Viscosity limit and deviations principles for a grade-two fluid driven by multiplicative noise

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Abstract In this paper we study a grade-two fluid driven by multiplicative Gaussian noise. Under appropriate assumptions on the initial condition and the noise, we prove a large and moderate deviations principle in the space $C([0,T];\mathbf{H}^m)$, $m \in \{2,3\}$, of the solution of our stochastic model as the viscosity ε converges to 0 and the coefficient of the noise is multiplied by $\varepsilon^{\frac{1}{2}}$. We present a unifying approach to the proof of the two deviations principles and express the rate function in term of the solution of the inviscid grade-two fluid which is also known as Lagrangian Averaged Euler equations. Our proof is based on the weak convergence approach to large deviations principle.

Keywords Grade-two fluids · Lagrangian Averaged Euler equations · Inviscid Limit · Moderate Deviations Principle · Large Deviations Principle · Weak convergence approach to LDP

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1 Introduction

In this paper we are interested in the stochastic version of the following system of partial differential equations (PDE)

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{u} - \alpha \Delta \mathbf{u}) - \varepsilon \Delta \mathbf{u} + \mathbf{rot}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} + \nabla \tilde{\mathfrak{p}} = \mathbf{f}, \\ \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.1}$$

which is a simplified system describing the motion of a homogeneous incompressible grade-two fluid with density $\rho = 1$, viscosity $\varepsilon > 0$ and stress moduli $\alpha > 0$. Hereafter we understand that in $\mathbb{R}^2$ the rotational of a vector $\mathbf{u} = (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ is a scalar function defined by

$$\mathbf{rot}\, \mathbf{u} = \frac{\partial \mathbf{u}^{(2)}}{\partial x_1} - \frac{\partial \mathbf{u}^{(1)}}{\partial x_2},$$

and for any vector and scalar functions $\mathbf{v} = (\mathbf{v}^{(1)}, \mathbf{v}^{(2)})$ and z

$$\mathbf{rot}\, \mathbf{u} \times \mathbf{v} = (-\mathbf{v}^{(2)}\, \mathbf{rot}\, \mathbf{u}, \mathbf{v}^{(1)}\, \mathbf{rot}\, \mathbf{u}),$$
$$\mathbf{rot}(z \times \mathbf{v}) = \mathbf{v} \cdot \nabla z := \mathbf{v}^{(1)} \frac{\partial z}{\partial x_1} + \mathbf{v}^{(2)} \frac{\partial z}{\partial x_2}.$$

The grade-two fluid is a particular case of differential fluids of complexity n; we refer to [58] for the definitions and the derivation of the above system. The system (1.1) is frequently used to describe fluid models in petroleum industry, polymer technology and suspensions of liquid crystals. It was also used in [53] to study the connection of Turbulence Theory to non-Newtonian fluids, especially fluids of differential type. When $\varepsilon = 0$, the system (1.1) reduces to what is known as the Lagrangian averaged Euler equations (LAEs) which appeared for the first time in the context of averaged fluid models in [31] and [32]. The derivation of LAEs used averaging and asymptotic methods in the variational formulation. The LAEs are also closely related to the following equation

$$u_t - u_{xxt} + 2\kappa u_x - 3uu_x = 2u_x u_{xx} + u_x u_{xxx},$$

where u_x, u_{xy}, ..., denote partial derivatives with respect to the variable x, x and then y, ... This equation was proposed by Camassa and Holm in [10] to describe a special model of shallow water. As in the case of the grade-two fluid this new model of shallow water also reduces to LAEs when

$\kappa = 0$ and in this case it was shown in [36] that it is the geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle. The articles [54] and [55] also contain interesting discussions concerning the grade-two fluids and the LAEs. The study of the physical properties, such as boundedness and stability, of the grade-two fluids based on (1.1) was initiated in [21], [22] and [25]. The first mathematical analysis was carried out in [13] and [14] where the first optimal result about the existence and uniqueness of weak solution was proved. Since then, the problem (1.1) has been the subject of intensive mathematical analysis which has generated several important results. We refer to [15], [1], [34], [44], [47], [28] and [27] for few interesting papers about the mathematical theory of grade-two fluids. For a detailed of past and recent results related to the deterministic grade-two fluid and the LAEs we refer to [26] and [46].

In this paper we are interested in a stochastic model for grade-two fluid in the two-dimensional torus $\mathbb{T}^2 = (0, 2\pi) \times (0, 2\pi)$. More precisely, we assume that a finite time horizon $[0, T]$, and an initial value ξ are given and we consider in $(0, T] \times \mathbb{T}^2$ the following stochastic system

$$d(\mathbf{u} - \alpha \Delta \mathbf{u}) + (-\varepsilon \Delta \mathbf{u} + \mathbf{rot}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} + \nabla \tilde{\mathfrak{p}})dt = \sqrt{\varepsilon} G(\mathbf{u}) dW \quad \text{in } \mathbb{T}^2 \times [0, T], \tag{1.2a}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{on} \quad \mathbb{T}^2 \times [0, T], \tag{1.2b}$$

$$\langle \mathbf{u} \rangle = 0, \tag{1.2c}$$

$$\mathbf{u}(0) = \xi, \tag{1.2d}$$

where $\mathbf{u} = (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ and $\tilde{\mathfrak{p}}$ represent the random velocity and the modified pressure, respectively, and $\langle u \rangle = \int_{\mathbb{T}^2} u(x) dx$ for an integrable function u. The stochastic process $\{W(t);\ t \in [0, T]\}$ is a cylindrical Wiener process evolving on a given separable Hilbert space $\mathscr{H}_0$.We assume that the initial condition ξ, the velocity field $\mathbf{u}$ as well as the pressure is periodic in the following sense

$$\mathbf{u}(x + 2\pi e_i, t) = \mathbf{u}(x, t), \quad x \in \mathbb{R}^2, t \in [0, T]. \tag{1.3}$$

where $e_1,\ e_2$ is the canonical basis of $\mathbb{R}^2$. In what follows when we refer to problem (1.2) we always mean the system (1.2) with the boundary condition (1.3).

Denoting by A the Stokes operator and $\mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{u})$ the projection of $\mathbf{rot}\,\mathbf{v} \times \mathbf{u}$ onto the space of square integrable, divergence free and periodic functions with zero mean, see Subsection 2.1, the problem (1.2) can be written as an abstract stochastic evolution equation of the form

$$\begin{cases} d\mathbf{v} + [\varepsilon \mathrm{A}\mathbf{u} + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{u})]dt = \varepsilon^{\frac{1}{2}} G(\mathbf{u})dW, \\ \mathbf{v} = \mathbf{u} + \alpha \mathrm{A}\mathbf{u}, \\ \mathbf{u}(0) = \xi. \end{cases}$$

In contrast to the deterministic result, there are only few works related to problem (1.2). By using the method elaborated in [13] the global existence of both martingale and strong (in the stochastic calculus sense) solutions were proved in [49], [51], [11]. Convergence of the solution of (1.2) to the weak martingale solution of the two dimensional stochastic Navier-Stokes equations was established in [50]. Existence of a global weak martingale solution for the grade-two fluids driven by external forcing of Lévy noise type is shown in [30]. Some important results related to the problem (1.2) were recently proved in [64], [59] and [65]. By Odasso's exponential mixing criterion [45] it was shown in [59] that the problem (1.2) has a unique invariant measure which is exponentially mixing. The large and moderate deviation estimates for the solution to (1.2) were respectively established in [64] and [65] by the weak convergence method of Budhiraja and Dupuis [5].

Our interest in this paper is related to Large Deviations Principle (LDP) and Moderate Deviations Principle (MDP) in small noise diffusion. Roughly speaking, in the study of MDP one is interested in probabilities of deviations of lower speed than in the classical LDP. In small diffusion (the coefficient of the noise is usually multiplied by $\varepsilon^{\frac{1}{2}}$) the speed for LDP is usually of order ε and the speed for MDP is of order $\lambda^2(\varepsilon)$ where $\lambda : (0, 1] \to (0, \infty)$ is a function satisfying

$$\lambda(\varepsilon) \to \infty \text{ and } \varepsilon^{\frac{1}{2}} \lambda(\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{1.4}$$

Observe that since since $\lambda(\varepsilon)$ converges to ∞ as slow as desired, then the MDP bridges the gap between the Central Limit Theorem and the LDP. We refer, for instance, to [29] and [35] for more detailed explanation and historical account of the MDP. We refer, for instance, to [4], [2], [3], [12], [64], [65] [5], [6], [12], [19], [37], [39], [62], [61], [60] and references therein for a small sample of results from the extensive literature devoted to MDP and LDP for stochastic differential equations with small noise.

In this paper we will study the LDP and MDP for the inviscid model of grade-two fluids. For this purpose we assume that the coefficient of the noise is multiplied by the square root of the viscosity which is denoted by ε and analyse the asymptotic behaviour, as $\varepsilon \to 0$, of the trajectories family of $(\mathbf{u}^\varepsilon)_{\varepsilon\in(0,1]}$ and $\left(\varepsilon^{-\frac{1}{2}}\lambda^{-1}(\varepsilon)[\mathbf{u}^\varepsilon - \mathbf{u}]\right)_{\varepsilon\in(0,1]}$ where $\lambda(\cdot)$ is a function satisfying (1.4) and $\mathbf{u}$ is the solution to the deterministic system

$$\partial_t \mathbf{v} + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{u}) = 0 \tag{1.5a}$$

$$\mathbf{v} = \mathbf{u} + \alpha \mathrm{A}\mathbf{u} \tag{1.5b}$$

$$\mathbf{u}(0) = \xi. \tag{1.5c}$$

System (1.5) is the inviscid model of the grade-two fluid and is known in the literature as the Lagrangian Averaged Euler (LAE) equations. Many prominent mathematicians have studied the LAE equations and their studies have generated several important results. Without being exhaustive we refer to [23], [33], [42], [63], [9], [7], [38], [40], [41] and references therein for the amount of mathematical results related to the theory of LAEs.

In order to describe the main results of the paper, let us denote by $\mathbf{V}$ the subspace of the Sobolev space $\mathbf{H}^1(\mathbb{T})$ consisting of periodic, divergence free functions that have zero mean, and by $\mathbf{W}$ the subspace of $\mathbf{V}$ consisting of functions $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{rot}(\mathbf{v} - \alpha\Delta\mathbf{v}) \in \mathbf{L}^2(\mathbb{T}^2)$. Roughly speaking, the main results in this paper can be summarized in the following theorem.

Theorem. *Let $s \in \{0, 1\}$.*

(LDP) If $\xi \in D(\mathrm{A}^{\frac{3+s}{2}})$ and $G : \mathbf{V} \to \mathscr{L}_2(\mathscr{H}_0, D(\mathrm{A}^{\frac{s+1}{2}}))$ is Lipschitz continuous with respect to the $\mathbf{L}^2$-norm. Then, the family of solutions $(\mathbf{u}^\varepsilon)_{\varepsilon\in(0,1]}$ to (1.2) *satisfies an LDP on $C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed ε^{-1}.*

(MDP) If $\xi \in D(\mathrm{A}^{2+\frac{s}{2}})$ and $G : \mathbf{V} \to \mathscr{L}_2(\mathscr{H}_0, D(\mathrm{A}^{\frac{s+1}{2}}))$ is Lipschitz continuous with respect to the $\mathbf{L}^2$-norm. Then, $\left(\varepsilon^{-\frac{1}{2}}\lambda^{-1}(\varepsilon)[\mathbf{u}^\varepsilon - \mathbf{u}]\right)_{\varepsilon\in(0,1]}$ satisfies an LDP on $C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed $\lambda^2(\varepsilon)$, where $\mathscr{L}_2(\mathscr{H}_0, D(\mathrm{A}^{\frac{s+1}{2}}))$ denotes the space of all Hilbert-Schmidt operators from $\mathscr{H}_0$ onto $D(\mathrm{A}^{\frac{s+1}{2}})$.

The first part of the theorem presents an LDP result and the second one is a MDP for the family of solutions $(\mathbf{u}^\varepsilon)_{\varepsilon\in(0,1]}$ to (1.2). We should note that since $\xi \in D(\mathrm{A}^{\frac{3+s}{2}})$ one would expect that the LDP should hold in the space $C([0,T]; D(\mathrm{A}^{\frac{3+s}{2}}))$, but we will explain in Remark 3.12 why we only have an LDP in $C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$ and why we are unable to treat the case $s \in (0,1)$.

