Role of Higher-Order Terms in Local Piston Theory

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I. Introduction

The use of second- and third-order classical piston theory [1] (CPT) is commonplace, with the role of the higher-order terms being well understood [2]. The advantages of local piston theory (LPT) relative to CPT have been demonstrated previously [3]. Typically, LPT has been used to perturb a mean-steady solution obtained from the Euler equations, and recently, from the Navier-Stokes equations [4]. The applications of LPT in the literature have been limited to first-order LPT [5–7]. The reasoning behind this has been that the dynamic linearization used assumes small perturbations. The present note clarifies the role of higher-order terms in LPT. It is shown that second-order LPT makes a non-zero contribution to the normal-force prediction, in contrast to second-order CPT.

II. Methodology

The simplest case of an inclined flat-plate in supersonic flow is considered in order to eliminate thickness effects. Furthermore, only steady perturbations are considered to eliminate dependencies on reduced frequency. This has been illustrated in Fig. 1. In the computations performed, a nominal Mach number of $M_\infty = 3$ is used. To facilitate comparison between CPT and LPT, the generalized formulation for piston theory of [8] is used. The pressure coefficient due to piston theory is given

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up to third-order by

$$C_p = C_{p(cyl)} + \frac{p_{cyl}}{p_\infty} \frac{2}{M_\infty^2} \left[c_1 \epsilon + c_2 \epsilon^2 + c_3 \epsilon^3 \right] , \tag{1}$$

where subscript "cyl" denotes cylinder conditions, subscript "∞" denotes freestream conditions, C_p is the pressure coefficient, p is pressure, M is Mach number, c_i are the coefficients due to various piston theory formulations [9], and ϵ is the piston downwash-Mach number, given by

$$\epsilon = M_{cyl} \tan \theta , \tag{2}$$

where θ is the inclination of plate surface to the cylinder orientation. In CPT, the cylinder is oriented normal to the freestream velocity. In LPT, the cylinder is oriented normal to the mean-steady surface. Suppose that LPT is applied about a mean-steady solution obtained at an angle-of-attack α_0. Consider a perturbation δ to the angle-of-attack, such that the incidence following a perturbation is given by

$$\alpha = \alpha_0 + \delta . \tag{3}$$

This relationship may be used to define the perturbation δ for a given incidence α in both CPT and LPT. The following relationship is then obtained for θ:

$$\theta = \mp \delta , \tag{4}$$

where the upper symbol is for the upper surface. The angles have been depicted in Fig. 1, in which the cylinder orientation has been defined as in LPT.

Returning to Eqs. (1–2), the cylinder conditions remain to be defined. In LPT, these are typically different between the lower surface (subscript "Lc") and the upper surface (subscript "Uc"). The definition of terms in the equations in this note for CPT and LPT is given in Table 1. The cylinder conditions in LPT are defined by the exact solution at the mean-steady incidence (α_0). For the upper surface (expansion), the exact solution is obtained using the Prandl-Meyer relations (subscript "PM"). For the lower surface (compression), the exact solution is obtained using the oblique shock relations (also known as the Rankine-Hugoniot equations, subscript "RH"). These familiar relations are not repeated here for brevity's sake.

Table 1: Definition of terms.

Term	CPT	LPT
α_0	0°	$\alpha_0\ (\neq 0^\circ)$
δ	α	$\alpha - \alpha_0$
θ	$\mp\alpha$	$\mp(\alpha - \alpha_0)$
M_{Lc}	M_∞	$M_{RH}(\alpha_0)$
M_{Uc}	M_∞	$M_{PM}(\alpha_0)$
p_{Lc}	p_∞	$p_{RH}(\alpha_0)$
p_{Uc}	p_∞	$p_{PM}(\alpha_0)$

