Analytic fragmentation semigroups and classical solutions to coagulation–fragmentation equations — a survey

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1. Introduction

Possibly the most important classical equations modelling the evolution of matter are the Navier-Stokes equation, the Boltzmann equation and the coagulation-fragmentation equation. The first describes the motion of a continuum, the second tells us about the behaviour of particles constituting this continuum, while the latter describes how fragments of matter are rearranged. Coagulation and fragmentation models often arise in natural sciences and engineering, where they describe processes ranging from the distribution of the sizes of animal groups, evolution of phytoplankton aggregates, blood agglutination, through planetesimals formation and rock crushing, to polymerization and de-polymerization, see e.g. [24, 40, 54, 23, 36, 16, 69, 70].

One of the most efficient approaches to modelling the dynamics of such processes is through the kinetic (rate) equation that describes the evolution of the distribution of interacting clusters with respect to their size (mass).

FIGURE 1. Marian Smoluchowski, 28 May 1872 – 5 September 1917 (Wikipedia)

The first equation of this kind was derived almost exactly 100 years ago by a Polish-Austrian physicists Marian von Smoluchowski to describe pure coagulation in the so-called discrete case; that is, when the ratio of the mass of the basic building block (monomer) to the mass of a typical cluster is positive and thus the mass of a cluster is a finite multiple of the mass of the monomer. We shall call a cluster of n monomers an n-mer and normalize the mass of monomer to one.

The Smoluchowski population balance equation, describing the time-evolution of the number density of n-mers of size $n \geq 2$, augmented here by the part describing breaking down of clusters into smaller pieces, is given by, [61, 62],

$$\begin{aligned} \frac{df_n}{dt}(t) &= -a_n f_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j(t) \\ &+ \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} f_{n-j}(t) f_j(t) - \sum_{j=1}^{\infty} k_{n,j} f_n(t) f_j(t), \quad n > 1, \quad (1) \end{aligned}$$

where $\boldsymbol{f}(t) = (f_n(t))_{n\in\mathbb{N}}$ is the number density of n-mers at time $t \geq 0$, a_n is the net rate at which n-mers break up, $b_{n,j}$ is the daughter distribution function that gives the average number of n-mers produced upon the break-up of a j-mer (and hence $b_{n,j} = 0$ for $j \leq n$), $k_{n,j} = k_{j,n}$ represents the coagulation rate of n-mers with j-mers. In this paper we shall assume that the fragmentation rates $(a_n)_{n\geq 1}$ are unbounded – otherwise the problem significantly simplifies.

Since monomers do not fragment and loss of monomers can only arise due to coagulation, for $n = 1$ we have

$$\frac{d}{dt}f_1(t) = \sum_{j=2}^{\infty} a_j b_{1,j} f_j(t) - \sum_{j=1}^{\infty} k_{1,j} f_1(t) f_j(t). \tag{2}$$

Important quantities in the process are:

$$\begin{aligned} M_0 &= \sum_{i=1}^{\infty} f_i, && \text{the number of aggregates,} \\ M_1 &= \sum_{i=1}^{\infty} i f_i, && \text{the total number of monomers.} \end{aligned} \tag{3}$$

The conservation of mass requires that the total mass of particles resulting from the break up of an n-mer should be n; mathematically, this is expressed as

$$\sum_{j=1}^{n-1} j b_{j,n} = n. \tag{4}$$

Due to this fact the process only consists in rearranging the monomers into clusters and thus it is expected that the mass of the whole ensemble will be conserved; that is,

$$\frac{d}{dt} M_1(t) = 0. \tag{5}$$

It turns out, however, that this not always holds.

In many applications it is advantageous to allow clusters to be composed of particles of any size $x > 0$. This leads to the continuous version of (1) that was derived by Müller in the pure coagulation case, [53], and extended to a coagulation–fragmentation version by [52]. In the notation proposed in [49]

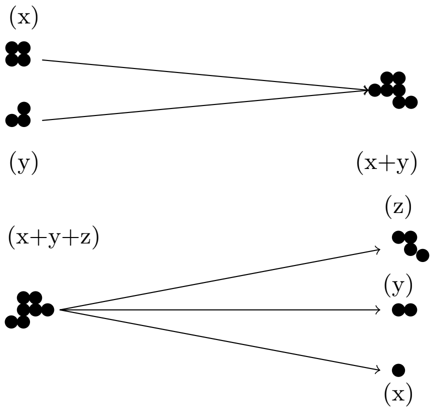

FIGURE 2. Coagulation (above) and fragmentation (below) processes

the full equation reads

$$\begin{aligned}\partial_t f(x,t) &= -a(x)f(x,t) + \int\limits_x^\infty a(y)b(x|y)f(y,t)dy \\ &\quad -f(x,t)\int\limits_0^\infty k(x,y)f(y,t)dy + \frac{1}{2}\int\limits_0^x k(x-y,y)f(x-y,t)f(y,t)dy,\end{aligned} \tag{6}$$

where $x \in \mathbb{R}_+ := (0,\infty)$ denotes the mass or size of a particle/cluster. Here f is the density of particles of mass/size x and, as in the discrete case, a is the fragmentation rate and b describes the distribution of masses x of the particles spawned by fragmentation of a particle of mass y. Further, $b \geq 0$ is assumed to be a measurable function of two variables satisfying $b(x|y) = 0$ for $x > y$; the continuous version of (4) is

$$\int\limits_0^y xb(x|y)dx = y, \quad y \in \mathbb{R}_+ \tag{7}$$

and the number of particles and the total mass the ensemble are given by the integral counterparts of (3). However, contrary to the discrete case, the total number of particles and the expected number of particles resulting from breaking up of a size y parent,

$$n_0(y) := \int\limits_0^y b(x|y)dx, \tag{8}$$

no longer are controlled by the respective masses and could be infinite.

The theories of discrete and continuous coagulation-fragmentation equations are to certain extent parallel and for the purpose of this paper we shall focus on the discrete case pointing out, however, if in a particular situation the results for the continuous case are qualitatively different.

2. Why do we need functional analysis in coagulation–fragmentation theory?

Due to their importance in applications, Eqns (1) and (6) have attracted much attention that resulted in a number of explicit solutions, see e.g. [68, 69, 49, 67, 39, 56, 52, 35, 51, 57, 25, 1, 17]. However, a number of them turned out to have properties that were undesirable from the physical point of view.

2.1. Breach of the mass conservation law

As mentioned above, due to the local conservation of mass, expressed in (4) and (7), the total mass of the ensemble should remain constant; that is, it should satisfy (5) (if it is differentiable). Consider, however, pure fragmentation version of (6) with $b(x,y) = 2/y$, $a(x) = 1/x$ and $k(x,y) = 0$. If we take the mono-disperse initial condition $f^{in}(x) = \delta(x-l)$, $l > 0$, (where δ denotes the Dirac delta), then the solution found in [49] is given by

$$f_l(t,x) = e^{-t/l}\left(\delta(x-l) + \frac{2t}{l^2} - \frac{t^2}{l^2}\left(\frac{1}{l} - \frac{1}{x}\right)\right), \quad x \le l, \tag{9}$$

and $f_l(t,x) = 0$ for $x > l$. Hence the total mass of the ensemble is given by

$$M(t) = e^{-t/l}\left(l + t + \frac{t^2}{2l}\right), \tag{10}$$

and clearly decreases monotonically in time. We also observe that the number of the particles is infinite for any $t > 0$ due to the non-integrability of the term $1/x$ at $x = 0$.

This phenomenon possibly was first observed for a probabilistic model of fragmentation in [30] and has been analysed in a series of more recent papers [15, 31, 38, 37, 41, 65]. It was termed *shattering* in [49, 68, 69]. It is worthwhile to note that shattering does not occur in discrete models. We shall return to this later but, intuitively, shattering is attributed to the creation of infinitely many 'zero' mass particles (dust) carrying nevertheless a nonzero mass and such a scenario cannot occur in discrete fragmentation, where the number of particles in an ensemble does not exceed its mass.

