A Notation for Sets, Sequences and Series

Possible Benefits for Understanding and Use

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Abstract. The use of a well-designed and uniform notational system is imperative to improve communication and understanding. Currently, a fairly consistent notational system is used to notate the elements of sets, but the notations for sequences, series and other quantifications often vary from one author to another. The most widely used notation for sets has limitations and cannot easily be generalised to sequences and series. Furthermore, the well-known sigma notation for series is ambiguous and difficult to grasp. This article proposes a clear notation that can be used for all of the above. We aim to test the hypothesis that this notation could aid an understanding of concepts which build on the use of this notation. As a first step, we conducted a pilot study to inform the design of an experiment to evaluate the effect of the proposed notation on understanding. Secondary-school learners who had never been exposed to the sigma notation participated in the pilot study.

1 Introduction

Mathematical notation is a language in its own right – unique and constantly evolving. Mathematicians, speaking a range of indigenous languages, use it to express mathematical concepts in a way which everyone can understand. The use of mathematical language has evolved over many centuries as users of this language share a mutual understanding of the meaning of its symbols, words and sentences. Good notation enhances the precision of expression and at the same time simplifies communication.

Mathematical notations are introduced to provide concise and accurate ways of communicating complex yet well-understood concepts. It is well known that brevity is the leading characteristic of mathematical elegance, but this is not the only requirement. Most mathematicians agree on the value of clever notations. Dijkstra and Van Gasteren [1] emphasise that the use of appropriate notation can make a difference in mathematical work. Lipton [2] gives an example of the European mathematicians who used Leibniz's $\frac{dx}{dt}$ differential notation, which enabled them to progress faster than their British counterparts who used Newton's $\dot{x}$ to express the same concept. To support mathematical thinking, notation should not only be designed to enhance the brevity of the text but should also control the number of rules governing the manipulation of expressions [1].

The influence of notational differences on the ease of mastering the underlying concepts, has been investigated in various contexts [3–5]. Chirume [6] found in a study involving secondary schools in Zimbabwe, that a clear notation plays a role at the initial stage of learning a new concept. He found that students might fail to grasp mathematical concepts because they take the symbols themselves as the objects of mathematics rather than the ideas and processes which they represent. These studies confirm that clear and unambiguous notation may promote better cognition and that issues of symbol familiarity and symbol density should be considered when teaching mathematics [7].

Electronic recognition of mathematical expression is a large area of research, one of which is the CROHME project that strives to build a comprehensive database of handwritten mathematical expressions which can be used for the online recognition of mathematical expressions. Other examples of such research are Simistira et al. [8] that propose an algorithm using probabilistic SVMs and stochastic context-free grammars to arrive at a marginal improvement in the recognition of mathematical expressions in order to produce MathML output. More specific research in the use of mathematical expressions online is that of Cuartero-Olivera et al. [9]. They found that a crucial factor in the use of online tools for mathematical expression is the time it takes to enter the expression. They suggest a speech-to-text tool to circumvent this issue.

One of the notations that has been identified as particularly cumbersome, when encountered for the first time, is Euler's sigma notation [10]. This notation is usually introduced for the first time to South African learners at the beginning of Grade 12 when covering the topic of sequences and series. This paper proposes a consistent notational system for sets, sequences and series, which has the potential to increase familiarity and reduce the introduction of additional symbols when teaching these topics. We aim to investigate the value of using the proposed notations: firstly to simplify the electronic formulation of the constructs and secondly, to promote an understanding of the concepts. The participants in the pilot study were secondary-school Grade 11 learners who had not previously been introduced to series or to the use of the sigma notation.

