

On the use of Lévy processes in option pricing

by

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Submitted in partial fulfillment of the requirements for the degree

Magister Scientiae

In the Department of Statistics

In the Faculty of Natural and Agricultural Sciences

University of Pretoria

November 2017



UNIVERSITEIT VAN PRETORIA
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I, *Clemence Rangarirai Kwinje*, declare that this mini-dissertation (100 credits), which I hereby submit for the degree Magister Scientiae in Mathematical Statistics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Acknowledgements

I hereby acknowledge the financial assistance from the Mastercard Foundation Scholarship Fund.

I also would like thank;

- my supervisor Dr. Jaco Visagie for his unwavering support. He helped me in situations that I never anticipated that he could. I was blessed to have him as my supervisor and I hope to work with him in my further studies.
- my parents for their love, support and care.
- my friends for allowing me to nag them when my codes were not running.
- Above all, I owe this dissertation to my Lord God and Savior, without Him I would not have reached this far.

Summary

In this dissertation, we fit various financial models to observed stock prices and we calculate the option prices under each of these models. All of the models considered are based on Lévy processes, which are processes with independent and identically distributed increments. The processes are popular in finance due to their flexibility and their desirable mathematical properties. The models considered include the celebrated Black-Scholes model, under which the log-returns are assumed to be driven by a Brownian motion. Two other classes of models are included in this study, both of which are generalizations of the Black-Scholes model. The first class is the geometric Lévy process models, of which the Black-Scholes is a special case. Two specific examples within this class are considered, the two models use the normal inverse Gaussian and Meixner processes to model log-returns. The second class of model considered generalizes the Black-Scholes while modeling the passing of time using an increasing stochastic process. The two specific examples considered model time using a Pareto and a lognormal process.

The aim of this dissertation is to explore the question of which model to use in a given financial market. To this end, we fit each of the models considered to observed log-returns. Following this step we calculate the prices of options available in this market. This is done in order to compare the prices calculated under the models to the prices observed in the market. In each case the Esscher transform is used in order to calculate the equivalent martingale measure used for the calculation of the option prices. Note that this is not the approach typically employed by financial practitioners. In practice these models are often calibrated to the observed option prices, meaning that the parameters of the models are chosen so as to minimise some distance measure between the observed and calculated option prices. In this dissertation we depart from this methodology in order to determine if the models fitted to the stock prices are capable of producing realistic option prices.

When analysing the results obtained we use a two fold approach. The first step is to determine which of the models considered provides the best fit to the observed log-returns (this is done by comparing the integrated squared errors between the resulting densities and a kernel density estimate), and the second step is to compare the calculated and

observed option prices (using the root mean square error calculated between the two sets of option prices). We conclude that, surprisingly, the model that fits the stock price data best often does not provide an adequate fit to the option prices, and vice versa.

Keywords : geometric Meixner model, exponential lognormal-normal model, exponential Pareto-normal model, geometric normal inverse Gaussian model, Black-Scholes model, martingale, arbitrage, Esscher transform method, Fourier method, Lévy processes, infinitely divisible.

Chapter 1

Introduction

In this chapter an overview of, and a motivation for this study are given. This is done in Section 1.1. The objectives of this study are listed in Section 1.2. In Section 1.3, we give brief summaries of the chapters to follow.

1.1 Overview and motivation

The trade in options constitutes a substantial proportion of all trades made in a financial market. An option is a financial asset deriving its value from some underlying assets. These assets are mostly stocks and bonds. There has been an increase in the recent studies on the quantification of the risk involved in holding these options, see [2]. The calculation of option prices is a non-trivial task, a great deal of research is done on this topic, see [1]. Various types of options are available. In this report we focus on the European call option. This option type, discussed in Section 2.3.1, is one of the most basic options available.

In this study, we consider various types of option price models based on the class of Lévy processes. The reason being that the Lévy processes have desirable mathematical properties, they are flexible and are often used in the financial literature. Within the class of Lévy processes, we consider most popular model which is the Black-Scholes model. This model assumes that the stock prices are from an exponential Brownian motion. Other financial models considered assume that the stock prices follow an exponential Lévy process with jumps, see [16]. These models include the geometric normal inverse Gaussian model and the geometric Meixner distribution.

There are also other models that generalize time as a stochastic process. These models that take into account the fact that the number of trades per given time-interval are not the same. They are called subordinated/time-changed models. In these models the number of trades is assumed to be a random variable which follow a particular distribution. The

models considered in this context are the exponential lognormal-normal model and the exponential Pareto-normal model.

We want to make predictions of option prices using different financial models. This is done by first estimating the parameters of the model by fitting the models to the observed stock price data. Based on these estimated parameters, we then calculate the option prices. In practice, the parameters are often attained through the method of calibration which entails minimising the discrepancy between the observed and calculated option prices. This method actually violates the economic theory because the models' parameters have specific interpretations.

Changes of probability measure form an integral part of option pricing. This is because pricing options using the real world probability measure allows for the existence of arbitrage. Arbitrage is not allowed in any market because it enables market participants to have a chance of enjoying risk-less profit. Therefore a new probability measure has to be derived so as to curb arbitrage. In this study we look at one of the methods of changing the probability measure called the Esscher transform method. We will also use two different methods of calculating option prices. These are numerical integration and the Monte Carlo simulation method.

In this study, we are going to compare models in two-folds. The first fold is to see which financial model fits the observed stock price data best. The second one is to see which model mimics the observed option prices closest. These ranks are subjective to the form of the data set being used. If the stock price data set being used is normally distributed then the Black-Scholes model will be the best one. If the data is non-normal, then other models will perform better than the Black-Scholes model.

We also want to investigate if the integrated square errors (ISEs) match up with the root mean square errors (RMSEs) of these financial models. This means that we want to see if it is always true that if the financial model fits the stock price data well then it will also predict the option prices very well. In other words, we want to investigate the existence of a discrepancy between the model fit and the option price calculation.

1.2 Objectives

The objectives of this study are to:

- Analyse and discuss different financial models.
- Discuss changes in probability measure brought about by the Esscher transform method.

- Fit the different financial models to observed stock price data.
- Calculate option prices under each of the models considered.
- Compare models to see which one fits the stock price data well.
- Compare models to see which one mimics the observed option prices well.

1.3 Outline of the dissertation

Chapter 2 : Financial markets

In this chapter we define the most important financial concepts used in the remainder of the study. We also discuss the assumptions used in this dissertation. The various types of different market participants are also briefly touched upon.

Chapter 3 : Lévy processes

A definition and properties of a Lévy process are discussed in this chapter. Three types of Lévy processes are discussed. These are the Brownian motion, normal inverse Gaussian and the geometric Meixner processes.

Chapter 4 : Option pricing models

Different option pricing models will be analysed and discussed in this chapter. Explanations of how the models' parameters change when we change the probability measure, are also included here. The disadvantages of using some of the models are also discussed.

Chapter 5 : Empirical results

Different option pricing models are fitted to the observed stock price data. The parameters of the models are also estimated. Option prices are calculated using these different models.

Chapter 6 : Conclusion

Comparison and analysis of the models to see which model fits the observed stock prices better, is done in this chapter. This is done by using the integrated squared error as our ranking criterion. Option prices calculated using these models are also compared using the root mean square error to see which model best mimics the observed option prices.

Chapter 2

Financial markets

2.1 A financial market

A financial market is a business arena where assets, which are monetary in nature or can easily be converted to money are traded, see [9]. These assets include stocks, bonds and derivatives, such as bonds. The definitions of these assets will be discussed shortly after the definition of the filtered probability space in which we will be working throughout. Below, we are mainly concentrating on stocks and bonds since derivatives are derived from the manipulation of these assets.

Throughout this study, we shall assume a filtered probability space; $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. \mathcal{F}_t is a filtration that is right-continuous and non-decreasing. Two processes are defined on this filtered probability space, a stock process and a bond process. These processes are $S = \{S_t : t \geq 0\}$ and $B = \{B_t : t \geq 0\}$, respectively. The process S represents the monetary value of a single share or a bundle (multiple shares) in a public traded investment. The filtration \mathcal{F}_t , is the sigma algebra generated by S_t , see [9]. The definition of sigma algebra is as follows:

\mathcal{F} is a σ – algebra if the following conditions are met:

1. $\Omega \in \mathcal{F}$. Ω comprises of all possible events in the space.
2. For every event $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$.
3. If $A_1, A_2, A_3, \dots \in \mathcal{F}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$.

Thus, \mathcal{F}_t contains the history of the process from time 0 until time t . We assume that there exists a time $T^* > t$ such that $\mathcal{F}_{T^*} = \mathcal{F}$ since most investments have a finite lifetime, see [19]. This means that \mathcal{F} contains the history of the whole process.

Continuing with our definition of the filtered probability space, we have P which is the probability measure that determines the behavior of our stock price process, S_t . The measure P is called the objective probability measure. This measure is crucial because it is used in the determination of the probability distribution function of S_t .

The process B_t is a risk-free financial security that pays a single bulk of money at a future date. A risk-free financial asset is one that has a probability of one that at the end of the specified period, it will accumulate profit in the form of interest gained. The value of the bond at time zero is equals to one, that is, $B_0 = 1$. This value is assumed to increase exponentially at the risk free interest rate, $r > 0$

$$B_t = e^{rt}, r > 0,$$

where $t \geq 0$. The interest rate r , is assumed to be constant, positive and known throughout. The relationship between the value of the bond at time $t > 0$ and its value at time $t = 0$ analysed through the manipulation of the risk-free interest rate.

2.2 Assumptions made in the financial market

In order to find a tractable model for the calculation of option prices that mimic what is happening in the real financial market, simplifying assumptions are needed. These are listed below:

1. Absence of market frictions:

This implies that there are no transaction costs, no bid-ask spread, no taxes, no margin requirement. The bid-ask spread is the difference between the amount that the seller of an asset is willing to accept for his asset (the ask price) and the amount the buyer is prepared to pay for the asset (the bid price), see [18]. In a real financial market, in order for a market participant to be allowed to start trading, he or she is required to have a certain amount of money in his account. This is called margin requirement. This is done to reduce the risk of default faced by both parties to a trade. In reality all financial markets has frictions but for simplicity, it is assumed that there is no type of any market friction, see [18].

2. Absence of default risk:

Default risk is one of the risks that is inherent when trading in any financial market. It is the risk that is caused by macro-economics adverse incidents, like economic depressions, see [18]. For instance, a depression can lead to traders not being able to

honor their payments which may adversely affect other market participants. So the assumption of the absence of default risk enables our financial market to be simple and easy to understand.

3. Absence of price increase due to increase in demand:

This means that a single trader can trade an unlimited amount of a single type of a security without causing a change in the security's price. This assumption is satisfied to a greater extent when working in a very large financial market, see [18]. In real markets, however, this does not hold even for large markets.

4. Rational subgroups:

This means that market participants are individuals who act rationally. For instance, investors are assumed to prefer a gain to a loss, see [18].

5. Short selling is allowed:

This means that it is possible for market participants to hold negative quantities of assets in their portfolio, see [5].

6. Perfect divisibility of assets:

Traders in the financial market are allowed to hold any number of assets. This means that market participants are allowed to hold not only integer values of assets but also fractions of assets, see [5].

7. There will be no dividends payable to shareholders.

Since we now have a market that is governed by the above assumptions, we can now give a formal definition of a portfolio. We shall use portfolios in the calculation of the option prices. We can buy stocks and bonds which has the value that is equivalent to the payoff of the derivative at time T , see [9]. If we can build a portfolio with the same value of the derivative at time $t = T$, then the expected value of that portfolio at time $t = 0$, is the same as the value of the derivative at that time as well. This expectation is calculated under the risk free probability measure.

Definition of a portfolio

A collection of different assets held by a single market participant is called a portfolio. A simple financial market may consist of two assets namely, bonds and stocks, see [9]. A particular portfolio can be defined as a combination of stocks and bonds held by a trader

in the market at a specific point in time. We shall use the following notation to denote a portfolio at time t , which is a 2 – dimensional process given as

$$p_t = (p_t^B, p_t^S),$$

where p_t^B and p_t^S represents the number of units held at time t of both bonds and stocks respectively, see [5]. In order to attain the value of the portfolio p_t , it is necessary to multiply the number of units of bonds by its price at time t , B_t , and to add the number of units of stock at time t multiplied by the stock price at that time, S_t . The value of the portfolio is given as

$$V_t^p = p_t^B B_t + p_t^S S_t.$$

An important concept that is derived from the definition above is the one of a self-financing portfolio. A self-financing portfolio strategy is one that does not require any additional funds when buying assets once the first investment is made, see [18]. A portfolio is said to be self-financing if

$$dV_t^p = p_t^B dB_t + p_t^S dS_t.$$

This means that if a portfolio is self-financing then the change in the portfolio's value is solely due to the change in the value of the assets that make up the portfolio, see [5]. Self-financing strategies are going to be of interest when calculating the price of any derivative. Throughout this study, the terms portfolio and strategy are going to be used interchangeably.

Definition of a derivative

A derivative is a financial instrument whose monetary value is dependent on the values of the other more basic assets, and other underlying variables, see [18]. The underlying variables that are used to construct derivatives are assets or a combination of assets being traded in the financial market. These assets includes bonds and stocks and other things. There are different kinds of derivatives but our main focus will be on options.

2.3 Options

There are many different kinds of options. Our main focus is on European options. The latter comprises of two different types namely, call and put option. There are two distinct parties involved in the writing of an option. These parties are the *buyer* and the *seller*, the latter is also called the *writer* of the option, see [18]. Formal definitions of European

call and put options are given below. Let K be defined as the strike price, which is the money to be paid for the European option at maturity time T . This price is stipulated at the beginning of the contract. Let S_T be the stock price at the time of the maturity of the option.

There are many groups of options. These classifications are based on whether or not $S_t > K$, where K is amount agreed on, to buy the option. The initial one is the group of options that brings an inflow of money. These are called *in the money*. The second group is the one with options that break even, thus they bring a zero gain. They are called *at the money*. The last group is the one with options that actually brings losses. They are called *out of money*, see [18].

2.3.1 European call options

A European call option is a contract that gives the holder the right (but not the obligation) to buy a predetermined number of units of specific underlying assets at the maturity time namely, T , at a price that is fixed at the outset, K , see [5]. The name *call* emanates from the fact that the holder of the contingent claim has the right to buy the asset from the seller, which is, to call the asset from the seller of the contract, see [18]. The time of the payoff of the European call option is fixed since the option can be exercised at the time to maturity date of the option. The payoff is given as follows

$$[S_T - K]^+ = \begin{cases} S_T - K, & S_T > K, \\ 0, & S_T \leq K. \end{cases}$$

2.3.2 European put options

A European put option is a contract that gives the holder the right (but not the obligation) to sell a predetermined number of units of specific underlying assets at the maturity time T , at a price that is fixed at the start of the contract, K , see [5]. The name *put* emanates from the fact that the holder of the contingent claim has the right to sell the asset, which is, to put the asset to the buyer of the contract, see [18]. The payoff is given as follows

$$[K - S_T]^+ = \begin{cases} K - S_T, & S_T \leq K, \\ 0, & S_T > K. \end{cases}$$

Let the price of the options held at a particular time t denoted by I_t . This means that all the prices of options are processes that depends on time. The introduction of this

new asset with price Π_t , leads to the extension of our financial market to contain three classes of assets. These are stocks, bonds and options. The new definition of the portfolio will be extended as follows

$$p_t = (p_t^{\Pi}, p_t^S, p_t^B),$$

where p_t^{Π} is the number of options that are contained in a certain portfolio of a given market participant at time t . The value of the portfolio needs to be amended because of the addition of another asset. The new value of the portfolio will now be as follows

$$V_t^p = p_t^{\Pi} \Pi_t + p_t^B B_t + p_t^S S_t.$$

The definition of a self-financing strategy will also change. It will now be characterized by the following equation,

$$dV_t^p = p_t^{\Pi} d\Pi_t + p_t^B dB_t + p_t^S dS_t.$$

The price that will be of interest is the price of the European call option at time $t = 0$, which is denoted by Π_0 . For the rest of the study, the subscript 0 will be eliminated. This leaves the prices of options at time $t = 0$, denoted just as Π .

2.4 Arbitrage and the definition of locally equivalent martingale measures

An arbitrage opportunity is a trading strategy that requires no money to construct at time $t = 0$, with a probability of losing money equal to zero, and the probability of making money at some point in the future, greater than zero, see [18]. Formally, the definition of arbitrage is as follows:

A portfolio strategy constitutes an arbitrage opportunity if;

1. the portfolio p_t is self financing.
2. the initial value of the portfolio p is zero, $V_0^p = 0$.
3. $P(V_T^p \geq 0) = 1$ and $P(V_T^p > 0) > 0$, where $T > 0$, see [19].

This implies that a market participant holding this portfolio has a probability of making money without injecting initial capital or without being exposed to any sort of risk, see [5]. Arbitrage is unrealistic and it allows traders to enjoy the possibility of unlimited risk-less

profit which is contrary to economic theory. Below is an example that shows how the elimination of arbitrage enforces a specific price for a certain derivative.

Consider a market participant who is selling a forward contract. He is required to, upon receiving of the agreed amount, deliver an asset (usually a stock) at maturity time T . The seller could borrow money amounting to S_0 (which is the stock price at time zero) and then buy the stock and hold this stock until the maturity of the contract. Upon the expiration of the contract, the seller should pay back the amount borrowed. The amount paid will include the original price as well as interest. The repaid amount at the maturity of the contract will be S_0e^{rT} . If the seller of the contract agreed to receive an amount less than S_0e^{rT} in exchange of the stock which costs S_0 at time 0, then the buyer will make a profit with a probability equal to one. This implies that the forward price has a lower bound which equates to S_0e^{rT} .

Now also, considering the context of the buyer of the forward contract. If, upon writing the contract, the buyer agrees to pay more than S_0e^{rT} then with probability equal to one, the seller makes a profit, see [5]. Therefore, again the forward price has an upper bound which is S_0e^{rT} . Let f_T denote the price of the contract above such that

$$S_0e^{rT} \leq f_T \leq S_0e^{rT},$$

Therefore, the unique arbitrage-free price of the forward contract is

$$f_T = S_0e^{rT}.$$

If a market participant strikes a different price than S_0e^{rT} , then it will be certain that someone in the market will exploit the opportunity of gaining risk-less profit, by buying and selling the derivative in large quantities. This would constitute an arbitrage opportunity. As a result, f_T is then called the arbitrage-free price, see [5].

There exists a close link between the concept of arbitrage-free and a locally equivalent martingale measure (LEMM). The definition of a locally equivalent martingale measure is provided in the next section.

2.4.1 Local equivalence

Consider two probability measures, P and P^* defined on a common probability space. These two measures of probability are said to be equivalent if and only if

$$P(G) > 0 \iff P^*(G) > 0,$$

for every measurable event G . Equivalence between the two measures P and P^* is denoted by $P \sim P^*$, see [19].

Define $P_n = P|_{\mathcal{F}_n}$ and $P_n^* = P^*|_{\mathcal{F}_n}$ to be the probability measures restricted to \mathcal{F}_n . The probability measures P and P^* are said to be locally equivalent for all $n = 1, 2, 3, \dots$, if

$$P_n(G) > 0 \iff P_n^*(G) > 0,$$

for all $G \in \mathcal{F}_n$, see [19].

Martingales

A stochastic process X_t is said to be a martingale with respect to the objective probability measure Q if and only if the following conditions are met:

1. $E_Q(|X_t|) < \infty$, for all t .
2. $E_Q(X_t | \mathcal{F}_s) = X_s$, for all $s \leq t$, see [5]. This means that the expected value of X_t conditioned on the filtration up to time s is equal to the value of the process X_s .

With the definition of a martingale and local equivalence given above, we can now join the two, to formally define a locally equivalent martingale measure as follows:

Definition of locally equivalent martingale measures

A probability measure, Q is said to be a locally equivalent martingale measure with respect to another probability measure P , if;

- Q is locally equivalent to P .
- $\frac{S_t}{B_t} = e^{-rt} S_t$ forms a Q – martingale, see [16]

The importance of this definition stems from the fact that option prices can be calculated as the expected value with respect to a locally equivalent martingale measure.

There are different methods in which we can calculate the locally equivalent martingale measure Q . We are going to use a method known as the Esscher transform. Application of the Esscher transform method gives rise to a unique locally equivalent martingale measure.

The Esscher transform

In models that are continuous, it is easy to obtain an equivalent martingale measure by changing the drift. This is not the case for models with jumps as there will be a number of

different equivalent measures obtained by altering the distribution of the jumps, see [9]. In order to obtain an equivalent martingale measure, we use the Esscher transform method defined below.