The prediction quantities of interest will then be taken as the normal-force coefficient C_N and its aerodynamic stiffness, given by $\partial C_N/\partial\delta$. The prediction of C_N by CPT is shown in Fig. 2. The force prediction due to LPT is shown in Fig. 3. Also of interest are C_P and its derivatives. All these quantities are obtained through manipulation of Eq. (1). In taking derivatives with respect to δ, the cylinder conditions are assumed to remain constant. The following results for the pressure coefficient are obtained:

$$C_p = C_{p(cyl)} + \left.\frac{\partial C_p}{\partial\delta}\right|_{\delta=0}\delta + \left.\frac{\partial^2 C_p}{\partial\delta^2}\right|_{\delta=0}\delta^2 + \dots\,, \tag{5}$$

$$\left.\frac{\partial C_p}{\partial\delta}\right|_{\delta=0} = \mp\frac{p_{cyl}}{p_\infty}\frac{2}{M_\infty^2}c_1 M_{cyl}\,, \tag{6}$$

$$\left.\frac{\partial^2 C_p}{\partial\delta^2}\right|_{\delta=0} = \frac{p_{cyl}}{p_\infty}\frac{4}{M_\infty^2}c_2 M_{cyl}^2\,. \tag{7}$$

The derivatives of C_p are shown in Figs. 4–7 for the upper and lower surfaces. The values have been computed with $\alpha_0 = \alpha$ (i.e., $\delta = 0^\circ$) to depict the accuracy of LPT in the immediate vicinity of the mean-steady solution. The corresponding results for the normal-force coefficient may be written as

$$C_N = C_{p(L)} - C_{p(U)} \tag{8}$$

$$C_N = C_{N(cyl)} + \left.\frac{\partial C_N}{\partial\delta}\right|_{\delta=0}\delta + \left.\frac{\partial^2 C_N}{\partial\delta^2}\right|_{\delta=0}\delta^2 + \dots\,, \tag{9}$$

$$\left.\frac{\partial C_N}{\partial\delta}\right|_{\delta=0} = \frac{2c_1}{p_\infty M_\infty^2}\left(p_{Lc}M_{Lc} + p_{U_c}M_{Uc}\right)\,, \tag{10}$$

$$\left.\frac{\partial^2 C_N}{\partial \delta^2}\right|_{\delta=0} = \frac{4c_2}{p_\infty M_\infty^2}\left(p_{Lc}M_{Lc}^2 - p_{U_c}M_{Uc}^2\right) , \tag{11}$$

where for simplicity's sake, it has been assumed that the coefficients c_i are constant and equal on both surfaces. This is the case for Lighthill's [10] coefficients $c_1 = 1$, $c_2 = (\gamma + 1)/4$, and $c_3 = (\gamma + 1)/12$, which have been used in the computations in this note. The derivatives of C_N are shown in Figs. 8 and 9.

III. Results and Discussion

A number of insightful conclusions may be drawn from inspection of Eqs. (9–11), which have been plotted in Figs. 3, 8 and 9, respectively. The linear aerodynamic stiffness is given by the Eq. (10), which receives modeling contributions from only the first-order piston theory term. Equation (11) gives the quadratic aerodynamic stiffness, and receives modeling contributions from only the second-order piston theory term. This highlights the role of the second-order piston theory term as introducing nonlinearity to the aerodynamic stiffness. At this point, the distinction between CPT and LPT yields further differences. In CPT, the cylinder conditions are equal on the upper and lower surfaces (being equal to freestream conditions). This yields the following results for CPT:

$$\left.\frac{\partial C_N}{\partial \delta}\right|_{\delta=0} = \frac{4c_1}{M_\infty} , \quad \left.\frac{\partial^2 C_N}{\partial \delta^2}\right|_{\delta=0} = 0 , \tag{12}$$

with the well-known result recovered that second-order CPT makes no contribution to the normal force. The difference between first-order and second-order CPT noted in Fig. 2 only arises due to vacuum pressure being reached on the upper surface. This is in contrast to the LPT result, which is given by Eq. (11) and is non-zero for $\alpha_0 \neq 0$. That is to say, the second-order term in LPT introduces a quadratic aerodynamic stiffness term which is absent in CPT. Both this term and the linear stiffness term due to LPT are functions of the mean-steady solution, and therefore vary with α_0. This shown in Figs. 8 and 9.

The accuracy of LPT in modeling the linear component of aerodynamic stiffness in the immediate vicinity of the mean-steady solution (at α_0) is effectively determined by the first-order term. This accuracy may be assessed by comparing the LPT result with the exact result at any given α in Figs. 4, 6 and 8. From Eq. [10] it is seen that the second-order LPT term plays no role in the accuracy of the linear stiffness prediction.