In this paper we present a unifying approach to the LDP and MDP for the solution to (1.2) instead of giving two separate proofs of these two results. To this end, we fix $\delta \in \{0,1\}$ and consider the following problem

$$d\mathbf{v}^{\varepsilon,\delta} + \left[\varepsilon \mathrm{A}\mathbf{u}^{\varepsilon,\delta} + \lambda_\delta(\varepsilon)\mathcal{C}(\mathbf{rot}\,\mathbf{v}^{\varepsilon,\delta}, \mathbf{u}^{\varepsilon,\delta}) + \delta(\mathcal{C}(\mathbf{rot}\,\mathbf{v}^{\varepsilon,\delta}, \mathbf{u}) + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{u}^{\varepsilon,\delta}))\right] dt$$

$$= \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\mathrm{A}\mathbf{u}\, dt + \varepsilon^{\frac{1}{2}}\lambda_\delta^{-1}(\varepsilon)G(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}^{\varepsilon,\delta})dW, \tag{1.6a}$$

$$\mathbf{v}_h^{\varepsilon,\delta} = (\mathbf{u}^{\varepsilon,\delta} + \alpha\mathrm{A}\mathbf{u}^{\varepsilon,\delta}), \tag{1.6b}$$

$$\mathbf{u}^{\varepsilon,\delta}(0) = (1-\delta)\xi, \tag{1.6c}$$

where $\mathbf{u}$ is the unique solution to (1.5), $\mathbf{v} = \mathbf{u} + \alpha\mathrm{A}\mathbf{u}$, and λ_δ, $\delta \in \{0,1\}$, is defined by

$$\lambda_\delta(\varepsilon) = \begin{cases} 1 \text{ if } \delta = 0 \\ \varepsilon^{\frac{1}{2}}\lambda(\varepsilon) \text{ if } \delta = 1. \end{cases} \tag{1.7}$$

The major part of the present paper is devoted to the proof of the following result.

Theorem. *Let $\delta, s \in \{0,1\}$, $\xi \in \mathbf{V} \cap \mathbf{H}^{3+s+\delta}$ and $G : \mathbf{V} \to \mathscr{L}(\mathscr{H}, D(\mathrm{A}^{\frac{s+1}{2}}))$ be a map which is Lipschitz continuous with respect to the $\mathbf{L}^2$-norm . Then, the family of solutions $(\mathbf{u}^{\varepsilon,\delta})_{\varepsilon\in(0,1]}$ to (1.6) satisfies an LDP on $C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed $\varepsilon^{-1}\lambda_\delta^2(\varepsilon)$.*

The items (LDP) and (MDP) in the former theorem follow from the latter theorem. In fact, for $\delta = 0$ problem (1.6) is exactly the problem (1.2) and, by the definition of λ_δ, the LDP speed $\varepsilon^{-1}\lambda_0^2(\varepsilon)$ is equal to ε^{-1}. For $\delta = 1$ problem (1.6) reduces to

$$d\eta^\varepsilon + \left[\varepsilon\mathrm{A}\mathbf{y}^\varepsilon + \varepsilon^{\frac{1}{2}}\lambda(\varepsilon)\mathcal{C}(\mathbf{rot}\,\eta^\varepsilon, \mathbf{y}^\varepsilon) + (\mathcal{C}(\mathbf{rot}\,\eta^\varepsilon, \mathbf{u}) + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{y}^\varepsilon))\right] dt$$

$$= \lambda^{-1}(\varepsilon)G(\mathbf{u} + \varepsilon^{\frac{1}{2}}\lambda(\varepsilon)\mathbf{y}^\varepsilon)dW, \tag{1.8a}$$

$$\eta^\varepsilon = (\mathbf{y}^\varepsilon + \alpha\mathrm{A}\mathbf{y}^\varepsilon), \tag{1.8b}$$

$$\mathbf{y}^\varepsilon(0) = 0, \tag{1.8c}$$

which has a unique solution $\mathbf{y}^\varepsilon = \varepsilon^{-\frac{1}{2}}\lambda^{-1}(\varepsilon)[\mathbf{u}^\varepsilon - \mathbf{u}]$. By the last theorem and the definition of $\lambda_\delta(\varepsilon)$ the family of random variables $\left(\varepsilon^{-\frac{1}{2}}\lambda^{-1}(\varepsilon)[\mathbf{u}^\varepsilon - \mathbf{u}]\right)_{\varepsilon\in(0,1]}$ satisfies an LDP on $C([0,T];\mathbf{W})$ with speed $\lambda^2(\varepsilon)$.

As in [64], [65], [2], [3] and [12] our proof is based on weak convergence approach to LDP and Budhiraja-Dupuis' results on representation of functionals of Brownian motion, see [5] and [6]. However, we should mention that in addition to the presentation of a unifying approach to LDP and MDP for the grade-two fluids our paper differ from [64] and [65] in the following respect:

1. in contrast to [65] our estimates are uniform with respect to the viscosity, and thus as in [2] and [3] we are allowed to let the fluid viscosity, which is denoted by ε, to converge to zero.
2. While the authors in [64] and [65] were only able to prove their LDP and MDP results on the bigger space $C([0,T];\mathbf{V})$, we are able to establish our results on the smaller space $C([0,T];D(\mathrm{A}))$ under the same assumptions as in the [64] and [65]. Imposing further regularity on the noise and the initial data we are able to prove the LDP and MDP on $C([0,T];\mathbf{W})$.
3. Finally, unlike in [64] and [65] we directly work with the infinite dimensional solution $\mathbf{u}^{\varepsilon,\delta}$ instead of using their finite-dimensional projection (Galerkin approximation).

We should also note that some parts of our proofs also differ to the approaches used in several papers that deal with the LDP for hydrodynamical models with small noise, see [12], [19], and other models [37], [39] and [62]. We postpone the description of this difference to Remark 3.17 as it requires the introduction of additional notation and technical terms. We are mainly inspired by [2] and [3], but our model does not fall in their frameworks. In fact, the viscous models considered in [2] and [3] are parabolic semilinear evolution equations and in this paper we are dealing with non-parabolic fully nonlinear PDEs. The main difference also lies in the techniques used for the derivations of uniform estimates. We finally note that we were also very much inspired by the recent paper [4].

To close this introduction we outline the structure of the paper. In section 2 we introduce the notations and recall or prove elementary results frequently used in the paper. The standing hypotheses, the main results and their proofs are given in Section 3. Sections 4 and 5 are devoted to the proofs of intermediary results that are needed to establish the main results. To keep the paper self-contained we recall or prove important theorems that are scattered in the literature.

2 Notations and auxiliary results

2.1 Notations: Functional spaces

We introduce necessary definitions of functional spaces frequently used in this work. Throughout this paper we denote by $L^p(\mathbb{T}^2)$ and $W^{m,p}(\mathbb{T}^2)$, $p \in [1,\infty]$, $m \in \mathbb{N}$, the Lebesgue and Sobolev spaces of functions defined on $\mathbb{T}^2$. The spaces of $u \in L^p(\mathbb{T}^2)$ and $W^{m,p}(\mathbb{T}^2)$ which are 2π-periodic in each direction $0x_i$, $i = 1, 2$, see for example [16], are denoted by $\mathbf{L}^p(\mathbb{T}^2)$ and $\mathbf{W}^{m,p}(\mathbb{T}^2)$, respectively. We simply write $\mathbf{L}^p$ (resp. $\mathbf{W}^{m,p}$) instead of $\mathbf{L}^p(\mathbb{T}^2)$ (resp. $\mathbf{W}^{m,p}(\mathbb{T}^2)$) when there is no risk of ambiguity. We will also use the notation $\mathbf{H}^m := \mathbf{W}^{m,2}$. For non integer $r > 0$ the Sobolev space $\mathbf{H}^r$ is defined by using classical interpolation method. The space $[\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^2)]^2 := \mathcal{C}^\infty_{\text{per}}(\mathbb{R}^2, \mathbb{R}^2)$ denotes the space of functions which are infinitely differentiable and 2π-periodic in each direction $0x_i$, $i = 1, 2$ (see also (1.3)).

In what follows we still denote by $\mathbf{X}$ the space of $\mathbb{R}^2$-valued functions such that each component belongs to $\mathbf{X}$. We also introduce the following spaces

$$\begin{aligned}
&\mathbf{H} = \left\{ \mathbf{u} \in \mathbf{L}^2(\mathbb{T}^2); \int_{\mathbb{T}^2} \mathbf{u}(x)dx = 0, \ \ \operatorname{div} \mathbf{u} = 0 \right\}, \\
&\mathbf{V} = \mathbf{H}^1(\mathbb{T}^2) \cap \mathbf{H}.
\end{aligned}$$

It is well-known, see [57], that $\mathbf{H}$ and $\mathbf{V}$ are the closure of

$$\mathcal{V} = \left\{ \mathbf{u} \in [\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^2)]^2; \int_{\mathbb{T}^2} \mathbf{u}(x)dx = 0, \ \ \operatorname{div} \mathbf{u} = 0 \right\},$$

with respect to the $\mathbf{L}^2$ and $\mathbf{H}^1$ norms. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbf{L}^2(\mathbb{T}^2)$ on $\mathbf{H}$, respectively. Let $\Pi : \mathbf{L}^2(\mathbb{T}^2) \to \mathbf{H}$ be the Helmholtz-Leray projection, and $\mathrm{A} = -\Pi\Delta$ be the Stokes operator with the domain $D(\mathrm{A}) = \mathbf{H}^2(\mathbb{T}^2) \cap \mathbf{H}$. It is well-known that A is a self-adjoint positive operator with compact inverse, see for instance [57, Chapter 1, Section 2.6]. Hence, it has an orthonormal sequence of eigenvectors $\{e_j;\ j \in \mathbb{N}\}$ with corresponding eigenvalues $0 < \lambda_1 < \lambda_2 < ...$ The domain of A^r, $r \in \mathbb{R}$ is characterized by

$$D(\mathrm{A}^r) = \mathbf{V} \cap \mathbf{H}^{2r},$$

see [16, page 43].

The inner product and the norm induced by that of $\mathbf{H}_0^1(\mathbb{T}^2)$ on $\mathbf{V}$ are denoted respectively by $((\cdot,\cdot)) := (\mathrm{A}^{\frac{1}{2}}\cdot\ ,\ \mathrm{A}^{\frac{1}{2}}\cdot)$ and $||\cdot|| := |\mathrm{A}^{\frac{1}{2}}\cdot|$. Observe that in the space $\mathbf{V}$, the norm $\|\cdot\|$ is equivalent to the norm generated by the following scalar product

$$((\mathbf{u},\mathbf{w}))_\alpha = (\mathbf{u},\mathbf{w}) + \alpha((\mathbf{u},\mathbf{w})), \text{ for any } \ \mathbf{u}\ \mathbf{w} \in \mathbf{V}. \tag{2.1}$$

More precisely, we have

$$\alpha||\mathbf{u}||^2 \leqslant \|\mathbf{u}\|_\alpha^2 \leqslant \left(\frac{1}{\lambda_1} + \alpha\right)||\mathbf{u}||^2, \forall \mathbf{u} \in \mathbf{V}. \tag{2.2}$$

From now on, we will equip $\mathbf{V}$ with the norm $\|\mathbf{u}\|_\alpha$ generated by the inner product defined in (2.1). Note that we also have the equivalence of norms

$$c_0|\mathbf{rot}\,\mathbf{u}| \leqslant \|\mathbf{u}\| \leqslant c_1|\mathbf{rot}\,\mathbf{u}|, \text{ for any } \mathbf{u} \in \mathbf{H}^1 \text{ with } \operatorname{div}\mathbf{u} = 0, \tag{2.3}$$

where as in [47] any two dimensional vector $\mathbf{u} = (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ is identified with the three dimensional vector $(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, 0)$, and$\mathbf{rot}\,\mathbf{u} = (0, 0, \frac{\partial \mathbf{u}^{(2)}}{\partial x_1} - \frac{\partial \mathbf{u}^{(1)}}{\partial x_2})$ is identified with the scalar

$$\mathbf{rot}\,\mathbf{u} = \frac{\partial \mathbf{u}^{(2)}}{\partial x_1} - \frac{\partial \mathbf{u}^{(1)}}{\partial x_2}.$$

We also introduce the following space

$$\mathbf{W} = \left\{\mathbf{u} \in \mathbf{V};\ \mathbf{rot}(\mathbf{u} + \alpha\mathrm{A}\mathbf{u}) \in \mathbf{L}^2(\mathbb{T}^2)\right\},$$

which is a Hilbert space equipped with the norm generated by the following scalar product

$$((\mathbf{u},\mathbf{v}))_{\mathbf{W}} = (\mathbf{rot}(\mathbf{u} + \alpha\mathrm{A}\mathbf{u}), \mathbf{rot}(\mathbf{v} + \alpha\mathrm{A}\mathbf{v})), \ \ \forall \mathbf{u},\ \mathbf{v} \in \mathbf{W},$$

see [15]. Note that for $\mathbf{v} \in \mathbf{V}$, $\alpha\,\mathbf{rot}\,\mathrm{A}\mathbf{v} \in \mathbf{L}^2(\mathbb{T}^2)$ is understood in the weak sense. We recall that there exist constants $c_2, c_3 > 0$ such that for all $\mathbf{u} \in \mathbf{W}$

$$c_2\|\mathbf{u}\|_{\mathbf{H}^3} \leqslant \|\mathbf{rot}(\mathbf{u} + \alpha\mathrm{A}\mathbf{u})\| \leqslant c_3\|\mathbf{u}\|_{\mathbf{H}^3}, \tag{2.4}$$

see [15].

For any Banach space $\mathbf{B}$ we denote its dual by $\mathbf{B}^*$ and by $\langle \mathbf{f}, \mathbf{v}\rangle$ the action of any element $\mathbf{f}$ of $\mathbf{B}^*$ on an element $\mathbf{v} \in \mathbf{B}$. By identifying $\mathbf{H}$ with its dual space $\mathbf{H}^*$ via the Riesz representation, we have the Gelfand-Lions triple

$$\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}^\star,$$

where each space is dense in the next one and the inclusions are continuous. It follows from the above identification that we can write

$$\langle \mathbf{v}, \mathbf{w}\rangle = (\mathbf{v}, \mathbf{w}), \tag{2.5}$$

for any $\mathbf{v} \in \mathbf{H}, \mathbf{w} \in \mathbf{V}$.