In this study, the Esscher transform method is used to construct the locally equivalent martingale measure for each of the models considered. The probability measure used uniquely identifies the density of the stochastic process at every point in time. Let f_t be the density function of the random process Y_t , under the objective probability measure P . For every $\tau \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \exp(\tau x) f_t(x) dx < \infty,$$

a unique density function can be obtained as follows

$$f_t^\tau(y) = \frac{\exp(\tau y) f_t(y)}{\int_{-\infty}^{\infty} \exp(\tau y) f_t(y) dy}. \quad (2.1)$$

Let the probability measure under which the density of Y_t is f_t^τ , be denoted by P^τ . Different probability measures are attained by changing the value of τ in equation (2.1). In order to apply this technique, we need a unique value of τ which ensures that $\exp(-rt)S_t$ forms a P^τ -martingale. A P^τ -martingale measure is formed if and only if the following equation is satisfied

$$\exp(r)\phi(-i\tau) = \phi(-i(1 + \tau)), \quad (2.2)$$

where ϕ is defined as the characteristic function of Y_t , see [16]. The characteristic function is defined as

$$\phi_y(u) = E(\exp(iuy)).$$

Equation (2.2) has a unique solution denoted as τ^* .

The Esscher transform, by definition, provides martingale probability measures. We now need to see if the probability measure obtained from the Esscher transform is equivalent to P by using the Radon-Nikodym theorem. This theorem states that, the probability measures P and Q are equivalent if the derivative $\frac{dQ}{dP}$ satisfies the following result, see [5]

$$0 < E_Q(Y_T) = E_P\left(\frac{dQ}{dP} Y_T\right) < \infty. \quad (2.3)$$

Equation (2.3) can only be positive if and only if $\frac{dQ}{dP} > 0$. Therefore, the derivative of P^τ with respect to P , given by

$$\frac{dP^\tau}{dP} = \frac{\exp(\tau y)}{\int_{-\infty}^{\infty} \exp(\tau y) f_t(y) dy}. \quad (2.4)$$

Equation (2.4) is strictly positive and bounded for all real values of y . As a result, P^τ and P are equivalent probability measures, see [5].

Since equivalence of measures implies local equivalence, the resulting probability measures can be used in the calculation of arbitrage-free option prices, see [5]. If the market is complete then, there exists only one locally equivalent martingale measure Q . In this case, the Esscher transform can be used to obtain this measure.

2.5 The calculation of European option prices

Considered below are three of the most important methods of calculating option prices which we call direct numerical integration, Monte Carlo simulation and the Fourier inversion method. Direct numerical integration is only implemented when the probability density function of the option price process or of other functions (mostly the log returns or just $\log(S_t)$ is known, see [9]. Monte Carlo simulation and the Fourier inversion method are mostly used when calculating option prices using complicated density functions.

The calculation of option prices is easier when the distribution function is known. For instance, the expected value with respect to a locally equivalent martingale measure (LEMM) Q , is the arbitrage free price of the call option. It is given as

$$\Pi = e^{-rT} E^Q [(S_T - K^+)], \quad (2.5)$$

which is the discounted value of the expected payoff.

2.5.1 Direct numerical integration

Let f_t be the density function of the logarithm of the stock process, $\log(S_t)$, for $t \geq 0$, under the measure of probability Q . If the density function at the maturity time, f_T is known then the price of a European call option can be calculated numerically as follows

$$\Pi = e^{-rT} \int_{\log(\frac{K}{S_0})}^{\infty} (Se^x - K) f_T(x) dx,$$

where T is the time when the contract expires, K is the amount agreed on to pay for the option and S_0 is the stock price at time $t = 0$.

2.5.2 Monte Carlo simulation

There are instances where the density function of the option prices process is not known or it is difficult to attain. In such situations, the Monte Carlo simulation is used to simulate S_T in equation (2.5). This means that, in order to calculate the option price, we simulate the payoffs then calculate their average. We then discount the average payoffs to the present value.

2.5.3 Fourier inversion

The Fourier inversion method uses the characteristic function of the given distribution. Let $\phi_N(t)$ represents the characteristic function. Using the Fourier inversion method we attain a new density function of the form,

$$f_{Y_T}^{(Q)}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_N(t) \exp(-ity) dt \quad (2.6)$$

The integration in (2.6) can be approximated using a Riemann sum since the integrand is continuous and smooth. This is done as

$$\frac{1}{2\pi} \sum_{t \in T} \phi_N(t) \exp(-ity) \Delta t, \quad (2.7)$$

where $T = \{\frac{t}{n} : |t| < nb\}$ for sufficiently large values of n and b . These values are obtained such that;

$$\phi_N(nb) \approx 0.$$

The interval width $\frac{1}{n}$ should be very small so as to approximate the characteristic function well.

2.6 The various kinds of market participants

Market participants trade in different assets in the market (stocks, bonds and all different types of derivatives). These traders behave in different manner which leads to classifications defined below.

2.6.1 Arbitrageurs

These traders are in the market only to search for risk less profit. They mostly combine markets by transacting in more than one market, see [18]. In every market they venture

in, arbitrage will rapidly vanish since through trading, the market will reach economic equilibrium, in which demand equals supply. Arbitrageurs are not only opportunists but they actually participate in the market on a full time basis.

2.6.2 Hedgers

Hedging is the way mostly used by many excellent businesses in order to ensure survival in the unpredictable and competitive trading arenas, see [18]. Hedgers basically utilize the markets in which derivatives are traded to insure themselves from unfavorable fluctuations of currencies, prices and, or interest rates. The main attempt of hedgers is to try minimize chances of being exposed to excessive risk.

2.6.3 Speculators

Speculators are the risk takers. They make use of the trading markets to make massive profits rapidly by taking risks that most rational market participants avoid, see [18]. Speculators are different from hedgers in the sense that, the former are actually opportunists. Their available funds are invested in such a way that for the majority of times, they actually retain large sums of profit with certainty. Generally, speculators enjoy large sums of profits because they invest in very high risk investments.

Chapter 3

Lévy processes

Lévy processes are named after a French mathematician by the name of Paul Lévy who spearheaded the field, see [16]. The definition of a Lévy process is given below. Let $Y = (Y_t : t \geq 0)$ be a process defined on a filtered probability space stated in Section 2.1

3.1 Definition of a Lévy process

A process $Y = (Y_t : t \geq 0)$ is said to be a Lévy process if, see [16] :

- Y is a stochastic process.
- $Y_0 = 0$.
- $Y_{t+s} - Y_t$, for every $s, t \geq 0$, is independent.
- The increments are stationary.

Infinitely divisible distributions and Lévy processes are closely related. For each infinitely divisible distribution, there exists a Lévy process and vice versa. The definition of an infinitely divisible distribution is given below.

A probability measure λ defined on \mathbb{R} is infinitely divisible if for every positive integer $n \geq 2$, there exists independent and identically distributed random variables X_1, X_2, \dots, X_n , such that, $\sum_{i=1}^n X_i$ has a probability measure, λ , see [9].

Since we now know what a Lévy process is, we can now go ahead and define a geometric Lévy process which is the exponent of the Lévy process

$$\exp(Y_t).$$

Throughout the study, we consider the financial market defined in Section 2.1. We add one more characteristic of the market, which is, under the objective probability measure P , the stock price process is now modeled as

$$S_t = S_0 \exp(Y_t),$$

where Y_t is a Lévy process and S_0 is the stock price at time $t = 0$. In this section we consider the case where Y_t is a normal inverse Gaussian model.

Different distributions discussed below are infinitely divisible which means that the corresponding stochastic processes are Lévy processes. Below we define a normal inverse Gaussian distribution and show that it is infinitely divisible, which makes it a Lévy process.

3.2 The normal inverse Gaussian distribution

A normal inverse Gaussian distribution has a density function given as

$$f(x; \Theta) = \frac{\alpha\delta}{\pi} \left[\exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1\left(\alpha\sqrt{\delta^2 + (x - \mu)^2}\right)}{\sqrt{\delta^2 + (x - \mu)^2}} \right], \quad -\infty < x < \infty. \quad (3.1)$$

The parameters of this distribution should be in such a way that $\alpha > 0$, $|\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$.

The random variable X that follows the normal inverse Gaussian distribution is denoted as $X \sim N \circ IG(\Theta)$. This distribution has a characteristic function given as

$$\phi(t; \Theta) = \exp\left(i\mu t - \delta\left(\sqrt{\alpha^2 - (\beta + it)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right).$$

The normal inverse Gaussian distribution has the mean, variance, skewness and kurtosis denoted as $m_1(\Theta)$, $m_2(\Theta)$, $m_3(\Theta)$ and $m_4(\Theta)$ respectively. These four moments are given

by

$$\begin{aligned}
m_1(\Theta) &= \mu + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}}, \\
m_2(\Theta) &= \frac{\alpha^2\delta}{\sqrt{(\alpha^2 - \beta^2)^3}}, \\
m_3(\Theta) &= \frac{3\beta}{\alpha\sqrt{\delta(\alpha^2 - \beta^2)^{0.5}}}, \\
m_4(\Theta) &= 3 \left(1 + \frac{\alpha^2 + 4\beta^2}{\alpha^2\delta\sqrt{\alpha^2 - \beta^2}} \right). \tag{3.2}
\end{aligned}$$

We now need to show that the normal inverse Gaussian distribution is an infinitely divisible distribution. We define a process which follows a normal inverse Gaussian distribution as

$$X = (X_t; t \geq 0), \tag{3.3}$$

where the increments are independent and identically normal inverse Gaussian distributed and $X_0 = 0$. Let $X_n = \sum_{j=0}^n \Delta X_j$ where $n \geq 2$ and the ΔX_j 's are independent increments which follow a normal inverse Gaussian distribution with parameter set $\Theta = (\alpha, \beta, \delta, \mu)$. The characteristic function of a sum of these random variables is given as

$$\begin{aligned}
\phi_{\sum_{j=0}^n \Delta X_j}(u) &= E \left[\exp \left(iu \sum_{j=0}^n \Delta X_j \right) \right] \\
&= E [\exp(iu\Delta X_1)] \cdot E [\exp(iu\Delta X_2)] \dots E [\exp(iu\Delta X_n)] \\
&= \phi_{\Delta X_1}(u) \cdot \phi_{\Delta X_2}(u) \dots \phi_{\Delta X_n}(u) \\
&= \prod_{j=1}^n \left\{ \exp \left(i\mu u - \delta \left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2} \right) \right) \right\} \\
&= \exp \left\{ \sum_{j=1}^n \left(i\mu u - \delta \left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2} \right) \right) \right\} \\
&= \exp \left(\sum_{j=1}^n i\mu u - \sum_{j=1}^n \delta \left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2} \right) \right) \\
&= \exp \left(i\mu n u - \delta n \left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2} \right) \right). \tag{3.4}
\end{aligned}$$

Equation (3.4) is the characteristic equation of a random variable which follows a normal inverse Gaussian distribution with parameter set $\Theta^* = (\alpha, \beta, \delta n, \mu n)$. This implies that

$$X_n \sim N \circ IG(\Theta^*),$$

under the objective probability measure. This proves that this distribution is infinitely divisible. Since it has been proved that the normal inverse Gaussian distribution is infinitely divisible, therefore, the process defined on (3.3) is indeed a Lévy process.

Below we define another Lévy process which is called the Brownian motion. The background and the properties of the Brownian motion are given below. Before we discuss about the Brownian motion we should define the normal distribution and its properties.

3.3 The normal distribution

One of the most important distributions that is used in many areas of statistics is the normal distribution. The normal distribution is denoted by $N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ is the mean and $\sigma^2 > 0$ is the variance of the distribution, see [16]. The normal distribution is defined on the real number line. The density function of the normal distribution is given as

$$f_X(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[-\frac{1}{2} \left(\frac{(x - \mu)}{\sigma} \right)^2 \right], \quad -\infty < x < \infty.$$

The normal distribution's characteristic function is

$$\phi(u; \mu, \sigma^2) = \exp \left(iu\mu - \frac{1}{2}\sigma^2 u^2 \right).$$

Properties of the normal distribution

The normal distribution has a mean and variance denoted as μ and σ^2 , respectively. The skewness of the normal distribution is zero. This is because the normal distribution is symmetrical.

3.4 Brownian motion

Brownian motion was first introduced in 1828, by Robert Brown, when he observed pollen particles in suspension under a microscope, see [9]. He observed that the particles moved in an irregular motion. It was in 1900 when Bachelier considered Brownian motion as one of the models that could be used for stock prices, see [16]. Later in 1905, it was considered that the Brownian motion was a model of suspended particles. Bachelier observed that, if the kinetic theory of fluids was correct, then the molecules of the fluid would shift indiscriminately and so a minute molecule would receive an arbitrary number of strikes of random vigor and from random directions in any short period of time, see [16]. This

would ensure that a very small particle would move in exactly the way discussed by Robert Brown in 1828.

Albert Einstein also used the Brownian motion theory to estimate parameters in most of his research, for example, when he invented the theory of relativity. In 1923, Norbert Wiener, mathematically formalized the definition of the Brownian motion for the first time. In 1965 Samuelson used Brownian motion as a tool to model stock prices in his work. The Brownian motion process is often denoted by W_t for $t \geq 0$, see [16].

The definition of a Brownian motion

The stochastic process $W = (W_t : t \geq 0)$ is a Brownian motion if the following conditions are met;

- $W_0 = 0$, and also, W_t is continuous for $t \geq 0$,
- W has stationary increments,
- the increments $W_{t+s} - W_t$ are normally distributed with mean 0 and variance s and is independent of the history of the process up until time t , see [16].

Properties of a Brownian motion

1. W is everywhere continuous but it is nowhere differentiable shown in figure 3.1, see [5]. This is because the jaggedness of the Brownian motion process never smooth out even when you zoom in the process. This is illustrated by the graph in Figure 3.1 below.
2. As time $t \rightarrow \infty$, W_t will hit every value on the real number line, regardless of how large or negative it is.
3. W_t has a scaling property which states that if $W = (W_t : t \geq 0)$ is a Brownian motion, then, for every constant $d \neq 0$

$$W^\bullet = \left(W_t^\bullet = dW_{\frac{t}{d^2}}, t \geq 0 \right),$$

is a Brownian motion as well, see [5].

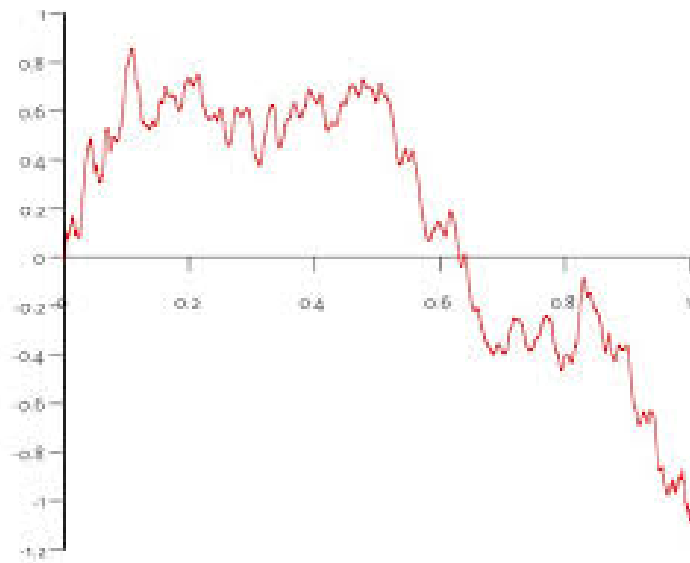


Figure 3.1: Standard Brownian motion getting to values below zero

From the properties of the Brownian motion and the graph given above, we notice that the standard Brownian motion also reaches negative values. This is not required when we use the Brownian motion to model the stock prices because stock prices will never be negative. We then need to manipulate the standard Brownian motion so that it does not get to negative values. This is done by applying an exponent function to the standard Brownian motion. The manipulated Brownian motion is called the geometric Brownian motion. This transformed Brownian motion is important because it gives the basis of the Black-Scholes model discussed in chapter 4 for option prices calculations.

The text below comes from a book called Financial derivatives pricing, applications and mathematics, see [6]. The formal definition of the geometric Brownian motion is given below.

3.4.1 Geometric Brownian motion

With the definition of the Brownian motion W , we can now define a stochastic process that is important for our calculation of option prices, an extension of the standard Brownian motion which is called the geometric Brownian motion. The evolution of a stock price process $S = (S_t : t \geq 0)$, under a particular model is as follows. We will look at how the stock price S will change in an interval of time, from t to $t + \Delta t$, where Δt , represents an infinitely small time change. Let ΔS_t , be defined as

$$\Delta S_t = S_{t+\Delta t} - S_t.$$

Thus, ΔS_t is the change in stock price from time t to $t + \Delta t$. The return of the stock in the time interval $[t, t + \Delta t]$ is

$$\frac{\Delta S_t}{S_t}.$$

Economically, it is reasonable to anticipate that the stock-return comprises of two parts which are; a random part and a systematic part, see [16]. Let us consider the random part first. Since the stock price fluctuates stochastically, a reliable assumption is that the variance of the return, $\frac{\Delta S_t}{S_t}$ over the time interval $[t, t + \Delta t]$ is proportional to the length of the interval, given as

$$\Delta t = t + \Delta t - t.$$

Therefore, the random part of the stock-return is modeled by $\sigma \Delta W_t$, see [16]. The variable ΔW_t is considered as the normally distributed noise term with the mean equal to zero and variance equal to Δt , see [16]. The scale parameter, $\sigma > 0$, describes how much effect the noise has. Thus it describes the magnitude of how the stock price fluctuates, see [5]. The variance of the return equals $\sigma^2 \Delta t$, in total. The variance of the stock-return is calculated as follows

$$\begin{aligned} \text{var}(\sigma \Delta W_t) &= \sigma^2 \text{var}(\Delta W_t) \\ &= \sigma^2 \Delta t, \end{aligned}$$

since $\Delta W_t \sim N(0, \Delta t)$. The volatility of the stock price is governed by σ , see [5].

For the systematic part, we assume that the length of the period considered is proportional to the stock's expected return over that period. This implies that, in an interval of length Δt , the expected stock-return is given by $\mu \Delta t S_t$, where μ is the drift-parameter representing the average rate of the stock-return. The deterministic part of the return of the stock is modeled by $\mu \Delta t$, see [16].

Combining the two components that model the returns of the stock, we attain the following equation

$$\begin{aligned} \Delta S_t &= \mu S_t \Delta t + \sigma \Delta W_t S_t \\ &= S_t (\mu \Delta t + \sigma \Delta W_t). \end{aligned}$$

The process ΔS_t is stochastic, therefore as $\Delta t \rightarrow 0$, we attain a stochastic differential equation given as

$$dS_t = S_t (\mu dt + \sigma dW_t),$$

where $S_0 > 0$. There is a unique solution for the stochastic differential equation given

above. It is given as

$$S_t = S_0 \exp(\gamma t + \sigma W_t),$$

where $S_0 > 0$ and $\gamma = \mu$, see [5]. The above exponential function is called the geometric Brownian motion with a drift and S_t is the stock price process.

If we apply the *log* function on the equation above we get

$$\begin{aligned} \log(S_t) - \log(S_0) &= \log \frac{S_t}{S_0} \\ &= \gamma t + \sigma W_t. \end{aligned} \tag{3.5}$$

From equation above, we note that $\log \left(\frac{S_t}{S_0} \right)$ follows a normal distribution with mean equal to γt and variance equal to $\sigma^2 t$, thus

$$\log \left(\frac{S_t}{S_0} \right) \sim N(\gamma t, \sigma^2 t),$$

see [16]. This implies that the stock price process ($S_t : t \geq 0$) follows a log normal distribution. The log normal distribution and the geometric Brownian form the basis of the Black-Scholes model for pricing options in the continuous time discussed in chapter 4. The logarithm of the stock price is a simple arithmetic of Brownian motion given in (3.5), therefore conditional probability density function of the logarithm of the stock price at time to maturity T is, see [6]

$$\log(S_T) | \log(S_0) \sim N(\log(S_0) + \gamma T, \sigma^2 T). \tag{3.6}$$

Below we define the Meixner distribution. We also show that its an infinitely divisible distribution meaning that the processes emanating from this distribution will be Lévy processes.

