot \mathbf{1}_{(M\times 1)}, \quad (1)$$

where $\mathbf{s}_{(1\times M)}$ is the steady-state initial probabilities vector and the rest of the elements of Equation (1) are explained in Part 1 of this work.

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In the area of basic synthetic and runs-rules $\bar{X}$ charts, most authors usually consider only zero-state mode, apart from Shongwe and Graham (2018a) (who considered the steady-state empirical analysis of all the four different types of these schemes) to the best of our knowledge, only Champ (1992), Khoo et al. (2012), Davis and Woodall (2002) and Machado and Costa (2014) have considered some empirical steady-state studies on the basic RR1, S1, S2 and S3, respectively. More importantly, and also what inspired this paper, the excellent account of theoretical discussion that was conducted by Champ (1992) for the RR1 scheme as well as Machado and Costa (2014) for the S1 and S3 schemes. Thus, to supplement on this work as well as the empirical analysis in Shongwe and Graham (2018a), in this paper, a contribution to the theory of synthetic and runs-rules $\bar{X}$ monitoring schemes is made by:

i. Derive the steady-state closed-form expressions of $\mathbf{s}_{(1\times M)}$ and $\boldsymbol{ARL}_{(M\times 1)}$, so that we formulate the *SSARL* and the overall performance expressions;
ii. Show that the three methods that are mostly used to compute the $\mathbf{s}_{(1\times M)}$, yield empirically and notionally different $\mathbf{s}_{(1\times M)}$'s; however, the resulting empirical *SSARL* values are approximately equal.

The rest of the paper is organized as follows: In Sections 2 and 3, the steady-state initial probability and *ARL* vectors for each of the schemes are derived, respectively. In Section 4, the corresponding *SSARL* expressions are computed. In Section 5, the overall run-length distribution properties that may be used as an alternative to design or to evaluate the schemes here are given. Finally, in Section 6, we give some concluding remarks.

2. Initial probabilities vectors

The steady-state probabilities vector (SSPV) is such that $\mathbf{s}_{(1\times M)} = (s_1, s_2, \ldots, s_M)$, with $\mathbf{s}\cdot\mathbf{Q} = \mathbf{s}$ subject to $\sum_{j=1}^{M} s_j = 1$. In the SPCM literature, there are three methods that are mostly used to compute the SSPV: the cyclical, simplified cyclical and conditional steady-state methods – denoted by SSPV1, SSPV2 and SSPV3, respectively. Note that some of these steady-state methods were discussed in Shongwe and Graham (2017). Next, these three methods are discussed in Sections 2.1 to 2.3. In Section 2.4, the general form of the initial probabilities vectors are given for each scheme.

2.1 The cyclical steady-state method (SSPV1)

The SSPV1 method by Crosier (1986) requires that $\mathbf{P}^*$ be computed first. The matrix $\mathbf{P}^*$ is obtained by altering $\mathbf{P}$ (in Equation (3) of Part 1 of this work) so that the control statistic is reset to the initial state whenever it goes into an OOC state. That is, the component with the value one on the last row of the TPM is altered so that the value of one is moved to the respective initial state. Then, once $\mathbf{P}^{*}$ has been determined, it can be used to find the $(M+1)\times 1$ probability vector $\mathbf{v}$ such that the following equation is satisfied

$$\mathbf{v} = \mathbf{P}^{*\prime}\mathbf{v} \quad \text{subject to} \quad \mathbf{1}'\mathbf{v} = 1. \tag{2}$$

where

$$\boldsymbol{P}^{*}_{(M+1)\times(M+1)} = \begin{pmatrix} \mathbf{Q}_{(M\times M)} & | & \boldsymbol{r}_{(M\times 1)} \\ - & - & - \\ \mathbf{u}_{j}{}'_{(1\times M)} & | & 0_{(1\times 1)} \end{pmatrix} \tag{3}$$

with the jth unit vector defined as

$$\mathbf{u}_j = \begin{cases} \mathbf{u}_1 & \text{for RR1} \\ \mathbf{u}_{\frac{M+1}{2}} & \text{for RR2, RR3, RR4} \\ \mathbf{u}_2 & \text{for S1} \\ \mathbf{u}_{M+1} & \text{for S2, S3, S4.} \end{cases} \tag{4}$$

Thus, the SSPV1 method is given by

$$\mathbf{s} = (\mathbf{1}'\mathbf{z})^{-1} \cdot \mathbf{z} \tag{5}$$

where $\mathbf{z}$ is the $M\times 1$ vector obtained from $\mathbf{v}$ by deleting the $(M+1)^{\text{th}}$ component associated with the absorbing state.