2.2 Some useful results and inequalities

In this subsection we will prove several results which are essential for the subsequent analysis.

Let p_i, $i = 1, 2, 3$, be three positive numbers satisfying $\sum_{i=1}^{3} p_i^{-1} = 1$. Let $c : \mathbf{L}^{p_1} \times \mathbf{L}^{p_2} \times \mathbf{L}^{p_3} \to \mathbb{R}$ be a trilinear form defined by

$$c(\psi, \phi, \varphi) = (\psi \times \phi, \varphi), \quad \psi \in \mathbf{L}^{p_1},\ \phi \in \mathbf{L}^{p_2},\ \varphi \in \mathbf{L}^{p_3}.$$

By Hölder's inequality, there exists a constant $K_0 > 0$ such that for any $\psi_i \in \mathbf{L}^{p_i}$,$i \in \{1, 2, 3\}$

$$|c(\psi_1, \psi_2, \psi_3)| \leqslant K_0 \prod_{i=1}^{3} \|\psi_i\|_{\mathbf{L}^{p_i}}.$$

In particular, we formulate the following lemma which is a consequence of the last observation and the definition of $c(\cdot, \cdot, \cdot)$.

Lemma 2.1. *One can find a continuous bilinear map* $\mathcal{C} : \mathbf{L}^2 \times \mathbf{V} \to \mathbf{V}^*$ *such that for any* $y \in \mathbf{L}^2$, $\mathbf{v} \in \mathbf{V}$, $\mathbf{w} \in \mathbf{V}$

$$\langle \mathcal{C}(y, \mathbf{v}), \mathbf{w}\rangle = (y \times \mathbf{v}, \mathbf{w}), \tag{2.6}$$

$$\langle \mathcal{C}(y, \mathbf{v}), \mathbf{v}\rangle = 0. \tag{2.7}$$

Proof. From the observation preceding the lemma the trilinear form $c : \mathbf{L}^2 \times \mathbf{L}^4 \times \mathbf{L}^4 \to \mathbb{R}$ is continuous, hence, by the continuous embedding $\mathbf{V} \subset \mathbf{L}^4$, the restriction, which is still denoted by c, of c on $\mathbf{L}^2 \times \mathbf{V} \times \mathbf{V}$ is continuous too. We now conclude the proof of the existence of the bilinear map $\mathcal{C} : \mathbf{L}^2 \times \mathbf{V} \to \mathbf{V}^*$ satisfying (2.6) by the celebrated Riesz representation, see [20, Theorem 5.5.1].

As in [47], we identify any two dimensional vector (f_1, f_2) with the three dimensional vector $(f_1, f_2, 0)$. With this in mind, we can easily prove that the vector $y \times \mathbf{v}$ is perpendicular to $\mathbf{v}$ in $\mathbb{R}^3$ from which follows the identity (2.7). □

Remark 2.2. (i) The bilinear map $\mathcal{C}(\cdot, \cdot)$ can be defined on various spaces, but the above definition is enough for our purpose.

(ii) Remark that the identity (2.7) implies that for any $y \in \mathbf{L}^2$, $\mathbf{v} \in \mathbf{V}$, $\mathbf{w} \in \mathbf{V}$

$$\langle \mathcal{C}(y, \mathbf{v}), \mathbf{w} \rangle = -\langle \mathcal{C}(y, \mathbf{w}), \mathbf{v} \rangle. \tag{2.8}$$

(iii) Observe also that due to the continuous embedding $\mathbf{H}^3 \subset \mathbf{L}^\infty$ $\mathcal{C}(\cdot, \cdot)$ continuously maps $\mathbf{L}^2 \times \mathbf{W}$ onto $\mathbf{L}^2$. More precisely, there exists a constant $K_1 > 0$ such that for any $\psi \in \mathbf{L}^2$ and $\phi \in \mathbf{W}$

$$|\mathcal{C}(\psi, \phi)| \leqslant K_1 |\psi| \|\phi\|_{\mathbf{W}}. \tag{2.9}$$

We will state and prove an important lemma later on, but for now let us introduce few additional notations. Let p_i be positive numbers satisfying $\sum_{i=1}^3 p_i^{-1} = 1$. We define the well-known trilinear form b used in the mathematical theory of the Navier-Stokes equation by

$$\begin{aligned} b(\psi, \phi, \varphi) &= (\psi \cdot \nabla \phi, \varphi) \\ &= \sum_{i,j=1}^2 \int_{\mathbb{T}^2} \psi_i(x) \partial_i \phi_j(x) \varphi_j(x) dx, \end{aligned}$$

for any $\psi \in \mathbf{L}^{p_1}$, $\phi \in \mathbf{W}^{1,p_2}$ and $\varphi \in \mathbf{L}^{p_3}$. We recall the following properties of b which can be proved first for smooth functions and then for less regular ones by using standard density argument, see for instance [57, Chapter 2, §1] and [16, Chapter 6].

1. Let p_i, $i \in \{1,2\}$, be numbers such that $p_1^{-1} + 2p_2^{-1} = 1$. If $\psi \in \mathbf{L}^{p_1}$ is divergence free and $\phi \in \mathbf{W}^{1,p_2}$, then

$$b(\psi, \phi, \phi) = 0. \tag{2.10}$$

The identity (2.10) implies that if p_i, $i \in \{1,2,3\}$, are numbers satisfying $\sum_{i=1}^{3} p_i^{-1} = 1$, and $\psi \in \mathbf{L}^{p_1}$ is divergence free, $\phi \in \mathbf{L}^{p_3}$ and $\varphi \in \mathbf{W}^{1,p_2}$, then

$$b(\psi, \phi, \varphi) = -b(\psi, \varphi, \phi). \tag{2.11}$$

2. There exists a constant $K_2 > 0$ such that

$$|b(\psi, \phi, \varphi)| \leqslant K_2 \begin{cases} \|\psi\|_{\mathbf{L}^{p_1}} \|\nabla\phi\|_{\mathbf{L}^{p_2}} \|\varphi\|_{\mathbf{L}^{p_3}}, \forall \psi \in \mathbf{L}^{p_1} \text{ with } \operatorname{div}\psi = 0, \phi \in \mathbf{W}^{1,p_2}, \varphi \in \mathbf{L}^{p_3}, \\ \|\psi\|_{\mathbf{L}^{p_1}} \|\nabla\varphi\|_{\mathbf{L}^{p_2}} \|\phi\|_{\mathbf{L}^{p_3}}, \forall \psi \in \mathbf{L}^{p_1} \text{ with } \operatorname{div}\psi = 0, \varphi \in \mathbf{W}^{1,p_2}, \phi \in \mathbf{L}^{p_3}. \end{cases} \tag{2.12}$$

We now give the following identity

$$((\mathbf{rot}\,\phi) \times \psi, \varphi) = b(\psi, \phi, \varphi) - b(\varphi, \phi, \psi), \qquad \psi \in \mathbf{L}^{p_1}, \phi \in \mathbf{W}^{1,p_2} \text{ and } \varphi \in \mathbf{L}^{p_3}. \tag{2.13}$$

The above identity is proved first for smooth functions and then extended to less regular functions by using standard density argument. We refer to [13] and [1] for the detail of the proofs.

We are now ready to state and prove the important lemma we alluded earlier.

Lemma 2.3. *There exists a constant $K_3 > 0$ such that*

$$|\langle \mathcal{C}(\mathbf{rot}(\phi - \alpha\Delta\phi), \psi), \varphi\rangle| \leqslant K_3 \|\phi\| \|\psi\| \|\varphi\|_{\mathbf{W}}, \text{ for any } \phi \in \mathbf{W}, \psi \in \mathbf{V}, \varphi \in \mathbf{W}, \tag{2.14}$$

$$|\langle \mathcal{C}(\mathbf{rot}(\phi - \alpha\Delta\phi), \psi), \varphi\rangle| \leqslant K_3 \|\phi\| \|\psi\|_{\mathbf{W}} \|\varphi\|, \text{ for any } \phi \in \mathbf{W}, \psi \in \mathbf{W}, \varphi \in \mathbf{V}. \tag{2.15}$$

Proof. The proofs of the two inequalities (2.14) and (2.15) are very similar, hence we will only prove (2.14). Let us fix $\phi \in \mathbf{W}$, $\psi \in \mathbf{V}$ and $\varphi \in \mathbf{W}$. From (2.6) and (2.13) we infer that

$$\langle \mathcal{C}(\mathbf{rot}(\phi - \alpha\Delta\phi), \psi), \varphi\rangle = b(\psi, \phi, \varphi) - b(\varphi, \phi, \psi) - \alpha(b(\psi, \Delta\phi, \varphi) + b(\varphi, \Delta\phi, \psi)).$$

We will only estimate the last two terms of the above identity since the first two terms can be estimated from above by the right-hand side of (2.14) by using (2.12). For this aim we observe that by an integration-by-parts

$$b(\psi, \Delta\phi, \varphi) = \sum_{\ell=1}^{2} [b(\partial_\ell \psi, \varphi, \partial_\ell \phi) + b(\psi, \partial_\ell \varphi, \partial_\ell \phi)],$$

from which along with Hölder's inequality we infer that there exists a constant $\tilde{K}_1 > 0$ such that

$$|b(\psi, \Delta\phi, \varphi)| \leqslant \tilde{K}_1 \sum_{\ell=1}^{2} |\partial_\ell \psi| \|\nabla \varphi\|_{\mathbf{L}^\infty} |\partial_\ell \phi| + \tilde{K}_1 \sum_{\ell=1}^{2} |\partial_\ell \psi| \|\nabla \varphi\|_{\mathbf{L}^4} \|\nabla(\partial_\ell \varphi)\|_{\mathbf{L}^4}.$$

Using Sobolev inequalities we readily infer that there exists a constant $\tilde{K}_2 > 0$ such that

$$|b(\psi, \Delta\phi, \varphi)| \leqslant \tilde{K}_2 \|\psi\| \|\phi\| \|\varphi\|_{\mathbf{H}^3}.$$

Similarly, there exists a constant $\tilde{K}_3 > 0$ such that

$$|b(\varphi, \Delta\phi, \psi)| \leqslant \tilde{K}_3 \|\psi\| \|\phi\| \|\varphi\|_{\mathbf{H}^3}.$$

Owing to the equivalence of the norms $\|\varphi\|_{\mathbf{H}^3}$ and $\|\varphi\|_{\mathbf{W}}$, see (2.4), we easily derive the estimate (2.14) from the last two estimates.

For $\phi \in \mathbf{W}$, $\psi \in \mathbf{W}$ and $\varphi \in \mathbf{V}$ we infer from (2.8) that

$$\langle \mathcal{C}(\mathbf{rot}(\phi - \alpha\Delta\phi), \psi), \varphi \rangle = -\langle \mathcal{C}(\mathbf{rot}(\phi - \alpha\Delta\phi), \varphi), \psi \rangle.$$

With this in mind we can prove (2.15) with the same argument as we used for the proof of (2.14). This completes the proof of the lemma. □

3 Main results

This section, which is divided in several subsections, is devoted to the statement and proof of our main results.

3.1 The standing hypotheses on the noise coefficient

Throughout we fix a complete filtered probability space $\mathscr{U} := (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = \{\mathscr{F}_t;\ \ t \in [0, T]\}$ satisfies the usual conditions. We also fix two separable Hilbert spaces $\mathscr{H}_0$ and $\mathscr{H}$ such that the canonical injection $\iota : \mathscr{H}_0 \to \mathscr{H}$ is Hilbert-Schmidt. The operator $Q = \iota\iota^*$, where ι^* is the adjoint of i, is symmetric, nonnegative and since ι is Hilbert-Schmidt it is also of trace class. Moreover, from [48, Corollary C.0.6] we infer that $\mathscr{H}_0 = Q^{\frac{1}{2}}(\mathscr{H})$. Now, let W be a cylindrical Wiener process evolving on $\mathscr{H}_0$. It is well-known, see [17, Theorem 4.5], that W has the following series representation

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\beta_j(t)h_j, \quad t \in [0,T],$$

where $\{\beta_j;\; j \in \mathbb{N}\}$ is a sequence of mutually independent and identically distributed standard Brownian motions, $\{h_j; j \in \mathbb{N}\}$ is an orthonormal basis of $\mathscr{H}$ consisting of eigenvectors of Q and $\{q_j;\; j \in \mathbb{N}\}$ is the family of eigenvalues of Q. It is also well-known, see [17, Section 4.1] and [48, Section 2.5.1], that W is an $\mathscr{H}$-valued Wiener process with covariance Q.