3.5 The Meixner distribution

The density function of the Meixner distribution is given as

$$f(x^*; \alpha, \beta, \delta) = \frac{(2 \cos(\frac{\beta}{2}))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x^*)}{\alpha}\right) \left| \Gamma\left(\delta + \frac{i(x^*)}{\alpha}\right) \right|^2, \tag{3.7}$$

where X^* is a random variables which follows the Meixner distribution, $-\pi < \beta < \pi$, $\delta > 0$, $\alpha > 0$ and Γ is a gamma function, see [16]. The gamma function is an installed

package in most of the statistical software packages, which makes it easier to compute. The Meixner distribution has a characteristic function as follows

$$\phi_{X^*}(u; \alpha, \beta, \delta) = \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha u - i\beta)}{2}\right)} \right)^{2\delta}. \quad (3.8)$$

We need to add a location parameter μ , in the distribution by redefining the random variable as

$$X = X^* + \mu.$$

We need to include the location parameter in the density function so that we can manipulate it in order to get to the Q martingale measure. The new random variable will have the following density function

$$f(x; \alpha, \beta, \delta) = \frac{(2\cos\left(\frac{\beta}{2}\right))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x - \mu)}{\alpha}\right) \left| \Gamma\left(\delta + \frac{i(x - \mu)}{\alpha}\right) \right|^2. \quad (3.9)$$

The characteristic function of the new random variable will be attained as

$$\begin{aligned} \phi_X(u; \alpha, \beta, \delta) &= E[e^{iuX}] \\ &= E[e^{iu(X+\mu)}] \\ &= E[e^{iu\mu} (e^{iu(X^*)})] \\ &= e^{iu\mu} E[(e^{iu(X^*)})] \\ &= e^{iu\mu} \phi_{X^*}(u; \alpha, \beta, \delta) \\ &= e^{iu\mu} \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha u - i\beta)}{2}\right)} \right)^{2\delta}. \end{aligned} \quad (3.10)$$

The Meixner distribution has the mean, variance, skewness and kurtosis denoted as $m_1(\alpha, \beta, \delta, \mu)$, $m_2(\alpha, \beta, \delta, \mu)$, $m_3(\alpha, \beta, \delta, \mu)$ and $m_4(\alpha, \beta, \delta, \mu)$ respectively. These four

moments are defined as follows

$$\begin{aligned}
m_1(u; \alpha, \beta, \delta, \mu) &= \mu + \alpha \delta \tan(\beta/2) \\
m_2(\alpha, \beta, \delta, \mu) &= \frac{\alpha^2 \delta}{2} \left(\frac{1}{\cos^2(\frac{\beta}{2})} \right) \\
m_3(\alpha, \beta, \delta, \mu) &= \sqrt{\frac{2}{\delta}} \sin\left(\frac{\beta}{2}\right) \\
m_4(\alpha, \beta, \delta, \mu) &= 3 + \left(\frac{(2 - \cos(\beta))}{\delta} \right). \tag{3.11}
\end{aligned}$$

We are now going to show that the Meixner distribution is a Lévy process. This is done by showing that the Meixner distribution is infinitely divisible. Let

$$X = (X_t; t \geq 0), \tag{3.12}$$

be a stochastic process that follows a Meixner distribution and $P = X_1 + X_2 + X_3 + \dots + X_n$, where X_t 's are independent and identically distributed, see [19]. The characteristic function of P is defined as

$$\begin{aligned}
\phi_P(u) &= E[\exp(iuP)] \\
&= E[\exp(iu(X_1 + X_2 + X_3 + \dots + X_n))] \\
&= E[\exp(iuX_1) \exp(iuX_2) \exp(iuX_3) \dots \exp(iuX_n)] \tag{3.13} \\
&= E[\exp(iuX_1)] \cdot E[\exp(iuX_2)] \cdot E[\exp(iuX_3)] \dots E[\exp(iuX_n)] \\
&= \phi_{\Delta X_1}(u) \cdot \phi_{\Delta X_2}(u) \cdot \phi_{\Delta X_3}(u) \dots \phi_{\Delta X_n}(u) \\
&= \prod_{j=1}^n \left\{ \exp^{iu\mu} \left(\frac{\cos(\frac{\beta}{2})}{\cosh\left(\frac{(\alpha u - i\beta)}{2}\right)} \right)^{2\delta} \right\} \\
&= \exp\left\{ \sum_{j=1}^n (i\mu u) \right\} \prod_{j=1}^n \left\{ \left(\frac{\cos(\frac{\beta}{2})}{\cosh\left(\frac{(\alpha u - i\beta)}{2}\right)} \right)^{2\delta} \right\} \\
&= \exp\left(\sum_{j=1}^n i\mu u \right) \left\{ \prod_{j=1}^n \left(\frac{\cos(\frac{\beta}{2})}{\cosh\left(\frac{(\alpha u - i\beta)}{2}\right)} \right)^{2\delta} \right\} \\
&= \exp(i\mu n u) \left(\frac{\cos(\frac{\beta}{2})}{\cosh\left(\frac{(\alpha u - i\beta)}{2}\right)} \right)^{2\delta n}. \tag{3.14}
\end{aligned}$$

Equation (3.14) is the characteristic function of the Meixner distribution with parameter set $(\alpha, \beta, \delta n, \mu n)$. This shows that the Meixner distribution is indeed an infinitely divisible distribution. Therefore, the process given in (3.12) is a Lévy process.

Chapter 4

Option pricing models

4.1 The Black-Scholes model

The Black-Scholes model for the calculation of option prices is one of the models which gives a closed-form solution, see [6]. The initial step towards the understanding of how to calculate the prices of options in more complicated markets, is the understanding of the Black-Scholes model because it gives the basis of option pricing. The Black-scholes model discussed is also a Lévy process since the normal distribution is infinitely divisible.

4.1.1 The Black-Scholes model for pricing options

The Black-Scholes model was initially developed by Fischer Black and Myron Scholes in 1969, see [9]. In 1970 Robert Merton, corrected an error that was made on the proof of the Black-Scholes model. Under this model, the stock price process is assumed to follow a geometric Brownian motion.

From the explanation given in Section 3.4.1, the geometric Brownian motioned is used to model the stock price process S_t for $0 \leq t \leq T$, see [15]. It is given as

$$S_t = S_0 \exp(\gamma t + \sigma W_t),$$

where $\gamma = \mu$ and W_t is a standard Brownian motion. As mentioned above, the parameters γ and σ reflect the drift and volatility of the process respectively. It is assumed that these parameters are constant. The second asset is the risk-free one and it is called a bond. The definition of the bond process B_t for $t \geq 0$, is given in Chapter 2 but for ease of reference, it is again given below

$$B_t = e^{rt}, r \geq 0,$$

where r is the continuously compounded risk-free interest rate, see [18].

The Black-Scholes model is said to be market complete. Using the Esscher transform method discussed in Section 2.4, we note that the risk-neutral stock price process $(S_t, t \geq 0)$ will still be a geometric Brownian motion with the same volatility σ but drift γ term is changed to

$$\mu = r - \frac{1}{2}\sigma^2.$$

The detailed reader is referred to the paper titled, Option pricing by Esscher transforms, see [17]. The stock process is given as

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right),$$

With the definition of the stock price in a risk-neutral setting, we can now calculate the fair prices of European call options.

Since we are going to be concentrating on European call options, below is the explicit formula of the calculation of the European call option using the Black-Scholes model. Note that the prices of other options are calculated in the same manner. The discussion below comes from a book called Financial derivatives, application and mathematics, see [6].

Explicit formulae for European call option price

There are two methods that can be used to calculate the price of the European call option using the Black-Scholes model. The first method is to directly use the the normal distribution function values. The second method is to use numerical integration, and it is also given below. Derivations of the two methods are given below.

The European call option has a payoff function given as

$$C(S_T) = \max[0, (S_T - K)]. \quad (4.1)$$

The price of the European call option at time $t = 0$, is given below

$$\begin{aligned} \Pi &= \exp(-rT) E_Q [C(S_T)] \\ &= \exp(-rT) E_Q [\max[0, (S_T - K)]]. \end{aligned} \quad (4.2)$$

Equation (4.2) gives the same answer as equation (2.5). In order to calculate the closed form solution for equation (4.2), we initially write equation (4.1), in terms of the logarithm-

mic function of S_T , see [6]. Thus,

$$C(S_T) = \max[0, (\exp(\log(S_T))) - K] = [S_T - K]^+. \quad (4.3)$$

To calculate equation (4.2), using the payoff transformed in terms of the logarithm of the stock price in (4.3), we use integration. Thus the expected value in (4.2) is expressed in terms of the integral of the conditional density function of the logarithm of stock price (3.6), see [6]. From the distribution (3.6), equation (4.2), and equation (4.3), we can calculate the European call option price as follows

$$\begin{aligned} \Pi &= \exp(-rT) E_Q[C(S_T)] \\ &= \exp(-rT) E_Q[\max[0, (\exp(\log(S_T))) - K]] \\ &= \exp(-rT) \int_{\log(K)}^{\infty} [\exp(\log(S_T)) - K] \frac{1}{\sqrt{2\pi\sigma^2T}} \\ &\quad \exp\left[-\frac{[\log(S_T) - \log(S_0) - (r - \frac{1}{2}\sigma^2)T]^2}{2\sigma^2T}\right] d\log(S_T) \\ &= \frac{1}{\sqrt{2\pi\sigma^2T}} \exp(-rT) \\ &\quad \int_{\log(K)}^{\infty} \exp\left[\log(S_T) - \frac{[\log(S_T) - \log(S_0) - (r - \frac{1}{2}\sigma^2)T]^2}{2\sigma^2T}\right] \\ &\quad d\log(S_T) - K \frac{1}{\sqrt{2\pi\sigma^2T}} \exp(-rT) \\ &\quad \int_{\log(K)}^{\infty} \exp\left[-\frac{[\log(S_T) - \log(S_0) - (r - \frac{1}{2}\sigma^2)T]^2}{2\sigma^2T}\right] d\log(S_T). \end{aligned} \quad (4.4)$$

From equation (4.4) we notice that there are two separate integrals which we can solve separately. We are going to manipulate the two integrals so that both of them will be in the form of the normal distribution function with mean equal to zero and variance equal to one, that is a standard normal distribution function, see [9]. We start with manipulating the first integral by letting

$$Z = \log\left(\frac{S_T}{S_0}\right),$$

and noting that;

$$dZ = d\log(S_T).$$

Given the above transformation, the first integral can be stated as

$$\frac{1}{\sqrt{2\pi\sigma^2T}} \exp(-rT) S_0 \int_{\log(\frac{K}{S_0})}^{\infty} \exp \left[Z - \frac{[Z - (r - \frac{1}{2}\sigma^2) T]^2}{2\sigma^2T} \right] dZ. \quad (4.5)$$

Expanding the expression given above and grouping like terms in the exponent, we get

$$\frac{1}{\sqrt{2\pi\sigma^2T}} \exp(-rT) S_0 \int_{\log(\frac{K}{S_0})}^{\infty} \exp \left[\frac{2Z\sigma^2T - Z^2 + 2Z(r - \frac{1}{2}\sigma^2)T - (r - \frac{1}{2}\sigma^2)^2T^2}{2\sigma^2T} \right] dZ.$$

To further simplify the integral above we use the method of completing the square on the expression on the numerator of the exponent. This is done by subtracting $2r\sigma^2T^2$ and adding the same term. This gives us the following integral

$$\frac{1}{\sqrt{2\pi\sigma^2T}} \exp(-rT) S_0 \int_{\log(\frac{K}{S_0})}^{\infty} \exp \left[rT - \frac{Z^2 - 2Z(r + \frac{1}{2}\sigma^2)T + (r + \frac{1}{2}\sigma^2)^2T^2}{2\sigma^2T} \right] dZ,$$

which implies to

$$\frac{1}{\sqrt{2\pi\sigma^2T}} S_0 \int_{\log(\frac{K}{S_0})}^{\infty} \exp \left[-\frac{[Z - (r + \frac{1}{2}\sigma^2)T]^2}{2\sigma^2T} \right] dZ. \quad (4.6)$$

Transforming the integral above again by letting

$$Y = \frac{Z - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and noting that

$$dZ = \sigma\sqrt{T}dY,$$

gives a simplified expression in the exponent function given below

$$S_0 \int_{\left(\frac{\log \frac{K}{S_0} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{Y^2}{2} \right] dY. \quad (4.7)$$

The integral (4.7), is the standard normal distribution that is valued at $-\frac{\log \frac{K}{S_0} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

[6]. Therefore equation (4.4) can be expressed as

$$\begin{aligned} \Pi = & \Phi \left(\frac{\log \frac{K}{S_0} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - K \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp(-rT) \\ & \int_{\log(K)}^{\infty} \exp \left[-\frac{[\log(S_T) - \log(S_0) - (r - \frac{1}{2}\sigma^2)T]^2}{2\sigma^2 T} \right] d\log(S_T). \end{aligned}$$

Since 4.7 is the of a standard normal distribution function, we use the fact that

$$1 - \Phi(x) = \Phi(-x). \quad (4.8)$$

The function Φ represents the distribution function of a standard normal.

We will now transform the second integral given on equation (4.4) in the similar way as we did for the first integral, see [6]. This is done such that the integral can be in terms of a standard normal distribution function. Let

$$Y = \frac{\log(S_T) - \log(S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

and

$$\sigma\sqrt{T}dY = d(\log(S_T)),$$

which gives the expression given below

$$K \exp(-rT) \int_{\left(\frac{\log \frac{K}{S_0} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{Y^2}{2} \right] dY.$$

The expression above is the transformed integral evaluated at $-\frac{\log \frac{K}{S_0} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$. Using the fact given in (4.8) and combining the solutions of the two integrals in (4.4), we can therefore give the price of the option as

$$\Pi = S_0 \left(\Phi \left(\frac{\log \left(\frac{K}{S_0} \right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right) - K \exp(-rT) \Phi \left(\frac{\log \left(\frac{K}{S_0} \right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right). \quad (4.9)$$

The second method of calculating the European call option prices using numerical integration is given below. Note that, the strike price, maturity time, interest rate and payoff are defined the same way as above. The stock price at time $T > 0$ is defined as

$$S_T = S_0 \exp(X_T),$$

where,

$$X_T = \mu T + \sigma W_T,$$

and W_T is a standard Brownian motion, μ is the drift parameter and σ is the volatility parameter, see [6]. After applying the Esscher transform method, the drift parameter should be equal to

$$\mu = r - \frac{1}{2}\sigma^2,$$

in order to move from the probability measure P to the locally equivalent martingale measure, Q . Thus,

$$\begin{aligned} \Pi &= e^{-rT} E_Q [(S_T - K)^+] \\ &= e^{-rT} E_Q [(S_T - K) | (S_T \geq K)] \end{aligned} \quad (4.10)$$

$$\begin{aligned} &= e^{-rT} E_Q [(S_T - K) | (S_0 \exp(X_T) \geq K)] \\ &= e^{-rT} E_Q [(S_T - K) | (S_0 \exp(\mu T + \sigma W_T) \geq K)] \\ &= e^{-rT} E_Q \left[(S_T - K) | \left((\mu T + \sigma W_T) \geq \log \left(\frac{K}{S_0} \right) \right) \right] \\ &= e^{-rT} E_Q \left[(S_T - K) | \left((\mu T + \sigma W_T) \geq \log \left(\frac{K}{S_0} \right) \right) \right] \\ &= e^{-rT} E_Q \left[(S_T - K) | \left(\frac{W_T}{\sqrt{T}} \geq \frac{\log \left(\frac{K}{S_0} \right) - \mu T}{\sigma \sqrt{T}} \right) \right] \end{aligned} \quad (4.11)$$

$$\begin{aligned} &= e^{-rT} E_Q \left[(S_T - K) | \left(Z \geq \frac{\log \left(\frac{K}{S_0} \right) - \mu T}{\sigma \sqrt{T}} \right) \right] \\ &= e^{-rT} E_Q \left[(S_0 \exp(\mu T + \sigma W_T) - K) | \left(Z \geq \frac{\log \left(\frac{K}{S_0} \right) - \mu T}{\sigma \sqrt{T}} \right) \right] \\ &= e^{-rT} E_Q \left[(S_0 \exp(\mu T + \sigma \sqrt{T} Z) - K) | \left(Z \geq \frac{\log \left(\frac{K}{S_0} \right) - \mu T}{\sigma \sqrt{T}} \right) \right] \end{aligned} \quad (4.12)$$

$$= e^{-rT} \int_{\frac{\log \left(\frac{K}{S_0} \right) - \mu T}{\sigma \sqrt{T}}}^{\infty} \left[(S_0 \exp(\mu T + \sigma \sqrt{T} z) - K) \right] \phi(z) dz \quad (4.13)$$

Numerical integration is performed on equation (4.13). On equation (4.10), we have a conditional expectation because the payoff of $(S_T - K)$ is only going to exist on the

condition that the stock price is greater than the strike price. We know that

$$W_T \sim N(0, T),$$

therefore standardizing W_T , gives

$$\begin{aligned} \frac{W_T - 0}{\sqrt{T}} &= \frac{W_T}{\sqrt{T}} \\ &= Z \sim N(0, 1) \end{aligned} \tag{4.14}$$

which is what was done on equation (4.11). Using (4.14), we get that

$$\frac{W_T}{\sqrt{T}} = Z,$$

which leads to

$$W_T = \sqrt{T}Z,$$

as shown in equation (4.12). The function $\phi(z)$ given in equation (4.13) is the density function of the standard normal random variable.

Equations (4.9) and (4.13) give the same value for the European call option price. In other financial models that are going to follow, we will be using numerical integration and the Monte Carlo simulation to calculate the option prices.

Even though the Black-Scholes model is one of the popular financial models used to calculate option prices, it has some shortcomings. These are discussed below.

4.1.2 Drawbacks of the Black-Scholes model

The Black-Scholes model assumes that the log-returns follow a normal distribution. Below, we consider two departures from normality in observed log-returns, thus in skewness and in kurtosis of the log-returns. As a result, the normal distribution may not be a realistic model for log-returns.

Skewness measures the extent of asymmetry of a given distribution. The normal distribution is symmetric, and therefore has a skewness of zero. This is not the case for the log-returns, which are typically negatively skewed.

Kurtosis measures how heavy or light the tails of a distribution are. It is defined as

$$\frac{E[Y - \mu]^4}{(\sigma^2)^2},$$

where μ is the mean, σ^2 is the variance of the random variable Y respectively. The log-returns data has heavier tails than those of the normal distribution. Therefore, assuming that the log returns of the stock price follow a normal distribution might not be the best solution.

We need to look for other models that best explains the distribution of the log-returns. These models should be flexible enough in order to cater for the excess kurtosis and heavy tails that are inherent in log returns data. Below we have some models that fits better the log-returns.

4.2 The geometric normal inverse Gaussian process model

The parameters $\alpha, \beta, \delta, \mu$ can be estimated from the observed data. These estimated parameters are under the objective probability measure P . Application of the Esscher transform method changes the parameters to

$$\alpha, \beta + \eta, \delta, \mu, \tag{4.15}$$

where η is the solution to the equation

$$r - \mu - \delta \left(\sqrt{\alpha^2 - (\beta + \eta)^2} - \sqrt{\alpha^2 - (\beta + \eta + 1)^2} \right) = 0,$$

where r is the compounded interest rate, see [16]. We can now calculate the arbitrage-free option price under the new probability measure that has the new set of parameters given in (4.15).

4.2.1 Numerical complications encountered

The density function of the normal inverse Gaussian distribution given in equation (3.1), includes a Bessel function, $K_1(z)$. This Bessel and the exponential function turns to infinity when α and δ become large values. The softwares like R or SAS, will quickly stop the density function calculations when they encounter parameter sets with these large values. This hinders the ability to calculate the best option prices due to the fact that we now have to restrict our parameter sets to be in the range that is accepted by the software.

4.3 The geometric Meixner process model

In order to calculate the arbitrage free option price, we need a locally equivalent martingale measure Q . This measure is attained through the Esscher transform method. Under this method, the parameters of the Meixner model are changed to

$$\alpha, \alpha\vartheta^* + \beta, \delta, \mu,$$

where ϑ^* is a solution of the the following equation

$$\vartheta^* = \frac{1}{\alpha} \left(\beta + 2 \arctan \left(\frac{-\cos\left(\frac{\alpha}{2}\right) + \exp\left(\frac{(\mu-r)}{2\delta}\right)}{\sin\left(\frac{\alpha}{2}\right)} \right) \right),$$

where r is the interest rate, see [16] and [11].

There are certain cases where the density function of the Meixner model is difficult to calculate. This happens when α turns to zero and β and δ turns to infinity. In such cases we use a method known as the Fourier inversion method (discussed in chapter 2) in order to approximate the density function.

4.4 Subordinated models for option pricing

The financial models discussed above are under the assumption that the number of trades on a particular stock are constant on specific time intervals, for instance hours. This is not always the case since the trading environment, for example, in the first hour, is not the same as the trading environment in the second hour. In this section, we are going to consider models that take into account the randomness of the number of trades per time interval. These are called subordinated (time changed) models.