The utility of the second-order LPT term may be illustrated by considering Eqs. [9] and [11] along with Figs. 3, 8 and 9. For a given mean-steady solution (depicted by the dot in Fig. 3), a constant linear aerodynamic stiffness is predicted by LPT. Referring to Eq. [9] and Fig. 3 it is seen that if the aerodynamic stiffness is nonlinear, the accuracy of LPT will progressively degrade with increasing perturbation size. However, if the second-order LPT term is included, then LPT may be applied further away from the mean-steady solution. The accuracy of LPT away from the mean-steady solution is then determined by how well the LPT prediction at α_0 for the second-derivative of normal force corresponds to the exact solution. The third-order term serves a similar role in extending the range in which LPT may be applied around the mean-steady solution.

From Fig. 4, it is seen that LPT gives poor prediction of even the linear stiffness component as shock detachment is approached. This is also reflected in the nonlinear stiffness prediction shown in Fig. 5 This error in prediction may be attributed to two main sources. The first source is the effect of entropy behind the oblique shock. The contribution of entropy is increases with the shock angle. It may thus be expected to influence the accuracy of LPT in the limit of shock-detachment at high Mach numbers and in general in the limit of lower supersonic Mach numbers.

The second source of error in LPT relates to the approximation of the fluid as plane slabs which are independent of one another. As noted in [11], the exact relations for oblique shocks and Prandtl-Meyer expansion may accurately be reduced to piston-like formulation provided the cylinder orientation is specified correctly. In the case of oblique-shock flows, the cylinder should be oriented perpendicular to the shock. In the limit of low Mach numbers or of shock-detachment, the angle subtended between the surface and the shock increases. This subsequently increases the error in the LPT assumption of the cylinder oriented normal to the surface. A similar error arises in the limit of low Mach numbers for expansion flows.

These remarks on the accuracy of LPT are intended to inform the discussion of the present consideration of its higher-order terms. A fuller treatment of the Mach and α_0 dependence of LPT's accuracy is outside the scope of the present note, and is reserved for further work.

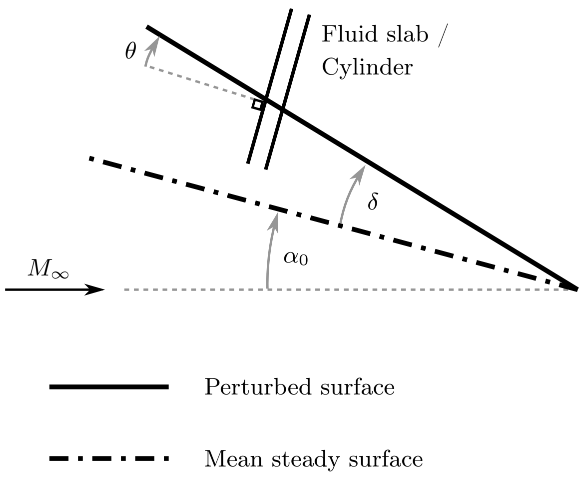

Fig. 1: Flat plate configuration.

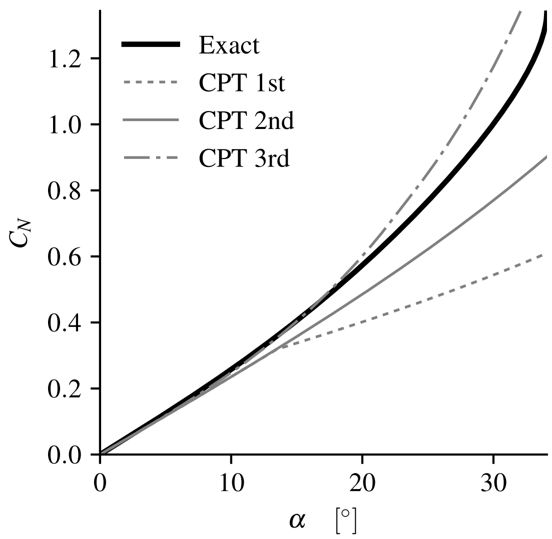

Fig. 2: Normal-force coefficient prediction by CPT.