Now, we recall few basic facts about stochastic integrals with respect to a cylindrical Wiener process evolving on $\mathscr{H}_0$. For this purpose, let K be a separable Banach space, $\mathscr{L}(\mathscr{H}_0, \mathrm{K})$ the space of all bounded linear K-valued operators defined on $\mathscr{H}_0$, and $\mathscr{M}^2_T(\mathrm{K}) := \mathscr{M}^2(\Omega \times [0,T]; \mathrm{K})$ the space of all equivalence classes of $\mathbb{F}$-progressively measurable processes $\Psi : \Omega \times [0,T] \to \mathrm{K}$ satisfying

$$\mathbb{E}\int_0^T \|\Psi(r)\|^2_{\mathrm{K}} dr < \infty.$$

We denote by $\mathscr{L}_2(\mathscr{H}_0, \mathrm{K})$ the Hilbert space of all operators $\Psi \in \mathscr{L}(\mathscr{H}_0, \mathrm{K})$ satisfying

$$\|\Psi\|^2_{\mathscr{L}_2(\mathscr{H}_0,\mathrm{K})} = \sum_{j=1}^{\infty} \|\Psi h_j\|^2_{\mathrm{K}} < \infty.$$

From the theory of stochastic integration on infinite dimensional Hilbert space, see [43, Chapter 5, Section 26] and [17, Chapter 4], for any $\Psi \in \mathscr{M}^2_T(\mathscr{L}_2(\mathscr{H}_0, \mathrm{K}))$ the process M defined by

$$M(t) = \int_0^t \Psi(r) dW(r), t \in [0,T],$$

is a K-valued martingale. Moreover, we have the following Itô isometry

$$\mathbb{E}\left(\left\|\int_0^t \Psi(r)dW(r)\right\|^2_{\mathrm{K}}\right) = \mathbb{E}\left(\int_0^t \|\Psi(r)\|^2_{\mathscr{L}_2(\mathscr{H}_0,\mathrm{K})} dr\right), \forall t \in [0,T], \tag{3.1}$$

and the Burkholder-Davis-Gundy's (BDG's) inequality

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}\left\|\int_0^s \Psi(r)dW(r)\right\|^q\right) \leqslant C_q\mathbb{E}\left(\int_0^t \|\Psi(r)\|^2_{\mathscr{L}_2(\mathscr{H}_0,\mathrm{K})} dr\right)^{\frac{q}{2}}, \forall t \in [0,T], \forall q \in (1,\infty). \tag{3.2}$$

We now give the standing hypotheses on the coefficient of the noise G.

($\mathbf{G}s$) Let $s \in \{0,1\}$ and $G : \mathbf{V} \to \mathscr{L}_2(\mathscr{H}_0; D(\mathrm{A}^{\frac{1+s}{2}}))$, be a map satisfying: there exists a constant $c_4 > 0$ such that for any $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathscr{L}_2(\mathscr{H}_0, D(\mathrm{A}^{\frac{1+s}{2}}))} \leqslant c_4 |\mathbf{u} - \mathbf{v}|. \tag{3.3}$$

Remark 3.1.

(a) Note that Assumption ($\mathbf{G}s$) implies that there exists a constant $c_5 > 0$ such that

$$\|G(\mathbf{u}) - G(\mathbf{v})\|_{\mathscr{L}_2(\mathscr{H}_0, \mathbf{V})} \leqslant c_5 \|\mathbf{u} - \mathbf{v}\|_\alpha,$$

$$\|G(\mathbf{u})\|_{\mathscr{L}_2(\mathscr{H}_0, \mathbf{V})} \leqslant c_5 (1 + \|\mathbf{u}\|_\alpha),$$

and

$$\|\operatorname{\mathbf{rot}}[G(\mathbf{u}) - G(\mathbf{v})]\|_{\mathscr{L}_2(\mathscr{H}_0, \mathbf{L}^2(\mathbb{T}^2))} \leqslant c_5 \|\mathbf{u} - \mathbf{v}\|_\alpha,$$

$$\|\operatorname{\mathbf{rot}} G(\mathbf{u})\|_{\mathscr{L}_2(\mathscr{H}_0, \mathbf{L}^2(\mathbb{T}^2))} \leqslant c_5 (1 + \|\mathbf{u}\|_\alpha),$$

for any $\mathbf{u}, \ \mathbf{v} \in \mathbf{V}$.

(b) Assumption ($\mathbf{G}s$) with $s = 1$ yields that there exists a number $c_6 > 0$ such that

$$\|\nabla \operatorname{\mathbf{rot}}[G(\mathbf{u}) - G(\mathbf{v})]\|_{\mathscr{L}_2(\mathscr{H}_0, \mathbf{L}^2(\mathbb{T}^2))} \leqslant c_6 \|\mathbf{u} - \mathbf{v}\|_\alpha,$$

$$\|\nabla \operatorname{\mathbf{rot}} G(\mathbf{u})\|_{\mathscr{L}_2(\mathscr{H}_0, \mathbf{L}^2(\mathbb{T}^2))} \leqslant c_6 (1 + \|\mathbf{u}\|_\alpha),$$

for any $\mathbf{u}, \ \mathbf{v} \in \mathbf{V}$.

(c) Owing to the first two items, if $\mathbf{u} \in \mathscr{M}_T^2(\mathbf{V})$, then $\operatorname{\mathbf{rot}} G(\mathbf{u})$, as well as $\nabla \operatorname{\mathbf{rot}} G(\mathbf{u})$, belongs to $\mathscr{M}_T^2(\mathscr{L}_2(\mathscr{H}_0, \mathbf{L}^2(\mathbb{T}^2)))$ and the stochastic integral $\int_0^t \mathscr{D} G(\mathbf{u}(r)) dW(r)$, $\mathscr{D} \in \{\operatorname{\mathbf{rot}}, \nabla \operatorname{\mathbf{rot}}\}$ is a well defined $\mathbf{L}^2(\mathbb{T}^2)$-valued martingale.

(d) Item (a) was very important in [52] for the proof of the existence of weak martingale solution to (1.2).

3.2 Statement of the main results

In order to state our main results we briefly recall few definitions and theorems from LDP theory. Let $\mathcal{E}$ be a Polish space and $\mathcal{B}(\mathcal{E})$ its Borel σ-algebra.

Definition 3.2. A function $I : \mathcal{E} \to [0, \infty]$ is a (good) rate function if it is lower semicontinuous and the level sets $\{e \in \mathcal{E}; I(e) \leqslant a\}$, $a \in [0, \infty)$, are compact subsets of $\mathcal{E}$.

Next let ϱ be a real-valued map defined on $[0, \infty)$ such that

$$\varrho(\varepsilon) \to \infty \text{ as } \varepsilon \to \infty.$$

Definition 3.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. An $\mathcal{E}$-valued random variable $(X_\varepsilon)_{\varepsilon \in (0,1]}$ satisfies the LDP on $\mathcal{E}$ with speed $\varrho(\varepsilon)$ and rate function I if and only if the following two conditions hold

(a) for any closed set $F \subset \mathcal{E}$

$$\limsup_{\varepsilon \to 0} \varrho^{-1}(\varepsilon) \log \mathbb{P}(X_\varepsilon \in F) \leqslant - \inf_{x \in F} I(x);$$

(b) for any open set $O \subset \mathcal{E}$

$$\liminf_{\varepsilon \to 0} \varrho^{-1}(\varepsilon) \log \mathbb{P}(X_\varepsilon \in O) \geqslant - \inf_{x \in O} I(x).$$

We also consider a function $\lambda : (0, 1] \to (0, \infty)$ satisfying (1.4), i.e.,

$$\lambda(\varepsilon) \to \infty \text{ and } \varepsilon^{\frac{1}{2}} \lambda(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

For $\delta \in \{0, 1\}$ we introduce the function λ_δ defined by

$$\lambda_\delta(\varepsilon) = \begin{cases} 1 \text{ if } \delta = 0 \\ \varepsilon^{\frac{1}{2}} \lambda(\varepsilon) \text{if } \delta = 1. \end{cases} \tag{3.4}$$

Remark 3.4. From the definition of λ_δ and the properties of λ we derive that for $\delta \in \{0, 1\}$

$$\lambda_\delta(\varepsilon) \to 1 - \delta \text{ as } \varepsilon \to 0. \tag{3.5}$$

These notations will be used to describe the unifying approach to the LDP and MDP results stated in the first theorem of the introduction.

In the following propositions we give some results related to the well-posedness and regularity of the solution to (1.5) and (1.6).

Proposition 3.5. *For any* $\xi \in \mathbf{W}$ *the problem* (1.5) *has a unique solution* $\mathbf{u} \in C([0,T];\mathbf{W})$ *satisfying* (1.5) *in the weak sense and*

$$\sup_{0\leqslant s\leqslant T} \|\mathbf{u}(r)\|_{\mathbf{W}} \leqslant R_0, \tag{3.6}$$

where $R_0 > 0$ *is a constant depending on* α, T *and* $\|\xi\|_{\mathbf{W}}$ *only.*

If in addition $\xi \in D(\mathrm{A}^{\frac{3+s}{2}})$, $s \geqslant 1$, *then* $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbf{H}^{3+s})$ *and there exists a constant* $R_1 > 0$ *(which may depend on* α, T *and* $\|\xi\|_{\mathbf{H}^{3+s}}$ *) such that*

$$\sup_{0\leqslant r\leqslant T} \|\mathbf{u}(r)\|_{\mathbf{H}^4} \leqslant R_1, \tag{3.7}$$

Proof. This proposition is a corollary of Theorem B.1 and Theorem B.2. In fact the existence and uniqueness of a solution $\mathbf{u} \in C([0,T];\mathbf{W})$ follows from setting $\boldsymbol{\varphi} = 0$, $\delta = 0$, $h = 0$ in Theorem B.1. The regularity result follows from Theorem B.2. □

Proposition 3.6. *Let* $\delta \in \{0,1\}$, $\xi \in D(\mathrm{A}^{\frac{3+\delta}{2}})$ *and* $p \in [1,\infty)$. *If Assumption (**G**s) is satisfied with* $s = 0$, *i.e.,(**G**0), then the problem* (1.6) *has a unique solution* $\mathbf{u}^{\varepsilon,\delta} \in C([0,T];\mathbf{W})$ *satisfying*

$$\mathbb{E} \sup_{0\leqslant s\leqslant T} \|\mathbf{u}^{\varepsilon,\delta}(r)\|^p_{\mathbf{W}} \leqslant R_2, \tag{3.8}$$

where $R_2 > 0$ *is a constant which may depend on* $\varepsilon \in (0,1)$.

Proof. For $\delta = 0$ the problem (1.6) reduces to the stochastic model for grade-two fluid. Under Assumption (**G**0) it is proved in [52], see also [49], that for $\delta = 0$ problem (1.6) has a weak martingale solution satisfying (3.8) and which is pathwise unique. Thus, by the Watanabe-Yamada's theorem, see [48], it has a unique strong solution; see also [51] for a direct proof of the existence and uniqueness of a strong solution.

For $\delta = 1$ the problem reduces to (1.8) whose existence of solution can be easily proved using Galerkin approximation as in [49] and [65]. Here the assumption $\xi \in D(\mathrm{A}^2)$ is necessary to ensure that $\mathbf{u} \in \mathbf{L}^\infty(0,T;D(\mathrm{A}^2))$ which in its turn enables us to rigorously justify all the required steps to derive the estimate (3.8) for the Galerkin solutions. We refer to [65, Lemma 5.2] for the stochastic case and Theorem B.1 for the deterministic case. Thanks to Assumption (**G**0) one can prove by arguing as in [52] that if it has a solution then it is pathwise unique. In fact, under the theorem assumption it is not difficult to check that $\mathbf{u}^{\varepsilon,1} := \varepsilon^{-\frac{1}{2}}\lambda^{-1}(\varepsilon)[\mathbf{u}^\varepsilon - \mathbf{u}]$ satisfies (1.8) and hence the only solution. □

Remark 3.7. (i) The estimate in Proposition 3.6 may explode as ε is approaching zero, but we will later on derive new and uniform estimates which are of the essence for our analysis.

(ii) We used the result from [65] to justify the existence of solution to (1.6) for $\delta = 1$, but our results does not follow from [65]. In fact, [65, Lemma 3.1] was crucial for the validity of the results in [65], but the proof of this crucial step depends on [15, Theorem 3.6] which in its turn relies on estimate which explode when the viscosity $\nu = \varepsilon \to 0$, see for instance [15, Eq. (4.11)].

(iii) The existence and uniqueness of a strong solution to (1.6) enables us to define a Borel measurable map $\varGamma_\xi^{\varepsilon,\delta} : C([0,T];\mathscr{H}) \to C([0,T];\mathbf{W})$ such that $\varGamma_\xi^{\varepsilon,\delta}(W)$ is the unique solution to (1.6) on the filtered probability space $(\varOmega,\mathscr{F},\mathbb{F},\mathbb{P})$ with the Wiener process W.

In order to describe the rate functions associated to the LDP and MDP results, we also need to introduce few additional notations and auxiliary problems. For fixed $M > 0$ we set

$$S_M = \Big\{h \in \mathrm{L}^2(0,T;\mathscr{H}_0) : \int_0^T \|h(r)\|^2_{\mathscr{H}_0} dr \leqslant M\Big\}.$$

The set S_M, endowed with the weak topology

$$d_1(h,k) = \sum_{k\geqslant 1} \frac{1}{2^k}\Big|\int_0^T \big(h(r) - k(r), \tilde{e}_k(r)\big)_{\mathscr{H}_0} dr\Big|, \tag{3.9}$$

where $(\tilde{e}_k, k \geqslant 1)$ is an orthonormal basis for $\mathrm{L}^2(0,T;\mathscr{H}_0)$, is a Polish (complete separable metric) space, see [6].