2.2 The simplified steady-state method (SSPV2)

In the synthetic literature, it has been erroneous stated the SSPV2 method by Champ (1992) is calculated using Equation (5); however, with

$$\mathbf{z}_{(M\times 1)} = (\mathbf{G} - \mathbf{Q}')^{-1} \cdot \mathbf{u} \text{ where } \mathbf{u}_{(M\times 1)} = (1\ 0\ 0\ \ldots\ 0)' \tag{6}$$

and

$$\mathbf{G}_{(M\times M)} = \begin{pmatrix} 2 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{7}$$

see for example, Khoo et al. (2012), Hu et al. (2015), Hu and Sun (2015), Shongwe and Graham (2016, 2018a). In fact, Equations (6) and (7) were supposed to be defined as

$$\mathbf{z}_{(M\times 1)} = (\mathbf{G} - \mathbf{Q}')^{-1} \cdot \mathbf{u}_j \text{ where } \mathbf{u}_j \text{ is as given in Equation (4) above} \tag{8}$$

and

$$\mathbf{G}_{(M\times M)} = \mathbf{u}_j \cdot \mathbf{1}' + \mathbf{I}_{(M\times M)}. \tag{9}$$

Redefining $\mathbf{G}$ as done in Equation (9) instead of Equation (7) ensures that the SSPV1 and SSPV2 methods are exactly equivalent, as proclaimed in general by Champ (1992).

2.3 The conditional steady-state method (SSPV3)

The SSPV3 method (also used by Davis and Woodall (2002) and Machado and Costa (2014)) is obtained by dividing each element of $\mathbf{Q}(\delta = 0)$ by its corresponding row sum, so that the '*new*' $\mathbf{Q}$ is called the conditional essential TPM, denoted by $\mathbf{Q}_{\boldsymbol{C}}$. That is, $\mathbf{Q}_{\boldsymbol{C}}$ is the altered version of $\mathbf{Q}$ so that the '*new*' essential TPM is ergodic. Thus, the SSPV3 method is a vector such that

$$\mathbf{s} \cdot \mathbf{Q}_{\boldsymbol{C}} = \mathbf{s} \text{ subject to } \textstyle\sum_{j=1}^{\tau} s_j = 1.$$

Unlike the SSPV1 and SSPV2 methods, the SSPV3 method is such that the dimension of the essential TPM of the corresponding synthetic and runs-rules schemes are equal (RR1 & S1, RR2 & S2, RR3 & S3, RR4 & S4) i.e. equal to τ rather than M. That is, the design of the SSPV3 method is such that it is assumed that the process has been IC for a very long time, hence, the effect of a head-start feature has disappeared. Removing the head-start feature in the synthetic design is similar to removing the shaded elements in the TPMs in Table 5 of Part 1 of this work. Consequently, this implies that the synthetic and runs-rules schemes in that Table 5 (a), (b), (c) and (d) are respectively equivalent.

2.4 General form of the initial probabilities vectors

Since the process must be IC when calculating $\mathbf{s}$, let: $p = p_O$, $p_1 = p_A = p_D$, $\gamma = \frac{1}{2}p$, $\varphi = \frac{2p}{1+p}$, $\vartheta = \frac{2\gamma}{1+2\gamma} = \frac{p}{1+p}$, $\lambda = \frac{1-p}{1+p}$. In Table 1, we give a summary of which schemes have explicitly easy $\mathbf{s}$ expressions for each SSPV1, SSPV2 and SSPV3 methods. Note that those with 'Not simple' do not necessarily mean the expression does not exist; instead as H becomes very large, they are complicated to write explicitly. Consequently, the SSPV expressions are shown in Tables 2 and 3, respectively (with the vector of the S2 and RR2 schemes using the dummy variables defined in Table 5, Panel (b) of Part 1 of this work). For the sake of consistency, we recommend the use of the SSPV3 method and consequently, moving forward we use the SSPV3 method.