2 Sets

A set is a collection of objects. One way of describing or specifying the members of a set is by extension, i.e. by listing each member of the set. When the members of a set follow a pattern, the set can be specified by intentional definition, i.e. by using a rule or semantic description as specified in the international standard ISO 80000-2:2009 [11]. The general format for the notation is:

$$\{x \mid P(x)\}$$

The symbol $\mid$ means *such that*. Therefore, the above means the set of all elements x *such that* $P(x)$ is a true statement. The property P can be expressed in either words or symbols. The variable on the left of the $\mid$ specifies a dummy whereas the expression on the right delineates the scope. One may use formulas to specify the dummy. For example, the set of odd natural numbers can be expressed as:

$$\{2t + 1 \mid t \in \mathbb{N}\}$$

Dijkstra [12] criticises this notation. He gives the following example which reveals an inherent flaw in this notation:

$$\{i^n \mid i < n\}$$

It can be interpreted as $\{1^n, 2^n, 3^n, \ldots, n^n\}$ or as $\{i^{i+1}, i^{i+2}, i^{i+3}, \ldots\}$. This ambiguity arises because the specification of the dummy is not separated from the description of the elements. He proposes a notational system to remedy this deficiency. This notation requires the clear separation of three aspects, namely (i) the dummy elements, (ii) the scope description, and (iii) the description of the elements of the set in terms of the dummy elements. Dijkstra's notation for the intentional definition of a set specifies these aspects separated by the : character and enclosed in angle brackets. This notation not only removes ambiguities, it is also a more versatile expression of the conditions for membership of a set. The following is the general format for Dijkstra's notation:

$$\langle\ x : P(x) : f(x)\ \rangle$$

Here x is the dummy, $P(x)$ is a predicate that specifies the scope and $f(x)$ is an expression that describes the elements of the set. More than one dummy, as well as more than one predicate to specify the scope, may be used. When doing so, they should be separated by commas. This allows a distinction between the following:

$$\langle\ i : i < n : i^n\ \rangle,\ \langle\ n : i < n : i^n\ \rangle \text{ and even } \langle\ i, n : i < n : i^n\ \rangle$$

Dijkstra states that he has no logical objection to declaring the type of the dummy when identifying the dummy. For example, the following are equivalent specifications of the set of even numbers less than 100 using Dijkstra's notation:

$$\langle\ i \in \mathbb{N} : i < 100 : 2 \times i\ \rangle$$
$$\langle\ i : i \in \mathbb{N}, i < 100 : 2 \times i\ \rangle$$

In this paper, we introduce a notation similar to Dijkstra's notation but closer to the ISO standard. The following is the general format for the proposed notation:

$$\{\ x \mid P(x) \mid f(x)\ \}$$

An obvious difference between this notation and Dijkstra's is the use of the punctuation prescribed in the ISO standard, i.e. $\{\ \mid\ \mid\ \}$ instead of $\langle\ :\ \ :\ \rangle$. To avoid further ambiguities the following restrictions are also proposed:

- Type specifications of dummy variables are not allowed.
- Multiple scope predicates are not allowed.

If needed, the type of a dummy variable may be specified by including it in the scope predicate. Limiting the specification of the scope to a single predicate is not a real restriction, because if both P_1 and P_2 should hold, it can just as well be specified with the single predicate $P_1 \wedge P_2$ where $\wedge$ is the symbol for the Boolean AND operation. The following is therefore the only legitimate specification of the set of even numbers less than 100, using the proposed notation:

$$\{\ i \mid (i \in \mathbb{N}) \wedge (i < 100) \mid 2 \times i\ \}$$

3 Sequences

A sequence is an ordered list of objects. A sequence differs from a set because the order of the objects matters. In addition, exactly the same elements can appear multiple times at different positions in the sequence. A sequence with n entries is called an *n-tuple*. As far as we know, no standard has been specified to notate an intentional definition of sequences. In this paper we propose a notation for cases where there is a relationship between i and the value of the i^{th} term in the sequence. This is an adaptation of the proposed notation for the intentional definition of sets in Sect. 2. Similar to the notation for sets, the three aspects — namely the dummy variables, the range predicate and the expression describing the entries — are specified and separated by the | character. This is a variation of the notation proposed by Pieterse [13]. Here we use the punctuation prescribed in the ISO standard for sequences.