These models were first introduced by Clark in 1973 when he modeled the first difference of the stock prices, thus

$$\Delta S_t = S_t - S_{t-1}. \quad (4.16)$$

On short time intervals, for instance hourly intervals, empirical evidence suggest that the following relations hold approximately

$$E[\Delta S_t] = 0 \quad (4.17)$$

and

$$E[\Delta S_t \Delta S_l] = 0 \quad (4.18)$$

for $t \neq l$. This means that the expected value of the change in stock price is equal to zero. Equation (4.18) implies that changes in stock price at different times are uncorrelated. This means that there exists independence at different time intervals.

Another fact often remarked in the financial literature is that the distribution of the returns is leptokurtic, meaning that it has a kurtosis that is greater than 3. This implies that ΔS_t is not normally distributed. Note that ΔS_t is a summation of a large number of smaller stock price changes. Since ΔS_t does not follow a normal distribution, it implies that some necessary conditions for the central limit theorem does not hold.

The suggestion that the subordinated models follow a random walk comes from the paper published by Clark in 1973, see [8].

4.4.1 A random walk model

A random walk model is suggested since it can cater for the properties of ΔS_t given in equation (4.18), see [8]. There are some theoretical justifications for why random walk is used to model ΔS_t and these are given in [4]. The author argued that if the changes in the stock prices are correlated then market participants can buy and sell large quantities of that particular stock, at correct times and make unlimited profit. For instance, if ΔS_{t-1} is positively correlated to ΔS_t , see [19]. A market participant who can access this information and observing that ΔS_{t-1} is negative will know that ΔS_t will likely be negative. A market participant will sell large quantities of stock at time $t - 1$. However, if it happens that many traders employ the same strategy then the demand of the stock decreases and the supply is increased. This leads to an increase in the stock price until the correlation is discarded.

From the above-mentioned arguments it seems valid to model ΔS_t as follows

$$\Delta S_t = \Delta S_{t-1} + \epsilon_t,$$

where

$$E[\epsilon_t] = 0,$$

and

$$E[\epsilon_l \epsilon_t] = 0,$$

for all $l \neq t$, where ϵ_t follows some heavy-tailed distribution.

The frequency at which the stock price changes is different given identical time intervals, see [8]. This implies that the distribution of ΔS_t is not normal. We can subordinate ΔS_t to normality. This is done by making the number of small price changes that makes

up ΔS_t , a random variable. Thus we no longer assume that there is a constant number of small price changes in every time period. In his paper, Clark hypothesise that, ϵ_t in the random walk model is the sum of a random number of random variables that are normally distributed. As mentioned above, the central limit theorem conditions are not met by ΔS_t but we can attain generalizations of the central limit theorem that can help us get to normality. These generalizations are special cases of Anscombe's theorem, see [3].

4.4.2 Generalization 1

Denote $N_n = N_1, N_2, N_3, \dots$ to be a sequence of positive integer random variables such that

$$\lim_{n \rightarrow \infty} \left(\frac{N_n}{n} \right) = 1.$$

Also, let $X_n = X_1, X_2, X_3, \dots$ be a sequence that is independent of N_n . Also, let the sequence X_1, X_2, X_3, \dots be random variables that are independent and identically distributed with mean equal to zero and variance equals to unity, see [3]. Then

$$D = \frac{Y_{N_n}}{\sqrt{n}},$$

converges to a standard normal distribution, as n turns to infinity, where

$$Y_{N_n} = \sum_{i=1}^{N_n} X_i.$$

Proof

Let the characteristic function of a random variable, Y be denoted by Φ_Y and

$$C_n(t) = \Phi_D(t).$$

In order to show that the random variable D follows a standard normal distribution, it is sufficient to show that

$$\lim_{n \rightarrow \infty} C_n(t) = \exp\left(-\frac{1}{2}t^2\right).$$

Now, consider

$$\begin{aligned}
C_n(t) &= \Phi_D(t) \\
&= \Phi_{\frac{Y_{N_n}}{\sqrt{n}}}(t) \\
&= E \left[\exp \left(it \frac{Y_{N_n}}{\sqrt{n}} \right) \right] \\
&= E \left[\exp \left(it \frac{\sum_{i=1}^{N_n} X_i}{\sqrt{n}} \right) \right] \\
&= E \left[E \left[\exp \left(it \frac{\sum_{i=1}^{N_n} X_i}{\sqrt{n}} \right) \middle| N_n \right] \right] \\
&= E \left[\left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right)^{N_n} \right] \\
&= E \left[\exp \left(\frac{N_n}{n} n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right) \right] \tag{4.19}
\end{aligned}$$

In the fourth equality, we applied the tower law of expected values. The fifth equality holds because the random variables X_1, X_2, X_3, \dots are independent and identically distributed. Applying limits on both sides of (4.19) we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} C_n(t) &= \lim_{n \rightarrow \infty} E \left[\exp \left(\frac{N_n}{n} n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right) \right] \\
&= E \left[\exp \left(\lim_{n \rightarrow \infty} \left\{ \frac{N_n}{n} \right\} \lim_{n \rightarrow \infty} \left\{ n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right\} \right) \right] \\
&= E \left[\exp \left(-\frac{1}{2} t^2 \right) \right] \\
&= \exp \left(-\frac{1}{2} t^2 \right), \tag{4.20}
\end{aligned}$$

From the second to the third equality, we used the fact that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left\{ n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right\} &= \lim_{n \rightarrow \infty} \log \left(\left(\Phi \left(\frac{t}{\sqrt{n}} \right) \right)^n \right) \\
&= \log \lim_{n \rightarrow \infty} \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right)^n \\
&= \exp \left[\lim_{n \rightarrow \infty} \log E \left[e^{\frac{itX}{\sqrt{n}}} \right]^n \right] \\
&= \exp \left[\lim_{n \rightarrow \infty} \log E \left[e^{\frac{it \sum_{j=1}^n X_j}{\sqrt{n}}} \right]^n \right] \\
&= \exp \left[\log \lim_{n \rightarrow \infty} E \left[e^{\frac{it \sum_{j=1}^n X_j}{\sqrt{n}}} \right]^n \right] \\
&= \log \left(\exp \left(-\frac{1}{2} t^2 \right) \right) \\
&= -\frac{1}{2} t^2,
\end{aligned}$$

which emanates from the central limit theorem. This is because, the third equality has the sum of a large number of independent and identically distributed characteristic functions, which in this case are the random variables, see [19]. The interchanging of the expected value and the limit is justified by Lebesgue's dominated convergence theorem since the $|\Phi(\cdot)| \leq 1$.

4.4.3 Generalization 2

Let $N_n = \lfloor Hn \rfloor$ for large values of n , where $\lfloor Hn \rfloor$ is the integer part of Hn . Thus $\lfloor Hn \rfloor \leq Hn$, see [3]. The variable H is random and it has a mean of one and variance that is greater than zero. Let the sequence X_1, X_2, X_3, \dots be independent random variables that are identically distributed with zero mean and variance equal to one. The random variable H and the sequence X_1, X_2, X_3, \dots are independent. Then, the distribution of

$$\frac{Y_{N_n}}{\sqrt{n}},$$

given H , converges to a normal distribution with zero mean and variance equal to H , where

$$Y_{N_n} = \sum_{i=1}^{N_n} X_i.$$

Proof

$$\begin{aligned}
C_n(t) &= \Phi_{\frac{Y_{N_n}}{\sqrt{n}}}(t) \\
&= E \left[\exp \left(it \frac{\sum_{i=1}^{N_n} X_i}{\sqrt{n}} \right) \right] \\
&= E \left[E \left[\exp \left(it \frac{\sum_{i=1}^{N_n} X_i}{\sqrt{n}} \right) \middle| N_n \right] \right] \\
&= E \left[\left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right)^{N_n} \middle| N_n \right] \\
&= E \left[\exp \left(\frac{N_n}{n} n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right) \middle| N_n \right] \\
\lim_{n \rightarrow \infty} C_n(t) &= \lim_{n \rightarrow \infty} E \left[\exp \left(\frac{N_n}{n} n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right) \middle| N_n \right] \\
&= E \left[\exp \left(\lim_{n \rightarrow \infty} \left\{ \frac{N_n}{n} \right\} \left[\lim_{n \rightarrow \infty} \left\{ n \log \left(\Phi_{X_1} \left(\frac{t}{\sqrt{n}} \right) \right) \right\} \right] \right) \middle| N_n \right] \\
&= \left[\exp \left(H \left[\lim_{n \rightarrow \infty} \left\{ n \log E \left[e^{\frac{itX}{\sqrt{n}}} \right] \right\} \right] \right) \middle| N_n \right] \tag{4.21} \\
&= \exp \left[H \lim_{n \rightarrow \infty} \log E \left[e^{\frac{itX}{\sqrt{n}}} \right]^n \middle| N_n \right] \\
&= \exp \left[H \lim_{n \rightarrow \infty} \log E \left[e^{\frac{it \sum_{j=1}^n X_j}{\sqrt{n}}} \right]^n \middle| N_n \right] \\
&= \exp \left[H \log \lim_{n \rightarrow \infty} E \left[e^{\frac{it \sum_{j=1}^n X_j}{\sqrt{n}}} \right]^n \middle| N_n \right] \\
&= \exp \left[H \log \left(e^{-\frac{1}{2}t^2} \right) \middle| N_n \right] \\
&= \exp \left(-\frac{1}{2} H t^2 \right) \tag{4.22}
\end{aligned}$$

On the seventh going to the eighth equality we used the fact that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left\{ \frac{N_n}{n} \right\} &= \lim_{n \rightarrow \infty} \frac{\lfloor Hn \rfloor}{n} \\
&= H
\end{aligned}$$

Both generalizations imply non-constant frequency of trades and we will use it to build the time-changed models below.

4.4.4 Time changed stochastic processes

Consider a stochastic process Y_1, Y_2, Y_3, \dots with the subscript of a sequence of increasing integers. We can change the subscript as follows $Y_{t_1}, Y_{t_2}, Y_{t_3}, \dots$ where t_1, t_2, t_3, \dots are realizations from a non-decreasing stochastic process, say T_t . A new stochastic process will be formed. This new process Y_{T_t} , is called a time-changed stochastic process, subordinated to Y_t . The stochastic process T_t is called the directing process. The distribution of ΔY_t is said to be subordinated by the distribution of $\Delta Y_{T_t} = Y_{T_t} - Y_{T_{t-1}}$.

Below, we consider the properties of the distribution of ΔY_{T_t}

Properties

Define Y_t and T_t to be two independent stochastic processes that have stationary and independent increments. Below, we consider some properties of the subordinated process obtained from these two processes under certain conditions. If the following four properties hold;

$$E[\Delta Y_t] = 0,$$

$$Var[\Delta Y_t] = \sigma^2,$$

$$E[\Delta T_t] = \mu,$$

$$T_{t+1} - T_t \geq 0,$$

for all $t \geq 0$, then the process Y_{T_t} has independent and stationary increments, see [19]. Also,

$$E[\Delta Y_{T_t}] = 0,$$

and

$$Var[\Delta Y_{T_t}] = \mu\sigma^2,$$

for all $t \geq 0$.

Proof

The independence and stationarity of the increments of Y_t and T_t lead to the independence and stationarity of the increments of ΔY_{T_t}

$$\begin{aligned}
E[\Delta Y_{T_t}] &= E[E[\Delta Y_t | T_{t+1}, T_t]] \\
&= E[\Delta Y_t] E[\Delta T_t] \\
&= 0 \cdot \mu \\
&= 0.
\end{aligned}$$

The second equality emanates from the evidence that the processes Y_t and T_t are independent. Consider,

$$\begin{aligned}
Var[\Delta Y_{T_t}] &= var[E[\Delta Y_t | T_{t+1}, T_t]] + E[var[\Delta Y_t | T_{t+1}, T_t]] \\
&= var[\Delta T_t E[\Delta Y_t]] + E[\Delta T_t var[\Delta Y_t]] \\
&= (E[\Delta Y_t])^2 var[\Delta T_t] + var[\Delta Y_t] E[\Delta T_t] \\
&= 0 \cdot var[\Delta T_t] + \sigma^2 \mu \\
&= \mu \sigma^2.
\end{aligned}$$

We can see that the $Var[\Delta Y_{T_t}]$ is not influenced by $var[\Delta T_t]$. This implies that we can attain many different distributions of ΔY_{T_t} with the same mean and variance stated above, by only changing $var[\Delta T_t]$.

As was discussed in chapter 4.1, the financial log-returns (or financial returns) have a kurtosis that is greater than 3. This means that the normal distribution does not capture all of the observed properties of the returns, and a distribution which can take the leptokurtic nature of the financial returns data into account is required. A theorem below suggest that we can fit the log-returns with any kurtosis value by changing the variance of ΔT_t .

Theorem 1

If Y_t and T_t are processes defined as above and ΔY_t is a random variable that follows a normal distribution then an increase in $var[\Delta T_t]$ will cause an increase in the kurtosis of ΔY_{T_t} , see [19].

Proof

$$\begin{aligned}
kurt [\Delta Y_{T_t}] &= \frac{E [[\Delta Y_{T_t}]^4]}{Var [\Delta Y_{T_t}]^2} \\
&= \frac{E [[Y_{T_{t+1}} - Y_{T_t}]^4]}{E^2 [[Y_{T_{t+1}} - Y_{T_t}]^2]} \\
&= \frac{1}{\mu^2 \sigma^4} E [E [(Y_{T_{t+1}} - Y_{T_t})^4 | T_{t+1}, T_t]] \\
&= \frac{3}{\mu^2 \sigma^4} E [E [(Y_{T_{t+1}} - Y_{T_t})^2 | T_{t+1}, T_t]] \\
&= \frac{3}{\mu^2} E [[\Delta T_t]^2] \\
&= \frac{3}{\mu^2} \{ E^2 [\Delta T_t] + var [\Delta T_t] \} \\
&= 3 \left(1 + \frac{var[\Delta T_t]}{\mu^2} \right).
\end{aligned}$$

From the third equality going forth, we used the fact that $[(Y_{T_{t+1}} - Y_{T_t}) | T_{t+1}, T_t]$ is a random variable that follows a normal distribution. Therefore,

$$\begin{aligned}
kurt [(Y_{T_{t+1}} - Y_{T_t}) | T_{t+1}, T_t] &= \frac{E [[(Y_{T_{t+1}} - Y_{T_t}) | T_{t+1}, T_t]^4]}{E^2 [[(Y_{T_{t+1}} - Y_{T_t}) | T_{t+1}, T_t]^2]} \\
&= 3.
\end{aligned}$$

This theorem helps us to come up with models that best fit the financial returns data since we can change the kurtosis of the distribution of ΔY_{T_t} by increasing the variance of ΔT_t .

4.4.5 Time-changed Brownian motions

Below we consider the general form of the density function of ΔY_{T_t} where Y_t is a Brownian motion. In addition to the general case, we consider some specific density functions.

Theorem 2

If T_t and Y_t are defined as independent processes with independent and stationary increments and ΔT_t follows a distribution with a density function denoted as u and $\Delta Y_t \sim$

$N(\mu, \sigma^2)$, see [19]. Then, the density function of ΔY_{T_t} is given as,

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(y - \mu x)^2}{2\sigma^2 x}\right) \frac{u(x)}{\sqrt{x}} dx.$$

Proof

Let F denote the probability distribution function of ΔY_{T_t} . Using the notation $Y_{\Delta T_t} = Y_{T_{t+1}} - Y_{T_t}$, we obtain

$$\begin{aligned} F(y) &= P((Y_{T_{t+1}} - Y_{T_t}) \leq y) \\ &= P(Y_{\Delta T_t} \leq y) \\ &= E[P(Y_{\Delta T_t} \leq y | \Delta T_t)] \\ &= E\left[P\left(\frac{Y_{\Delta T_t} - \mu\Delta T_t}{\sigma\sqrt{\Delta T_t}} \leq \frac{y - \mu\Delta T_t}{\sigma\sqrt{\Delta T_t}} \mid \Delta T_t\right)\right] \\ &= E\left[P\left(Z \leq \frac{y - \mu\Delta T_t}{\sigma\sqrt{\Delta T_t}} \mid \Delta T_t\right)\right] \\ &= E\left[\int_{-\infty}^{\frac{y - \mu\Delta T_t}{\sigma\sqrt{\Delta T_t}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz\right] \\ &= E\left[\Phi\left(\frac{y - \mu\Delta T_t}{\sigma\sqrt{\Delta T_t}}\right)\right]. \end{aligned} \tag{4.23}$$

In the fourth equality, we are standardizing since $Y_{\Delta T_t} \sim N(\mu\Delta T_t, \sigma^2\Delta T_t)$. On the second equality $Y_{\Delta T_t}$ and ΔY_{T_t} are equal because they converge to the same distribution. The standard normal distribution function is denoted by Φ in the sixth equality. In order to attain the density function of ΔY_{T_t} , we partially differentiate (4.23) with respect to y as

follows

$$\begin{aligned}
f(y) &= \frac{\partial}{\partial y} E \left[\Phi \left(\frac{y - \mu \Delta T_t}{\sigma \sqrt{\Delta T_t}} \right) \right] \\
&= E \left[\frac{\partial}{\partial y} \Phi \left(\frac{y - \mu \Delta T_t}{\sigma \sqrt{\Delta T_t}} \right) \right] \\
&= E \left[\phi \left(\frac{y - \mu \Delta T_t}{\sigma \sqrt{\Delta T_t}} \right) \frac{1}{\sigma \sqrt{\Delta T_t}} \right] \\
&= \int_0^\infty \frac{1}{\sigma \sqrt{x}} \phi \left(\frac{y - \mu x}{\sigma \sqrt{x}} \right) u(x) dx \\
&= \int_0^\infty \frac{1}{\sigma \sqrt{2\pi x}} \exp \left(-\frac{1}{2} \left(\frac{y - \mu x}{\sigma \sqrt{x}} \right)^2 \right) u(x) dx \\
&= \frac{1}{\sigma \sqrt{2\pi}} \int_0^\infty \exp \left(-\frac{(y - \mu x)^2}{2\sigma^2 x} \right) \frac{u(x)}{\sqrt{x}} dx. \tag{4.24}
\end{aligned}$$

The density function of the standard normal distribution is denoted by ϕ . On the fourth equality we used the fact that,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx,$$

where $f(x)$ is the density function of the random variable X .

The directing processes used to model the time process T_t are the lognormal and Pareto distributions. The resulting density functions of the time changed stochastic processes are given in the following two corollaries.

Corollary 1

The density function of ΔY_{T_t} , given that ΔT_t follows a lognormal distribution with parameters α and β , is

$$f(y) = \frac{1}{2\pi\beta\sigma} \int_0^\infty x^{-\frac{3}{2}} \exp \left[-\frac{(y - \mu x)^2}{2x\sigma^2} - \frac{(\log(x) - \alpha)^2}{2\beta^2} \right] dx. \tag{4.25}$$

The distribution of Y_{T_t} is called the lognormal-normal.

Proof

$$\begin{aligned}
f(y) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) \frac{u(x)}{\sqrt{x}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) \frac{1}{x\beta\sqrt{2\pi x}} \exp\left(\frac{(\log(x)-\alpha)^2}{2\beta^2}\right) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{\beta\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) \frac{1}{x^{\frac{3}{2}}} \exp\left(\frac{(\log(x)-\alpha)^2}{2\beta^2}\right) dx \\
&= \frac{1}{\beta\sigma 2\pi} \int_0^\infty x^{-\frac{3}{2}} \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) \exp\left(\frac{(\log(x)-\alpha)^2}{2\beta^2}\right) dx. \tag{4.26}
\end{aligned}$$

Corollary 2

If ΔT_t follows a Pareto distribution with parameters a and g , then ΔY_{T_t} has the following density function,

$$f(y) = \frac{ab^a}{\sigma\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{x}(a+x)^{(b+1)}} \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) dx.$$

The distribution of ΔY_{T_t} is called the Pareto-normal.

Proof

$$\begin{aligned}
f(y) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) \frac{u(x)}{\sqrt{x}} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) \left(\frac{ab^a}{\sqrt{x}(a+x)^{(b+1)}}\right) dx \\
&= \frac{ab^a}{\sigma\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{x}(a+x)^{(b+1)}} \exp\left(-\frac{(y-\mu x)^2}{2\sigma^2 x}\right) dx \tag{4.27}
\end{aligned}$$

4.4.6 Subordinated Lévy process models

Under the subordinated exponential Lévy process models, we define our stock process as follows,

$$S_t = S_0 \exp(Y_{T_t}),$$

where Y_t is a Lévy process that is independent of the subordinator process T_t .