Table 1: Summary of which schemes have easy explicit expressions of the initial probability vectors

	RR1	**RR2**	**RR3**	**RR4**	**S1**	**S2**	**S3**	**S4**
SSPV1 **SSPV2**	Table 2	Not simple	Table 2	Table 2	Table 2	Not simple	Not simple	Not simple
SSPV3	Table 3							

Table 2: The closed-form expressions of some of the synthetic and runs-rules $\bar{X}$ schemes using the SSPV1 and SSPV2 methods

(a) RR1	(b) S1	(c) RR3	(d) RR4
$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-1}\\ s_H\\ s_{H+1}\end{pmatrix}' = \frac{1}{2-p_O^H}\begin{pmatrix}1\\ 1-p_O\\ p_O(1-p_O)\\ p_O^2(1-p_O)\\ \vdots\\ p_O^{H-2}(1-p_O)\\ p_O^{H-1}(1-p_O)\end{pmatrix}'$	$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-1}\\ s_H\\ s_{H+1}\end{pmatrix}' = \begin{pmatrix}p_O^H\\ 1-p_O\\ p_O(1-p_O)\\ p_O^2(1-p_O)\\ \vdots\\ p_O^{H-2}(1-p_O)\\ p_O^{H-1}(1-p_O)\end{pmatrix}'$	$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-2}\\ s_{H-1}\\ s_H\\ s_{H+1}\\ s_{H+2}\\ s_{H+3}\\ s_{H+4}\\ \vdots\\ s_{2H-1}\\ s_{2H}\\ s_{2H+1}\end{pmatrix}' = \frac{1}{1+p_1\sum_{i=0}^{H-1}p_O^I}\begin{pmatrix}p_O^{H-1}p_1\\ p_O^{H-2}p_1\\ p_O^{H-3}p_1\\ \vdots\\ p_O^2p_1\\ p_Op_1\\ p_1\\ 1-p_1\sum_{i=0}^{H-1}p_O^I\\ p_1\\ p_Op_1\\ p_O^2p_1\\ \vdots\\ p_O^{H-3}p_1\\ p_O^{H-2}p_1\\ p_O^{H-1}p_1\end{pmatrix}'$	$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-2}\\ s_{H-1}\\ s_H\\ s_{H+1}\\ s_{H+2}\\ s_{H+3}\\ s_{H+4}\\ \vdots\\ s_{2H-1}\\ s_{2H}\\ s_{2H+1}\end{pmatrix}' = \frac{1}{1+p_1\sum_{i=0}^{H-1}\gamma^I}\begin{pmatrix}\gamma^{H-1}p_1\\ \gamma^{H-2}p_1\\ \gamma^{H-3}p_1\\ \vdots\\ \gamma^2p_1\\ \gamma p_1\\ p_1\\ 1-p_1\sum_{i=0}^{H-1}\gamma^I\\ p_1\\ \gamma p_1\\ \gamma^2p_1\\ \vdots\\ \gamma^{H-3}p_1\\ \gamma^{H-2}p_1\\ \gamma^{H-1}p_1\end{pmatrix}'$

Table 3: The closed-form expressions of the initial probabilities vectors of the synthetic and runs-rules $\bar{X}$ schemes using SSPV3 method