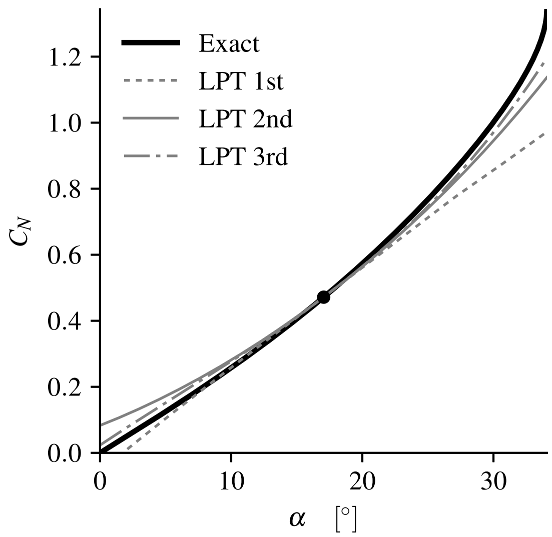

Fig. 3: Normal-force coefficient prediction by LPT with mean-steady solution indicated by dot.

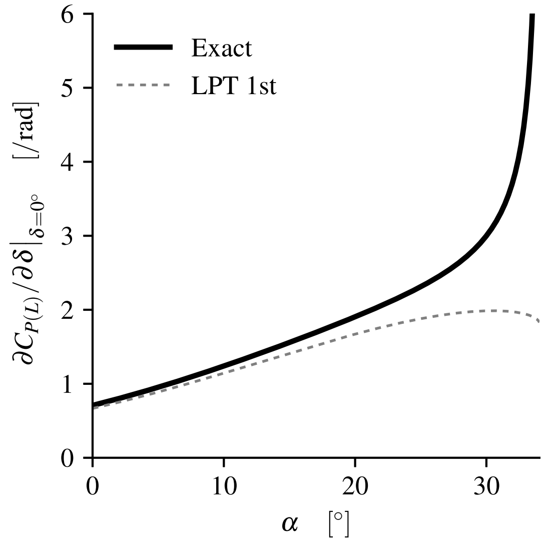

Fig. 4: First-derivative of pressure-coefficient on the lower (compression) surface.

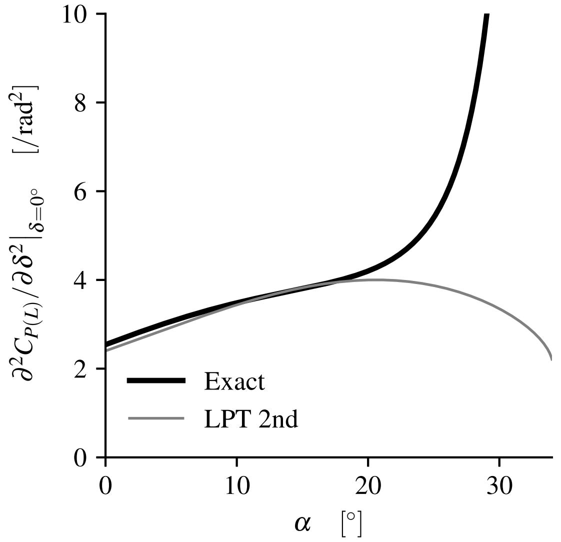

Fig. 5: Second-derivative of pressure-coefficient on the lower (compression) surface.

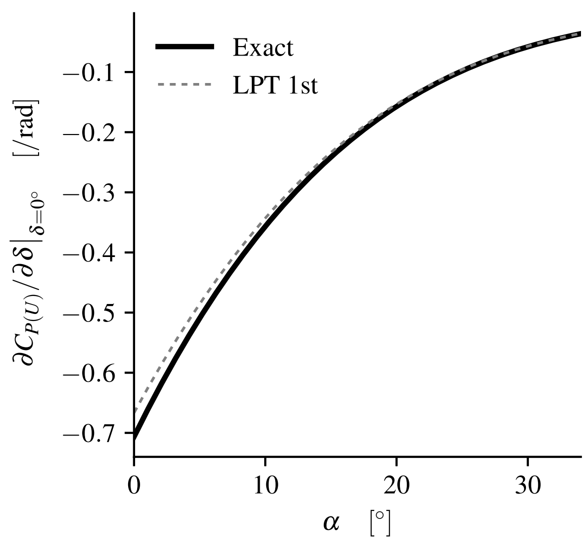

Fig. 6: First-derivative of pressure-coefficient on the upper (expansion) surface.