On the other hand, both in continuous and discrete coagulation systems there can occur mass loss due to a reverse process, called 'gelation', that is the formation of an infinite particle ('gel') of finite mass, see e.g. [19, 22, 29, 46, 66]. An example of such a solution for (1) with no fragmentation part and $k_{nj} = nj, n, j \geq 1$, and the initial condition $\boldsymbol{f}(0) = (1, 0, \ldots)$ is given in [46]:

$$\begin{aligned} f_n(t) &= \frac{n^{n-3}}{(n-1)!} t^{n-1} e^{-nt}, \quad t \leq 1, \\ f_n(t) &= t^{-1} f_n(1), \quad t > 1. \end{aligned} \tag{11}$$

2.2. Non-uniqueness

2.2.1. Exponentially growing solutions. Consider the fragmentation part of (1) with $a_n = n-1$ and $b_{n,j} = 2/(n-1)$:

$$\frac{df_n}{dt}(t) = -(n-1) f_n(t) + 2 \sum_{j=n+1}^{\infty} f_j(t), \quad f_n(0) = \mathring{f}_n, \quad n \geq 1. \tag{12}$$

One can show, [59], that the mass conserving solution is given by

$$f_n(t) = e^{-(n-1)t} \mathring{f}_n + e^{-(n-1)t} (1 - e^{-t})^2 \sum_{k=n+1}^{\infty} k \mathring{f}_k \tag{13}$$

$$+ e^{-(n-1)t} (1 - e^{-2t} - n(1 - e^{-t})^2) \sum_{k=n+1}^{\infty} \mathring{f}_k, \quad n \geq 1. \tag{14}$$

However, using separation of variables in (12) we obtain other solutions, given by

$$f_n(t) = e^{\lambda t} \frac{\lambda(\lambda+1)(\lambda+2)}{(\lambda+n-1)(\lambda+n)(\lambda+n+1)}, \quad n \geq 1, \lambda > 0. \tag{15}$$

These are differentiable solutions to (12) but they are not mass-conserving.

2.2.2. Mass conserving solutions. In [27] the authors found the solution to (12) in the form $\boldsymbol{f}(t) = (f_n(t))_{n\in\mathbb{N}}$, where

$$f_n(t) = (1-e^{-t})^2 e^{-t(n-1)}, \quad n \geq 1. \tag{16}$$

This solution is mass conserving for $t > 0$,

$$\sum_{n=1}^{\infty} n f_n(t) = e^t(1-e^{-t})^2 \sum_{n=1}^{\infty} n e^{-kt} = 1. \tag{17}$$

However, $f_n(0) = 0$ for $n \geq 1$ so this solution describes instantaneous creation of mass from nothing.

The above examples show that it is important to have a precise definition of the solution to eliminate 'formal solutions' with physically undesirable properties.

2.3. Some history

Coagulation-fragmentation problems can be considered either from probabilistic or deterministic point of view. One should mention here the seminal paper [30] that provided a theoretical foundation for analysis of fragmentation processes. An exhaustive exposition of probabilistic theory can be found in [16].

An exhaustive exposition of the deterministic description, expressed through equation (1), can be found in the forthcoming monograph [**?**]. Here we sketch two main approaches to this problem.

Deterministic equations received much attention in the late 80s and early 90s with a systematic analysis beginning with the seminal work [3], where the authors developed the basic methodology used in many subsequent papers, see e.g. [20, 21] and references therein. The analysis of *op. cit.* was confined to binary fragmentation processes. The approach was to truncate (1) and to show that the solutions of the resulting finite dimensional systems form a weakly compact set in an appropriate space from which can extract a subsequence converging to an appropriately defined solution to (1).

The extension of these results along similar lines to the continuous problem (6) has proved to be nontrivial due to a more complex nature of weak compactness in the space $X_1 = L_1(\mathbb{R}_+, xdx)$ (natural to (6) due to the conservation of mass property), as compared with the space $l_1^1 = \{\boldsymbol{f};\ \sum_{n=1}^{\infty} n|f_n| < \infty\}$, natural to (1) due to (3). The first result in this direction was established in [63]. The approach was later refined in e.g. [26, 43, 45] and more recently used in e.g. [32, 33]. In general, it yields local in time existence of weak solution to either (1), or (6). On the other hand, the uniqueness of solutions and global existence in time of mass conserving solutions have proved to be more difficult. For instance, for discrete problems the uniqueness was addressed for a special class of solutions which are limits of mass conserving solutions of truncations of (1), called admissible solutions, [18, 21]. The authors often imposed the so-called strong fragmentation assumption which ensured that a significant amount of mass after fragmentation is concentrated in small particles. Global existence results have been extended to the multiple fragmentation case in [44] for arbitrary fragmentation rates and coagulation rates satisfying

$$0 \leq k_{i,j} \leq K(i+j), \qquad 1 \leq i, j < +\infty, \tag{18}$$

by a refinement of the truncation approach. All the above results have been obtained in the space l_1^1. However, in [20, 44] the authors proved global solvability in a smaller space l_U^1 of sequences integrable with a weight function U that was supposed to be convex and to satisfy certain growth conditions. On the other hand, it follows that if the coagulation rates grow at a superlinear rate then, in general, mass conservation breaks and a gelation occurs (or solutions do not exist), see e.g. [3, 19] in the discrete case, or [29] in the continuous one.

The weak compactness approach has proved itself very effective in dealing with pure coagulation problems. However, for the full coagulation–fragmentation equation, the fragmentation part is required to be in some way subordinated to the coagulation kernel (see the discussion in [10]). This has meant that the truncation/compactness method yielded results for a restricted class of fragmentation rates, see e.g. [34], where in fact the fragmentation is required to be binary with linear growth at $x = 0$ and $x \to \infty$.

Another approach, that is the subject of this paper, is based on semigroup theory, see e.g. [8], and consists in looking at (1) (or (6)) as a perturbation of the linear fragmentation semigroup by the nonlinear coagulation operator, denoted by $\mathcal{C}$. This approach, initiated in [1], has been well developed for continuous coagulation-fragmentation models (though see [48, 59, 60] for the discrete version of the results), allowing for handling more singular fragmentation rates than the previously described method. However, since in a standard approach the nonlinear perturbation $\mathcal{C}$ is required to be Lipschitz continuous in the chosen state space, originally only bounded coagulation kernels were considered, see e.g. [50, 9, 7, 10, 59].

On the other hand, it has been known, see e.g. [55, 58], that if the fragmentation semigroup was analytic, then one could allow $\mathcal{C}$ only to be Lipschitz continuous from the domain of a fractional power of the generator. This would accommodate a class of unbounded coagulation kernels provided one could prove that the fragmentation operator was sectorial. Since, however, in general no simple description of the generator's domain was known, [8, Remark 8.16] or [4], the analyticity of the fragmentation semigroup was not studied. Fortunately, recently, [60], it has been proved that a simple fragmentation operator with uniform binary fragmentation generates an analytic semigroup in l^1_1. This prompted interest in the topic and, subsequently, it was shown that a class of fragmentation operators, that includes physically relevant binary and homogeneous fragmentations, both in the discrete and continuous case, is sectorial albeit in a smaller space of densities that have finite higher moments (the space $l^1_p := l^1_U$ with $U = i^p, p > 1$, or its equivalent $X_{0,p} = L_1(\mathbb{R}_+, (1 + x^p)dx)$ in the continuous case), [6, 11]. Moreover, for the so-called power law fragmentation it was proven that analyticity holds also in the basic space $X_{0,1}$, [14]. These results also allowed for an explicit characterization of the domains of the fragmentation operator. We observe, however, that there are fragmentation kernels b for which the fragmentation semigroup is not analytic, [5, 6].

For analytic fragmentation semigroups we can define their fractional powers and, using real interpolation methods, we were able to provide an explicit characterization of their domains and hence to prove the existence of unique, positive, classical, local in time solutions to (1) (respectively (6)) in l^1_p (resp. $X_{0,p}$), $p > 1$, provided the coagulation kernel is controlled by a sublinear power of the fragmentation rate. Under additional assumption, that essentially amounts to the coagulation kernel having a sublinear growth with respect to the particle mass, this solution is global in time.