(a) S1 or RR1	(c) S3 or RR3	(d) S4 or RR4
$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-1}\\ s_H\\ s_{H+1}\end{pmatrix}' = \frac{1}{1+H(1-p)}\begin{pmatrix}1\\ 1-p\\ 1-p\\ \vdots\\ 1-p\\ 1-p\\ 1-p\end{pmatrix}'$	$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-2}\\ s_{H-1}\\ s_H\\ s_{H+1}\\ s_{H+2}\\ s_{H+3}\\ s_{H+4}\\ \vdots\\ s_{2H-1}\\ s_{2H}\\ s_{2H+1}\end{pmatrix}' = \frac{1}{2\sum_{i=0}^{H-1}\varphi^i+2\varphi^H(1-p)^{-1}}\begin{pmatrix}\varphi^{H-1}\\ \varphi^{H-2}\\ \varphi^{H-3}\\ \vdots\\ \varphi^2\\ \varphi\\ 1\\ 2\varphi^H(1-p)^{-1}\\ 1\\ \varphi\\ \varphi^2\\ \vdots\\ \varphi^{H-3}\\ \varphi^{H-2}\\ \varphi^{H-1}\end{pmatrix}'$	$\begin{pmatrix} s_1\\ s_2\\ s_3\\ \vdots\\ s_{H-2}\\ s_{H-1}\\ s_H\\ s_{H+1}\\ s_{H+2}\\ s_{H+3}\\ s_{H+4}\\ \vdots\\ s_{2H-1}\\ s_{2H}\\ s_{2H+1}\end{pmatrix}' = \frac{1}{2\sum_{i=0}^{H-1}\vartheta^i+\left(2\sum_{i=1}^{H-1}\vartheta^i+4\vartheta^H\right)(1-2\gamma)^{-1}}\begin{pmatrix}\vartheta^{H-1}\\ \vartheta^{H-2}\\ \vartheta^{H-3}\\ \vdots\\ \vartheta^2\\ \vartheta\\ 1\\ \left(2\sum_{i=1}^{H-1}\vartheta^i+4\vartheta^H\right)(1-2\gamma)^{-1}\\ 1\\ \vartheta\\ \vartheta^2\\ \vdots\\ \vartheta^{H-3}\\ \vartheta^{H-2}\\ \vartheta^{H-1}\end{pmatrix}'$

Table 3: (continued)

(b) S2 or RR2: $\tau = H^2 + H + 1$

$$
\begin{pmatrix}
\mathrm{s}_1 \\ \mathrm{s}_2 \\ \vdots \\ \mathrm{s}_{a-2} \\ \mathrm{s}_{a-1} \\ \mathrm{s}_a \\ \mathrm{s}_{a+1} \\ \mathrm{s}_{a+2} \\ \vdots \\ \mathrm{s}_{b-1} \\ \mathrm{s}_b \\ \mathrm{s}_{b+1} \\ \vdots \\ \mathrm{s}_{c-1} \\ \mathrm{s}_c \\ \vdots \\ \vdots \\ \vdots \\ \mathrm{s}_{l-11} \\ \mathrm{s}_{l-10} \\ \mathrm{s}_{l-9} \\ \mathrm{s}_{l-8} \\ \mathrm{s}_{l-7} \\ \mathrm{s}_{l-6} \\ \mathrm{s}_{l-5} \\ \mathrm{s}_{l-4} \\ \mathrm{s}_{l-3} \\ \mathrm{s}_{l-2} \\ \mathrm{s}_{l-1} \\ \mathrm{s}_l \\ \mathrm{s}_{l+1} \\ \mathrm{s}_{l+2} \\ \mathrm{s}_{l+3} \\ \mathrm{s}_{l+4} \\ \mathrm{s}_{l+5} \\ \mathrm{s}_{l+6} \\ \mathrm{s}_{l+7} \\ \mathrm{s}_{l+8} \\ \mathrm{s}_{l+9} \\ \mathrm{s}_{l+10} \\ \mathrm{s}_{l+11} \\ \vdots \\ \vdots \\ \vdots \\ \mathrm{s}_x \\ \mathrm{s}_{x+1} \\ \vdots \\ \mathrm{s}_{y-1} \\ \mathrm{s}_y \\ \mathrm{s}_{y+1} \\ \vdots \\ \mathrm{s}_{z-2} \\ \mathrm{s}_{z-1} \\ \mathrm{s}_z \\ \mathrm{s}_{z+1} \\ \mathrm{s}_{z+2} \\ \vdots \\ \mathrm{s}_{\tau-1} \\ \mathrm{s}_\tau
\end{pmatrix}' = \frac{1}{2\left(H + \lambda \sum_{i=1}^{H-1}(H-i) + \varphi(1-p)^{-1}\right)}
\begin{pmatrix}
\lambda \\ \lambda \\ \vdots \\ \lambda \\ \lambda \\ 1 \\ \lambda \\ \lambda \\ \vdots \\ \lambda \\ 1 \\ \lambda \\ \vdots \\ \lambda \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ \lambda \\ \lambda \\ \lambda \\ 1 \\ \lambda \\ \lambda \\ 1 \\ \lambda \\ 1 \\ 1 \\ 2\varphi(1-p)^{-1} \\ 1 \\ 1 \\ \lambda \\ 1 \\ \lambda \\ \lambda \\ 1 \\ \lambda \\ \lambda \\ \lambda \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ \lambda \\ \vdots \\ \lambda \\ 1 \\ \lambda \\ \vdots \\ \lambda \\ \lambda \\ 1 \\ \lambda \\ \lambda \\ \vdots \\ \lambda \\ \lambda
\end{pmatrix}'
$$