To indicate that it is a series and not a set, parentheses are used instead of curly braces, for example, the following specifies the sextuple $(3, 5, 7, 9, 11, 13)$. The formula for the i^{th} term in this sextuple is $2 \times i + 3$. It can therefore be described by using the following intentional definition:

$$(\ i \mid i \in \mathbb{N}_6 \mid 2 \times i + 3 \ )$$

4 Series

A series is the sum of the terms of a sequence. The well-known sigma notation for the summation of sequences was introduced by Leonard Euler in 1755 [14]. Euler's notation and modern refinements of this notation are well established in the mathematical community. When using Euler's notation, $3^2+4^2+5^2+6^2+7^2$ is written as:

$$\sum_{m=3}^{7} m^2$$

Wees [10] contends that students find Euler's sigma notation difficult to understand at first. He attributes this to the complexity of the expression, which is a function that takes as many as four parameters, all of which have to be understood at once. He proposes a programming-like notation as a substitute for Euler's sigma notation. His notation requires descriptive names for the parameters. Though his proposal contributes significantly to the clarity of the expression, it loses two essential attributes of viable notational systems, namely conciseness and independence from natural language.

Dijkstra [12] points out a flaw in Euler's notation which is beyond the problem observed by Wees, namely an inherent semantic ambiguity resulting from uncertainty related to the extent of the description of the elements in the series, for example,

$$\sum_{m=3}^{7} m^2 + 1 \qquad \text{can be interpreted as} \qquad \left(\sum_{m=3}^{7} m^2\right) + 1 \qquad \text{or as} \qquad \sum_{m=3}^{7} (m^2 + 1)$$

This uncertainty can be resolved by introducing parentheses. Dijkstra, however, proposes that the use of an adaptation of his set notation should replace Euler's notation. Dijkstra's proposed notation enforces careful thought which would eliminate possible ambiguities, whereas Euler's notation allows ambiguous expression, placing the onus on the writer to solve possible ambiguities. Dijkstra follows Euler's idea of using the Σ symbol to indicate summation. In Dijkstra's proposal, this symbol is specified along with the dummy variable. The above-mentioned ambiguous specification of a series can therefore only be specified as one of the following unambiguous expressions:

$$\langle\ \Sigma m : 3 \leq m \leq 7 : m^2 + 1\ \rangle \qquad \text{or as} \qquad \langle\ \Sigma m : 3 \leq m \leq 7 : m^2\ \rangle + 1$$

Besides promoting unambiguity of expression, Dijkstra's notation reduces the need for spatial and size differences to convey meaning in comparison with Euler's notation. Dijkstra's notation has the advantage of clearly distinguishing the description of the terms in the sequence from the surrounding text. The linearity of the Dijkstra notation will make electronic recognition simpler and faster. It does, however, have drawbacks, namely the introduction of an unfamiliar symbol, overloading the purpose of a section in the expression, and applying the operation in the wrong context. We discuss these problems and state how we avoid each of these drawbacks when proposing our own notation.

4.1 Introduction of an Unfamiliar Symbol

Both Euler and Dijkstra use the Σ symbol to indicate summation. This is an entirely new and unfamiliar character for learners who have not been introduced to this topic. We avoid the introduction a new symbol by simply using the $+$ symbol. The learners are already familiar with the symbol and its meaning.

4.2 Overloading the Purpose of a Section in the Expression

Dijkstra overloads the purpose of the first section of the expression when requiring the specification of an operation along with the dummy variable in this section. We suggest that the symbol that specifies the operation should be placed as a prefix operation instead of embedding it in the specification.

4.3 Applying the Operation in the Wrong Context

Dijkstra amended the expression for a set to serve as an expression of a series. Interpreting the series as the summation of a set poses some problems. Consider the following series:

$$\sum_{i=1}^{10} (i \ \% \ 2)$$

where % is the symbol for the modulus operation, namely to determine the remainder when one divides the value of the left expression by the value of the right expression. The value of the this series is 5. This is determined by evaluating the following expression:

$$1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0$$

When representing this series using Dijkstra's notation, one would write

$$\langle \ \sum i : 1 \leq i \leq 10 : i \ \% \ 2 \ \rangle$$

When interpreting the expression – assuming that the summation is over the set which may only have unique values – the expression is equivalent to the summation of the elements of $\{0, 1\}$. The resulting value is 1 instead of the intended 5. This error in interpretation arises from the assumption that the specified terms are elements of a set, instead of the intended meaning, namely to be the terms in a sequence, in which the same element may appear multiple times. To remedy this problem, our proposed notation amends the expression for a sequence to serve as an expression of a series.