In order to calculate arbitrage free option price, we need to change from the objective probability measure P to an equivalent martingale measure Q . This is done by calculating the value of τ (from the Esscher transform) that ensures that the discounted stock price

process follows a Q martingale. This happens if and only if,

$$\begin{aligned}
 \exp(-r)E[S_1] &= S_0 \\
 S_0E[\exp(Y_1)] &= \exp(r)S_0 \\
 E[\exp(Y_1)] &= \exp(r) \\
 \int_{-\infty}^{\infty} \exp(y)f_{Y_1}(y)dy &= \exp(r) \\
 \int_{-\infty}^{\infty} \frac{\exp(y)f(y)\exp(\tau y)}{\int_{-\infty}^{\infty} \exp(\tau x)f_X(x)dx} dy &= \exp(r) \\
 \int_{-\infty}^{\infty} \frac{\exp((\tau+1)y)f(y)}{\int_{-\infty}^{\infty} \exp(\tau x)f_X(x)dx} dy &= \exp(r) \\
 \frac{1}{\int_{-\infty}^{\infty} \exp(\tau x)f_X(x)dx} \int_{-\infty}^{\infty} \exp((\tau+1)y)f(y)dy &= \exp(r) \tag{4.28}
 \end{aligned}$$

We used subscript $t = 1$ from the first equality through out the proof. We can use numerical integration to solve (4.28) in order to get the value of τ that makes $\exp(-r)S_t$ a martingale. Our density function under the martingale measure Q will be attained by substituting the value of τ in the formula (2.1). Once the martingale measure Q is obtained then we can now calculate the arbitrage free option prices.

Chapter 5

Empirical results

5.1 Introduction

In this chapter we fit each of the models discussed in the previous chapter to the observed stock price data. Following this step, we calculate the corresponding option prices under the probability measure obtained using the Esscher transform method. All the programming is done in R. The reason we fit the models first is because we need to see how much of the observed stock data is being explained by the models. We are going to compare the models and see which one best fits the stock price data. This is going to be done using integrated square error (ISE). The integrated square error is defined as,

$$ISE = \int_{-\infty}^{\infty} (f_h(y) - f_e(y))^2 dy,$$

where f_h is the kernel density estimate and f_e is the estimated distribution. The expression above can be calculated using numerical integration as an approximation as it is not given in closed form. The ISE considered here is a discretised approximation. This approximation is obtained by evaluating the function provided above in 512 equally spaced points determined by a subroutine in R. The model that gives the smallest ISE value is considered to be the best model.

We use maximum likelihood estimation to estimate the parameters of the models. We also employ an optimisation procedure in R called the Nelder-Mead. This is usually used when we are changing the probability measures using the Esscher transform method. For f_h we used the Gaussian kernel density estimate and the bandwidth is chosen according to the Silverman's rule of thumb. For a review of the choice of bandwidth, see [12].

We are also going to calculate the option prices using these models and see which model predicts the observed option prices better than the others. The criterion used to

compare the option prices is the root mean square error denoted as RMSE. It is defined as,

$$RMSE = \sqrt{\frac{1}{n} \sum_{k=1}^n (\pi_k^o - \pi_k^e)^2}$$

where π^o and π^e are the observed and estimated option prices, respectively and n is the number of the option prices considered. The subscript k indicates the k^{th} observed or calculated option price. The model that gives the least RMSE is considered to be the best model.

We are considering two different data sets in our analysis. The first data set is from S&P 500 in the year of 2002. The data set consist of 126 observed option prices and 74 observed stock prices. The second data is from Eurostoxx. It has 254 observed stock prices and 144 observed option prices. The reason we used these data sets is because there are accessible easily and they give different results discussed at the end of the next chapter . Current option prices data are difficult and expensive to get.

5.2 Model fitting

In this section we are fitting the financial models to the the S&P 500 data set.

5.2.1 The Black-Scholes model

Fitting the Black-Scholes models entails fitting a normal distribution to the observed log-returns. When fitting the normal distribution we obtain the following estimates;

$$\hat{\mu} = 0.0011 \text{ and } \hat{\sigma}^2 = 0.0124.$$

Figure 5.1 shows a kernel density estimate of the log-returns (in green). In order to aid comparison, the estimated normal density is superimposed (in red) in the figure. The integrated square error is given to be;

$$ISE = 0.08703349.$$

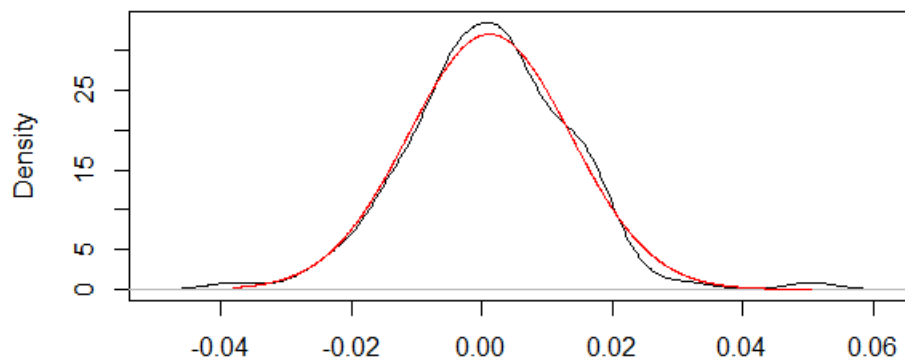


Figure 5.1: The Kernel estimated density of the observed log-returns (green) with fitted normal density (red) superimposed

5.2.2 The geometric normal inverse Gaussian process model

From Figure 5.2 we see that the geometric normal inverse Gaussian model explains the variation in the observed log-returns well but not as well as the Black-Scholes model. When fitting the geometric normal inverse Gaussian model we obtain the following estimates;

$$\hat{\alpha} = 106.1905, \hat{\beta} = 2.1024, \hat{\mu} = 0.000825 \text{ and } \hat{\delta} = 0.0171.$$

Figure 5.2 shows a kernel density estimate of the log-returns (in green). In order to aid comparison, the estimated geometric normal inverse Gaussian density is superimposed (in red) in the figure. The integrated square error is given to be;

$$ISE = 0.4501165.$$

We can see that the integrated square error value of the geometric normal inverse Gaussian model is greater than that of the Black-Scholes model. This means that the Black-Scholes model fits the observed log-returns better.

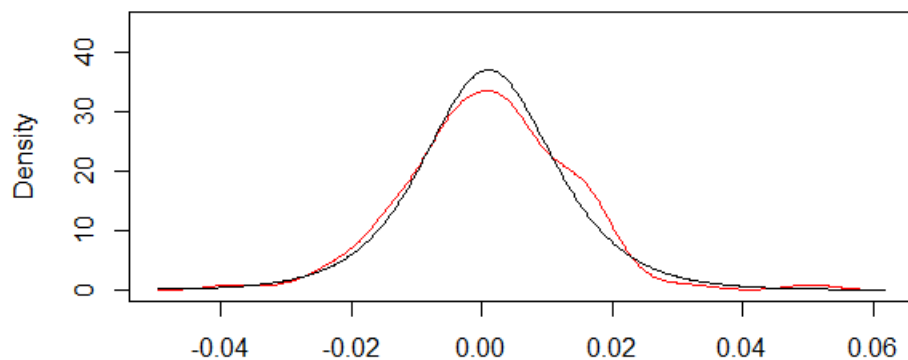


Figure 5.2: The Kernel estimated density of the observed log-returns (green) with fitted geometric normal inverse Gaussian density (red) superimposed

5.2.3 The geometric Meixner process model

Figure 5.3 shows that the geometric Meixner model fits the stock price data better than the geometric normal inverse Gaussian model. The estimated parameters of the model are;

$$\hat{\alpha} = 0.00209, \hat{\beta} = 1.8089, \hat{\mu} = -0.0639 \text{ and } \hat{\delta} = 24.1465.$$

The integrated square error is given to be;

$$ISE = 0.3143507.$$

This value is smaller than the one for the geometric normal inverse Gaussian model.

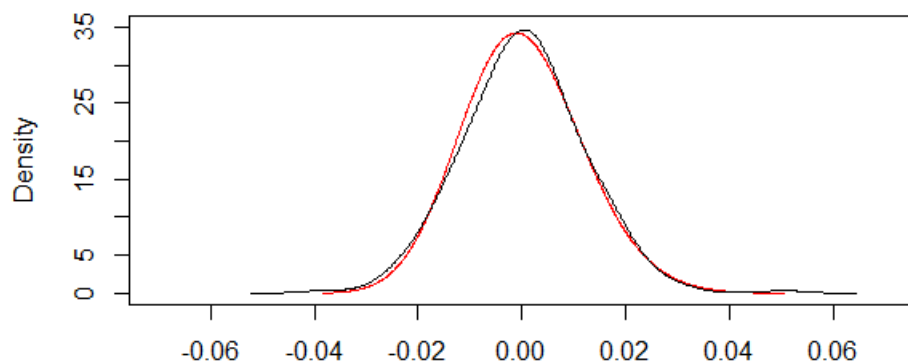


Figure 5.3: The Kernel estimated density of the observed log-returns (green) with fitted geometric Meixner density (red) superimposed

5.2.4 The exponential lognormal-normal process model

Figure 5.4 shows that the lognormal-normal model fits the stock price data better than the geometric normal inverse Gaussian model and the geometric Meixner model. The estimated parameters of the model are;

$$\hat{\alpha} = 0.2995, \hat{\beta} = 0.4889, \hat{\mu} = 0.000253 \text{ and } \hat{\sigma}^2 = 0.00937.$$

The integrated square error is given to be;

$$ISE = 0.1150531.$$

This value is lower than that of the geometric normal inverse Gaussian model but greater than that of the Black-Scholes model.

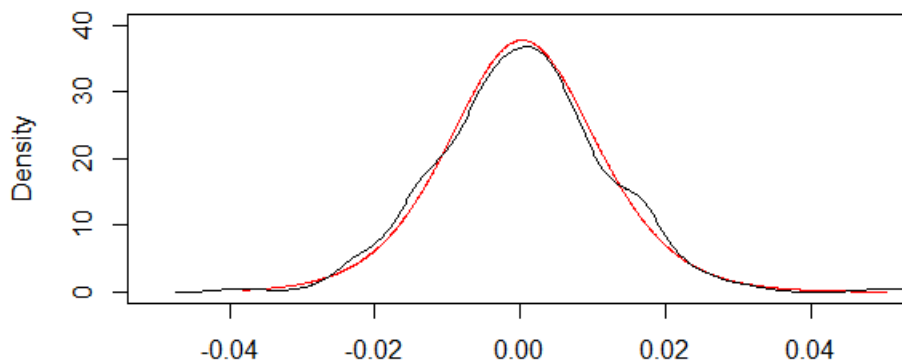


Figure 5.4: The Kernel estimated density of the observed log-returns (green) with fitted exponential lognormal-normal density (red) superimposed

5.2.5 The exponential Pareto-normal process model

From Figure 5.5 we see that the exponential Pareto-normal model does not explain the variation in the observed log-returns well. When fitting the exponential Pareto-normal model we obtain the following estimates;

$$\hat{g} = 0.0219, \hat{h} = 44.7359, \hat{\mu} = -0.2553 \text{ and } \hat{\sigma}^2 = 0.5656.$$

Figure 5.5 shows a kernel density estimate of the log-returns (in green). In order to aid comparison, the estimated exponential Pareto-normal density is superimposed (in red) in

the figure. The integrated square error is given to be;

$$ISE = 1.346161.$$

This is highest ISE value attained so far which means that the exponential Pareto-normal model is the model that least fits the observed log-returns of the S&P 500 data.

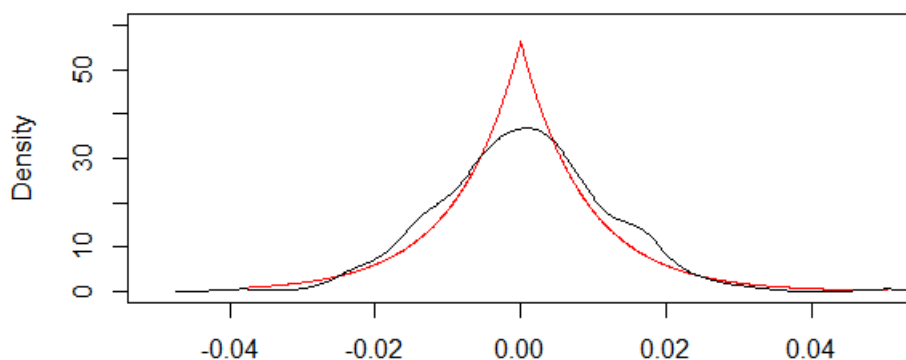


Figure 5.5: The Kernel estimated density of the observed log-returns (green) with fitted exponential Pareto-normal density (red) superimposed

5.3 Calculation of the empirical option prices

In this section we are going to use the models discussed above to calculate the option prices. We will also compare these empirical option prices with the observed ones to see which model best mimics the real world option prices. As discussed in Section 5.1 we will be using the root mean square error (RMSE) as the ranking criterion.

5.3.1 The Black-Scholes model

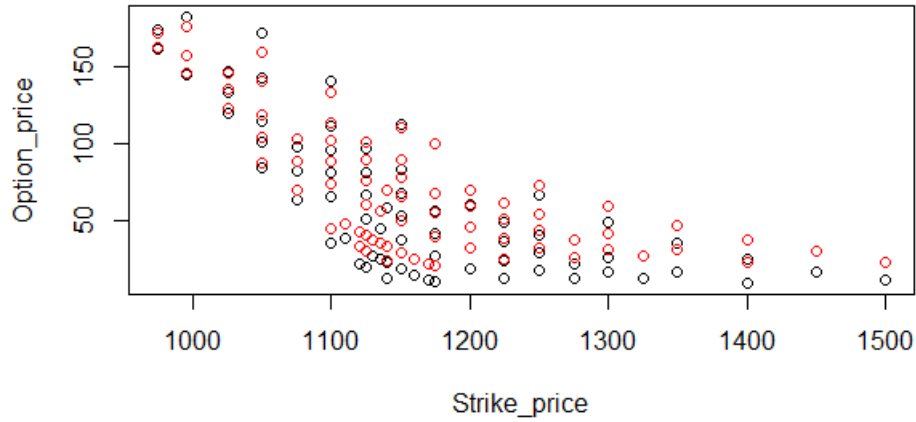


Figure 5.6: Empirical (red) and observed (black) option prices

The graph above shows that the option prices calculated from the Black-Scholes model are almost the same as the observed ones. Option prices with strike prices that are greater than \$100, are being over-estimated by the empirical option prices. The root mean square error value of the Black-Scholes model is given below,

$$RMSE = 10.17715.$$

5.3.2 The geometric normal inverse Gaussian process model

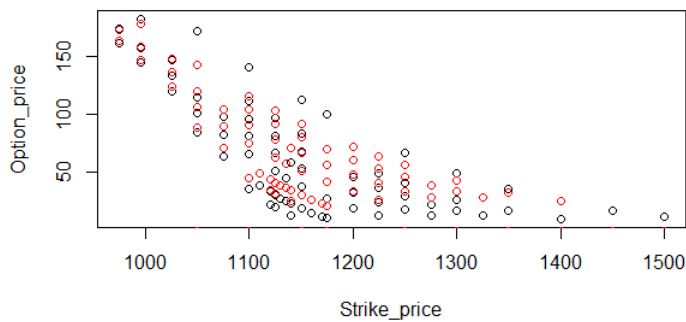


Figure 5.7: Empirical (red) and observed (black) option prices

The geometric normal inverse Gaussian model option prices are under-estimating the observed option prices. This is seen above as most of the black small circles are above the corresponding red ones. The graph shown in Figure (5.2) shows that the geometric normal inverse Gaussian model does not fit the stock price data well which will not be a

surprise when the model gives option prices that under-estimates the observed ones. We can already expect the root mean square value to be higher than that of the Black-Scholes model,

$$RMSE = 16.45665.$$

The root mean square error value of the geometric normal inverse Gaussian model is greater than the one for the Black-Scholes model. This implies that the Black-Scholes model mimics the real world option prices better.

5.3.3 The exponential lognormal-normal process model

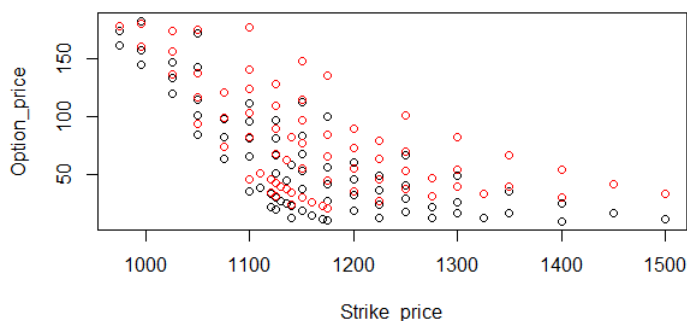


Figure 5.8: Empirical (red) and observed (black) option prices

The exponential lognormal-normal model over-estimates the observed option prices. This is seen from Figure (5.8) as the empirical option prices plots above the corresponding observed option prices. The root mean square error value of the exponential lognormal-normal model is given below,

$$RMSE = 23.0822.$$

The above RMSE value is the greatest so far which means that the exponential lognormal-normal is, so far, the least in mimicking the real world option prices. From Figure (5.4) we see that the model fits the stock price data very well. However, it still gave us an over-estimation of the option prices. This means that it is not always the case that a model which fits the stock prices well gives the best option prices.

5.3.4 The exponential Pareto-normal process model

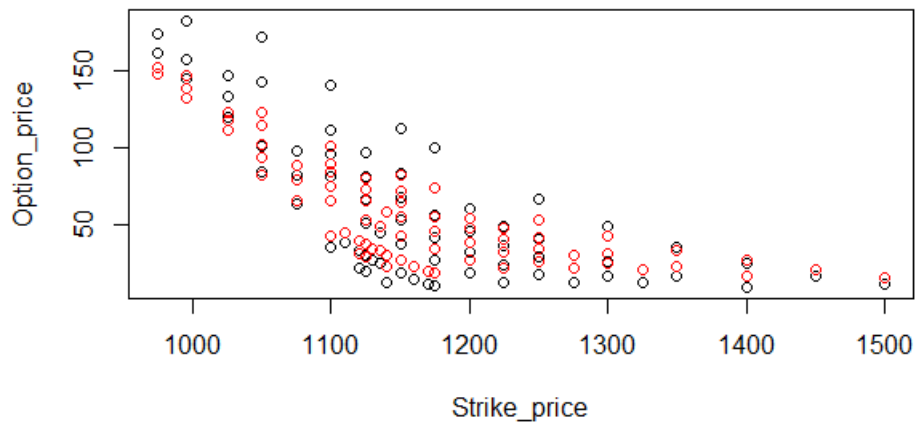


Figure 5.9: Empirical (red) and observed (black) option prices

From Figure (5.9) we see that the option price calculated using the exponential Pareto-normal model under-estimates the observed option prices. This is further seen in Figure (5.5) as the density of the exponential Pareto-normal has a greater kurtosis than the real world stock price data. The root mean square error is given below,

$$RMSE = 13.29202$$

This RMSE value is smaller than that of the other models except the one for the Black-Scholes model. This means that even though the exponential Pareto-normal under-estimates the option prices, it is the second best so far.

5.3.5 The geometric Meixner process model

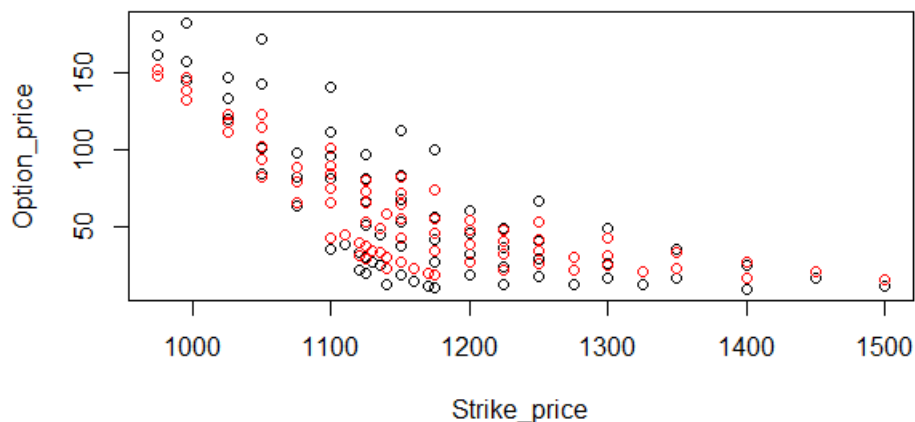


Figure 5.10: Empirical (black) and observed (red) option prices

From Figure (5.10) we see that this model mimics the observed option price better than most models mentioned above. This is seen as most of the empirical option prices are so close to the corresponding observed option prices. Further evidence is attained from Figure (5.3), which shows that the density of the model fits the stock price well. The root mean square error is given below,

$$RMSE = 11.47488.$$

This value is the second smallest compared to the ones above.