We note that, in contrast to the former method, the semigroup approach produces unique and differentiable solutions and allows for a greater range of fragmentation processes but it is more restrictive with respect to the coagulation kernels.

In the remaining part of the paper we shall describe the semigroup approach in more detail.

3. The Linear Part

3.1. Semigroups

3.1.3. Basic properties and generation. We consider the evolution equation

$$\begin{aligned} \partial_t f(t) &= [\mathcal{K} f](t), \quad t > 0, \\ f(0) &= \mathring{f}, \end{aligned} \tag{19}$$

where $\mathcal{K}$ is a differential, integral or functional expression. We describe the evolution by a family of operators $(S(t))_{t\geq 0}$ on an appropriately chosen state space X that here is assumed to be a Banach space. Operators $S(t), t \geq 0$, map the initial state of the system onto all subsequent states in the evolution, $f(t) = S(t)\mathring{f}$. If the evolution problem is well-posed, then the family $(S(t))_{t\geq 0}$ consists of continuous operators, is continuous (strongly) in time and satisfies the flow property $S(t+s) = S(t)S(s)$; it is then called a C_0-*semigroup*, e.g. [55].

It is important to realize that, in general, $(S(t))_{t\geq 0}$ does not provide the solution to (19) but to

$$\begin{aligned} \partial_t f(t) &= [Kf](t), \quad t > 0, \\ \lim_{t\to 0^+} f(t) &= \mathring{f}, \end{aligned} \tag{20}$$

where K is called the (infinitesimal) generator of $(S(t))_{t\geq 0}$ and is defined as $Kf = \left.\frac{dS(t)f}{dt}\right|_{t=0}$ on the domain consisting of those f for which the derivative exists. In general, $K \neq \mathcal{K}$ but it is a certain realization of $\mathcal{K}$.

It is important to note that while $t \to S(t)\mathring{f}$ is defined for any $\mathring{f} \in X$, in general it only is differentiable, with $S(t)\mathring{f} \in D(K)$, if $\mathring{f} \in D(K)$, and hence only for such initial conditions it is a solution to (20).

By the Hille-Yosida theorem, [55], an operator K is the generator of a C_0-semigroup if and only if it is closed, densely defined and for some $M > 0$ and $\omega \in \mathbb{R}$ its resolvent $R(\lambda, K) := (\lambda I - K)^{-1}$ exists and for $n \geq 1$

$$\|R^n(\lambda, K)\| \leq M(\lambda - \omega)^{-n}, \quad \lambda > \omega. \tag{21}$$

3.1.4. Analytic semigroups. If

$$\|R(\lambda)\| \leq \frac{C}{|\lambda|}, \tag{22}$$

in some sector $S_{\frac{\pi}{2}+\delta} = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$, then K is the generator of a semigroup that can be extended to an analytic function and has the integral representation

$$S(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t} R(\lambda, K)\mathrm{d}\lambda, \tag{23}$$

where Γ is an unbounded smooth curve in $S_{\frac{\pi}{2}+\delta}$.

A great advantage of analytic semigroups is that, in contrast to the general case, $t \to S(t)$ has derivatives of arbitrary order on $]0,\infty[$. This shows that $t \to S(t)\mathring{f}$ solves the Cauchy problem (20) and $S(t)\mathring{f} \in D(K^n)$ for any $n \geq 0$ and $t > 0$ whenever $\mathring{f} \in X$.

3.1.5. Extrapolation – Sobolev towers. Since $S(t)D(K) \subset D(K)$, it follows that the restriction $(S(t))_{t\geq 0}$ to $D(K)$ is a C_0-semigroup on $D(K)$ (equipped with the graph norm) generated by the part of K in $D(K)$. This idea lies behind the construction of the so-called Sobolev towers, see [28, pp. 124–129], that also allows for an extension of $(S(t))_{t\geq 0}$ to larger spaces. To simplify the notation, we assume that the semigroup $(S(t))_{t\geq 0}$ generated by K is of negative type so that $K^{-1} \in \mathcal{L}(X)$ (this always can be achieved by considering, instead of K^{-1}, the resolvent $R(\lambda, K)$ for sufficiently large λ). Then we define spaces $X_n = (D(K^n), \|\cdot\|_n), n \in \mathbb{N}$, where $\|x\|_n = \|K^n x\|$, that are called the associated Sobolev spaces (of order n). The semigroup $(S_n(t))_{t\geq 0}$ consisting of restrictions of $S(t)$ to X_n is a C_0-semigroups on X_n, generated by the part K_n of K in X_n. In particular, we have $(K_n, D(K_n)) = (K, D(K^{n+1}))$.

We can invert the procedure to obtain X_n as the completion of X_{n+1} with respect to the norm $\|x\|_n = \|K^{-1}_{n+1}\|$. For example, the space X_{-1} is obtained as the completion of X with respect to the norm

$$\|x\|_{-1} = \|K^{-1}x\|. \tag{24}$$

In particular, the generator $(K_{-1}, D(K_{-1}))$ of $(S_{-1}(t))_{t\geq 0}$, where $D(K_{-1}) = X$, is the unique extension by density of $(K, D(K))$.

It is easy to see that the functional theoretic properties of $(S_n(t))_{t\geq 0}$ are the same for $n \in \mathbb{Z}$ hence, in particular, $(S_n(t))_{t\geq 0}$ is analytic whenever $(S(t))_{t\geq 0}$ is. Thus,

$$S(t) \bigcup_{n\in\mathbb{Z}} D(K^n) \subset \bigcap_{n\in\mathbb{Z}} D(K^n), \quad t > 0. \tag{25}$$

3.1.6. Substochastic semigroups and their perturbations. Checking condition (21), or even (22), is often infeasible in practice. Thus we often try to break $\mathcal{K}$ into a sum so that (19) can be written as

$$\partial_t u = \mathcal{A}u + \mathcal{B}u, \tag{26}$$

where a realization $(A, D(A))$ of $\mathcal{A}$ can be easily proven to generate a semigroup; then the generation for a suitable realization of $\mathcal{A} + \mathcal{B}$ is achieved by one of the perturbation theorems, [55]. In the context of fragmentation problems possibly the most useful perturbation theorem is the one concerning substochastic semigroups, whose origins go back to the paper [42] that concerns the Kolmogorov equation.

Let $X = L_1(\Omega, \mathrm{d}\mu)$, where $d\mu$ is a measure on a suitable σ-algebra of subsets of Ω. We emphasize that Ω could be a discrete set in which case all integrals below should be replaced by relevant series. The space X equipped with partial order, $f \geq 0$ if and only if $f(x) \geq 0$ for μ almost all $x \in \Omega$, is a Banach lattice; by X_+ we denote the cone of nonnegative elements of X. We say that a C_0-semigroup $(S(t))_{t\geq 0}$ on X is *substochastic* if it is positive and contractive (that is, for each $t \geq 0$ and $f \in X_+$ we have $S(t)f \geq 0$ and $\|S(t)f\| \leq \|f\|$). We assume that $(A, D(A))$ generates a substochastic semigroup $(S_A(t))_{t\geq 0}$, $B := \mathcal{B}|_{D(A)}$ is positive on $D(A)_+ := D(A) \cap X_+$ and

$$\int\limits_{\Omega} (A + B)u \,\mathrm{d}\mu = c_0(u) - c(u), \qquad u \in D(A)_+, \tag{27}$$

where $|c_0(u)| \leq \gamma\|u\|$ and c is a positive functional on $D(A)_+$.