Note that Machado and Costa (2014) were the first to use the SSPV3 method to compute the initial probabilities vectors expressions for the S1 and S3 schemes. In this paper, it is pointed out that these also correspond to those of the *2-of-*(*H+1*) RR1 and RR3 schemes. In addition, the initial probabilities expressions for the S2 / RR2 and S4 / RR4 schemes are introduced; not only by using SSPV3 method, by also illustrating how to correctly use SSPV1 and SSPV2 methods to calculate the steady-state initial probability vectors.

3. Average run-length vectors

Since we opted to use the SSPV3 method moving forward, here, we derive the closed-form expressions of the $\boldsymbol{ARL}_{(\tau\times 1)}$, for each of the schemes. This procedure is conducted recursively and based on the patterns of these expressions as H increases, we use mathematical or complete induction to formulate the general form of the *ARL* vectors – these are shown in Table 4, Panel (a) to (c), for the S1 / RR1, S3 / RR3 and S4 / RR4 schemes; where

$$\mathcal{F}_1 = 1 - p_O - p_O^H + p_O^{H+1},$$

$$\mathcal{F}_2 = 1 - p_O - p_A p_O^H - p_D p_O^H - p_A p_D \sum_{i=0}^{2H-1} p_O^i,$$

$$\mathcal{F}_3 = 1 - p_A\left(p_C + p_D + p_D p_C \sum_{i=0}^{H-1} p_C^i\right) - p_B\left(1 + p_D \sum_{i=0}^{H-1} p_C^i\right) - p_C - p_A p_B^H\left(1 + p_D \sum_{i=0}^{H-1} p_C^i\right) - p_D p_C^H - p_A \sum_{j=1}^{H-1} p_B^j \left(p_C + p_D + p_D p_C \sum_{i=0}^{H-1} p_C^i\right).$$

Note that unlike the S1 / RR1, S3 / RR3 and S4 / RR4 schemes, the $\boldsymbol{ARL}_{(\tau\times 1)}$ of the S2 / RR2 scheme does not have a recursive pattern. We believe this is partly caused by the dimension of the TPMs as shown in Equation (6) of Part 1 of this work. That is, while those of the other schemes increase in a linear manner, the dimension of the S2 / RR2 schemes increase in a quadratic manner.

Table 4: The closed-form expressions of the *ARL* vectors of the synthetic and runs-rules $\bar{X}$ schemes