We are now going to analyse the data set from Eurostoxx to see if we will get the same results. We will apply the same criteria as above to see which model fits the stock price data best and which model mimics the real world option prices.

5.4 Model fitting

5.4.1 The Black-Scholes model

Fitting the Black-Scholes models entails fitting a normal distribution to the observed log-returns. When fitting the normal distribution we obtain the following estimates;

$$\hat{\mu} = 0.0006214 \text{ and } \hat{\sigma}^2 = 0.02017.$$

Figure 5.11 shows a kernel density estimate of the log-returns (in green). In order to aid comparison, the estimated normal density is superimposed (in red) in the figure. The integrated square error is given to be;

$$ISE = 0.2015346.$$

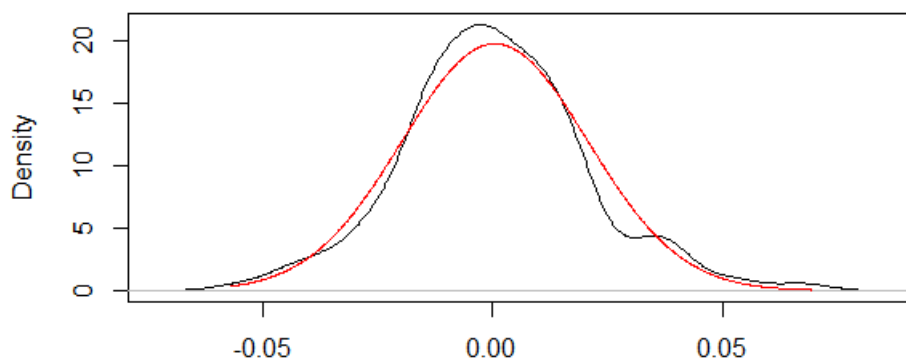


Figure 5.11: The Kernel estimated density of the observed log-returns (green) with fitted normal density (red) superimposed

5.4.2 The geometric normal inverse Gaussian process model

From Figure 5.12 we see that the geometric normal inverse Gaussian model explains the variation in the observed log-returns well better than the Black-Scholes model. When fitting the geometric normal inverse Gaussian model we obtain the following estimates;

$$\hat{\alpha} = 49.0987, \hat{\beta} = 3.2245, \hat{\mu} = -0.001253 \text{ and } \hat{\delta} = 0.0223.$$

Figure 5.12 shows a kernel density estimate of the log-returns (in green). In order to aid comparison, the estimated geometric normal inverse Gaussian density is superimposed (in red) in the figure. The integrated square error is given to be;

$$ISE = 0.1359597.$$

We can see that the integrated square error value of the geometric normal inverse Gaussian model is smaller than that of the Black-Scholes model. This means that the geometric normal inverse Gaussian model fits the observed log-returns better.

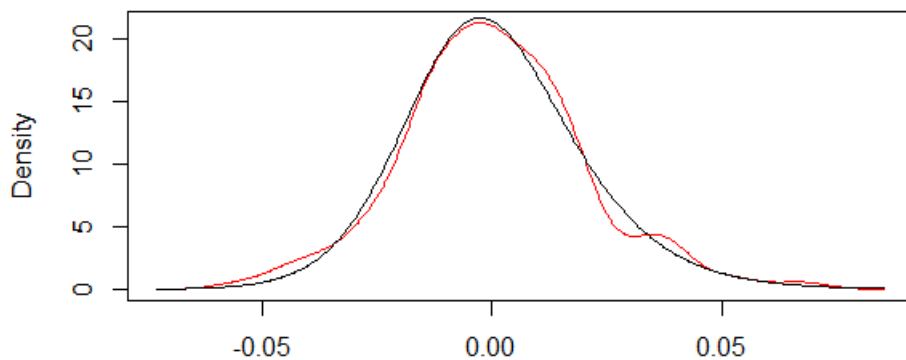


Figure 5.12: The Kernel estimated density of the observed log-returns (green) with fitted geometric normal inverse Gaussian density (red) superimposed

5.4.3 The geometric Meixner process model

Figure 5.13 shows that the geometric Meixner model does not fit the observed log-returns data better than the geometric normal inverse Gaussian model. The estimated parameters of the model are;

$$\hat{\alpha} = 0.006044, \hat{\beta} = 1.0494, \hat{\mu} = -0.05739 \text{ and } \hat{\delta} = 16.5452.$$

The integrated square error is given to be;

$$ISE = 0.198733.$$

This value is greater than the ISE value of the geometric normal inverse Gaussian models.

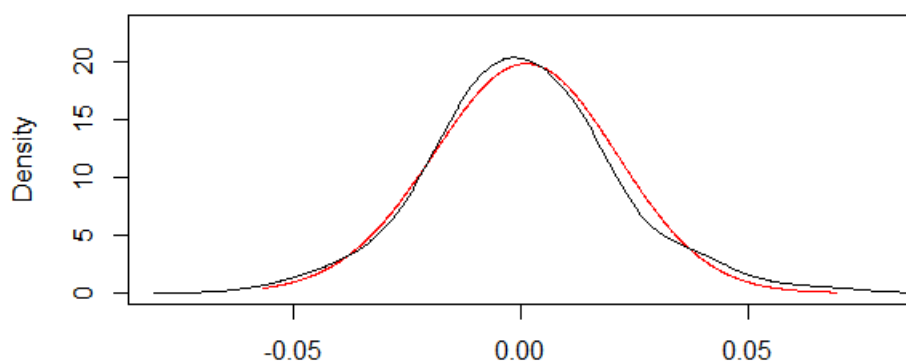


Figure 5.13: The Kernel estimated density of the observed log-returns (green) with fitted geometric Meixner density (red) superimposed

5.4.4 The exponential lognormal-normal process model

Figure 5.14 shows that the exponential lognormal-normal model fits the stock price data better than the geometric normal inverse Gaussian model. The estimated parameters of the model are;

$$\hat{\alpha} = 1.2209, \hat{\beta} = 0.6295, \hat{\mu} = 0.0001277 \text{ and } \hat{\sigma}^2 = 0.009969.$$

The integrated square error is given to be;

$$ISE = 0.1008621.$$

This value is lower than that of the geometric normal inverse Gaussian model and that of the Black-Scholes model.

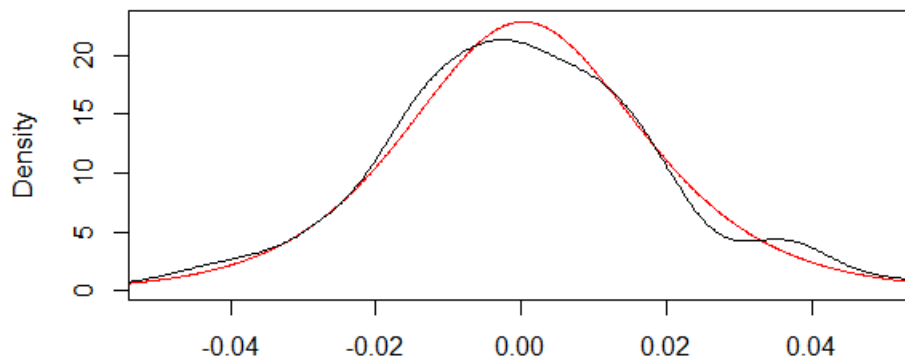


Figure 5.14: The Kernel estimated density of the observed log-returns (green) with fitted exponential lognormal-normal density (red) superimposed

5.4.5 The exponential Pareto-normal process model

From Figure 5.15 we see that the exponential Pareto-normal model does not explain the variation in the observed log-returns well. When fitting the exponential Pareto-normal model we obtain the following estimates;

$$\hat{g} = 0.01596, \hat{h} = 23.5141, \hat{\mu} = 0.864 \text{ and } \hat{\sigma}^2 = 0.8281.$$

Figure 5.15 shows a kernel density estimate of the log-returns (in green). In order to aid comparison, the estimated exponential Pareto-normal density is superimposed (in red) in

the figure. The integrated square error is given to be;

$$ISE = 0.7625883.$$

This is the highest ISE value attained so far which means that the exponential Pareto-normal model is the model that least fits the observed log-returns.

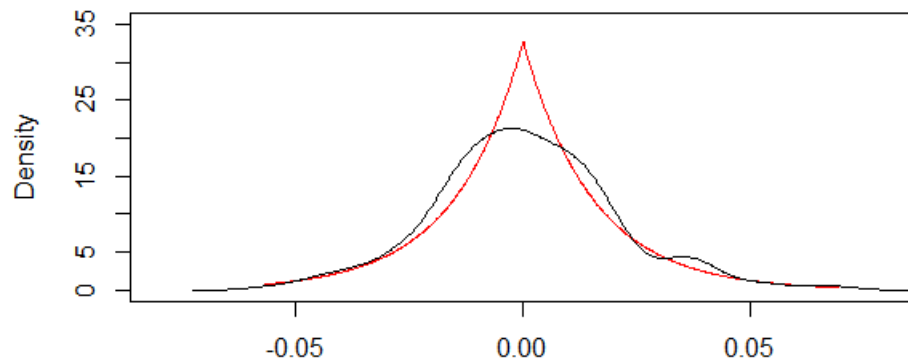


Figure 5.15: The Kernel estimated density of the observed log-returns (green) with fitted exponential Pareto-normal density (red) superimposed

5.5 Empirical option prices calculation

We are going to calculate the empirical option prices using the above-mentioned financial models. We are also going to plot both the empirical and the observed option prices to clearly see how well the models can estimate the option prices. The root mean square error is going to be used as a ranking criterion to see which model mimics best the real world option prices.

5.5.1 The Black-Scholes model

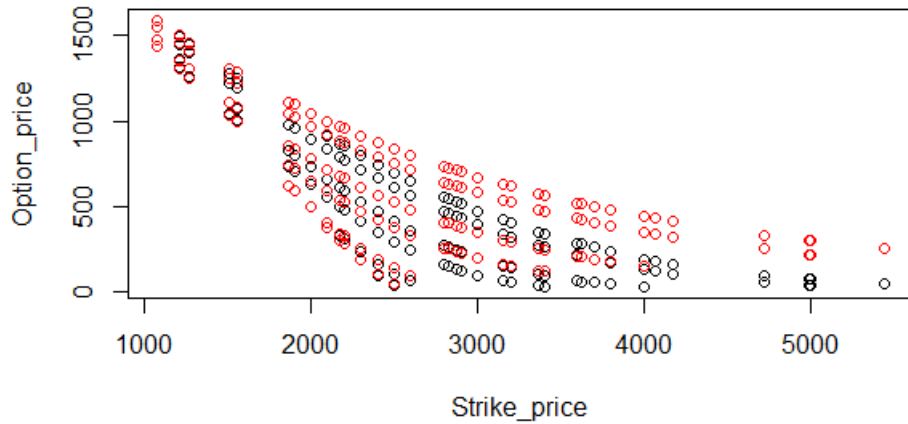


Figure 5.16: Empirical (red) and observed (black) option prices

From Figure (5.16) we see that the Black-Scholes model estimates the observed option prices well for options with the strike price between \$1 050 and \$1 600. Option prices with the strike price greater than \$1 600 are over-estimated. This shown with the red circles higher than the black ones. Further evidence is attained from Figure (5.11), which depicts the density of the Black-Scholes model having a lower kurtosis value than that of the stock price data. The root mean square error is given below,

$$RMSE = 131.6587.$$

5.5.2 The geometric normal inverse Gaussian process model

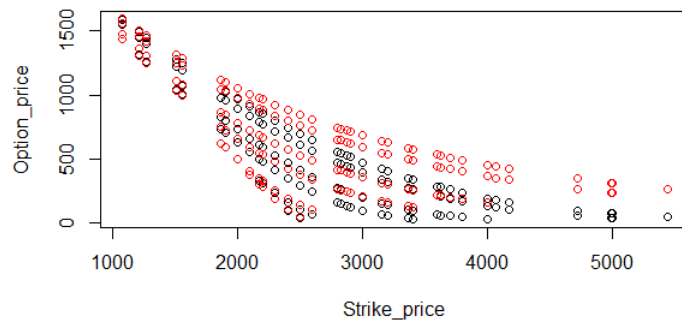


Figure 5.17: Empirical (red) and observed (black) option prices

Overall, the geometric normal inverse Gaussian model over-estimates the observed stock prices. Option prices with strike prices between \$3 100 and \$3 300 were estimated well.

The root mean square error value is given below,

$$RMSE = 151.1887$$

This value is greater than that of the Black-Scholes model. This means that the Black-Scholes model is better in terms of mimicking the observed option prices.

5.5.3 The exponential lognormal-normal process model

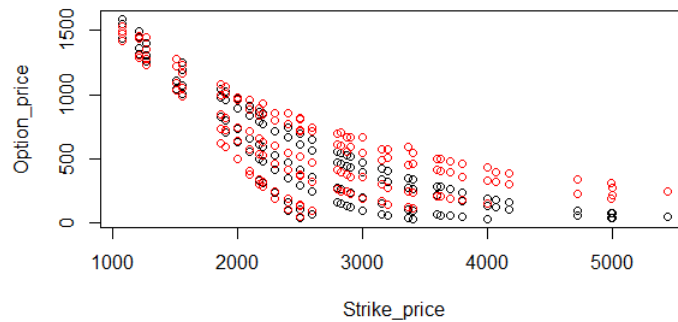


Figure 5.18: Empirical (red) and observed (black) option prices

Option prices with the strike price below \$1 400 are being under-estimated by the model. From the strike price greater than \$1 400, the model over-estimates the observed option prices. The extent of over-estimation is better than that of the Black-Scholes model. This is seen by the root mean square value of the exponential lognormal-normal model being smaller than that of the Black-Scholes. The RMSE value is given below,

$$RMSE = 117.0116.$$

5.5.4 The exponential Pareto-normal process model

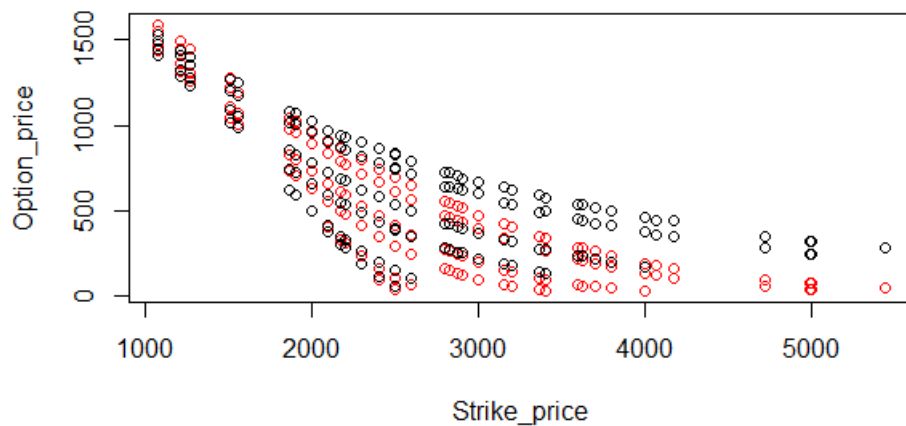


Figure 5.19: Empirical (red) and observed (black) option prices

The exponential Pareto-normal model under-estimates all the observed option prices. This is expected as its density has a kurtosis greater than that of the observed stock prices. The model only mimics very well, the option price with a strike price of about \$5 600. The root mean square error is given as;

$$RMSE = 120.7591.$$

The RMSE value given above is greater than the exponential lognormal-normal one but less than the Black-Scholes one. This means that the exponential Pareto-normal model is better than the Black-Scholes model.

5.5.5 The geometric Meixner process model

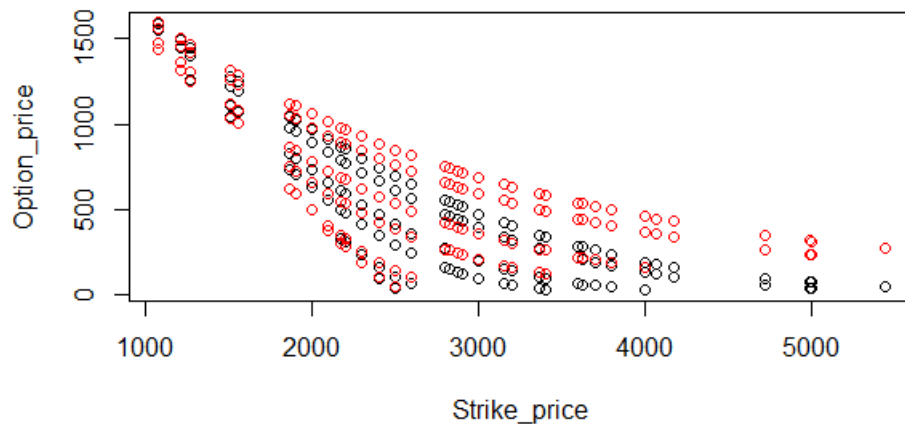


Figure 5.20: Empirical (red) and observed (black) option prices

The option prices calculated using the geometric Meixner model over-estimates the observed option prices. This is seen in Figure (5.20), where the red circles are plotting above the black ones. This implies that though the model fits the stock price data well, it does not guarantee that the empirical option prices will also be good estimates of the observed option prices. Figure (5.13) shows a good fit of the model to stock price data. The level of error is quantified by the RMSE given below.

$$RMSE = 129.1696.$$

The above RMSE value is the second largest from the ones calculated so far. This means that the geometric Meixner model is the second least in mimicking the real world option prices.

Chapter 6

Conclusion

Below we compare the results obtained in the previous chapter. This is done in order to see which model fits the log-returns best. We use the integrated square error as our ranking criterion. We also compare the different root mean square errors for different models to see which model mimics the observed option prices best. In Section 6.1 we analyse the results from the first data set. Analysis of the second data set is going to be done in Section 6.2. Overall conclusion is going to be done in Section 6.3.

We analysed two different sets of observed log-returns and the corresponding observed option prices. One data set is from the S&P 500 and the second one is from Eurostoxx. We specifically chose these data sets because they are readily available.

6.1 Analysis of the results attained from the S&P 500 data set

Below, is a compilation of the integrated square errors (ISE) attained from different financial models using the first data set.

Financial model	Integrated square error
the Black-Scholes model	0.0870
the geometric normal inverse Gaussian process model	0.4501
the exponential lognormal-normal process model	0.1151
the exponential Pareto-normal process model	1.346161
the geometric Meixner process model	0.3144

Table 6.1: The integrated square errors for different financial models

From the table above we see that the Black-Scholes model has the smallest integrated square error value. This means that the Black-Scholes model is the one that fits the

log returns data best. The exponential Pareto-normal model has the highest ISE value meaning that it gives least fit to the log returns data. We are now going to see which financial model gives better empirical option prices which mimics the real world option prices. This is done through the root mean square errors given below;

Financial model	Root mean square error
the Black-Scholes model	10.1772
the geometric normal inverse Gaussian process model	16.4567
the exponential lognormal-normal process model	23.0822
the exponential Pareto-normal process model	13.292
the geometric Meixner process model	11.4749

Table 6.2: The root mean square errors for different financial models

From the above table we see that the Black-Scholes model has the smallest root mean square error value. This means that it is the best model that mimics the observed option prices. This is not surprising because the Black-Scholes model fit the log returns data better than the other models. The exponential lognormal-normal model has the largest RMSE value. Note that the exponential Pareto-normal model performs quite well in terms of the RMSE in spite of providing a substantially higher ISE than any of the other models. We now analyse the second data set.

6.2 Analysis of the results attained from the Eurostoxx data set

The integrated square errors calculated from the second data set are given below.

Financial model	Integrated square error
the Black-Scholes model	0.2015
the geometric normal inverse Gaussian process model	0.136
the exponential lognormal-normal process model	0.1009
the exponential Pareto-normal process model	0.7626
the geometric Meixner process model	0.1987

Table 6.3: The integrated square errors for different financial models

The exponential lognormal-normal model has the smallest ISE value. This means that it fits the log returns data better than the other models. The exponential Pareto-normal model has the largest integrated square error. The table of the RMSE values is given below.

Financial model	Root mean square error
the Black-Scholes model	131.6587
the geometric normal inverse Gaussian process model	151.1887
the exponential lognormal-normal process model	117.0116
the exponential Pareto-normal process model	120.7591
the geometric Meixner process model	129.1696

Table 6.4: The root mean square errors for different financial models

From the table above, we see that the exponential lognormal-normal model has the smallest root mean square error value. This implies that it estimates the observed option prices better than the other models. The geometric normal inverse Gaussian model has the largest RMSE value, meaning that it is the least one.