Theorem 3.1. [8] *Assume that (27) is satisfied. Then there exists a smallest positive semigroup $(S_K(t))_{t\geq 0}$ generated by an extension K of the operator $A + B$. If $c_0 = 0$, $(S_K(t))_{t\geq 0}$ is substochastic. Furthermore, c extends from $D(A)$ to $D(K)$ by continuity (in the $D(K)$ norm) and from $D(A)_+$ to $D(K)_+$ by monotonic limits and*

$$\int\limits_{\Omega} Ku \, d\mu \leq c_0(u) - c(u), \qquad u \in D(K)_+. \tag{28}$$

3.1.7. Characterization of the generator. The main problem with the above result is that it does not provide any explicit characterization of the generator and this has serious consequences for the physical well-posedness of the model. More precisely, for fragmentation models we expect the conservation of mass equation, Eqn. (5), to hold. It can be obtained from (1) by summing up the equations multiplied by the weight n. Then (5) follows by the cancellation of the positive and negative parts of the equation provided, however, that one can sum them up separately. This is not always possible – each part may sum up to infinity – and this is why solutions that do not conserve mass, such as (9) and (11), do exist. In general, we do not have to restrict ourselves to conservative problems and we expect that the integration of the derivative of the solution yields the same result as the formal integration of the equation. More precisely, we call the semigroup $(S_K(t))_{t\geq 0}$ honest if

$$\frac{d}{dt}\|S_K(t)\mathring{f}\| = \int\limits_{\Omega} KS_K(t)\mathring{f}\, d\mu = c_0(S_K(t)\mathring{f}) - c(S_K(t)\mathring{f}),$$

for any $\mathring{f} \in D(K)_+$; otherwise $(S_K(t))_{t\geq 0}$ is called dishonest. Clearly, the mass losing solution (9) shows that the semigroup solving the fragmentation equation with $a(x) = 1/x$ is not honest.

It turns out that the problems of honesty, as well as the existence of multiple solutions such as (15), are closely related to how big the domain of the generator is. Remembering that K is a realisation of the expression $\mathcal{K} = \mathcal{A} + \mathcal{B}$, we observe that two obvious realizations of $\mathcal{K}$ in X_1 are:

- the minimal operator $K_{\min}$ defined as $\mathcal{K}|_{D(A)}$;
- the maximal operator $K_{\max}$ defined as $\mathcal{K}|_{D(K_{\max})}$, where

$$D(K_{\max}) = \{f \in X;\ \mathcal{A}f, \mathcal{B}f \text{ are finite a.e}, \mathcal{K}f \in X\}.$$

By [8, Theorem 6.20],

$$K_{\min} \subset K \subset K_{\max}.$$

Where K is situated between $K_{\min}$ and $K_{\max}$ determines the well-posedness of problem (26). Clearly, if $K = K_{\min}$, $(S_G(t))_{t\geq 0}$ is honest by (27). Also, since then $Af \in X$ for $f \in D(K)_+$, only a finite mass undergoes the fragmentation per unit time. On the other hand, by [8, Lemma 3.50 & Proposition 3.52], $K_{\max}$ is closed and

$$D(K_{\max}) = D(K) \oplus Ker(\lambda I - K_{\max}),$$

where Ker denotes the null-space of the operator. Hence if $D(K_{\max}) \neq D(K)$, then there exist strong exponentially growing solutions originating from $Ker(\lambda I - K_{\max})$, such as (15).

In general, the following cases are possible:

1. $K_{\min} = K = K_{\max}$ – uniqueness, honesty, finite rates of break-ups,
2. $K_{\min} \subsetneq K = \overline{K_{\min}} = K_{\max}$ – uniqueness, honesty, infinite rates,
3. $K_{\min} = K \subsetneq K_{\max}$ – nonuniqueness, honesty, finite rates,
4. $K_{\min} \subsetneq K = \overline{K_{\min}} \subsetneq K_{\max}$ – nonuniqueness, honesty, infinite rates,
5. $\overline{K_{\min}} \subsetneq K \subsetneq K_{\max}$ – nonuniqueness, dishonesty, infinite rates of events.

Interestingly enough, the other case of nonuniqueness, (16), is related to another phenomenon but its explanation requires introducing analytic fragmentation semigroups.

3.2. Fragmentation equation in higher moment spaces

If the functional c in (27) is nontrivial, then Theorem 3.1 has a great potential, not really recognized earlier. Namely, the domain of the monotone extension of c provides an additional information about the domain of the generator K. It is particularly useful if c does not change its form under monotonic limits – due to the Lebesgue Monotone Convergence Theorem this is the case, if c is defined by an integral or a series, see the application of this observation in Theorem 3.2 below. However, since the fragmentation process is conservative in l^1_1 (X_1), $c = 0$, and hence there is a need to consider smaller spaces l^1_p ($X_{0,p}$) introduced in Section 2.3 (in line with the restrictions introduced in e.g. [20, 44]).

In what follows, we shall focus on the discrete equation posed in the space l^1_p with the norm

$$\|\boldsymbol{f}\|_p = \sum_{i=1}^{\infty} i^p |f_i|, \qquad p > 1, \tag{29}$$

the continuous theory in $X_{0,p}$ being analogous, [6, 11]. We recall, that we use the operators $A_p = \mathcal{A}|_{D(A_p)}$, where $\mathcal{A}\boldsymbol{f} = \{0, -a_2f_2, -a_3f_3 \ldots\}$ and $D(A_p) = \{\boldsymbol{f} \in l^1_p;\ \mathcal{A}\boldsymbol{f} \in l^1_p\}$, and $B_p\boldsymbol{f} = \left(\sum\limits_{i=n+1}^{\infty} a_ib_{n,i}f_i\right)_{n\geq 1}$ on $D(A_p)$.

We define

$$\Delta_n^{(p)} := n^p - m_n^{(p)} := n^p - \sum_{k=1}^{n-1} k^p b_{k,n}, \quad n \geq 2, p \geq 0. \tag{30}$$

Then, for $n \geq 2$,

$$\Delta_n^{(0)} = 1 - m_1^0 \leq 0, \quad \Delta_n^{(1)} = 0, \quad \Delta_n^{(p)} \geq 0, \qquad p \geq 1. \tag{31}$$

Theorem 3.2. [6]

1. *The closure* $(F_p, D(F_p)) := \overline{(A_p + B_p, D(A_p))}$ *generates a positive semigroup of contractions, say* $(S_{F_p}(t))_{t\geq 0}$, *that satisfies*

$$\frac{d}{dt}\|S_{F_p}(t)\boldsymbol{f}\|_p = -\sum_{i=2}^{\infty} a_if_i\Delta_i^{(p)} =: -c_p(\boldsymbol{f}), \quad \boldsymbol{f} \in D(F_p)_+.$$

2. *If*

$$\liminf_{n\to\infty} \frac{\Delta_n^{(p)}}{n^p} > 0, \tag{32}$$

then $D(F_p) = D(A_p)$ *and* $(S_{F_p}(t))_{t\geq 0}$ *is analytic.*

3. *If* (32) *holds for some* p_0, *then it holds for all* $p \geq p_0$.

Sketch of the proof. Statement 1. follows from Theorem 3.1. In this case, (28) takes the form

$$\sum_{n=1}^{\infty} n^p[(A_p + B_p)\boldsymbol{f}]_n = -\sum_{n=2}^{\infty} a_nf_n\Delta_n^{(p)} = -c_p(\boldsymbol{f}) \leq 0, \tag{33}$$

where c_p is a nonnegative functional for $p \geq 1$. That $F_p = \overline{A_p + B_p}$ follows from a standard application of the theory of extensions.

If (32) is satisfied then, since c_p extends to $D(F_p)_+$ by monotonic limits,

$$\sum_{n=n_0}^{\infty} a_nn^pf_n \leq M \sum_{n_0=1}^{\infty} a_n\Delta_n^{(p)}f_n < \infty, \quad \boldsymbol{f} \in D(F_p)_+,$$

for some constant M by (32), hence $D(F_p) \subset D(A_p)$. Thus $D(F_p) = D(A_p)$. Hence, analyticity follows from the Arendt–Rhandi theorem:

Theorem 3.3. [2] *Assume that X is a Banach lattice, $(A, D(A))$ is a resolvent positive operator which generates an analytic semigroup and $(B, D(A))$ is a positive operator. If $(\lambda_0 I - (A+B), D(A))$ has a nonnegative inverse for some λ_0 larger than the spectral bound of A, then $(A+B, D(A))$ generates a positive analytic semigroup.*

Thus the result follows as $(A_p, D(A_p))$ is the generator of a positive analytic semigroup. □

We note that (32) cannot hold for $p = 1$ as $\Delta^{(1)} = 0$. Nevertheless, there are analytic fragmentation semigroups in l_1^1 and $X_{0,1}$, see [14, 60], but the proofs in these spaces require direct estimates of the resolvent that is not explicitly available in general.