(a) RR1 or S1	(b) RR3 or S3	(c) RR4 or S4
$\begin{pmatrix}\varsigma_1(\delta)\\ \varsigma_2(\delta)\\ \varsigma_3(\delta)\\ \varsigma_4(\delta)\\ \vdots\\ \varsigma_{H-2}(\delta)\\ \varsigma_{H-1}(\delta)\\ \varsigma_H(\delta)\\ \varsigma_{H+1}(\delta)\end{pmatrix} = \frac{1}{\mathcal{F}_1}\begin{pmatrix} 2-p_O^H\\ 1\\ 1+p_O^{H-1}-p_O^H\\ 1+p_O^{H-2}-p_O^H\\ \vdots\\ 1+p_O^4-p_O^H\\ 1+p_O^3-p_O^H\\ 1+p_O^2-p_O^H\\ 1+p_O-p_O^H\end{pmatrix}$	$\begin{pmatrix}\varsigma_1(\delta)\\ \varsigma_2(\delta)\\ \vdots\\ \varsigma_{H-3}(\delta)\\ \varsigma_{H-2}(\delta)\\ \varsigma_{H-1}(\delta)\\ \varsigma_H(\delta)\\ \varsigma_{H+1}(\delta)\\ \varsigma_{H+2}(\delta)\\ \varsigma_{H+3}(\delta)\\ \varsigma_{H+4}(\delta)\\ \varsigma_{H+5}(\delta)\\ \vdots\\ \varsigma_{2H}(\delta)\\ \varsigma_{2H+1}(\delta)\end{pmatrix} = \frac{1}{\mathcal{F}_2}\begin{pmatrix} \left(1+p_A p_O\sum_{i=0}^{H-2} p_O^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_O^i\right)\\ \left(1+p_A p_O^2\sum_{i=0}^{H-3} p_O^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_O^i\right)\\ \vdots\\ \left(1+p_A p_O^{H-3}\sum_{i=0}^{2} p_O^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_O^i\right)\\ \left(1+p_A p_O^{H-2}\sum_{i=0}^{1} p_O^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_O^i\right)\\ (1+p_A p_O^{H-1})\left(1+p_D\sum_{i=0}^{H-1} p_O^i\right)\\ 1+p_D\sum_{i=0}^{H-1} p_O^i\\ \left(1+p_A\sum_{i=0}^{H-1} p_O^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_O^i\right)\\ 1+p_A\sum_{i=0}^{H-1} p_O^i\\ (1+p_D p_O^{H-1})\left(1+p_A\sum_{i=0}^{H-1} p_O^i\right)\\ \left(1+p_D p_O^{H-2}\sum_{i=0}^{1} p_O^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_O^i\right)\\ \left(1+p_D p_O^{H-3}\sum_{i=0}^{2} p_O^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_O^i\right)\\ \vdots\\ \left(1+p_D p_O^2\sum_{i=0}^{H-3} p_O^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_O^i\right)\\ \left(1+p_D p_O\sum_{i=0}^{H-2} p_O^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_O^i\right)\end{pmatrix}$	$\begin{pmatrix}\varsigma_1(\delta)\\ \varsigma_2(\delta)\\ \vdots\\ \varsigma_{H-3}(\delta)\\ \varsigma_{H-2}(\delta)\\ \varsigma_{H-1}(\delta)\\ \varsigma_H(\delta)\\ \varsigma_{H+1}(\delta)\\ \varsigma_{H+2}(\delta)\\ \varsigma_{H+3}(\delta)\\ \varsigma_{H+4}(\delta)\\ \varsigma_{H+5}(\delta)\\ \vdots\\ \varsigma_{2H}(\delta)\\ \varsigma_{2H+1}(\delta)\end{pmatrix} = \frac{1}{\mathcal{F}_3}\begin{pmatrix} \left(1+p_A p_B\sum_{i=0}^{H-2} p_B^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_C^i\right)\\ \left(1+p_A p_B^2\sum_{i=0}^{H-3} p_B^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_C^i\right)\\ \vdots\\ \left(1+p_A p_B^{H-3}\sum_{i=0}^{2} p_B^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_C^i\right)\\ \left(1+p_A p_B^{H-2}\sum_{i=0}^{1} p_B^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_C^i\right)\\ (1+p_A p_B^{H-1})\left(1+p_D\sum_{i=0}^{H-1} p_C^i\right)\\ 1+p_D\sum_{i=0}^{H-1} p_C^i\\ \left(1+p_A\sum_{i=0}^{H-1} p_B^i\right)\left(1+p_D\sum_{i=0}^{H-1} p_C^i\right)\\ 1+p_A\sum_{i=0}^{H-1} p_B^i\\ (1+p_D p_C^{H-1})\left(1+p_A\sum_{i=0}^{H-1} p_B^i\right)\\ \left(1+p_D p_C^{H-2}\sum_{i=0}^{1} p_C^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_B^i\right)\\ \left(1+p_D p_C^{H-3}\sum_{i=0}^{2} p_C^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_B^i\right)\\ \vdots\\ \left(1+p_D p_C^2\sum_{i=0}^{H-3} p_C^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_B^i\right)\\ \left(1+p_D p_C\sum_{i=0}^{H-2} p_C^i\right)\left(1+p_A\sum_{i=0}^{H-1} p_B^i\right)\end{pmatrix}$