4.4 Proposed notation

Our notation is similar to Dijkstra's notation, yet it has the following differences:

- It uses the + symbol instead of the Σ symbol to denote summation.
- The operation symbol is written as a prefix to the sequence instead of placing it along with the specification of the dummy variable.
- It uses parentheses where Dijkstra's notation uses angle brackets. This notation therefore applies explicitly to series (not sets) and uses the punctuation prescribed for sequences in the ISO standard.

The following is the general format of the proposed notation for series:

$$+(\ x \mid P(x) \mid f(x) \ )$$

The meaning of the parameters is the same as in the notation for sequences: x is the dummy, $P(x)$ is a predicate that specifies the scope and $f(x)$ is an expression that describes the terms in the series. For example, when using our notation, the expression

$$\sum_{m=3}^{7} (m^2 + 1) \qquad \text{is written as} \qquad +(\ m \mid 3 \leq m \leq 7 \mid m^2 + 1 \ ).$$

The proposed notation is more versatile than other notations. It does not introduce a new symbol and can therefore be used without adaptation for quantifications involving operations other than +. Other notations require the introduction of additional symbols when used for quantifications involving other operations. For example, when using Euler's notation, the symbol Π is introduced to indicate multiplication over a series using, but our notation simply uses $\times$.

Owing to its linear nature and the absence of non-standard keyboard characters the proposed notation may pose fewer problems when used electronically.

5 Pilot Study

We designed a pilot study [15]. The purpose of the pilot study was to gain insight into the use of the notation in the field and to evaluate the feasibility of conducting a full-scale investigation to determine the effect of the use of our proposed notation on the participants' comprehension of the mathematical concepts underlying sequences and series.

The participants in the study were Grade 11 learners who had never been exposed to the topics in question. It is unlikely that these learners would have been exposed to the sigma notation elsewhere, since the target school follows a rigid syllabus and the first introduction of the topic is in Grade 12. We divided the learners randomly in two groups.

We used the same basic material from the Grade 12 mathematics textbook published by Siyavula[1] for both groups. The one group's material was modified to use the Dijkstra notation but used curly brackets instead of angle brackets. The same teacher presented the material to both groups to avoid differences in presentation. We did not introduce more notations because we are not aware of any other notations for these constructs and we did not have enough participants to conduct a meaningful experiment involving more notations.

The experiment was conducted over a period of three days with a work period of approximately three hours every day. The teacher was an experienced lecturer who had taught mathematics in secondary school for more than five years, was involved in teacher training and was familiar with both the notations. The same teaching aids were used in both classes. The lessons involved verbal explanations combined with questions to explain the concepts. Each lesson was concluded with use of worksheets containing exercises which the learners did on an individual basis. The teacher moved from one classroom to the next, presenting the same topic but using a different notation. While one group was receiving a lesson, the other group was doing exercises.

After all the material had been covered, both groups wrote the same test (Test 1). We then inverted the groups, giving the first group a view of the Dijkstra notation and the second group a view of the traditional (Sigma) notation, and again gave the same test to both groups (Test 2), requesting that they use the notation of their choice to write their answers. The final test was intended to ensure that the learners were able to use both notations.

At the end of the three days, an opinion poll was held, involving the participants who had remained in the experiment until the end. We asked four questions in this opinion poll:

1. Which notation is easier to write?
2. Which notation is easier to read?
3. Which notation is easier to understand?
4. Which notation do you prefer?

[1] https://www.siyavula.com/maths/grade-12

5.1 Learners' Performance

Table 1 shows the descriptive statistics of the marks that the learners achieved in the tests, while the individual performance of each participant is shown in detail in Fig. 1 and Fig.2.