We also introduce the class $\mathscr{S}$ as the set of $\mathscr{H}_0-$valued $(\mathscr{F}_t)-$predictable stochastic processes h such that $\int_0^T \|h(r)\|^2_{\mathscr{H}_0} dr < \infty$, a.s. For $M > 0$ we set

$$\mathscr{S}_M = \{h \in \mathscr{S} : h \in S_M \ a.s.\}. \tag{3.10}$$

Now, for $\delta \in \{0,1\}$, $\xi \in \mathbf{W}$ and $h \in \mathbf{L}^2(0,T;\mathscr{H}_0)$ we consider the following problem

$$\partial_t \mathbf{v}_h^\delta + (1-\delta)\mathcal{C}(\mathbf{rot}\,\mathbf{v}_h^\delta, \mathbf{u}_h^\delta) + \delta(\mathcal{C}(\mathbf{rot}\,\mathbf{v}_h^\delta, \mathbf{u}) + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{u}_h^\delta)) = G(\delta\mathbf{u} + (1-\delta)\mathbf{u}_h^\delta)h, \tag{3.11a}$$

$$\mathbf{v}_h^\delta = \mathbf{u}_h^\delta + \alpha \mathrm{A}\mathbf{u}_h^\delta, \tag{3.11b}$$

$$\mathbf{u}_h^\delta(0) = (1-\delta)\xi, \tag{3.11c}$$

where $\mathbf{u} \in C([0,T];\mathbf{W})$ is the unique solution to (1.5) and $\mathbf{v} = \mathbf{u} + \alpha \mathrm{A}\mathbf{u}$.

Proposition 3.8. *Let $\delta, s \in \{0,1\}$, $\xi \in D(\mathrm{A}^{\frac{3+s+\delta}{2}})$ and Assumption (**G**s) hold. Then for any $h \in \mathrm{L}^2(0,T;\mathscr{H}_0)$ the system (3.11) has a unique solution $\mathbf{u}_h^\delta \in C([0,T];\mathbf{W}) \cap \mathrm{L}^\infty(0,T;D(\mathrm{A}^{\frac{3+s}{2}}))$. Moreover, if $h \in \mathscr{S}_M$, $M > 0$, then there exists a deterministic positive constant $R_3 > 0$ (which may depend on M, T, α and $\|\xi\|_{D(\mathrm{A}^{\frac{3+s+\delta}{2}})}$) such that*

$$\sup_{t\in[0,T]} \|\mathbf{u}_h^\delta\|_{D(\mathrm{A}^{\frac{3+\delta+s}{2}})} \leqslant R_3 \quad \mathbb{P}-a.s.. \tag{3.12}$$

Proof. Observe that under the assumptions of the present proposition, the unique solution $\mathbf{u}$ to (1.5) belongs to $\mathrm{L}^\infty(0,T;\mathbf{H}^{3+s+\delta}) \cap C([0,T];\mathbf{W})$. With this in mind, the existence and uniqueness of a solution $\mathbf{u}_h^\delta \in C([0,T];\mathbf{W})$ satisfying $\mathbf{u}_h^\delta \in \mathrm{L}^\infty(0,T;D(\mathrm{A}^{\frac{3+s}{2}}))$ follows from setting $\boldsymbol{\varphi} = \mathbf{u}$ in Theorem B.1. □

The above proposition enables us to define a map $\Gamma_\xi^{0,\delta} : C([0,T];\mathscr{H}_0) \to C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$, $s \in \{0,1\}$, by setting

- $\Gamma_\xi^{0,\delta}(x)$ is the unique solution $\mathbf{u}_h^\delta$ to (3.11) if $x = \int_0^\cdot h(r)dr$, $h \in \mathrm{L}^2(0,T;\mathscr{H}_0)$;
- $\Gamma_\xi^{0,\delta}(x) = 0$ otherwise.

We will see later on, see Remark 3.15, that this map is in fact Borel measurable.

We are now ready to state our main results.

Theorem 3.9. *Let $\delta, s \in \{0,1\}$, $\xi \in D(\mathrm{A}^{\frac{3+s+\delta}{2}})$ and Assumption (**G**s) hold. Then, the family $(\mathbf{u}^{\varepsilon,\delta})_{\varepsilon\in(0,1]}$ satisfies an LDP on $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed $\varepsilon^{-1}\lambda_\delta^2(\varepsilon)$ and rate function I_δ given by*

$$I_\delta(x) = \inf_{\{h\in\mathrm{L}^2(0,T;\mathscr{H}_0):x=\Gamma_\xi^{0,\delta}(\int_0^\cdot h(r)dr)\}} \left\{\frac{1}{2}\int_0^T \|h(r)\|_{\mathscr{H}_0}^2 dr\right\}. \tag{3.13}$$

As usual, we understand that $\inf \varnothing = \infty$.

Proof. The proof of this theorem wil be given in the next subsection. □

We can divide the result in the above theorem into two parts which will form the following two corollaries. They give the LDP and MDP on $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$, $s \in \{0,1\}$, for the solution $\mathbf{u}^\varepsilon$ to (1.2).

Corollary 3.10. *Let $s \in \{0,1\}$, $\xi \in D(\mathrm{A}^{\frac{3+s}{2}})$ and G satisfies Assumption (**G**s). Then, the family of solutions $(\mathbf{u}^\varepsilon)_{\varepsilon\in(0,1]}$ to (1.2) satisfies an LDP on $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed ε^{-1} and rate function I_0 given by*

$$I_0(x) = \inf_{\{h\in \mathrm{L}^2(0,T;\mathscr{H}_0):x=\Gamma_\xi^{0,0}(\int_0^\cdot h(r)dr)\}} \left\{ \frac{1}{2}\int_0^T \|h(r)\|^2_{\mathscr{H}_0} dr \right\}. \tag{3.14}$$

Corollary 3.11. *If $\xi \in D(\mathrm{A}^{2+\frac{s}{2}})$ and G satisfies Assumption (**G**s), then $\left(\varepsilon^{-\frac{1}{2}}\lambda^{-1}(\varepsilon)[\mathbf{u}^\varepsilon - \mathbf{u}]\right)_{\varepsilon\in(0,1]}$ satisfies an LDP on $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed $\lambda^2(\varepsilon)$ and rate function I_1 given by*

$$I_1(x) = \inf_{\{h\in \mathrm{L}^2(0,T;\mathscr{H}_0):x=\Gamma_\xi^{0,1}(\int_0^\cdot h(r)dr)\}} \left\{ \frac{1}{2}\int_0^T \|h(r)\|^2_{\mathscr{H}_0} dr \right\}. \tag{3.15}$$

In both corollaries we understand that $\inf \varnothing = \infty$. To close this subsection we should state the following important remark.

Remark 3.12. We only considered the case $s \in \{0,1\}$ because we are unable to prove the results in Proposition 3.5 for $s \in (0,1)$. In particular, we are unable to prove that the solution $\mathbf{u}$ to (1.5) belongs to $C([0,T];\mathbf{W}) \cap \mathrm{L}^\infty(0,T;\mathbf{H}^{3+s})$ for $s \in (0,1)$. Here, the main issue is that the proof of Proposition 3.5 uses the results of Theorem B.1 and B.2 which in their turn depend on the commutator estimate (B.11) which to our knowledge is only true for $s \geqslant 1$.

Since $\xi \in D(\mathrm{A}^{\frac{3+s}{2}})$ one would expect that the LDP result should hold in $C([0,T];D(\mathrm{A}^{\frac{3+s}{2}}))$, but we are facing two major issues in getting the LDP in $C([0,T];D(\mathrm{A}^{\frac{3+s}{2}}))$:

(i) the proof of the LDP relies in particular on Proposition 3.14 with $\delta = 0$ which requires that $\xi \in D(\mathrm{A}^{\frac{3+s}{2}})$ for a compactness of the level sets K_M in $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$. This is more transparent in the proof of the case $s = 0$, see proof of (5.6) on page 32.

(ii) For the case $s = 1$ we do not know whether the solution of the non-viscous model belongs to $C([0,T];D(\mathrm{A}^2))$. In fact, we only know that it is an element of $C([0,T];D(\mathrm{A}^{\frac{3}{2}})) \cap \mathrm{L}^\infty(0,T;\mathbf{H}^4)$.

3.3 Proof of Theorem 3.9

This subsection contains the proof of the main result stated in Theorem 3.9. The proof will use Budhiraja and Dupuis' representation of functional of Brownian motion and the weak convergence approach to LDP, see [5], [6] and Appendix A, and requires several auxiliary results whose proof will be postponed to subsequent sections or subsections.

Let $\delta \in \{0,1\}$, $\xi \in \mathbf{W}$ and $h \in \mathscr{S}$. We consider the following stochastic system

$$d\mathbf{v}_h^{\varepsilon,\delta} + \Big[\varepsilon \mathrm{A}\mathbf{u}_h^{\varepsilon,\delta} + \lambda_\delta(\varepsilon)\mathcal{C}(\mathbf{rot}\,\mathbf{v}_h^{\varepsilon,\delta}, \mathbf{u}_h^{\varepsilon,\delta}) + \delta(\mathcal{C}(\mathbf{rot}\,\mathbf{v}_h^{\varepsilon,\delta}, \mathbf{u}) + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \mathbf{u}_h^{\varepsilon,\delta}))\Big]\,dt$$
$$= \Big[\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\mathrm{A}\mathbf{u} + G(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}_h^{\varepsilon,\delta})h\Big]\,dt + \varepsilon^{\frac{1}{2}}\lambda_\delta^{-1}(\varepsilon)G(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}_h^{\varepsilon,\delta})dW, \tag{3.16a}$$

$$\mathbf{v}_h^{\varepsilon,\delta} = (\mathbf{u}_h^{\varepsilon,\delta} + \alpha \mathrm{A}\mathbf{u}_h^{\varepsilon,\delta}), \tag{3.16b}$$

$$\mathbf{u}_h^{\varepsilon,\delta}(0) = (1-\delta)\xi \ \text{ in } \ \mathbb{T}^2, \tag{3.16c}$$

where $\mathbf{u}$ is the solution to (1.5) and $\mathbf{v} = \mathbf{u} + \alpha\mathrm{A}\mathbf{u}$.

Proposition 3.13. *Let $\delta \in \{0,1\}$, $\xi \in \mathbf{V} \cap D(\mathrm{A}^{\frac{3+\delta}{2}})$, $p \in [1,\infty)$ and $h \in \mathscr{S}$. If Assumption (**G**s) is satisfied, then the stochastic controlled system* (3.16) *has a unique solution* $\mathbf{u}_h^{\varepsilon,\delta} \in \mathrm{L}^p(\Omega; C([0,T];\mathbf{W}))$ *such that*

$$\mathbf{u}_h^{\varepsilon,\delta} = \Gamma_\xi^{\varepsilon,\delta}\left(W + \varepsilon^{-\frac{1}{2}}\lambda_\delta(\varepsilon)\int_0^{\cdot} h(r)dr\right).$$

Furthermore, if $h \in \mathscr{S}_M$ for a fixed $M > 0$, then there exists a constant R_4 (which may depend on $\|\xi\|_{D(\mathrm{A}^{\frac{3+\delta}{2}})}$, M, T, α and p) such that

$$\sup_{\varepsilon\in(0,1]} \mathbb{E} \sup_{t\in[0,T]} \|\mathbf{u}_h^{\varepsilon,\delta}(t)\|_{\mathbf{W}}^{2p} \leqslant R_4. \tag{3.17}$$

The proof of Theorem 3.9 heavily relies on the following two propositions.

Proposition 3.14. *Let $\delta, s \in \{0,1\}$ and $\xi \in D(\mathrm{A}^{\frac{3+s+\delta}{2}})$. Then, the sets $K_M = \{\mathbf{u}_h^{\delta}, h \in S_M\}$, $M > 0$, are compact sets of $C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$.*

Proof. The proof of this proposition will be given in Section 5. □

Remark 3.15. The above proposition amounts to say that if $(h_n)_{n\in\mathbb{N}} \subset S_M$, $M > 0$, is a sequence that converges weakly to $h \in S_M$, then $\Gamma_\xi^{0,\delta}(\int_0^{\cdot} h_n(r)dr)$ strongly converges to $\Gamma_\xi^{0,\delta(\int_0^{\cdot} h(r)dr)}$ in $C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$. This implies in particular that the map $S_M \ni h \mapsto \Gamma_\xi^{0,\delta}(\int_0^{\cdot} h(r)dr) \in C([0,T];\mathbf{W})$ is Borel measurable.