3.3. Physical meaning of (32)

To better understand (32) we note the following lemma, [6],

Lemma 3.1. *Let $M_{r_n}^{(n)} := \sum_{k=1}^{r_n} kb_{k,n}$. Assume that for each n there is $1 < r_n < n-1$ such that*

$$\liminf_{n\to\infty} \frac{M_{r_n}^{(n)}}{n} > 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_n}{n} < 1. \tag{34}$$

Then (32) holds.

In other words, (32) holds if a significant fraction of daughters' mass is concentrated away from the parent's mass.

Example 1. Homogeneous-like fragmentation. *Assume that $b_{k,n}$ can be written as*

$$b_{k,n} = \zeta_n h\left(\frac{k}{n}\right), \quad 1 \le k \le n-1, n \in \mathbb{N}, \tag{35}$$

where h is a nonnegative function such that the Riemann integral $\int_0^1 zh(z)dz$ exists and is positive, and ζ_n is an arbitrary positive sequence. Since

$$n = \sum_{k=1}^{n-1} kb_{k,n}$$

we can write

$$1 = \zeta_n(n-1)\sum_{k=1}^{n-1}\frac{k}{n}h\left(\frac{k}{n}\right)\frac{1}{n-1}.$$

Since

$$\frac{k-1}{n-1} \leq \frac{k}{n} \leq \frac{k}{n-1}$$

for $1 \leq k \leq n$*, we have*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}h\left(\frac{k}{n}\right)\frac{1}{n-1} = \int\limits_0^1 zh(z)dz$$

and thus

$$\lim_{n\to\infty}(n-1)\zeta_n = \frac{1}{\int\limits_0^1 zh(z)dz}.$$

Therefore

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^p b_{k,n} \quad = \quad \frac{\int\limits_0^1 z^p h(z)dz}{\int\limits_0^1 zh(z)dz} < 1$$

and

$$\liminf_{n\to\infty}\frac{\Delta_n^{(p)}}{n^p} = \lim_{n\to\infty}\frac{n^p - \sum\limits_{k=1}^{n-1}k^p b_{k,n}}{n^p} > 0.$$

*Hence (*32*) is satisfied for any* $p > 1$*.*

*Condition (*35*) is satisfied, in particular, by the uniform binary fragmentation*

$$b_{k,n} = \frac{2}{n-1}, \quad k = 1,\ldots,n-1$$

as well as when the binary fragmentation is described by a symmetric infinite matrix $(\psi_{i,j})_{i,j\geq 1}$ *through*

$$\frac{du_i}{dt} = -\frac{1}{2}u_i\sum_{j=1}^{i-1}\psi_{j,i-j} + \sum_{j=1}^{\infty}\psi_{j,i}u_{i+j}, \quad i \geq 1,$$

see [3, 18, 19, 21, 44]*. The cases particularly important in polymer degradation,* [68]*, are*

$$\psi_{i,j} = (i+j)^\beta, \quad \psi_{i,j} = (ij)^\beta, \quad \beta > -2.$$

In our notation, the first case gives $a_n = n^\beta(n-1)/2$ *and* $b_{i,n} = 2/(n-1)$ *and hence it is a uniform binary fragmentation. In the second case we have*

$$b_{i,n} = \frac{i^\beta(n-i)^\beta}{a_n} = \frac{n^{2\beta}}{a_n}\left(\frac{i}{n}\right)^\beta\left(1-\frac{i}{n}\right)^\beta,$$

hence (35) *is satisfied with*

$$\zeta(n) = \frac{n^{2\beta}}{a_n} \quad \text{and} \quad h(z) = z^\beta(1-z)^\beta.$$

Example 2. *Let us consider a fragmentation process in which a particle of mass* n *splits into two particles with masses 1 and* $n-1$*. In this case*

$$\begin{aligned} b_{1,2} &= 2, \quad \text{and} \quad b_{1,n} = b_{n-1,n} = 1, \\ b_{k,n} &= 0, \quad n \geq 2, 2 \leq k \leq n-2. \end{aligned} \tag{36}$$

Then

$$\lim_{n\to\infty} \frac{\Delta_n^{(p)}}{n^p} = \frac{n^p - (1+(n-1)^p)}{n^p} = 0$$

and, in fact, one can prove that

$$(F_p, D(F_p)) = \overline{(A_p + B_p, D(A_p))} \neq (A_p + B_p, D(A_p))$$

and $(S_{F_p}(t))_{t\geq 0}$ *is not analytic for any* $p \geq 1$, [5, 12].

3.4. Nonuniqueness of type (16)

Let $\boldsymbol{f}(t)$ be the solution defined by (16). Using (17), we see that for $t > 0$

$$\|\boldsymbol{f}(t)\|_{l_p^1} = (1-e^{-t})^2 e^t \sum_{n=1}^\infty n^p e^{-nt} \geq (1-e^{-t})^2 e^t \sum_{n=1}^\infty n e^{-nt} = \|\boldsymbol{f}\|_{l_1^1} = 1,$$

and thus $\boldsymbol{f}$ is not a solution of the problem in the sense of semigroup in l_p^1. Consider, however, the resolvent of F_p. Using formula (14) for $(S_{F_p}(t))_{t\geq 0}$, we find

$$\begin{aligned} [R(\lambda, F_p)\boldsymbol{f}]_n &= \left[\int_0^\infty e^{-\lambda t} S_F(t)\boldsymbol{f}dt\right]_n = \frac{1}{\lambda - 1 + n} f_n \\ &\quad + \frac{2}{(\lambda-1+n)(\lambda+n)(\lambda+n+1)} \sum_{k=n+1}^\infty (\lambda+k) f_k. \end{aligned}$$

Since $\boldsymbol{f}(t) \in D(F_p)$ for $t > 0$, $\boldsymbol{f}(t) = \boldsymbol{f}(t' + t_0) = S_{F_p}(t')\boldsymbol{f}(t_0)$ and thus

$$R(1, F_p)\boldsymbol{f}(t) = \int\limits_0^\infty e^{-s} S_{F_p}(s)\boldsymbol{f}(t)ds = \int\limits_0^\infty e^{-s}\boldsymbol{f}(s+t)ds = e^t \int\limits_t^\infty e^{-\sigma}\boldsymbol{f}(\sigma)d\sigma$$

and $\lim_{t\to 0^+} R(1, F_p)\boldsymbol{f}(t)$ exists if an only if $\int\limits_0^\infty e^{-\sigma}\boldsymbol{f}(\sigma)d\sigma$ exists in l_p^1 and

$$\lim_{t\to 0^+} R(1, F_p)\boldsymbol{f}(t) = \int\limits_0^\infty e^{-\sigma}\boldsymbol{f}(\sigma)d\sigma =: \boldsymbol{y}.$$

By (24), $\boldsymbol{f}(t)$ converges to $\mathring{\boldsymbol{f}}$ in the first Sobolev tower $l_{p,-1}^1$ that here, thanks to $D(F_p) = D(A_p)$, can be identified with the weighted space l_U^1, where $U = (n^p(1 + a_n)^{-1})_{n\geq 1}$. In our case, denoting $\boldsymbol{y} = (y_n)_{n\geq 1}$, we have

$$y_n = \int\limits_0^\infty (1 - e^{-s})^2 e^{-ns} ds = \frac{2}{n(n+1)(n+2)} \in l_p^1$$

provided $p < 2$. Thus, the initial condition in $l_{p,-1}^1$ is given by $\mathring{\boldsymbol{f}} = (1 - F_{p,-1})\boldsymbol{y} = (1 - (\mathcal{A} + \mathcal{B}))\boldsymbol{y}$.