Table 1. Descriptive statistics

	Test 1			Test 2		
	Average	Median	Std Dev	Average	Median	Std Dev
Sigma group	37.00%	37.70%	0.09	20.71%	19.19%	0.09
Dijkstra group	17.21%	13.93%	0.17	13.30%	7.07%	0.11

The results of the final test were lower than the first, which is unexpected, since one would imagine that learners would perform better after having had more exposure to the underlying concepts. We can only theorise that the lack of performance was due to the learners having had little incentive to complete the tests, so the learners did not put much effort into the final test.

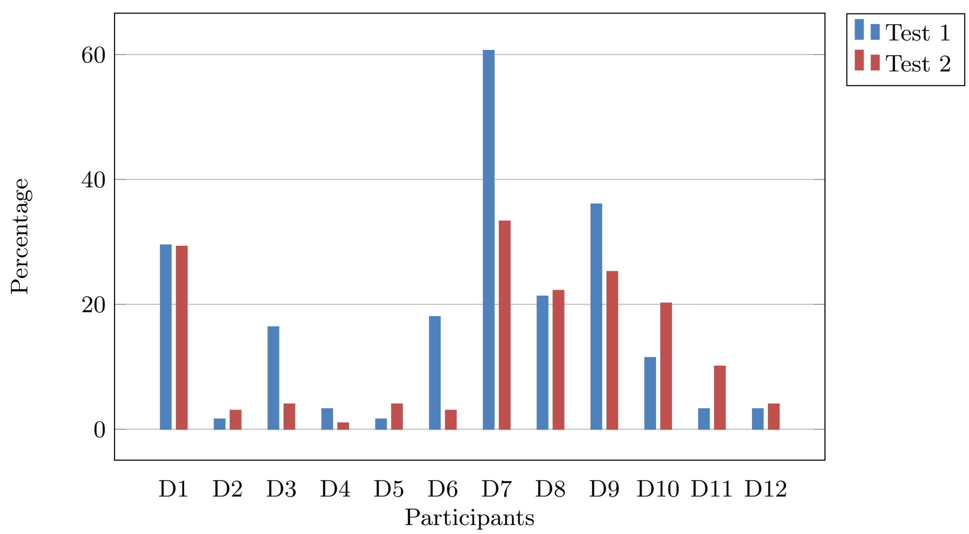

Fig. 1. Dijkstra Group Test Results

Statistical analysis is not feasible in this case, due to the small number of participants and the large deviation in the results. The one group was clearly stronger than the other. Evidently our selection method was flawed as it did not render comparable groups.