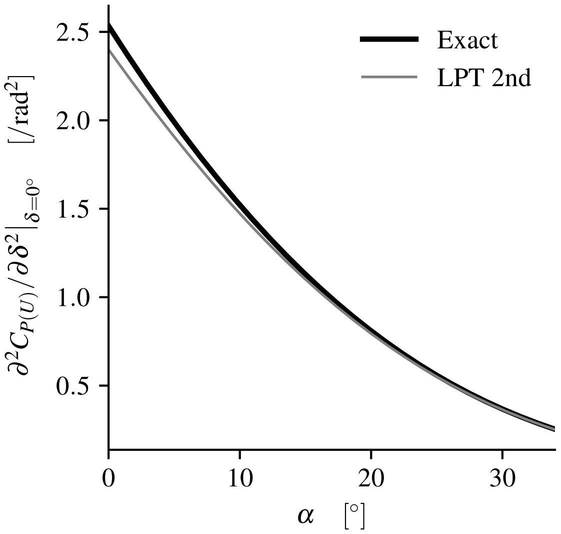

Fig. 7: Second-derivative of pressure-coefficient on the upper (expansion) surface.

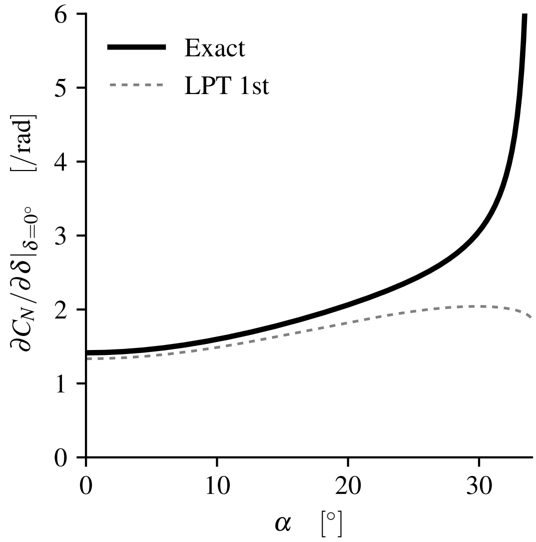

Fig. 8: First-derivative of normal-force coefficient.

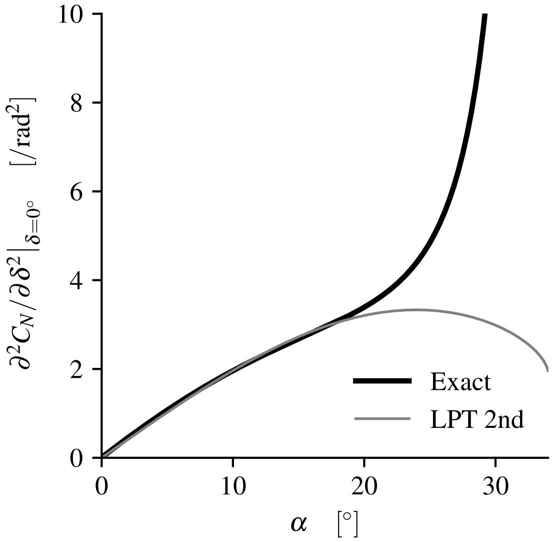

Fig. 9: Second-derivative of normal-force coefficient.

IV. Conclusions

An analysis of the role of higher-order terms in piston theory in predicting aerodynamic stiffness has been conducted. The familiar result of zero-contribution from second-order classical piston theory (CPT) has been recovered. It has been shown that second-order local piston theory (LPT) makes a non-zero contribution to the normal-force coefficient and the aerodynamic stiffness. In particular, it contributes a quadratic aerodynamic stiffness term. The utility of higher-order terms in allowing for LPT to be applied to larger perturbations about the mean-steady solution has been demonstrated for a simple test case. It is expected that this utility should hold in flows which do not exhibit strong aerodynamic non-linearities. A brief discussion on the sources of error in LPT has also been given.

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