To give some meaning to this initial condition, let us consider a sequence of initial conditions corresponding to a unit mass concentrated in m-clusters: $\boldsymbol{x}^m = m^{-1}(\delta_{n,m})_{n\geq 1}$. To simplify calculations we denote $\boldsymbol{f}^m = \frac{m}{m+1}\boldsymbol{x}^m$. Convergence of these two sequences is equivalent in any norm topology. Now,

$$(R(1, F_p)\boldsymbol{f}^m)_n = \begin{cases} \frac{2}{n(n+1)(n+2)} & \text{if} \quad n < m, \\ \frac{1}{m(m+1)} & \text{if} \quad n = m, \\ 0 & \text{if} \quad n > m, \end{cases}$$

so that

$$[R(1, F_{p,-1})(\boldsymbol{f}^m - \mathring{\boldsymbol{f}})]_n = \begin{cases} 0 & \text{if} \quad n < m, \\ \frac{1}{m(m+1)} - \frac{2}{m(m+1)(m+2)} & \text{if} \quad n = m, \\ -\frac{2}{n(n+1)(n+2)} & \text{if} \quad n > m, \end{cases}$$

and hence

$$\begin{aligned}\|\boldsymbol{f}^m - \mathring{\boldsymbol{f}}\|_{l^1_U} &= \|R(1, F_{p,-1})(\boldsymbol{f}^m - \mathring{\boldsymbol{f}})\|_{l^1_p} \\ &= m^p \left(\frac{1}{m(m+1)} - \frac{2}{m(m+1)(m+2)}\right) \\ &\quad + \sum_{n=m+1}^{\infty} \frac{n^{p-1}}{(n+1)(n+2)} \to 0\end{aligned}$$

as $m \to \infty$, provided $p < 2$. Thus, one can say that (16) is the solution corresponding to an infinitely large cluster of unit mass, originating (for $p < 2$) from the first Sobolev tower $l^1_{p,-1}$. At the same time for $p > 1$ (actually, for $p \geq 1$, see [60], but the proof for $p = 1$ is different) the semigroup $(S_{F_p}(t))_{t\geq 0}$ is analytic, and hence is $(S_{F_{p,-1}}(t))_{t\geq 0}$. Due to the immediate regularization property (25), the solution $t \to S_{F_{p,-1}}(t)\mathring{\boldsymbol{f}}$ immediately appears in l^1_p so that it has nonzero components. At the same time, no fraction of mass of $\mathring{\boldsymbol{f}}$ is concentrated in finite clusters explaining thus the fact that the initial value of the solution is zero coordinate-wise.

3.5. Immediate moment regularization

Proposition 1. [13] *Let (32) be satisfied for any $p > 1$ (so that $(S_{F_p}(t))_{t\geq 0}$ is analytic) and $a_n = O(n^\delta)$ for some $\delta > 0$. If $\mathring{\boldsymbol{f}} \in l^1_p$ (that is, if the initial condition has finite moment of order higher than 1), then $S_{F_p}(t)\mathring{u}$ immediately has finite moments of any order.*

This result immediately follows from the fact that $S_{F_p}(t)\mathring{\boldsymbol{f}} \in \bigcap_{n=1}^{\infty} D(F_p^n) = \bigcap_{n=1}^{\infty} D(A_p^n)$ for $\mathring{\boldsymbol{f}} \in l^1_p$ and $t > 0$. Thus, $S_{F_p}(t)\mathring{\boldsymbol{f}} \in l^1_{p+n\delta}$ for any $n \geq 1$.

4. The Nonlinear Part

4.1. Semilinear problems

We consider the following semilinear perturbation of (20),

$$\begin{aligned}\partial_t f &= Kf + g(t, f), \quad t > 0, \\ f(0) &= \mathring{f}, \end{aligned} \tag{37}$$

where K is the generator of a C_0-semigroup $(S_K(t))_{t\geq 0}$ in a Banach space X and $g : [0, T] \times X \to X$ is a known function. Typically such problems are converted to the integral equation

$$f(t) = S_K(t)\mathring{f} + \int\limits_0^t S_K(t-s)g(s, f(s))ds \tag{38}$$

by using the variation of constants formula and then analysed by an appropriate fixed point theorem such as the Banach contraction principle or the Schauder fixed point theorem. If $(S_K(t))_{t\geq 0}$ is a C_0-semigroup, then for the Banach principle to work, g must be Lipschitz continuous in f on X. However, if $(S_K(t))_{t\geq 0}$ is an analytic semigroup, this requirement can be significantly weakened.

4.1.8. Fractional powers and interpolation. If K is the generator of an analytic semigroup then, using the Dunford type functional calculus based on (23), we can define bounded operators $(-K)^{-\alpha}$ and, by inversion, unbounded operators $(-K)^{\alpha}$. A useful formula for fractional powers, that avoids complex integration, [55, Chapter 2, Eq. (6.9)], is

$$(-K)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S_K(t)dt. \tag{39}$$

We denote $X_\alpha = D((-K)^\alpha)$, the domain of $(-K)^\alpha$ equipped with the graph norm. It follows that for any $0 < \alpha < 1$,

$$D(K) \subset D((-K)^\alpha) \subset X$$

with dense embeddings.

Example 3. *One of the most important operators in fragmentation problems is the loss operator, defined by*

$$A_p\boldsymbol{f} = (0, -a_2f_2, \ldots, -a_nf_n \ldots), \quad \boldsymbol{f} \in D(A_p) = \{\boldsymbol{f} \in l^1_p;\ A_p\boldsymbol{f} \in l^1_p\}, \tag{40}$$

where $(a_n)_{n\in\mathbb{N}}$ is a given sequence of positive numbers, in the discrete case and

$$[A_{0,p}\boldsymbol{f}] = -a(x)f(x), \quad f \in D(A_p) = \{\boldsymbol{f} \in X_{0,p};\ A_{0,p}f \in X_{0,p}\}, \tag{41}$$

in the continuous case.

Let us focus on the discrete case, the continuous one being analogous. Due to $a_1 = 0$, $0 \notin \rho(A_p)$ *and hence we consider the shifted operator* $A_{p,\omega} := -\omega I + A_p$, $\omega > 0$, *so that* $0 \in \rho(A_{p,\omega})$. *Clearly*

$$[S_{A_{p,\omega}}(t)\boldsymbol{f}]_n = e^{-a_{\omega,n}t} f_n, \quad n \geq 1,$$

where $a_{\omega,1} = \omega$ *and* $a_{\omega,n} = \omega + a_n, n > 1$, *is an analytic semigroup and we have*

$$\begin{aligned} [(-A_{p,\omega})^{-\alpha}\boldsymbol{f}]_n &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-a_{\omega,n}t}f_n dt \\ &= \frac{a_{\omega,n}^{-\alpha}f_n}{\Gamma(\alpha)}\int\limits_0^\infty \sigma^{\alpha-1}e^{-\sigma}d\sigma = a_{\omega,n}^{-\alpha}f_n. \end{aligned} \tag{42}$$

Hence, in particular,

$$l^1_{p,\alpha} := D((-A_{p,\omega})^\alpha) = \left\{\boldsymbol{f};\ \sum_{n=1}^\infty a_n^\alpha n^p |f_n|\right\}. \tag{43}$$

If $(S_K(t))_{t\geq 0}$ is an analytic semigroup, then for every $t > 0$ and $\alpha \geq 0$ the operator $(-K)^\alpha S_K(t) (= S_K(t)(-K)^\alpha)$ is bounded and

$$\|t^\alpha(-K)^\alpha S_K(t)\| \leq M_\alpha \tag{44}$$

for some constant M_α, [55, Theorem 2.6.13], so that $t \to (-K)^\alpha S_K(t)$ is integrable close to 0. Then one can write

$$\begin{aligned} &(-K)^\alpha f(t) \\ &\qquad = S_K(t)(-K)^\alpha \mathring{f} + \int\limits_0^t (-K)^\alpha S_K(t-s)g(s,(-K)^{-\alpha}(-K)^\alpha f(s))ds \end{aligned}$$

and consider the problem for $h(t) = (-K)^\alpha u(t)$. Then it would be sufficient if $g(t,(-K)^{-\alpha}\cdot)$ was Lipschitz continuous or, in other words, that g was Lipschitz continuous with respect to f as a function from X_α to X.