Proposition 3.16. *Let $M>0$, $(h_n)_{n\in\mathbb{N}}\subset \mathscr{S}_M$, $h\in\mathscr{S}_M$, and $(\varepsilon_n)_{n\in\mathbb{N}}\subset(0,1]$ be a sequence converging to 0. Also, let $\delta,s\in\{0,1\}$ and $\xi\in D(\mathrm{A}^{\frac{3+s+\delta}{2}})$. If Assumption (**G**s) hold and h_n converges in distribution to h as $\mathscr{S}_M$-valued random variable, then the process $\Gamma_\xi^{\varepsilon_n,\delta}\left(W+\varepsilon_n^{-\frac{1}{2}}\lambda_\delta(\varepsilon_n)\int_0^\cdot h_n(r)dr\right)$ converges in distribution to $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right)$ as $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$-valued random variables.*

Proof. The proof will be given in Section 5. □

Remark 3.17. In many papers such as [2, 3, 64, 65] the proof of this proposition is based on showing that the solution $\mathbf{u}_{h_n}^{\varepsilon_n}$ is tight on appropriate Polish space, say $\mathcal{X}$, and then invoking Prokhorov's and Skorokhod's theorems to construct new filtered probability space on which is defined a (sub)sequence $\tilde{\mathbf{u}}_{h_n}^{\varepsilon_n}$ and a random variable $\tilde{\mathbf{u}}$ such that

$$\mathrm{law}_{\mathcal{X}}(\mathbf{u}_{h_n}^{\varepsilon_n}) = \mathrm{law}_{\mathcal{X}}(\tilde{\mathbf{u}}_{h_n}^{\varepsilon_n}),$$

$$\mathbf{u}_{h_n}^{\varepsilon_n}\to\tilde{\mathbf{u}} \text{ strongly in } \mathcal{X} \text{ a.s. on the new probability space.}$$

Finally, one shows that

$$\tilde{\mathbf{u}}=\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right),$$

which completes the proof.

Our approach differs to this method as we first show that $\Gamma_\xi^{\varepsilon_n,\delta}\left(W+\varepsilon_n^{-\frac{1}{2}}\lambda_\delta(\varepsilon_n)\int_0^\cdot h_n(r)dr\right)$ converges in probability to $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h_n(r)dr\right)$ as $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$-valued random variables, see Lemma 5.1. Then, we prove that $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h_n(r)dr\right)$ converges in distribution to $\Gamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right)$ as $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$-valued random variables, see Lemma 5.2. We learn this approach from the recent paper [4] and we believe it is much shorter and simpler than the one we outlined above.

Now, we give the promised proof of our theorem.

Proof of Theorem 3.9. Owing to Propositions 3.14 and 3.16 the assumptions (**A**1) and (**A**2) of Theorem A.2 are satisfied on $\mathcal{E}_s=C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$, $s\in\{0,1\}$. Thus, we infer that for δ and $s\in\{0,1\}$ the solution $\mathbf{u}^{\varepsilon,\delta}$ to (1.6) satisfies an LDP on $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$ with speed $\varepsilon^{-1}\lambda_\delta^2(\varepsilon)$ and rate function I_δ. This completes the proof of Theorem 3.9. □

Before we proceed to the next section we state the following remarks.

4 Qualitative studies of the stochastic controlled model: proof of Proposition 3.13

This section is devoted to the proof of Proposition 3.13 which will be divided into parts.

Proof of Proposition 3.13. Part I: Well-posedness of problem (3.16).

Since $h \in \mathscr{S}$ we have

$$\mathbb{E}\exp\left(\frac{1}{2}\varepsilon^{-1}\lambda_\delta^2(\varepsilon)\int_0^T \|h(r)\|^2 dr\right) < \infty.$$

Thus, by Girsanov's theorem there exists a probability measure $\mathbb{P}_h$ such that

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} = \exp\left(\frac{1}{2}\varepsilon^{-1}\lambda_\delta(\varepsilon)^2\int_0^T \|h(r)\|_{\mathscr{H}_0}^2 dr - \varepsilon^{-\frac{1}{2}}\lambda_\delta(\varepsilon)\int_0^T h(r)dW(r)\right),$$

and the stochastic process $\tilde{W}(\cdot) := W(\cdot) + \varepsilon^{-\frac{1}{2}}\lambda(\varepsilon)\int_0^{\cdot} h(r)dr$ defines a cylindrical Wiener process evolving on $\mathscr{H}_0$ and defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}_h)$. We now infer from Proposition 3.6 that on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}_h)$ the problem (1.6) with driving noise $\tilde{W}$ admits a unique strong solution $\mathbf{u}_h^{\varepsilon,\delta}$. By Remark 3.7(iii) we have $\mathbf{u}_h^{\varepsilon,\delta} = \Gamma_\xi^{\varepsilon,\delta}(\tilde{W})$ on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}_h)$ which reads

$$\mathbf{u}_h^{\varepsilon,\delta} = \Gamma_\xi^{\varepsilon,\delta}\left(W(\cdot) + \varepsilon^{-\frac{1}{2}}\lambda(\varepsilon)\int_0^{\cdot} h(r)dr\right) \text{ on } (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}).$$

Part II: Proof of the uniform estimates (3.17)

In order to complete the proof of Proposition 3.13, we need to establish (3.17). For this aim we fix $M > 0$ and $h \in \mathscr{S}_M$ and we prove the following result.

Claim 1. There exists a constant $R_5 > 0$, which depends only on M, p, T, α and $\|\xi\|_{\mathbf{W}}$, such that for any $\varepsilon \in (0,1]$

$$\mathbb{E}[\sup_{t\in[0,T]}\|\mathbf{u}_h^{\varepsilon,\delta}(t)\|_\alpha^{2p} + 2p\varepsilon\int_0^T \|\mathbf{u}_h^{\varepsilon,\delta}(t)\|^{2p-2}\|\mathbf{u}_h^{\varepsilon,\delta}(t)\|^2 dt] \leqslant R_5. \tag{4.1}$$

Proof of Claim 1. To alleviate the notation we will denote by $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ the solution to (3.16) and (3.11), respectively. We will also set

$$z^\varepsilon = \mathbf{rot}(\mathbf{u}^\varepsilon + \alpha \mathrm{A}\mathbf{u}^\varepsilon),$$

$$z = \mathbf{rot}(\mathbf{u} + \alpha \mathrm{A}\mathbf{u}),$$

$$\xi_\delta = (1-\delta)\xi.$$

By denoting the identity map on $\mathbf{V}$ by Id and using the bilinear map defined in Lemma 2.1, we can rewrite the first identity in (3.16) in the following form

$$\begin{aligned}(\mathrm{Id}+\alpha\mathrm{A})\mathbf{u}^\varepsilon(t)+\int_0^t\left(\varepsilon\mathrm{A}\mathbf{u}^\varepsilon(r)+\lambda_\delta(\varepsilon)\mathcal{C}(z^\varepsilon(r),\mathbf{u}^\varepsilon(r))+\delta[\mathcal{C}(z^\varepsilon(r),\mathbf{u}(r))+\mathcal{C}(z(r),\mathbf{u}^\varepsilon(r))]\right)dr\\ =\xi_\delta+\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\int_0^t\mathrm{A}\mathbf{u}(r)dr+\int_0^t G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))dr\\ +\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\int_0^t G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))dW(r),\end{aligned}\tag{4.2}$$

for any $t\in[0,T]$. For the sake of simplicity, we will set

$$\mathscr{A}:=(\mathrm{Id}+\alpha\mathrm{A})^{-1}\circ\mathrm{A},$$

$$\mathscr{C}:=(\mathrm{Id}+\alpha\mathrm{A})^{-1}\circ\mathcal{C},$$

$$\mathscr{G}=(\mathrm{Id}+\alpha\mathrm{A})^{-1}\circ G.$$

With this in mind, we derive from (4.2) that any solution $\mathbf{u}^\varepsilon$ of the problem (3.16) with the initial condition $\xi_\delta\in\mathbf{W}$ satisfies, $\mathbb{P}$ a.s.

$$\begin{aligned}\mathbf{u}^\varepsilon(t)+\int_0^t\left(\lambda_\delta(\varepsilon)\mathscr{C}(z^\varepsilon(r),\mathbf{u}^\varepsilon(r))+\delta[\mathscr{C}(z^\varepsilon(r),\mathbf{u}(r))+\mathscr{C}(z(r),\mathbf{u}^\varepsilon(r))+\varepsilon\lambda_\delta^{-1}(\varepsilon)\mathscr{A}\mathbf{u}(r)]\right)dr\\ =(\mathrm{Id}+\alpha\mathrm{A})^{-1}\xi_\delta+\int_0^t\mathscr{G}(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))h(r)dr-\int_0^t\varepsilon\mathscr{A}\mathbf{u}^\varepsilon(r)dr\\ +\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\int_0^t\mathscr{G}(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))dW(r),\end{aligned}\tag{4.3}$$

for any $t\in[0,T]$. Let $(\tau_\ell)_{\ell\in\mathbb{N}}$ be the sequence of stopping times defined by

$$\tau_\ell=\inf\{t\geqslant 0:|\mathbf{rot}(\mathbf{u}^\varepsilon(t)+\alpha\mathrm{A}\mathbf{u}^\varepsilon(t))|\geqslant\ell\}\wedge T.$$

Observe that thanks to (3.8) we infer that $\tau_\ell\to T$ a.s. as $\ell\to\infty$.

By the application of Itô formula, see [43, Theorem 26.5], to $\|\mathbf{u}^\varepsilon(t\wedge\tau_\ell)\|_\alpha^2$ and then to $\left(\|\mathbf{u}^\varepsilon(t)\|_\alpha^2\right)^p$ we obtain

$$\begin{aligned}&\|\mathbf{u}^\varepsilon(t\wedge\tau_\ell)\|_\alpha^{2p}+2p\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}\left[\varepsilon\|\mathbf{u}^\varepsilon\|^2+\delta[\langle\mathcal{C}(z^\varepsilon,\mathbf{u}),\mathbf{u}^\varepsilon\rangle+\varepsilon\lambda_\delta^{-1}(\varepsilon)((\mathbf{u},\mathbf{u}^\varepsilon))]\right](r)dr\\ &=p\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}\left(\varepsilon\lambda_\delta^{-2}(\varepsilon)\|G(\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\|^2_{\mathscr{L}_Q(\mathscr{H}_0,\mathbf{V})}+2(G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h,\mathbf{u}^\varepsilon)\right)(r)dr\\ &\quad-2p\varepsilon\int_0^{t\wedge\tau_\ell}(\|\mathbf{u}^\varepsilon\|_\alpha^{2p-2}\|\mathbf{u}^\varepsilon\|^2)(r)dr+f(p)\varepsilon\lambda_\delta^{-2}(\varepsilon)\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon\|_\alpha^{2p-4}\|\mathscr{G}^*(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon\|^2_{\mathscr{H}_0}(r)dr\\ &\qquad+2p\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}(\mathbf{u}^\varepsilon(r),G(\delta\mathbf{u}(r)+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))dW(r))+\|\xi_\delta\|_\alpha^{2p},\end{aligned}$$

where $f(p) = 2p(p-1)$. Note that we have used the identity

$$(((\mathrm{Id}+\alpha \mathrm{A})^{-1}\mathbf{f},\mathbf{v}))_\alpha = \langle \mathbf{f},\mathbf{v}\rangle \text{ for any } \mathbf{f}\in \mathbf{V}^*,$$

and (2.7) to justify that

$$((\mathscr{C}(z^\varepsilon,\mathbf{u}^\varepsilon)-\delta\mathscr{C}(z,\mathbf{u}^\varepsilon),\mathbf{u}^\varepsilon))_\alpha = 0.$$

Using Cauchy-Schwarz's inequality, Young's inequality and (3.6) we show that there exists a constant $C_0>0$ such that for any $\varepsilon>0$ and $\ell\geqslant 1$

$$\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|^{2p-2}((\mathbf{u}(r),\mathbf{u}^\varepsilon(r)))dr \leqslant C_0\left(TR_0^{2p}+\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}dr\right).$$

From (2.15) and (3.6) we infer that there exists a constant $C_1>0$ such that for any $\varepsilon>0$ and $\ell\geqslant 1$

$$\begin{aligned}\int_0^{t\wedge\tau_\ell}\langle \mathcal{C}(z^\varepsilon(r),\mathbf{u}(r)),\mathbf{u}^\varepsilon(r)\rangle dr &\leqslant C_1\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}\|\mathbf{u}(r)\|_{\mathbf{W}}dr\\ &\leqslant C_1R_0\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}dr\end{aligned}$$

Using Assumption ($\mathbf{G}s$) with $s=0$, Young's inequality and (3.6) we can find constants $C_2>0$ and $C_3>0$ such that for any $\varepsilon>0$ and $\ell\geqslant 1$

$$\begin{aligned}\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}\|G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)(r)\|^2_{\mathscr{L}_Q(\mathscr{H}_0,\mathbf{V})}dr &\leqslant C_2(1+\lambda_\delta^2(\varepsilon))\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}dr\\ &+C_2T\left(1+\delta^{2p}R_0^{2p}\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-4}\|\mathscr{G}^*(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon(r)\|^2_{\mathscr{H}_0}dr &\leqslant C_3(1+\lambda_\delta^2(\varepsilon))\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}dr\\ & C_3T\left(1+\delta^{2p}R_0^{2p}\right).\end{aligned}$$