Compare the performance of the geometric normal inverse Gaussian model and the exponential Pareto-normal model. The geometric normal inverse Gaussian model provides a small ISE and a large RMSE while the opposite is true for the exponential Pareto-normal model

6.3 Conclusion

In this study, we discuss five different financial models which are Lévy processes. The first of these models is the Black-Scholes model, under which the log-return process is assumed to be a Brownian motion. Other financial models assume that the stock returns are from a Lévy process that has jumps. These models are the geometric normal inverse Gaussian, geometric Meixner and the time-changed models. The time-changed models assume the number of trades per given time interval to be a random variable which follow a specific distribution. These models are the exponential Pareto-normal and the exponential lognormal-normal models.

We also showed how we manipulate the parameters of these models when moving from the objective probability measure to the equivalent martingale measure. This is done so as to ensure that the option prices calculated under these models are arbitrage-free. We also compared the different financial models to see which model fits the log return data better than the others and which model best mimics the real world option prices. Before we calculated the option prices we first had to estimate the parameters using the observed log returns data. From the analysis done above, we saw that the best model fit depends the nature of the data set. If the log-returns are normally distributed then the Black-Scholes model tends to be the best model in both cases. This is shown in the analysis of the S&P 500 data set given in Section 6.1. If the log-returns are not normally distributed then

other models will be the best fit. This shown in Section 6.2 as we analysed the Eurostoxx data set when the exponential lognormal-normal model became the best fit.

Again, from the analysis done above we saw that sometimes when the model fits the log returns data very well then it will also mimic the observed option prices well. This is shown when the Black-Scholes and the exponential lognormal-normal models were the best in fitting both the log returns data and the observed option prices. However, this is not always the case. Some models fit the log returns data well but still give bad empirical option prices. This is shown in Section 6.2 as the geometric normal inverse Gaussian model was the second best in fitting the log returns data but the fifth in estimating the option prices. This situation is also elaborated on Section 6.1 as the exponential lognormal-normal model was the second best in fitting the log-returns, but performed worst in estimating the observed option prices. This means that there exists some level of discrepancy between the model fit and the option price calculation.

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Appendix

Algorithms for calculating option prices using the different financial models

In this chapter we are going to list the algorithms used to fit the financial models to the log returns of the stock prices.

The Black-Scholes model

- Calculate the log-returns from the stock prices.
- Estimate the parameters mean and variance using the log returns. This is because the Black-Scholes model follows the normal distribution when using the log returns
- Generate the normal density function values with the mean and variance estimated above.
- Plot the log returns and the generated normal density values on the same axes to see which if it fits the data well.
- Use the Black-Scholes formula to calculate the option prices. The formula is given as;

$$\Pi = S_0 \left(\Phi \left(\frac{\log \frac{K}{S_0} + (r + \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}} \right) \right) - K \exp(-rT) \Phi \left(\frac{\log \frac{K}{S_0} + (r - \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}} \right).$$

where Φ represents the normal distribution function.

- Repeat the previous step for different values of K and T .

The geometric normal inverse Gaussian process model

- Calculate the log returns from the stock prices.

- Estimate the parameters $(\alpha, \beta, \mu, \delta)$ using the maximum likelihood estimation.
- Generate the density function values with the parameters estimated above.
- Plot the log returns and the generated density values on the same axes to see which if it fits the data well.
- Esscher transform method is used to change from the objective probability measure to a martingale measure by manipulating the estimated parameters. In this instance only the parameter β is affected.
- There is no precise formula for the calculation of the option price using the geometric normal inverse Gaussian model. This leads to the usage of numerical integration. In this instance only the parameter β is affected.
- Numerical integration is performed on the following equation;

$$e^{-rT} \int_{\left(\frac{\log \frac{K}{S_0} - \mu T}{\sigma \sqrt{T}}\right)}^{\infty} \left[\left(S_0 \exp(\mu T + \sigma \sqrt{T} Z) - K \right) \right] \phi(z) dz$$

where ϕ represents the density function of the geometric normal inverse Gaussian model.

- There are instances where the parameters gets so large which leads to difficulties in the density function calculation. This problem can be circumvented by using the Fourier method discussed in section (3.4.3).

The exponential lognormal-normal process model

- Calculate the log returns from the stock prices.
- Estimate the parameters $(\alpha, \beta, \mu, \sigma)$ using the maximum likelihood estimation.
- Generate the density function values with the parameters estimated above.
- Plot the log returns and the generated density values on the same axes to see which if it fits the data well.
- Esscher transform method is used to change from the objective probability measure to a martingale measure by manipulating the estimated parameters. This is done

by optimising the following equation to get the value of τ ;

$$\frac{1}{\int_{-\infty}^{\infty} \exp(\tau x) f_X(x) dx} \int_{-\infty}^{\infty} \exp((\tau + 1)y) f(y) dy = \exp(r).$$

The martingale density function will be $f(y + \tau)$.

- Calculation of the option prices is done using the numerical integration on the following equation;

$$e^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} (S_0 e^{Y_{T_i}} - K) f(y + \tau; \mu, \sigma^2, \alpha, \beta) dy.$$

The exponential Pareto-normal process model

- Calculate the log returns from the stock prices.
- Estimate the parameters (g, h, μ, σ) using the maximum likelihood estimation.
- Generate the density function values with the parameters estimated above.
- Plot the log returns and the generated density values on the same axes to see which if it fits the data well.
- Esscher transform method is used to change from the objective probability measure to a martingale measure by manipulating the estimated parameters. This is done by optimising the following equation to get the value of τ ;

$$\frac{1}{\int_{-\infty}^{\infty} \exp(\tau x) f_X(x) dx} \int_{-\infty}^{\infty} \exp((\tau + 1)y) f(y) dy = \exp(r).$$

The martingale density function will be $f(y + \tau)$.

- Calculation of the option prices is done using the numerical integration on the following equation;

$$e^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} (S_0 e^{Y_{T_i}} - K) f(y + \tau; \mu, \sigma^2, g, h) dy.$$

The geometric Meixner process model

- Calculate the log returns from the stock prices.
- Estimate the parameters $(\alpha, \beta, \mu, \delta)$ using the maximum likelihood estimation.

- Generate the density function values with the parameters estimated above.
- Plot the log returns and the generated density values on the same axes to see which if it fits the data well.
- Esscher transform method is used to change from the objective probability measure to a martingale measure by manipulating the estimated parameters. In this instance only the parameter β is affected.
- There is no precise formula for the calculation of the option price using the geometric normal inverse Gaussian model. This leads to the usage of numerical integration. In this instance only the parameter β is affected.
- Numerical integration is performed on the following equation;

$$e^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} [(S_0 \exp(Y_T) - K)] f_{Y_T}^{(Q)}(y) dy$$

where $f_{Y_t}^{(Q)}(y)$ represents the density function of the geometric Meixner model.

- There are instances where the parameters gets so large which leads to difficulties in the density function calculation. This problem can be circumvented by using the Fourier method discussed in section (3.4.3).

R-codes for the S&P 500 data set

Black-Scholes model

```
#####
```

```
#Entering the data
```

```
set.seed(12345)
```

```
data = read.csv("S&P 500 2002 DATA.csv")
```

```
prices = data[,5]
```

```
prices = prices[1:126]
```

```
prices = prices[length(prices):1]
```

```
plot(prices,type = "l")
```

```
#####
```

```
# Enter the option price data
```



```
Data <- read.csv("C:/Users/Clemence R Kwinje/Desktop/R Files/OptionData/Schoutens/Schout
T    <- Data[,1]
Strike_price    <- Data[,2]
Option_price <- Data[,3]
n2 = length(Strike_price)
plot(Strike_price,Option_price)
```

```
#####
```

```
#Calculating log-returns
```

```
n = length(prices)-1
logrets = 1:n*0
for (j in 1:n){
  logrets[j] = log(prices[j+1]/prices[j])
}
plot(logrets,type = "l")
plot(hist(logrets))
lines(density(logrets))
```

```
#####
```

```
# Fit normal distribution
```

```
muHat    = mean(logrets)
sigmaHat = sd(logrets)*sqrt((n-1)/n)
```

```
#sum(log(dnorm(logrets,muHat,sigmaHat)))
```

```
#install.packages("nortest")
```

```
#library(nortest)
```

```
#cvm.test(logrets)
```

```
plot(density(logrets))
```

```
x_min = min(logrets)
```

```
x_max = max(logrets)
```

```
x      = seq(x_min,x_max,(x_max-x_min)/999)
```

```
plot(density(logrets))
```

```

lines(x,dnorm(x,muHat,sigmaHat),col = "red",type = "l")
#####
#ISE:

xx = density(logrets)$x
yy = density(logrets)$y
yy2 = dnorm(xx,muHat,sigmaHat)

plot(xx,yy,type="l",col="red")
lines(xx,yy2)

plot(xx,(yy-yy2)^2,type="l",col="red")

ISE = sum((yy-yy2)^2)*(xx[2]-xx[1])
ISE

#####
# Calculating option prices
sig=sigmaHat
S=1124.47
r=0.007/252

BlackScholesPrice <- function(sig,S,T,r,Strike_price){

m=exp(-r*T)

a1=(log(S/Strike_price)+(r+0.5*sig^2)*T)/(sig*sqrt(T))
a2=(log(S/Strike_price)+(r-0.5*sig^2)*T)/(sig*sqrt(T))

b1=pnorm(a1, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
b2=pnorm(a2, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)

OpPrice=S*b1-Strike_price*m*b2
return(OpPrice)

}

```

```

CalcP <- BlackScholesPrice(sig,S,T,r,Strike_price)

plot(Strike_price,CalcP,col="red")
#####
# Comparing option prices
plot(Strike_price,Option_price,col="black")
points(Strike_price,CalcP,col="red")

RMSE = sqrt(mean((Option_price - CalcP)^2))
RMSE
#####

```

The geometric normal inverse Gaussian process model

```

#####
#Entering the data

set.seed(12345)
data = read.csv("S&P 500 2002 DATA.csv")
prices = data[,5]
prices = prices[1:126]
prices = prices[length(prices):1]
plot(prices,type = "l")

#####
# Enter the option price data

Data <- read.csv("C:/Users/Clemence R Kwinje/Desktop/R Files/OptionData/Schoutens/Schout
T <- Data[,1]
Strike_price <- Data[,2]
Option_price <- Data[,3]
n2 = length(Strike_price)
plot(Strike_price,Option_price)

#####
#Calculating log-returns

```

```
n = length(prices)-1
logrets = 1:n*0
for (j in 1:n){
logrets[j] = log(prices[j+1]/prices[j])
}
plot(logrets,type = "l")
plot(hist(logrets))
lines(density(logrets))

#####
#NoIG density

alpha = 2
beta = 1
mu = 1
delta = 1

#install.packages("Bessel")
library(Bessel)

f_NoIG <- function(x,alpha,beta,mu,delta){
f = alpha*delta/pi*exp(delta*sqrt(alpha^2-beta^2)+beta*(x-mu))*BesselK(alpha*sqrt(delta^2-beta^2),alpha*sqrt(delta^2-beta^2)+beta*(x-mu))
return(f)
}

x_min = -1
x_max = 5
x = seq(x_min,x_max,(x_max-x_min)/999)
y = f_NoIG(x,alpha,beta,mu,delta)

plot(x,y,type="l")

#####
# NoIG likelihood
```

```
LLvec = f_NoIG(logrets, alpha, beta, mu, delta)
LLvec = log(LLvec)
LL     = sum(LLvec)

if (alpha>0 & alpha>abs(beta) & delta>0){
LLvec = f_NoIG(logrets, alpha, beta, mu, delta)
LLvec = log(LLvec)
LL     = sum(LLvec)
} else {
LL = -Inf
}

parms = c(alpha, beta, mu, delta)

minLL_NoIG <- function(parms){
alpha = parms[1]
beta  = parms[2]
mu    = parms[3]
delta = parms[4]
if (alpha>0 & alpha>abs(beta) & delta>0){
LLvec = f_NoIG(logrets, alpha, beta, mu, delta)
LLvec = log(LLvec)
LL     = sum(LLvec)
} else {
LL = -Inf
}
minLL = -LL
return(minLL)
}

parms = c(alpha, beta, mu, delta)
mLL   = minLL_NoIG(parms)

#####
# Starting values
```

```
n_startvals = 1000

alpha_s <- runif(n_startvals,1,100)
beta_s  <- runif(n_startvals,-80,80)
mu_s    <- runif(n_startvals,-10,10)
delta_s <- runif(n_startvals,1,100)

startvals = rep(0,4)
startevals = 1:n_startvals*0
besteval   = Inf
for (k in 1:n_startvals){
  sparms      <- c(alpha_s[k],beta_s[k],mu_s[k],delta_s[k])
  startevals[k] <- minLL_NoIG(sparms)
  if (is.finite(startevals[k]) & startevals[k]<besteval){
    startvals = sparms
    besteval  = startevals[k]
  }
}

besteval
startvals
minLL_NoIG(startvals)

optm <- optim(startvals,minLL_NoIG)

alphaHat = optm$par[1]
betaHat  = optm$par[2]
muHat    = optm$par[3]
deltaHat = optm$par[4]

x_grid = density(logrets)$x
Density = density(logrets)$y

plot(x_grid,Density,type="l",col="red",ylim=c(0,45))
```

```

fx2_grid = f_NoIG(x_grid,alphaHat,betaHat,muHat,deltaHat)

lines(x_grid,fx2_grid,type="l")
#####
#ISE:

xx = density(logrets)$x
yy = density(logrets)$y
yy2 = f_NoIG(xx,alphaHat,betaHat,muHat,deltaHat)

plot(xx,yy,type="l",col="red")
lines(xx,yy2)

plot(xx,(yy-yy2)^2,type="l",col="red")

ISE = sum((yy-yy2)^2)*(xx[2]-xx[1])
ISE

#####

r=0.007/252

obj <- function(theta){
T1 = muHat
T2 = sqrt(alphaHat^2-(betaHat+theta)^2)
T3 = sqrt(alphaHat^2-(betaHat+theta+1)^2)
objf = abs(T1 + deltaHat*(T2-T3) - r)
return(objf)
}

optm = optimize(obj,c(-20,20))
theta = optm$min

betaHat = betaHat + theta

```

```
obj(theta)
```

```
#####
```

```
#Option Price Calculation
```

```
NoIG_density <- function(x){
```

```
  alpha = parms[1]
```

```
  beta  = parms[2]
```

```
  mu    = parms[3]*Tj
```

```
  delta = parms[4]*Tj
```

```
  f     = alpha*delta/pi*exp(delta*sqrt(alpha^2-beta^2)+beta*(x-mu))*BesselK(alpha*sqrt(de
```

```
  return(f)
```

```
}
```

```
S0    = 1124.41
```

```
parms = c(alphaHat,betaHat,muHat,deltaHat)
```

```
#plot(x_grid,NoIG_density(x_grid),type="l")
```

```
integrand <- function(x){
```

```
  intg = (S0*exp(x)-Kj)*NoIG_density(x)
```

```
  return(intg)
```

```
}
```

```
OptPrice = rep(0,length(Option_price))
```

```
for (j in 1:length(Option_price)){
```

```
  Kj      = Strike_price[j]
```

```
  Tj      = T[j]
```

```
  OptPrice[j] = exp(-r*Tj)*integrate(integrand,log(Kj/S0),3)$value
```

```
}
```

```
plot(Strike_price,Option_price,col="black")
```

```
points(Strike_price,OptPrice,col="red")
```

```
#####
```



```
# Comparing option prices
RMSE = sqrt(mean((Option_price-OptPrice)^2))
RMSE
```

The exponential lognormal-normal process model

```
#####
```

```
#Entering the data
rm(list=ls())
set.seed(12345)
data = read.csv("S&P 500 2002 DATA.csv")
prices = data[,5]
prices = prices[length(prices):1]
plot(prices,type = "l")
```

```
#####
```

```
# Enter the option price data
```

```
Data <- read.csv("C:/Users/Clemence R Kwinje/Desktop/R Files/OptionData/Schoutens/Schout
T <- Data[,1]
Strike_price <- Data[,2]
Option_price <- Data[,3]
n2 = length(Strike_price)
plot(Strike_price,Option_price)
```

```
#####
```

```
#Calculating log-returns
```

```
n = length(prices)-1
logrets = 1:n*0
for (j in 1:n){
  logrets[j] = log(prices[j+1]/prices[j])
}
plot(logrets,type = "l")
plot(hist(logrets))
lines(density(logrets))
```

```
#####
#Pareto-normal density

alpha    = 4.11
beta     = 3.44
mu       = 0
sigma    = 0.008

integrand <- function(y){
  T1 = (y)^(3/2)
  T2 = -(x-mu*y)^2/(2*sigma^2*y)
  T3 = -(log(y)-alpha)^2/(2*beta^2)
  f   = 1/T1*exp(T2+T3)
  return(f)
}

#x = 1
#y = seq(0.01,3,0.01)
#plot(integrand(y),type="l")

x_grid = seq(-0.05,0.05,0.0001)

fx = rep(0,length(x_grid))
for (j in 1:length(x_grid)){
  x      = x_grid[j]
  fx[j] = 1/(sigma*2*pi*beta)*integrate(integrand,0,Inf)$value
}

plot(x_grid,fx,type="l")

sum(fx)*(x_grid[2]-x_grid[1])

f_PN <- function(x_grid,alpha,beta,mu,sigma){

  integrand <- function(y){
    T1 = (y)^(3/2)
```

```

    T2 = -(x-mu*y)^2/(2*sigma^2*y)
    T3 = -(log(y)-alpha)^2/(2*beta^2)
    f = 1/T1*exp(T2+T3)
    return(f)
}

fx = rep(0,length(x_grid))
for (j in 1:length(x_grid)){
  x = x_grid[j]
  fx[j] = 1/(sigma*2*pi*beta)*integrate(integrand,0,Inf)$value
}
return(fx)
}

fx = PN_density(x_grid,alpha,beta,mu,sigma)
plot(x_grid,fx,type="l")

#####
# Pareto-normal likelihood

LLvec = f_PN(logrets,alpha,beta,mu,sigma)
LLvec = log(LLvec)
LL = sum(LLvec)

if (alpha>0 & beta>0 & sigma>0){
  LLvec = f_PN(logrets,alpha,beta,mu,sigma)
  LLvec = log(LLvec)
  LL = sum(LLvec)
} else {
  LL = -Inf
}

parms = c(alpha,beta,mu,sigma)

minLL_PN <- function(parms){
  alpha = parms[1]

```

```

beta    = parms[2]
mu      = parms[3]
sigma   = parms[4]
if (alpha>0 & beta>0 & sigma>0){
  LLvec = f_PN(logrets,alpha,beta,mu,sigma)
  LLvec = log(LLvec)
  LL     = sum(LLvec)
} else {
  LL = -Inf
}
minLL = -LL
return(minLL)
}

parms = c(alpha,beta,mu,sigma)
mLL   = minLL_PN(parms)

#####
# Starting values

n_startvals = 1000

alpha_s     <- runif(n_startvals,0.01,1)
beta_s      <- runif(n_startvals,0.01,1)
mu_s        <- runif(n_startvals,-0.1,0.1)
sigma_s     <- runif(n_startvals,0.01,1)

startvals  = rep(0,4)
startevals = 1:n_startvals*0
besteval   = Inf
pb = winProgressBar(title="Calculating starting values", label="0% done")
for (k in 1:n_startvals){
  sparms      <- c(alpha_s[k],beta_s[k],mu_s[k],sigma_s[k])
  startevals[k] <- minLL_PN(sparms)
  if (is.finite(startevals[k]) & startevals[k]<besteval){
    startvals = sparms
  }
}

```

```

    besteval = startevals[k]
  }
  info <- sprintf("%d%% done", floor((k/n_startvals*100)))
  setWinProgressBar(pb, k/n_startvals, label=info)
}
close(pb)

besteval
startvals
minLL_PN(startvals)

#####
#Optimisation
optm <- optim(startvals,minLL_PN)

alphaHat = optm$par[1]
betaHat = optm$par[2]
muHat = optm$par[3]
sigmaHat = optm$par[4]

x_min = min(logrets)
x_max = max(logrets)

x = seq(x_min,x_max,(x_max-x_min)/999)
Density = f_PN(x,alphaHat,betaHat,muHat,sigmaHat)

plot(x,Density,col="red",type="l",ylim=c(0,40),xlim=c(-0.05,0.05))
lines(density(logrets,adjust=1))
#####
#ISE:

xx = density(logrets)$x
yy = density(logrets)$y
yy2 = f_PN(xx,alphaHat,betaHat,muHat,sigmaHat)

plot(xx,yy,type="l",col="red")