We observe that the spaces $D((-K)^\alpha)$ form an important class of intermediate spaces between $D(K)$ and X. However, in some situations they are not sufficient as they are not a priori independent of the form of K and there is no explicit method to evaluate the norm. However, there is a parallel theory, [47], leading to analogous results, in which X_α can be any (reasonable) interpolation space. To explain this in more detail, first we observe that (44) can be written as

$$\|t^{1-\alpha} K S_K(t)(-K)^\alpha x\| \in L_\infty(I),$$

whenever $x \in D((-K)^\alpha)$, where $I := (0,1)$. Taking this formula as the starting point, let $(K, D(K))$ be the generator of an analytic semigroup $(S_K(t))_{t\geq 0}$ on a Banach space X. Then we construct families of intermediate spaces, $D_K(\alpha, r)$, $0 < \alpha < 1$, $1 \leq r \leq \infty$, in the following way:

$$\begin{aligned} D_K(\alpha, r) &:= \{x \in X;\ t \to v(t) := \|t^{1-\alpha-1/r} K S_K(t) x\|_X \in L_r(I)\}, \\ \|x\|_{D_K(\alpha,r)} &:= \|x\|_X + \|v(t)\|_{L_r(I)}, \end{aligned} \tag{45}$$

see [47, p.45]. It turns out that these spaces can be identified with real interpolation spaces between X and $D(K)$ and one can use the rich theory of them. In particular, by [47, Corollary 2.2.3], these spaces do not depend explicitly on K, but only on $D(K)$ and its graph norm. This is in contrast to the property of $D((-K)^\alpha)$ mentioned above, where we only have

$$D_K(\alpha, 1) \subset D((-K)^\alpha) \subset D_K(\alpha, \infty) \tag{46}$$

so in general $D((-K)^\alpha)$ may depend on the particular form of K.

In what follows we use $X_\alpha = D_K(\alpha, 1), 0 < \alpha < 1$, (though the statements below are valid for arbitrary r).

What makes the spaces $D_K(\alpha, 1)$ as useful as $D((-K)^\alpha)$ in dealing with the semigroup generated by K is the fact that the operations in (45) commute with $(S_K(t))_{t\geq 0}$, hence with $R(\lambda, K)$, and this leads to

$$\begin{aligned}\|R(\lambda,K)x\|_{D_K(\alpha,1)} &= \|R(\lambda,K)x\|_X + \int_0^1 \|s^{-\alpha} K S_K(s) R(\lambda,K)x\|_X ds \\ &\leq \|R(\lambda,K)\|_{\mathcal{L}(X)} \left(\|x\|_X + \int_0^1 \|s^{-\alpha} K S_K(s)x\|_X ds \right) \\ &\leq \|R(\lambda,K)\|_{\mathcal{L}(X)} \|x\|_{D_K(\alpha,1)}.\end{aligned}$$

This leads to the following observation.

Proposition 2. *Let $K_{\alpha,1}$ be the part of K in $D_K(\alpha,1)$. Then $\rho(K_{\alpha,1}) \subset \rho(K)$, $\|R(\lambda, K_{\alpha,1}\|_{\mathcal{L}(D_K(\alpha,1))} \leq \|R(\lambda,K)\|_{\mathcal{L}(X)}$ for $\lambda \in \rho(K)$. Consequently, $K_{\alpha,1}$ is a sectorial operator in $D_K(\alpha,1)$.*

Example 4. *Let us consider again $A_{p,\omega}$, considered in Example 3, and the space $D_{A_{p,\omega}}(\alpha,1)$. In this case we obtain a very useful identification as*

$$\begin{aligned}\|v(t)\|_{L_1(I)} &= \int_0^1 \left(\sum_{n=1}^{\infty} t^{-\alpha} a_{\omega,n} e^{-a_{\omega,n}t} |f_n| n^p \right) dt \\ &= \sum_{n=1}^{\infty} \left(a_{\omega,n} |f_n| n^p \right) \int_0^1 t^{-\alpha} e^{-a_{\omega,n}t} dt \Big) \\ &= \sum_{n=1}^{\infty} \left(a_{\omega,n}^{\alpha} |f_n| n^p \int_0^{a_{\omega,n}} \sigma^{-\alpha} e^{-\sigma} d\sigma \right)\end{aligned}$$

by

$$0 < \int_0^{\omega} \sigma^{-\alpha} e^{-\sigma} d\sigma \leq \int_0^{a_{\omega,n}} \sigma^{-\alpha} e^{-\sigma} \mathrm{d}\sigma \leq \int_0^{\infty} \sigma^{-\alpha} e^{-\sigma} \mathrm{d}\sigma < \infty,$$

yields

$$D_{A_{p,\omega}}(\alpha,1) = D((-A_{p,\omega})^{\alpha}) = \left\{ \boldsymbol{f};\ \sum_{n=1}^{\infty} a_n^{\alpha} n^p |f_n| \right\}.$$

An exhaustive characterization of real interpolation spaces between various l_p and L_p spaces with weights can be found in [64, Sections 1.18.1 & 1.18.5].

4.1.9. Solvability of (37). Let $(K, D(K))$ be the generator of an analytic semigroup and let X_α be any intermediate space satisfying $D_K(\alpha, 1) \subset X_\alpha \subset D_K(\alpha, \infty)$ and such that the part of K in X_α is sectorial. Combining Theorem 7.1.2 and Propositions 7.1.8, 7.1.10 and 7.1.11 of [47], we arrive at the following fundamental result.

Theorem 4.4. *Let $0 < \alpha < 1$ and $g : \mathbb{R} \times X_\alpha \to X$ be such that for all $(t, x) \in \mathbb{R} \times X_\alpha$ there exist $V \ni (t, x), L > 0, 0 < \theta \leq 1$ such that for all $(t_i, x_i) \in V$*

$$\|g(t_1, x_1) - g(t_2, x_2)\| \leq L(|t_1 - t_2|^\theta + \|x_1 - x_2\|_{X_\alpha}).$$

If $\mathring{f} \in X$, then there is $t_1 = t_1(\mathring{f})$ such that (37) has a unique classical solution $f \in C([0, t_1], X) \cap C((0, t_1], D(A)) \cap C^1((0, t_1], X)$. Furthermore, if $\mathring{f} \in X_\alpha$, then additionally $f \in C([0, t_1], X_\alpha)$ and it continuously depends on the initial data in the norm X_α on $[0, t_1]$. The solution is not global in time if its X_α norm blows up in finite time.

Another general problem we face is that the perturbation g may be not nonnegative, as in (1), and thus it is not immediate that the solution f is nonnegative. The strategy we employ here is to find a nonnegative linear operator M such that $Mu + f(t, u) \geq 0$ at least on some subset of $\mathbb{R}_+ \times X$ and then write (37) in the equivalent form

$$\begin{aligned} \partial_t f &= Kf - Mf + Mf + g(t, f), \quad t > 0, \\ f(0) &= \mathring{f}. \end{aligned} \tag{47}$$

If we can simultaneously ensure that $(S_{K-M}(t))_{t \geq 0}$ is a nonnegative semigroup, then we have

$$f(t) = S_{K-M}(t)\mathring{f} + \int_0^t S_{K-M}(t - s)(Mf(s) + g(s, f(s)))ds \tag{48}$$

and the nonnegativity of f will follow from Picard iterations used in the proof of Theorem 4.4.

4.2. Back to the fragmentation-coagulation equation.

As before, we focus on the discrete case, the continuous one being similar but technically more involved due to the necessity of separately controlling the zeroth moment of the solution; that is, the number of particles. Thus, the analysis in the continuous case is done in the spaces $L_1(\mathbb{R}_+, (1+x^p)dx)$ and requires some additional technical assumptions, [11].

Let us fix $p > 1$ for which (32) is satisfied. For the coagulation coefficients we further assume that there are $K > 0, 0 < \alpha < 1$, such that for all $i, j \in \mathbb{N}$ the following estimate holds

$$k_{i,j} \leq K(a_{1,i}^{\alpha} + a_{1,j}^{\alpha}), \tag{49}$$

see Example 3.