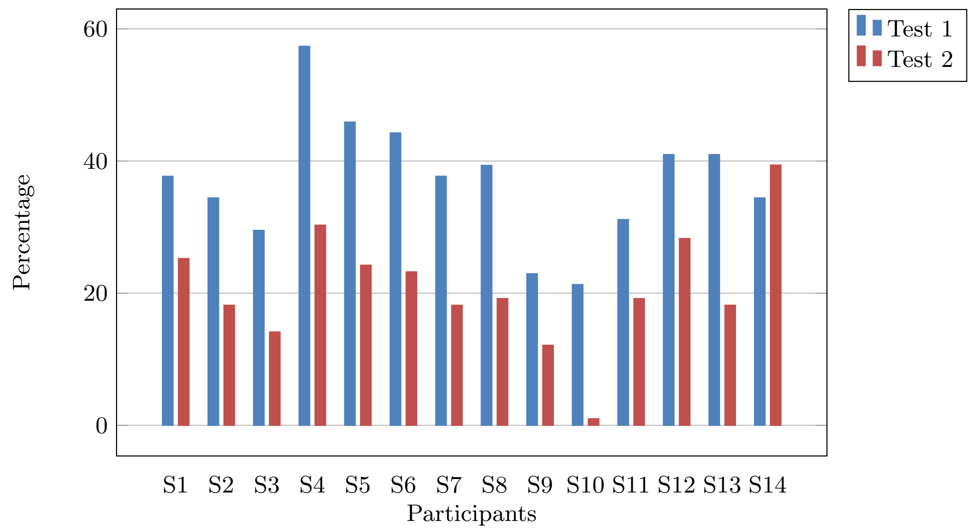

Fig. 2. Sigma Group Test Results

A factor which might have played a role is that the two groups were not equal in gender distribution. To make matters worse, the dropout rate over the three days skewed the gender distribution even more. The group with a majority of girls performed much better on average than the group with predominantly boys. The difference in the performance of the groups is so obvious that it is unlikely to be coincidental. The reason for this difference is, however, unclear. We are inclined to attribute it to gender differences rather than the way in which the material was presented in this experiment. This opinion is a topic for another investigation, which is beyond the scope of the present research.

Table 2. Start and Finish Number of Participants

	Start			Finish		
	Girls	Boys	Total	Girls	Boys	Total
Sigma group	11	5	16	10	4	14
Dijkstra group	5	12	17	4	8	12
Total	16	17	33	14	12	26

5.2 Opinion Poll Results

The results of the opinion poll were inconclusive but quite interesting, since the stronger academic group had a higher number of learners who preferred the new notation, although they had been primarily instructed in using the Sigma notation. This indicates that there might be merit in conducting a full-scale experiment to obtain more reliable results.

Table 3. Opinion Poll Results

	Sigma notation	Dijkstra notation	No preference	Sigma notation	Dijkstra notation	No preference
	Easier to write			Easier to read		
Sigma group	8	5	1	10	4	0
Dijkstra group	10	1	1	7	4	1
Total	18	6	2	17	8	1
	Easier to understand			Preferred		
Sigma group	9	4	1	10	4	0
Dijkstra group	8	3	1	8	4	0
Total	17	7	2	18	8	0

Table 3 shows the notational preferences of the 26 participants who completed the experiment. The majority of the learners favoured the sigma notation. Regardless of which notation they were taught at first, 18 preferred the Sigma notation and 8 the Dijkstra notation.
The following can be observed:

– 35% of the group instructed by using the Sigma notation felt that the Dijkstra notation was easier to write.
– 66% of the group instructed by using the Dijkstra notation preferred the Sigma notation.

All the tests and exercises were done in a hand-written format. It is clear that the participating learners preferred the sigma notation when they had to use pen and paper. This could be due to the easier visual separation of the elements in this notation but more research would have to be conducted before we could draw conclusions about this preference.

5.3 General Comments

The logistical issues involved in organising a study of this kind are daunting. Permission has to be obtained from governing bodies and teachers, and the parents also have to give their consent. The permissions alone could take several months. Since the school where we conducted our research, is semi-private and dependent on parental funding and goodwill, it is understandable that the schools was reluctant to introduce anything that could be seen as remotely controversial.

Once all the stakeholders had been persuaded and the requisite permissions obtained, a suitable time slot had to be found in the school's extremely packed agenda. For our experiment, this date ended up being after the final exams just before the summer holidays. In the South African school system, learners who have completed their end-of-year examinations do not wait for the official closing date of the schools to go on holiday. This meant that our pool of available learners was small and that the participating learners' motivation to sit through lessons and tests was very low. This could also explain the high dropout rate.

The participating learners had on average a scanty understanding of the material presented and none of the results could be used to draw conclusions about the influence that the use of the different notations had on the ease of comprehension of the underlying concepts.

6 Future Research

In the process of analysing the results of this experiment, several observations were made that warrant further research into the potential benefits of a new notation for the sum of sequences.

We have decided to conduct our full-scale research experiment online. This will enable us to reach a wider audience and to track learner progress more easily. Taking the research online will also allow us to improve some of the aspects of classroom teaching as experienced during this research.

7 Conclusion

Inventing a useful notational system that is economical and aesthetic is an art. A notational system should be equally convenient to write by hand and to typeset with the use of contemporary tools. If it is not promoted in the right place at the right time, it may remain unnoticed and unused. Abadir [16] concedes that it is likely that authors will not adhere to proposed notational standards. He uses the example of Bernoulli [17] who did not adopt the = sign for equality 150 years after it had been proposed, even though many other mathematicians used it.