Since $h\in\mathscr{S}_M$, applications of Cauchy-Schwarz's inequality, Assumption ($\mathbf{G}s$) with $s=0$, Young's inequality and (3.6) imply that there exists a constant $C_4>0$ such that for any $\varepsilon>0$ and $\ell\geqslant 1$

$$\begin{aligned}\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}(G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h(r),\mathbf{u}^\varepsilon(r))dr &\leqslant C_4(1+\lambda_\delta(\varepsilon))\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}\|h(r)\|_{\mathscr{H}_0}dr\\ &+C_4M^{\frac12}T^{\frac12}\left(1+\delta^{2p}R_0^{2p}\right).\end{aligned}$$

Collecting all these inequalities together yields

$$\|\mathbf{u}^\varepsilon(t\wedge\tau_\ell)\|_\alpha^{2p}+2p\varepsilon\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}\varepsilon\|\mathbf{u}^\varepsilon(r)\|^2dr\leqslant\Phi^{\varepsilon,\delta}+\int_0^{t\wedge\tau_\ell}\Psi^{\varepsilon,\delta}(r)\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}dr$$
$$+\mathscr{M}^{\varepsilon,\delta}(t\wedge\tau_\ell),$$

with

$$\mathscr{M}^{\varepsilon,\delta}(t)=2p\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\sup_{r\in[0,t]}\left|\int_0^s\|\mathbf{u}(r)\|_\alpha^{2p-2}(\mathbf{u}^\varepsilon(r),G(\delta\mathbf{u}(r)+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))dW(r))\right|,$$

$$\Phi^{\varepsilon,\delta}=\|\xi_\delta\|_\alpha^{2p}+C_5T\left(\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)+\delta^{2p}[\varepsilon\lambda_\delta^{-2}(\varepsilon)+M^{\frac12}T^{\frac12}]\right)+C_6T\varepsilon\lambda_\delta^{-2}(\varepsilon),$$

$$\Psi^{\varepsilon,\delta}(r)=C_7\delta(\varepsilon\lambda_\delta^{-1}(\varepsilon)+1)+C_8(\varepsilon\lambda_\delta^{-2}(\varepsilon)+\varepsilon)+C_9(1+\lambda_\delta(\varepsilon))\|h(r)\|_{\mathscr{H}_0},$$

where C_i, $i=5,\dots,9$, are positive constants which may depend on $\|\xi\|_{\mathbf{W}}$, T, α, p, but not on ε, $\ell\geqslant1$ and δ.

From Burkholder-Davis-Gundy's (BDG's) inequality, Assumption ($\mathbf{G}s$) with $s=0$, Cauchy-Schwarz's and Young's inequalities we infer that there exists a positive constant C_{10} such that for any $\varepsilon>0$, $\ell\geqslant1$ and $\theta>0$ we have

$$\mathbb{E}\mathscr{M}^{\varepsilon,\delta}(t\wedge\tau_\ell)\leqslant9\theta^{-1}p^2C_{10}\varepsilon\lambda_\delta^{-2}(\varepsilon)\left(T+(1+\lambda_\delta^2(\varepsilon))\mathbb{E}\int_0^{t\wedge\tau_\ell}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}dr+\delta^{2p}\right)$$
$$+\theta\mathbb{E}\sup_{r\in[0,t\wedge\tau_\ell]}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p}. \tag{4.4}$$

Hence, for $t\in[0,T]$ we have, with probability 1,

$$X(t\wedge\tau_\ell)+2p\varepsilon Y(t\wedge\tau_\ell)\leqslant\Phi^{\varepsilon,\delta}+\int_0^{t\wedge\tau_\ell}\Psi^{\varepsilon,\delta}(r)X(r)dr+\mathscr{M}^{\varepsilon,\delta}(t),$$
$$\mathbb{E}\mathscr{M}^{\varepsilon,\delta}(t\wedge\tau_\ell)\leqslant\theta\mathbb{E}X(t\wedge\tau_\ell)+\theta^{-1}C_{11}^{\varepsilon,\delta}\mathbb{E}\int_0^{t\wedge\tau_\ell}X(r)dr+C_{12}^{\varepsilon,\delta},$$

where we have set

$$X(t):=\sup_{r\in[0,t]}\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p},\qquad Y(t):=\int_0^t\|\mathbf{u}^\varepsilon(r)\|_\alpha^{2p-2}\|\mathbf{u}^\varepsilon(r)\|^2dr,$$

$$C_{11}^{\varepsilon,\delta}:=9p^2C_{10}\varepsilon\lambda_\delta^{-2}(\varepsilon)(1+\lambda_\delta^2(\varepsilon)),$$

$$C_{12}^{\varepsilon,\delta}:=C_{11}^{\varepsilon,\delta}(1+\lambda_\delta^2(\varepsilon))^{-1}(T+\delta^{2p}).$$

Now, observe that by the definition of $\lambda_\delta(\varepsilon)$, $\varepsilon\lambda_\delta^{-\ell}(\varepsilon)\to0$, $\ell\in\{1,2\}$, as $\varepsilon\to0$. Hence, for any $\varepsilon_0>0$ one can find positive constants $\tilde C_0,\tilde C_1$, $\tilde C_2$ and $\tilde C_3$ such that for any $\varepsilon>0$ and $\ell\geqslant1$

$$\int_0^T\Psi^{\varepsilon,\delta}(r)dr\leqslant\tilde C_0\text{ with probability 1},\quad\Phi^{\varepsilon,\delta}\leqslant\tilde C_1,\quad C_{11}^{\varepsilon,\delta}\leqslant\tilde C_2,\quad\text{and }C_{12}^{\varepsilon,\delta}\leqslant\tilde C_3.$$

The constants $\tilde{C}_i$, $i \in \{0,\dots,3\}$, may depend on $\|\xi\|_{D(\mathrm{A}^{\frac{3+\delta}{2}})}$, ε_0, p, T, and δ, but they do not depend on $\varepsilon > 0$ and $\ell \geqslant 1$. Now, choosing the constant $\theta > 0$ so that $2\theta e^{\tilde{C}_0} \leqslant 1$ and applying [12, Lemma A.1] yield

$$\mathbb{E}[X(t\wedge\tau_\ell) + 2p\varepsilon Y(t\wedge\tau_\ell)] \leqslant R_5, \tag{4.5}$$

where $R_5 = R_5(\|\xi\|_{\mathbf{W}}, p, \varepsilon_0, T)$ does not depend on $\varepsilon \in (0,1]$, $h \in \mathscr{S}_M$ and $\ell \in \mathbb{N}$. Letting $\ell \to \infty$ now completes the proof of Claim 1. □

We now proceed to the proof of (3.17). We keep the notations in the proof of Claim 1 and we also set

$$w^\varepsilon = \mathbf{rot}\,\mathbf{u}^\varepsilon \text{ and } w = \mathbf{rot}\,\mathbf{u}.$$

With these notations in mind we observe that z^ε satisfies

$$\begin{aligned}
&dz^\varepsilon + \Big(-\varepsilon\Delta w^\varepsilon + \lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon\cdot\nabla z^\varepsilon + \delta[\mathbf{u}\cdot\nabla z^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla z] - \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\Delta w + \tilde{G}(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h\Big)\,dt\\
&\qquad = \varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\tilde{G}(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)dW,\\
&z^\varepsilon(0) = z_0^\delta,
\end{aligned}$$

where

$$\tilde{G} = \mathbf{rot}\,G \text{ and } z_0^\delta := (1-\delta)\,\mathbf{rot}(\xi + \alpha\mathrm{A}\xi).$$

Now, let $\varrho \in C^\infty(\mathbb{T}^\in)$ be an even, smooth function such that its support is compact and lies within a ball of $\mathbb{T}^2$, and the ball lifts homeomorphically to the universal covering $\mathbb{R}^2$. We also assume that $\int_{\mathbb{T}^2}\varrho(x)dx = 1$. For each $k \in \mathbb{N}$ we set $\varrho_k(\cdot) = k^2\varrho(k\cdot)$ and define the convolution operator J_k by $J_k f = \varrho_k * f$. We refer to Appendix C for several important properties of J_k. For the sake of simplicity we set $u_k = J_k u$ for any distribution u. The process z_k^ε satisfies

$$\begin{aligned}
&dz_k^\varepsilon + \Big(\varepsilon\alpha^{-1}z_k^\varepsilon + \lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon\cdot\nabla z_k^\varepsilon + \delta(\mathbf{u}\cdot\nabla z_k^\varepsilon + J_k(\mathbf{u}^\varepsilon\cdot\nabla z)) + R_k^{\varepsilon,\delta} - \varepsilon\alpha^{-1}w_k^\varepsilon\Big)\,dt\\
&\qquad = \Big(\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\Delta w_k + J_k\tilde{G}(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h\Big)\,dt + \varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)J_k\tilde{G}(\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)dW,\\
&z_k^\varepsilon(0) = (1-\delta)\,\mathbf{rot}(\xi_k + \alpha\mathrm{A}\xi_k),
\end{aligned}$$

where

$$R_k^{\varepsilon,\delta} := \lambda_\delta(\varepsilon)[J_k(\mathbf{u}^\varepsilon\cdot\nabla z^\varepsilon) - \mathbf{u}^\varepsilon\cdot\nabla z_k^\varepsilon] + \delta[J_k(\mathbf{u}\cdot\nabla z^\varepsilon) - \mathbf{u}\cdot\nabla z_k^\varepsilon].$$

We now apply Itô's formula to the function $x\mapsto x^2$ and z_k^ε to obtain

$$\begin{aligned} d|z_k^\varepsilon|^2 + 2[\varepsilon\alpha^{-1}|z_k^\varepsilon|^2 + \left(R_k^{\varepsilon,\delta} + \delta J_k(\mathbf{u}^\varepsilon\cdot\nabla z), z_k^\varepsilon\right) - 2\left(\varepsilon\alpha^{-1}w_k^\varepsilon + \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\Delta w_k, z_k^\varepsilon\right)]dt \\ = [2\left(J_k\tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h, z_k^\varepsilon\right) + \varepsilon\lambda_\delta^{-2}(\varepsilon)\|J_k\tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\|^2_{\mathscr{L}_Q(\mathscr{H}_0,\mathbf{L}^2)}]dt \\ +2\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)(z_k^\varepsilon, J_k\tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)dW), \end{aligned}$$

where we have used (2.10) to justify that

$$(\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon\cdot\nabla z_k^\varepsilon + \delta\mathbf{u}\cdot\nabla z_k^\varepsilon, z_k^\varepsilon) = 0.$$

Thanks to Lemma C.1, Propositions C.1 and C.3 we can argue as in [52, Proof of Theorem 2.9(b), page 60] and prove that

$$\begin{aligned} d|z^\varepsilon|^2 + 2\left[\varepsilon\alpha^{-1}|z^\varepsilon|^2 + \left(\delta\mathbf{u}^\varepsilon\cdot\nabla z - \left[\varepsilon\alpha^{-1}w^\varepsilon + \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\Delta w + \tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h\right], z^\varepsilon\right)\right]dt \\ = \varepsilon\lambda_\delta^{-2}(\varepsilon)\|\tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\|^2_{\mathscr{L}_Q(\mathscr{H}_0,\mathbf{L}^2)} + 2\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)(z^\varepsilon, \tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)dW). \end{aligned}$$

Now, applying Itô's formula to the map $x\mapsto x^p$ and $|z^\varepsilon|^2$ yields

$$\begin{aligned} d|z^\varepsilon|^{2p} + 2p[\varepsilon\alpha^{-1}|z^\varepsilon|^{2p} - f(p)\varepsilon\lambda_\delta^{-2}(\varepsilon)\|\tilde G^*(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon\|^2_{\mathscr{H}_0}|z^\varepsilon|^{2p-4}]dt \\ = p|z^\varepsilon|^{2p-2}\Big[2\left(\varepsilon\alpha^{-1}w^\varepsilon + \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\Delta w + \tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon), z^\varepsilon\right) \\ +\varepsilon\lambda_\delta^{-2}(\varepsilon)\|\tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)\|^2_{\mathscr{L}_2(\mathscr{H}_0,\mathbf{L}^2)} - 2\delta\left(\mathbf{u}^\varepsilon\cdot\nabla z, z^\varepsilon\right)\Big] \\ +2p\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)|z^\varepsilon|^{2p-2}(z^\varepsilon, \tilde G(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)dW), \end{aligned} \tag{4.8}$$

where $\tilde G^*(\cdot)$ is the adjoint of $\tilde G(\cdot)$ and $f(p) := 2p(p-1)$. Using the Cauchy-Schwarz and Young inequalities, the norms equivalence (2.3), the Sobolev embedding $\mathbf{V}\subset\mathbf{W}$ and (3.6) we show that there exists a constant $k_0>0$ such that for any $\varepsilon>0$ and $\ell\geqslant 1$

$$\begin{aligned} 2p\int_0^{t\wedge\tau_\ell}\left(\varepsilon\alpha^{-1}(w^\varepsilon(r), z^\varepsilon(r)) + \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)(\Delta w(r), z^\varepsilon(r))|z^\varepsilon(r)|^{2p-2}\right)dr \leqslant 2pk_0\Bigg(\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)R_0 \\ +[\varepsilon\alpha^{-1} + \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)]\int_0^{t\wedge\tau_\ell}|z^\varepsilon(r)|^{2p}dr\Bigg) \end{aligned}$$