Theorem 4.5. *Assume that* (32) *and* (49) *hold. Then, for each* $\overset{\circ}{\boldsymbol{f}} \in l^1_{p,\alpha,+}$, *there is* $\tau > 0$ *such that the initial value problem* (1) *has a unique nonnegative classical solution* $\boldsymbol{f} \in C([0,\tau], l^1_{p,\alpha}) \cap C^1((0,\tau], l^1_p) \cap C((0,\tau], D(A_p))$.

Sketch of the proof. Following the considerations above, we convert (1) to the equivalent problem

$$\begin{aligned}
\frac{d}{dt} f_n &= -a_{1,n} + \gamma a_{1,n}^{\alpha} f_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \\
&\quad +(1+\gamma a_{1,n}^{\alpha}) f_n - f_n \sum_{j=1}^{\infty} k_{n,j} f_j \\
&\quad +\frac{1}{2}\sum_{j=1}^{n-1} k_{j,n-j} f_j(t) f_{n-j}(t), \\
&=: [F_\gamma \boldsymbol{f}]_n + [C_{\gamma,1}\boldsymbol{f}]_n + [C_{\gamma,2}\boldsymbol{f}]_n, \\
f_n(0) &= \overset{\circ}{f}_n, \quad n \geq 1,
\end{aligned}$$

where γ is a constant such that $C_{\gamma,1}\boldsymbol{f} = ((1+\gamma a_{1,n}^{\alpha}) f_n - f_n \sum\limits_{j=1}^{\infty} k_{n,j} f_j)_{n\geq 1} \geq 0$ on $B_+ := \{\boldsymbol{f} \in l^1_{p,\alpha,+};\ \|\boldsymbol{f}\|_{l^1_p} \leq 1+b\}$. It follows, [6, 11], that it suffices to take $\gamma \geq 2K(1+b)$, where K is defined in (49). Hence, $C_\gamma = C_{\gamma,1} + C_{\gamma,2}$ is nonnegative on B_+. Further, F_γ is a perturbation of the sectorial operator $F_p - I$

by the diagonal operator of multiplication $\boldsymbol{f} \to (-\gamma a_{1,n}^{\alpha} f_n)_{n\geq 1}$ that, since $0 < \alpha < 1$, is relatively bounded with respect to $F_p - I$ with relative bound equal to 0 and hence it also generates an analytic semigroup $(S_{F_\gamma}(t))_{t\geq 0}$ on l_p^1, [55, Corollaries 3.2.3 & 3.2.4]. It is easy to see that $(S_{F_\gamma}(t))_{t\geq 0}$ is a positive, analytic semigroup on $l_{p,\alpha}^1$. Moreover, it satisfies $S_{F_\gamma}(t) \leq S_{F_p - I}(t)$ hence, since $l_{p,\alpha}^1$ is a Banach lattice,

$$\|S_{F_\gamma}(t)\|_{l_{p,\alpha}^1} \leq \|S_{F_p - I}(t)\|_{l_{p,\alpha}^1} \leq c$$

for some constant c. Note that the importance of the above chain of inequalities lies in the fact that F_γ does not commute with F_p and thus a direct estimate of $(S_{F_\gamma}(t))_{t\geq 0}$ in the norm (45) is difficult.

Then the mild solution to (1) is obtained by the application of the Banach contraction principle to

$$\boldsymbol{f} = S_{F_\gamma}(t)\mathring{\boldsymbol{f}} + \int_0^t S_{F_\gamma}(t-s) C_\gamma[\boldsymbol{f}](s)ds$$

on $C([0,\tau], B_+)$ for sufficiently small τ, that is possible as C_γ is Lipschitz continuous as a quadratic operator. The existence of the fixed point and further regularity analysis leading to the existence of the classical solution are done as in [55, Section 6.3] or [47, Section 7.1].

Theorem 4.6. *Let the assumptions of Theorem* 4.5 *hold. If there is* $s > 0$ *such that for* $n \geq 1$

$$a_n \leq L n^s, \tag{50}$$

where $L > 0$ *is a constant and* $\alpha s \leq 1$, *then any local solution defined in Theorem* 4.5 *is global in time.*

Sketch of the proof. Let us denote by M_r the r-th moment of the solution,

$$M_r(t) := \sum_{i=1}^{\infty} i^r f_i(t).$$

Due to the last part of Theorem 4.4 and (50), the proof of this theorem will be accomplished if we show that the moment $M_{p+\alpha s}$ does not blow up in finite time. Unfortunately, the l^1_p regularity obtained in Theorem 4.5 is insufficient as it does not ensure the differentiability of $M_{p+\alpha s}$. However, using point 3 of Theorem 3.2, we see that Theorem 4.5 is valid in the scale of spaces l^1_r with $r \geq p$ provided, of course, $\mathring{\boldsymbol{f}} \in l^1_{r,\alpha}$. Since $l^1_{r,\alpha}$ is continuously embedded in $l^1_{p,\alpha}$ for $r \geq p$, the solutions emanating from the same initial value $\mathring{\boldsymbol{f}} \in l^1_{r,\alpha} \subset l^1_{p,\alpha}$ in each space coincide. Let us consider then $\mathring{\boldsymbol{f}} \in l^1_{p+s\alpha,\alpha} \subset l^1_{p+s\alpha} \subset l^1_{p,\alpha}$, where the last inclusion is due to (50). Thus, the local solution satisfies $\boldsymbol{f} \in C([0,\tau), l^1_{p+s\alpha,\alpha}) \cap C^1((0,\tau), l^1_{p+s\alpha}) \cap C((0,\tau), D(A_{p+s\alpha}))$, with possibly different, but still nonzero, τ; $(0,\tau)$ can be considered to be the maximal interval of existence of the solution. Hence, using the inequality

$$(i+j)^r - i^r - j^r \leq (2^r - 1)(i^{r-1}j + j^{r-1}i) = C_r(i^{r-1}j + j^{r-1}i),$$

for $r \geq 1$, $i, j \in \mathbb{N}$ and some C_r, established in [10], from (1) we get

$$\frac{d}{dt}M_{p+s\alpha} \leq 2C_{p+s\alpha}K(M_{p+2s\alpha-1}M_1 + M_{p+s\alpha-1}M_{1+s\alpha}). \tag{51}$$

This is an infinite system of inequalities which can be closed if $p + 2s\alpha - 1 \leq p + s\alpha$, or $s\alpha \leq 1$. Then

$$\frac{d}{dt}M_{p+s\alpha} \leq 2C_{p+s\alpha}KM_{p+s\alpha}(M_1 + M_2). \tag{52}$$

From (51) for $p + s\alpha = 2$, we have

$$\frac{d}{dt}M_2 \leq 2K(M_{1+s\alpha}M_1 + M_1M_{1+s\alpha}) \leq 4KM_2M_1$$

and since $M_1(t) \leq M_1(0)$, M_2 is globally defined and thus $M_{p+s\alpha}$ also exists globally in time. This ensures global existence of solutions emanating from any initial datum $\mathring{\boldsymbol{f}} \in l^1_{p+s\alpha,\alpha}$. However, by Theorem 4.4 the solutions depend continuously in $l^1_{p,\alpha}$ on the initial data on their common interval of existence. Since $l^1_{p+s\alpha,\alpha}$ is dense in $l^1_{p,\alpha}$, no solution with $\mathring{\boldsymbol{f}} \in l^1_{p,\alpha}$ can blow up in finite time.

Remark. The same results hold for the continuous coagulation–fragmentation equation in the scale of spaces $L_1(\mathbb{R}_+, (1 + x^p)dx)$ provided there exist $j \in$

$(0,\infty), l \in [0,\infty)$ and $a_0, b_0 \in \mathbb{R}_+$ such that, for any $x \in \mathbb{R}_+$,

$$a(x) \leq a_0(1+x^j), \quad n_0(x) \leq b_0(1+x^l),$$

see (8), with analogous growth conditions for $k(x,y)$ and $p \geq \max\{j+l, p_0\}$, where (32) holds for $p = p_0$, [11].

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