```

```

lines(xx,yy2)

plot(xx,(yy-yy2)^2,type="l",col="red")

ISE = sum((yy-yy2)^2)*(xx[2]-xx[1])
ISE

#####
#Determining the value of lambda in the Esscher transform

f_PN_parm <- function(x_grid){
  alpha    = parms[1]
  beta     = parms[2]
  mu       = parms[3]
  sigma    = parms[4]
  integrand <- function(y){
    T1 = (y)^(3/2)
    T2 = -(x-mu*y)^2/(2*sigma^2*y)
    T3 = -(log(y)-alpha)^2/(2*beta^2)
    f = 1/T1*exp(T2+T3)
    return(f)
  }

  fx = rep(0,length(x_grid))
  for (j in 1:length(x_grid)){
    x      = x_grid[j]
    fx[j] = 1/(sigma*2*pi*beta)*integrate(integrand,0,Inf)$value
  }
  return(fx)
}

obj <- function(lambda){
  integrand1 <- function(x){
    intg = exp((1+lambda)*x)*f_PN_parm(x)
    return(intg)
  }

```

```

integrand2 <- function(x){
  intg = exp(lambda*x)*f_PN_parm(x)
  return(intg)
}
I1 = integrate(integrand1,-50,50)$value
I2 = integrate(integrand2,-50,50)$value
objf = abs(I1/I2-exp(r))
return(objf)
}

r = 0.007/252
parms = c(alphaHat,betaHat,muHat,sigmaHat)
optm = optimize(obj,c(-1,1))
lambda = optm$min

#####
#Calculating density under the measure Q

parm = c(alphaHat,betaHat,muHat,sigmaHat)

Q_density <- function(x){
  integrnd <- function(x){
    intg = exp(lambda*x)*f_PN_parm(x)
    return(intg)
  }
  T1 = f_PN_parm(x)*exp(lambda*x)
  T2 = integrate(integrnd,-10,10)$value
  f = T1/T2
  return(f)
}

df = (x_max-x_min)
x = seq(x_min-df,x_max+df,(x_max-x_min)*3/999)
y1 = f_PN(x,alphaHat,betaHat,muHat,sigmaHat)
y2 = Q_density(x)

```

```
#####
#Comparison of densities under P and Q (these should be close)

plot(x,y1,col="red")
lines(x,y2)
sum(y2)*(x[2]-x[1])

#####
#Calculate distribution function under Q

Fx = cumsum(y2)*(x[2]-x[1])
plot(x,Fx,type="l")

#####
#Simulating from the distribution under Q

library("pracma")

PN_sim <- function(nsim){
  U          = sort(runif(nsim))
  U[U<min(Fx)] = min(Fx)
  U[U>max(Fx)] = max(Fx)
  X          = interp1(Fx,x,U,method="linear")
  return(X)
}

PN_sim(20)

#####
#Comparing simulated and calculated densities

df = (x_max-x_min)
x = seq(x_min-df,x_max+df,(x_max-x_min)*3/999)
y = Q_density(x)
X = PN_sim(1e6)
plot(density(X),lwd=3)
```



```

lines(x,y,col="red")

#####
#Simulating a stock price at time Tj

S0 = 1124.47
Tj = 50
S0*exp(sum(PN_sim(Tj)))

#####
#Estimating option prices using simulation

npaths = 100000
OptPrice = rep(0,length(Option_price))
pb = winProgressBar(title="Calculating option prices", label="0% done")
for (j in 1:length(Option_price)){
  Kj = Strike_price[j]
  Tj = T[j]
  payoff = rep(0,npaths)
  for (k in 1:npaths){
    payoff[k] = max(S0*exp(sum(PN_sim(Tj)))-Kj,0)
  }
  OptPrice[j] = exp(-r*Tj)*mean(payoff)
  info <- sprintf("%d%% done", floor((j/length(Option_price)*100)))
  setWinProgressBar(pb, j/length(Option_price), label=info)
}
close(pb)

plot(Strike_price,OptPrice)

#####
# Comparing option prices
plot(Strike_price,Option_price,col="black")
points(Strike_price,OptPrice,col="red")

RMSE = sqrt(mean((Option_price - OptPrice)^2))

```

RMSE

```
#####
```

The exponential Pareto-normal process model

```
#####
```

```
#Entering the data
rm(list=ls())
set.seed(12345)
data = read.csv("S&P 500 2002 DATA.csv")
prices = data[,5]
prices = prices[length(prices):1]
plot(prices,type = "l")
```

```
#####
```

```
# Enter the option price data
```

```
Data <- read.csv("C:/Users/Clemence R Kwinje/Desktop/R Files/OptionData/Schoutens/Schout
T <- Data[,1]
Strike_price <- Data[,2]
Option_price <- Data[,3]
n2 = length(Strike_price)
plot(Strike_price,Option_price)
```

```
#####
```

```
#Calculating log-returns
```

```
n = length(prices)-1
logrets = 1:n*0
for (j in 1:n){
logrets[j] = log(prices[j+1]/prices[j])
}
plot(logrets,type = "l")
plot(hist(logrets))
lines(density(logrets))
```

```
#####
#Pareto-normal density

g      = 4.11
h      = 3.44
mu     = 0
sigma  = 0.008

integrand <- function(y){
T1 = sqrt(y)*(g+y)^(h+1)
T2 = -(x-mu*y)^2/(2*sigma^2*y)
f   = 1/T1*exp(T2)
return(f)
}

#x = 1
#y = seq(0.01,3,0.01)
#plot(integrand(y),type="l")

x_grid = seq(-0.05,0.05,0.0001)

fx = rep(0,length(x_grid))
for (j in 1:length(x_grid)){
x     = x_grid[j]
fx[j] = h*g^h/(sigma*sqrt(2*pi))*integrate(integrand,0,Inf)$value
}

plot(x_grid,fx,type="l")

sum(fx)*(x_grid[2]-x_grid[1])

f_PN <- function(x_grid,g,h,mu,sigma){

integrand <- function(y){
T1 = sqrt(y)*(g+y)^(h+1)
T2 = -(x-mu*y)^2/(2*sigma^2*y)
```

```

f = 1/T1*exp(T2)
return(f)
}

fx = rep(0,length(x_grid))
for (j in 1:length(x_grid)){
  x      = x_grid[j]
  fx[j] = h*g^h/(sigma*sqrt(2*pi))*integrate(integrand,0,Inf)$value
}
return(fx)
}

fx = PN_density(x_grid,g,h,mu,sigma)
plot(x_grid,fx,type="l")

#####
# Pareto-normal likelihood

LLvec = f_PN(logrets,g,h,mu,sigma)
LLvec = log(LLvec)
LL     = sum(LLvec)

if (g>0 & h>0 & sigma>0){
  LLvec = f_PN(logrets,g,h,mu,sigma)
  LLvec = log(LLvec)
  LL     = sum(LLvec)
} else {
  LL = -Inf
}

parms = c(g,h,mu,sigma)

minLL_PN <- function(parms){
  g      = parms[1]
  h      = parms[2]
  mu     = parms[3]

```

```

sigma = parms[4]
if (g>0 & h>0 & sigma>0){
LLvec = f_PN(logrets,g,h,mu,sigma)
LLvec = log(LLvec)
LL    = sum(LLvec)
} else {
LL = -Inf
}
minLL = -LL
return(minLL)
}

parms = c(g,h,mu,sigma)
mLL    = minLL_PN(parms)

#####
# Starting values

n_startvals = 1000

g_s      <- runif(n_startvals,0.01,1)
h_s      <- runif(n_startvals,0.01,1)
mu_s     <- runif(n_startvals,-0.1,0.1)
sigma_s  <- runif(n_startvals,0.01,1)

startvals = rep(0,4)
startevals = 1:n_startvals*0
besteval   = Inf
pb = winProgressBar(title="Calculating starting values", label="0% done")
for (k in 1:n_startvals){
  sparms      <- c(g_s[k],h_s[k],mu_s[k],sigma_s[k])
  startevals[k] <- minLL_PN(sparms)
  if (is.finite(startevals[k]) & startevals[k]<besteval){
    startvals = sparms
    besteval  = startevals[k]
  }
}

```

```

info <- sprintf("%d%% done", floor((k/n_startvals*100)))
setWinProgressBar(pb, k/n_startvals, label=info)
}
close(pb)

besteval
startvals
minLL_PN(startvals)

#####
#Optimisation
optm <- optim(startvals,minLL_PN)

gHat      = optm$par[1]
hHat      = optm$par[2]
muHat     = optm$par[3]
sigmaHat  = optm$par[4]

x_min = min(logrets)
x_max = max(logrets)

x = seq(x_min,x_max,(x_max-x_min)/999)
Density = f_PN(x,gHat,hHat,muHat,sigmaHat)

plot(x,Density,col="red",type="l",ylim=c(0,60),xlim=c(-0.05,0.05))
lines(density(logrets,adjust=1))
#####
#ISE:

xx = density(logrets)$x
yy = density(logrets)$y
yy2 = f_PN(xx,gHat,hHat,muHat,sigmaHat)

plot(xx,yy,type="l",col="red")
lines(xx,yy2)

```

```

plot(xx, (yy-yy2)^2, type="l", col="red")

ISE = sum((yy-yy2)^2)*(xx[2]-xx[1])
ISE

#####
#Determining the value of lambda in the Esscher transform

f_PN_parm <- function(x_grid){
  g      = parms[1]
  h      = parms[2]
  mu     = parms[3]
  sigma  = parms[4]
  integrand <- function(y){
    T1 = sqrt(y)*(g+y)^(h+1)
    T2 = -(x-mu*y)^2/(2*sigma^2*y)
    f   = 1/T1*exp(T2)
  }
  return(f)
}

fx = rep(0, length(x_grid))
for (j in 1:length(x_grid)){
  x      = x_grid[j]
  fx[j] = h*g^h/(sigma*sqrt(2*pi))*integrate(integrand, 0, Inf)$value
}
return(fx)
}

obj <- function(lambda){
  integrand1 <- function(x){
    intg = exp((1+lambda)*x)*f_PN_parm(x)
  }
  return(intg)
}
integrand2 <- function(x){
  intg = exp(lambda*x)*f_PN_parm(x)
}
return(intg)
}

```

```

}
I1 = integrate(integrand1, -50, 50)$value
I2 = integrate(integrand2, -50, 50)$value
objf = abs(I1/I2-exp(r))
return(objf)
}

r = 0.007/252
parms = c(gHat, hHat, muHat, sigmaHat)
optm = optimize(obj, c(-1, 1))
lambda = optm$min

#####
#Calculating density under the measure Q

parm = c(gHat, hHat, muHat, sigmaHat)

Q_density <- function(x){
  integrnd <- function(x){
    intg = exp(lambda*x)*f_PN_parm(x)
    return(intg)
  }
  T1 = f_PN_parm(x)*exp(lambda*x)
  T2 = integrate(integrnd, -10, 10)$value
  f = T1/T2
  return(f)
}

df = (x_max-x_min)
x = seq(x_min-df, x_max+df, (x_max-x_min)*3/999)
y1 = f_PN(x, gHat, hHat, muHat, sigmaHat)
y2 = Q_density(x)

#####
#Comparison of densities under P and Q (these should be close)

```



```

plot(x,y1,col="red")
lines(x,y2)
sum(y2)*(x[2]-x[1])

#####
#Calculate distribution function under Q

Fx = cumsum(y2)*(x[2]-x[1])
plot(x,Fx,type="l")

#####
#Simulating from the distribution under Q

library("pracma")

PN_sim <- function(nsim){
  U          = sort(runif(nsim))
  U[U<min(Fx)] = min(Fx)
  U[U>max(Fx)] = max(Fx)
  X          = interp1(Fx,x,U,method="linear")
  return(X)
}

PN_sim(20)

#####
#Comparing simulated and calculated densities

df = (x_max-x_min)
x = seq(x_min-df,x_max+df,(x_max-x_min)*3/999)
y = Q_density(x)
X = PN_sim(1e6)
plot(density(X),lwd=3)
lines(x,y,col="red")

#####

```

```

#Simulating a stock price at time Tj

S0 = 1124.47
Tj = 50
S0*exp(sum(PN_sim(Tj)))

#####
#Estimating option prices using simulation

npaths = 100000
OptPrice = rep(0,length(Option_price))
pb = winProgressBar(title="Calculating option prices", label="0% done")
for (j in 1:length(Option_price)){
  Kj = Strike_price[j]
  Tj = T[j]
  payoff = rep(0,npaths)
  for (k in 1:npaths){
    payoff[k] = max(S0*exp(sum(PN_sim(Tj)))-Kj,0)
  }
  OptPrice[j] = exp(-r*Tj)*mean(payoff)
  info <- sprintf("%d%% done", floor((j/length(Option_price)*100)))
  setWinProgressBar(pb, j/length(Option_price), label=info)
}
close(pb)

plot(Strike_price,OptPrice)

#####
# Comparing option prices
plot(Strike_price,Option_price,col="black")
points(Strike_price,OptPrice,col="red")

RMSE = sqrt(mean((Option_price - OptPrice)^2))
RMSE

#####

```

The geometric Meixner process model

```
#####
#Entering the data
rm(list=ls())
set.seed(12345)
data = read.csv("S&P 500 2002 DATA.csv")
prices = data[,5]
prices = prices[length(prices):1]
plot(prices,type = "l")

#####
# Enter the option price data

Data <- read.csv("C:/Users/Clemence R Kwinje/Desktop/R Files/OptionData/Schoutens/Schout
T <- Data[,1]
K <- Data[,2]
ObsP <- Data[,3]
n2 = length(K)
plot(K,ObsP)

#####
#Calculating log-returns
n = length(prices)-1
logrets = 1:n*0
for (j in 1:n){
logrets[j] = log(prices[j+1]/prices[j])
}
plot(logrets,type = "l")
plot(hist(logrets))
lines(density(logrets))

#####
#Meixner density

#install.packages("pracma")
library(pracma)
```

```

f_Meixner <- function(x,alpha,beta,mu,delta){

T1 = ((2*cos(beta/2))^(2*delta))
T2 = (2*alpha*pi*gamma(2*delta))
T3 = (beta*(x-mu)/alpha)
T4 = abs(gammaz(delta+1i*((x-mu)/alpha)))

f_Meixner = T1/T2*exp(T3)*T4^2;
return(f_Meixner)
}

alpha = 2
beta = 1
mu = 0
delta = 1

x_min = -3
x_max = 8
x = seq(x_min,x_max,(x_max-x_min)/999)
y = f_Meixner(x,alpha,beta,mu,delta)

plot(x,y,type="l")

#####
# Meixner likelihood

LLvec = f_Meixner(logrets,alpha,beta,mu,delta)
LLvec = log(LLvec)
LL = sum(LLvec)

if (alpha>0 & abs(beta)<pi & delta>0){
LLvec = f_Meixner(logrets,alpha,beta,mu,delta)
LLvec = log(LLvec)
LL = sum(LLvec)
} else {

```

```
LL = -Inf
}

parms = c(alpha,beta,mu,delta)

minLL_Meixner <- function(parms){
  alpha = parms[1]
  beta  = parms[2]
  mu    = parms[3]
  delta = parms[4]
  if (alpha>0 & abs(beta)<pi & delta>0){
    LLvec = f_Meixner(logrets,alpha,beta,mu,delta)
    LLvec = log(LLvec)
    LL    = sum(LLvec)
  } else {
    LL = -Inf
  }
  minLL = -LL
  return(minLL)
}

parms = c(alpha,beta,mu,delta)
mLL    = minLL_Meixner(parms)

#####
# Starting values

n_startvals = 100000

alpha_s <- runif(n_startvals,0.01,100)
beta_s  <- runif(n_startvals,-pi,pi)
mu_s    <- runif(n_startvals,-10,10)
delta_s <- runif(n_startvals,0.01,100)

pb = winProgressBar(title="Calculating starting values", label="0% done")
startvals = rep(0,4)
```

```

startevals = rep(0,n_startvals)
besteval   = Inf
for (k in 1:n_startvals){
  sparms      <- c(alpha_s[k],beta_s[k],mu_s[k],delta_s[k])
  startevals[k] <- minLL_Meixner(sparms)
  if (is.finite(startevals[k]) & startevals[k]<besteval){
    startvals = sparms
    besteval  = startevals[k]
  }
  info <- sprintf("%d%% done", floor((k/n_startvals*100)))
  setWinProgressBar(pb, k/n_startvals, label=info)
}
close(pb)

besteval
startvals
minLL_Meixner(startvals)

optm <- optim(startvals,minLL_Meixner)

alphaHat = optm$par[1]
betaHat  = optm$par[2]
muHat    = optm$par[3]
deltaHat = optm$par[4]

x_min = min(logrets)
x_max = max(logrets)

x = seq(x_min,x_max,(x_max-x_min)/999)
Density = f_Meixner(x,alphaHat,betaHat,muHat,deltaHat)

plot(x,Density,col="red",type="l",ylim=c(0,35),xlim=c(-0.07,0.07))
lines(density(logrets,adjust=1.5))

#####
#ISE:

```

```

xx = density(logrets)$x
yy = density(logrets)$y
yy2 = f_Meixner(xx,alphaHat,betaHat,muHat,deltaHat)

plot(xx,yy,type="l",col="red")
lines(xx,yy2)

plot(xx,(yy-yy2)^2,type="l",col="red")

ISE = sum((yy-yy2)^2)*(xx[2]-xx[1])
ISE

#####
r=0.007/252

T1a = -cos(alphaHat/2)+exp((muHat-r)/(2*deltaHat))
T1b = sin(alphaHat/2)
T1 = T1a/T1b
theta = -1/alphaHat*(betaHat+2*atan(T1))

betaHat = betaHat + alphaHat*theta
#####
#Option Price Calculation

Meixner_density_parms <- function(x){
alpha = parms[1]
beta = parms[2]
mu = parms[3]*Tj
delta = parms[4]*Tj
T1 = ((2*cos(beta/2))^(2*delta))
T2 = (2*alpha*pi*gamma(2*delta))
T3 = (beta*(x-mu)/alpha)
T4 = abs(gammaz(delta+1i*((x-mu)/alpha)))

f_Meixner = T1/T2*exp(T3)*T4^2;

```

```
return(f_Meixner)
}

Meixner_density_FI <- function(x){
  integrand <- function(t){
    alpha = parms[1]
    beta  = parms[2]
    mu    = parms[3]*Tj
    delta = parms[4]*Tj
    phi   = exp(1i*mu*t)*(cos(beta/2)/cosh((alpha*t-1i*beta)/2))^(2*delta)
    intg  = Re(phi*exp(-1i*t*x))
    return(intg)
  }

  f_Meixner = 1/(2*pi)*integrate(integrand, -Inf, Inf)$value
  return(f_Meixner)
}

parms = c(alphaHat,betaHat,muHat,deltaHat)
Tj    = 1;

#Comparisons between the
x_grid = seq(-0.02,0.02,0.0001)
fx1    = rep(0,length(x_grid))
fx2    = rep(0,length(x_grid))

for (j in 1:length(x_grid)){
  fx1[j] = Meixner_density_parms(x_grid[j])
  fx2[j] = Meixner_density_FI(x_grid[j])
}

plot(x_grid,fx1,col="red")
lines(x_grid,fx2)

S0 = 2476.61
```



```

vMeixner_density_FI = Vectorize(Meixner_density_FI)

#Meixner_density_parms(0)
#Meixner_density_FI(0)
#vMeixner_density_FI(0)
#vMeixner_density_FI(c(0,0.01))

integrand1 <- function(x){
  intg1 = (S0*exp(x)-Kj)*vMeixner_density_FI(x)
  return(intg1)
}

#Kj = 1
#integrand1(c(0,0.01,0.02))

OptPrice = rep(0,length(ObsP))

pb = winProgressBar(title="Calculating option prices", label="0% done")
for (j in 1:length(ObsP)){
  Kj          = K[j]
  Tj          = T[j]
  OptPrice[j] = exp(-r*Tj)*integrate(integrand1,log(Kj/S0),3)$value
  info        = sprintf("%d%% done", floor((j/length(ObsP)*100)))
  setWinProgressBar(pb, j/length(ObsP), label=info)
}
close(pb)

#####
# Comparing option prices
plot(K,ObsP,col="red")
points(K,OptPrice)

RMSE = sqrt(mean((ObsP - OptPrice)^2))
RMSE

```

The code for the Eurostoxx data set is the same as the code used for the S&P 500 data set.