4. Average run-length expressions

The *SSARL* is the product of the initial probabilities vectors in Table 3 and the $\boldsymbol{ARL}$s in Table 4, that is, $SSARL(\delta) = \mathbf{s}_{1\times\tau} \cdot \boldsymbol{ARL}_{\tau\times 1}(\delta)$ yields the closed-form expressions that are given by:

$$\text{RR1 or S1:} \quad \frac{1}{1+H(1-p)}\varsigma_1(\delta) + \frac{1-p}{1+H(1-p)}\sum_{i=2}^{H+1}\varsigma_i(\delta), \tag{10a}$$

$$\text{RR2 or S2:} \quad \mathbf{s}_{1\times(H^2+H+1)} \cdot \left(\mathbf{I} - \mathbf{Q}_{\left((H^2+H+1)\times(H^2+H+1)\right)}(\delta)\right)^{-1} \cdot \mathbf{1}, \tag{10b}$$

$$\text{RR3 or S3:} \quad s_{H+1}^{\#}\varsigma_{H+1}^{\#}(\delta) + \sum_{i=1}^{H} s_i^{\#}\left(\varsigma_i^{\#}(\delta) + \varsigma_{(2H+2)-i}^{\#}(\delta)\right), \tag{10c}$$

$$\text{RR4 or S4:} \quad s_{H+1}\varsigma_{H+1}(\delta) + \sum_{i=1}^{H} s_i\left(\varsigma_i(\delta) + \varsigma_{(2H+2)-i}(\delta)\right). \tag{10d}$$

To avoid confusion between the S3 / RR3 and S4 / RR4 expressions, as they have similar structure but different values, from here onwards, the hash tag (i.e. '#') is used for the S3 / RR3 *SSARL* expressions. As explained in Section 4, the closed-form expression of the $\boldsymbol{ARL}_{\tau\times 1}(\delta)$ for the S2 / RR2 schemes does not exist and thus we need to use Equation (10b) directly for each different value of *H* by using some sophisticated statistical software.

Although not shown here, due to space limitation, it is observed that, for each *H*, multiplying the SSPV1, SSPV2 and SSPV3 vectors with their corresponding *ARL* vectors in Section 4, yields *SSARL* values that are approximately equal to each other – this was illustrated in Shongwe and Graham (2017) for the S1 scheme. Thus, it follows that as long as a user of the synthetic or runs-rules $\bar{X}$ scheme selects one of the SSPV methods and uses it throughout, the resulting conclusion will be the same.

5. Other run-length performance measures

As an extension to Section 6 of Part 1 of this work, in the steady-state mode, the EWRL expressions for the four types of synthetic and runs-rules $\bar{X}$ charts are given by:

$$\text{RR1 or S1:} \quad \int_{\delta_{min}}^{\delta_{max}} w(\delta) \times \left(\frac{1}{1+H(1-p)}\varsigma_1(\delta) + \frac{(1-p)}{1+H(1-p)}\sum_{i=2}^{H+1}\varsigma_i(\delta)\right) \times f(\delta)\, d\delta,$$

RR2 or S2: $$\int_{\delta_{min}}^{\delta_{max}} w(\delta) \times \left(\mathbf{s}_{1\times(H^2+H+1)} \cdot \left(\mathbf{I} - \mathbf{Q}_{\left((H^2+H+1)\times(H^2+H+1)\right)}(\delta) \right)^{-1} \cdot \mathbf{1} \right) \times f(\delta)\, d\delta,$$

RR3 or S3: $$\int_{\delta_{min}}^{\delta_{max}} w(\delta) \times \left(s^{\#}_{H+1}\varsigma^{\#}_{H+1}(\delta) + \sum_{i=1}^{H} s^{\#}_{i} \left(\varsigma^{\#}_{i}(\delta) + \varsigma^{\#}_{(2H+2)-i}(\delta) \right) \right) \times f(\delta)\, d\delta,$$

RR4 or S4: $$\int_{\delta_{min}}^{\delta_{max}} w(\delta) \times \left(s_{H+1}\varsigma_{H+1}(\delta) + \sum_{i=1}^{H} s_{i} \left(\varsigma_{i}(\delta) + \varsigma_{(2H+2)-i}(\delta) \right) \right) \times f(\delta)\, d\delta.$$

6. Concluding remarks

This paper provides the steady-state closed-form expressions of the four different types of the *2-of-*(*H+1*) synthetic and runs-rules $\bar{X}$ monitoring schemes. Since some authors are not familiar with Markov chain and / or Monte Carlo simulation, the steady-state expressions derived here will ease the implementation as well as further academic research in the area of Shewhart-type synthetic and runs-rules $\bar{X}$ monitoring schemes.

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