In this paper, we propose an alternative notation for series to replace the widely used yet cumbersome sigma notation. Our notation is an amendment of a notation designed by Dijkstra [12]. He proposed a set notation to avoid some flaws that are inherent in the standard set notation. He re-appropriated his set notation to specify series. We discuss problems with Dijkstra's notations and propose our own notations for sets, sequences and series, which we believe are more elegant, user friendly and robust than other known notations for these constructs. The notation might enhance understanding of the underlying mathematical concepts.

The results of our pilot study were inconclusive. There could be several reasons for this but we believe it was mainly because we had too little time to convey the topics sufficiently well for the learners to grasp the subject meaningfully.

We are still hopeful that additional research will show that the new notation is beneficial to learning and also more practical in electronic use and e-learning programs.

References

1. Dijkstra, E.W., van Gasteren, A.J.M.: On notation. http://www.cs.utexas.edu/users/EWD/ewd09xx/EWD950a.PDF (January 1986) [Accessed 2013-11-27].
2. Lipton, R.: Notation and thinking. http://rjlipton.wordpress.com/2010/11/30/notation-and-thinking/ (November 2010) [Accessed 2016-05-12].
3. Hoch, M., Dreyfus, T.: Structure sense in high school algebra: The effect of brackets. In: Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen University College (2004)
4. Kisiel, V.M.: I saw the sign and it opened up my eyes i saw the sign! a study ofthe impacts ofthe use of different multiplication symbols in the mathematics classroom. Master's thesis, Department of Mathematical Sciences at the State University of New York at Fredonia, Fredonia, New York (July 2010)
5. Torigoe, E., Gladding, G.: Symbols: Weapons of math destruction. AIP Conference Proceedings **951**(1) (2007) 200–203
6. Chirume, S.: How does the use of mathematical symbols influence understanding of mathematical concepts by secondary school students. International J. Soc. Sci. and Education 2012 Vol. 3 Issue 1, ISSN: 2223-4934 E and 2227-393X **3**(1) (2012)
7. Bardini, C., Pierce, R.: Assumed mathematics knowledge: the challenge of symbols. International Journal of Innovation in Science and Mathematics Education (formerly CAL-laborate International) **23**(1) (2015) 1–9
8. Simistira, F., Katsourosa, V., Carayannis, G.: Recognition of online handwritten mathematical formulas using probabilistic svms and stochastic contextfree grammars. Pattern Recognition letters(53) (2014)
9. Cuartero-Olivera, J., Hunter, G., Prez-Navarro, A.: Reading and writing mathematical notation in e-learning environments. ELC Research Paper Series Issue 4 (2012)
10. Wees, D.: Mathematical notation is broken. http://davidwees.com/content/mathematical-notation-broken (May 2012) [Accessed 2013-11-27].
11. ISO/TC 12: ISO 80000-2:2009 Quantities and units — Part 2: Mathematical signs and symbols to be used in the natural sciences and technology. http://www.iso.org/iso/catalogue_detail.htm?csnumber=31887 (November 2009) [Accessed 2013-12-22].
12. Dijkstra, E.W.: EWD1300: The Notational Conventions I Adopted, and Why. Formal Aspects of Computing **14** (2002) 99–107
13. Pieterse, V.: Topic Maps for Specifying Algorithm Taxonomies: A Case Study using Transitive Closure Algorithms. PhD thesis, University of Pretoria (2017)
14. Euler, L.: Leonhard Euler, Foundations of differential calculus. Transl. by JD Blanton. New York: Springer (1755/2000)
15. du Plessis, S.H., Pieterse, V.: Die effek vann alternatiewe notasie op die begrip van wiskundige konsepte vir graad 12 leerders. Suid-Afrikaanse Tydskrif vir Natuurwetenskap en Tegnologie **36**(1) 1–2
16. Abadir, K., Magnus, J.: Notation in econometrics: a proposal for a standard. Econometrics Journal **5**(1) (2002) 76–90
17. Bernoulli, J.: Ars conjectandi. Impensis Thurnisiorum, fratrum (1713)