Using (2.12) the Sobolev embedding $\mathbf{W} \subset \mathbf{L}^\infty$ and (3.7) we prove that there exists a constant $k_1 > 0$ such that

$$2p\delta \int_0^{t\wedge\tau_\ell} (\mathbf{u}^\varepsilon \cdot \nabla z, z^\varepsilon)(r)|z^\varepsilon(r)|^{2p-2}dr \leqslant 2pk_1 \int_0^{t\wedge\tau_\ell} |\nabla z(r)||z^\varepsilon(r)|^{2p-1}\|\mathbf{u}^\varepsilon(r)\|_{\mathbf{L}^\infty}dr$$
$$\leqslant 2p\delta k_1 R_1 \int_0^{t\wedge\tau_\ell} |z^\varepsilon(r)|^{2p}dr.$$

From Remark 3.1(a), Young's inequality and (3.6) we derive that

$$p\varepsilon\lambda_\delta^{-2}(\varepsilon)\int_0^{t\wedge\tau_\ell} \|\tilde{G}(\delta\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)(r)\|^2_{\mathscr{L}_Q(\mathscr{H}_0,\mathbf{L}^2)}|z^\varepsilon(r)|^{2p-2}dr$$
$$\leqslant k_3 p\varepsilon\lambda_\delta^{-2}(\varepsilon)\biggl[(1+\lambda_\delta^2(\varepsilon))\int_0^{t\wedge\tau_\ell}|z^\varepsilon(r)|^{2p}dr + \delta^{2p}T(R_0^{2p}+1)\biggr],$$

for some constant $k_3 > 0$ which does not depends on $\varepsilon \in (0,1]$, $\delta \in \{0,1\}$ and $\ell \geqslant 1$. In an almost similar way we prove that there exists a constant $k_4 > 0$ such that for any $\varepsilon \in (0,1]$ and $\ell \geqslant 1$

$$2p(p-1)\varepsilon\lambda_\delta^{-2}(\varepsilon)\int_0^{t\wedge\tau_\ell} \|\tilde{G}^*(\delta\mathbf{u}(r)+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))z^\varepsilon(r)\|^2_{\mathscr{H}_0}|z^\varepsilon(r)|^{2p-2}dr$$
$$\leqslant 2k_4 p(p-1)\varepsilon\lambda_\delta^{-2}(\varepsilon)\biggl[(1+\lambda_\delta^2(\varepsilon))\int_0^{t\wedge\tau_\ell}|z^\varepsilon(r)|^{2p}dr + \delta^{2p}T(R_0^{2p}+1)\biggr].$$

Finally, there exists a constant $k_5 > 0$ such that for any $\varepsilon \in (0,1]$ and $\ell \geqslant 1$

$$\int_0^{t\wedge\tau_\ell} |z^\varepsilon(r)|^{2p-2}(\tilde{G}(\mathbf{u}+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon)h(r), z^\varepsilon(r))dr \leqslant k_5 M^{\frac12}T^{\frac12}\left(1+\delta^{2p}R_0^{2p}\right)$$
$$+c_5(1+\lambda_\delta(\varepsilon))\int_0^{t\wedge\tau_\ell}|z^\varepsilon(r)|^{2p}\|h(r)\|_{\mathscr{H}_0}dr.$$

From these inequalities and (4.8) we infer that

$$|z^\varepsilon(t\wedge\tau_\ell)|^{2p} + 2p\varepsilon\alpha^{-1}\int_0^{t\wedge\tau_\ell}|z^\varepsilon(r)|^{2p}dr \leqslant \tilde{\Phi}^{\varepsilon,\delta} + \int_0^{t\wedge\tau_\ell}\tilde{\Psi}^{\varepsilon,\delta}(r)|z^\varepsilon(r)|^{2p}dr + \tilde{\mathscr{M}}^{\varepsilon,\delta}(t\wedge\tau_\ell),$$

with

$$\tilde{\mathscr{M}}^{\varepsilon,\delta}(t) = 2p\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\sup_{r\in[0,t]}\left|\int_0^s |z^\varepsilon(r)|^{2p-2}(z^\varepsilon(r), \tilde{G}(\mathbf{u}(r)+\lambda_\delta(\varepsilon)\mathbf{u}^\varepsilon(r))dW(r))\right|$$
$$+ 2p\tilde{c}_0(\varepsilon\alpha^{-1}+\delta\varepsilon\lambda_\delta^{-1}(\varepsilon))\int_0^t |w^\varepsilon(r)|^{2p}dr,$$

$$\Psi^{\varepsilon,\delta}(r) = \tilde{c}_1(\varepsilon\alpha^{-1}+\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)) + \tilde{c}_2\delta|\nabla z(r)| + +\tilde{c}_3\varepsilon\lambda_\delta^{-2}(\varepsilon)(1+\lambda_\delta^2(\varepsilon)) + \tilde{c}_4(1+\lambda_\delta(\varepsilon))\|h(r)\|_{\mathscr{H}_0},$$

$$\tilde{\Phi}^{\varepsilon,\delta} = |z_0^\delta|^{2p} + \tilde{c}_5\delta^{2p}(\varepsilon\lambda_\delta^{-2}(\varepsilon)T + T^{\frac12}),$$

where $\tilde{c}_i$, $i = 0,\dots,5$, are positive constants which may depend on $\|\xi\|_{D(\mathrm{A}^{\frac{3+\delta}{2}})}$, T, α and p, but not on ε and $\ell \geqslant 1$.

Using the same argument as in the proof of (4.4) and (3.6) we infer that for any $\theta > 0$

$$\tilde{\mathscr{M}}^{\varepsilon,\delta}(t\wedge\tau_\ell) \leqslant 9\theta^{-1}\tilde{c}_7\varepsilon\lambda_\delta^{-2}(\varepsilon)\left(T + (1+\lambda_\delta^2(\varepsilon))\mathbb{E}\int_0^{t\wedge\tau_\ell}|z^\varepsilon(r)|^{2p}dr + R_0^{2p}\right)$$
$$+\theta\mathbb{E}\sup_{r\in[0,t\wedge\tau_\ell]}|z^\varepsilon(r)|^{2p} + R_5,$$

where R_5 is the constant from (4.5) and $\tilde{c}_7$ is a positive constant which does not depend on ε and ℓ. Thus, setting

$$\tilde{X}^\varepsilon(t) := \sup_{0\leqslant r\leqslant t}|z^\varepsilon(r)|^{2p} \text{ and } \tilde{Y}^\varepsilon(t) := \int_0^t |z^\varepsilon(r)|^{2p}dr,$$

we see that for $t\in[0,T]$ we have with probability 1

$$\tilde{X}^\varepsilon(t\wedge\tau_\ell) + 2p\varepsilon\alpha^{-1}\tilde{Y}^\varepsilon(t\wedge\tau_\ell) \leqslant \tilde{\Phi}^{\varepsilon,\delta} + \int_0^{t\wedge\tau_\ell}\tilde{\Psi}^{\varepsilon,\delta}(r)\tilde{X}^\varepsilon(r)dr + \tilde{\mathscr{M}}^{\varepsilon,\delta}(t\wedge\tau_\ell),$$
$$\mathbb{E}\tilde{\mathscr{M}}^{\varepsilon,\delta}(t\wedge\tau_\ell) \leqslant \theta\mathbb{E}\tilde{X}^\varepsilon(t\wedge\tau_\ell) + \theta^{-1}\tilde{K}_0^{\varepsilon,\delta}\mathbb{E}\int_0^{t\wedge\tau_\ell}\tilde{X}^\varepsilon(r)dr + \tilde{K}_1^{\varepsilon,\delta},$$

where $\theta > 0$ is an arbitrary constant and

$$\tilde{K}_0^{\varepsilon,\delta} = 9\tilde{c}_7\varepsilon\lambda_\delta^{-2}(\varepsilon)(1+\lambda_\delta^2(\varepsilon)),$$
$$\tilde{K}_1^{\varepsilon,\delta} = \theta^{-1}\tilde{c}_7\varepsilon\lambda_\delta^{-2}(\varepsilon)(T+R_0^{2p}) + R_5.$$

Observe that from Proposition 3.5 we can find a deterministic positive constant $\tilde{K}_2$ such that for any $\varepsilon\in(0,1]$

$$\int_0^T \tilde{\Psi}^{\varepsilon,\delta}(r)dr \leqslant \tilde{K}_2 \text{ with probability 1.}$$

We now argue as in the proof of (4.5) and infer that one can find a positive constant $\tilde{K}_3$, which may depend on p, T, α and $\|\xi\|_{D(\mathrm{A}^{\frac{3+\delta}{2}})}$, but not on ε such that

$$\mathbb{E}\left(\tilde{X}^\varepsilon(t) + 2p\varepsilon\alpha^{-1}\tilde{Y}^\varepsilon(t)\right) \leqslant \tilde{K}_3(1+(1-\delta)|\,\mathbf{rot}(\xi+\alpha\mathrm{A}\xi)|^{2p}). \tag{4.9}$$

This completes the proof of (3.17) and Proposition 3.6. □

5 Proof of Propositions 3.14 & 3.16

This section is devoted to the proof of the crucial Propositions 3.14 and 3.16.

5.1 Proof of Proposition 3.14

We will now give the proof of Proposition 3.14. For this purpose we fix $\delta \in \{0,1\}$, a constant $M>0$ and consider a sequence $(h_n)_{n\in\mathbb{N}} \subset S_M$ which weakly converges to $h \in S_M$. Let $\mathbf{u}_n^\delta$ (resp. $\mathbf{u}^\delta$) be the solution to (3.11) corresponding to h_n (resp. h). By Proposition 3.8 we have

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\mathbf{u}_n^\delta(t)\|_{\mathbf{W}} \leqslant R_0, \tag{5.1}$$

Using (2.9) and Assumption ($\mathbf{G}s$) we easily derive that there exists a constant $\tilde{R}_1>0$ such that for any $n \in \mathbb{N}$

$$|\partial_t \mathbf{v}_n^\delta(\cdot)| \leqslant \tilde{R}_1 \|\mathbf{u}_n^\delta\|_{\mathbf{W}} \left((1-\delta)\|\mathbf{u}_n^\delta\|_{\mathbf{W}} + \delta\|\mathbf{u}\|_{\mathbf{W}}\right) + \tilde{R}_1\|h_n\|_{\mathscr{H}_0}(1+\delta\|\mathbf{u}\|_\alpha + (1-\delta)\|\mathbf{u}_n^\delta\|_\alpha),$$

which along with (5.1) and (3.6) we infer that there exists a constant $R_6^\delta>0$ such that

$$\sup_{n\in\mathbb{N}} \|\partial_t \mathbf{u}_n^\delta(\cdot)\|_{\mathrm{L}^2(0,T;D(\mathrm{A}))} \leqslant R_6^\delta. \tag{5.2}$$

These two estimates, Banach-Alaoglu's theorem and the celebrated Aubin-Lions-Simon compactness theorem, see [56, Corollary 4], implies that one can extract a subsequence, which is not relabeled, from $(\mathbf{u}_n^\delta)_{n\in\mathbb{N}}$ such that

$$\mathbf{u}_n^\delta \rightharpoonup \bar{\mathbf{u}}^\delta \text{ weak-}* \text{ in } \mathrm{L}^\infty(0,T;\mathbf{W}), \tag{5.3}$$

$$\mathbf{u}_n^\delta \to \bar{\mathbf{u}}^\delta \text{ strong in } C([0,T];D(\mathrm{A}^{1-\frac{\theta}{2}})), \tag{5.4}$$

for any $\theta \in (0,1]$. Arguing as in [14] or [15] we can show that $\bar{\mathbf{u}}^\delta$ solves (3.11) and by uniqueness of solution we infer that $\bar{\mathbf{u}}^\delta = \mathbf{u}^\delta$. The uniqueness of solution also implies that the whole sequence strongly converges to $\mathbf{u}^\delta$ in $C([0,T];D(\mathrm{A}^{1-\frac{\theta}{2}}))$, $\theta\in(0,1]$. Note also that from (5.1) and (5.3) we infer that the function $z_n^\delta := \mathbf{rot}(y_n^\delta + \alpha \mathrm{A} y_n^\delta)$, where $y_n^\delta := \mathbf{u}_n^\delta - \mathbf{u}^\delta$, satisfies

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} |z_n^\delta(t)| \leqslant \tilde{R}_3, \tag{5.5}$$

for a constant $\tilde{R}_3>0$ independent of n.

Now we will divide the proof into two parts: the case $s=0$ and $s=1$.

Case $s = 0$**.** We infer from Sobolev interpolation inequality that

$$|\mathrm{A}y_n^\delta| = \|\mathrm{A}^{\frac{1}{2}}y_n^\delta\| \leqslant \tilde{R}_5 |\mathrm{A}^{\frac{1}{2}}y_n^\delta|^{\frac{1}{2}} |\mathrm{A}^{\frac{3}{2}}y_n^\delta|^{\frac{1}{2}},$$

which along with the estimate (5.5) and the convergence (5.4) easily implies

$$y_n^\delta := \mathbf{u}_n^\delta - \mathbf{u}^\delta \to 0 \text{ strongly in } C([0,T]; D(\mathrm{A})). \tag{5.6}$$

This completes the proof of Proposition 3.14 for the case $s = 0$.

Case $s = 1$**.** To prove the Proposition for the case $s = 1$ we shall show that

$$z_n^\delta \to 0 \text{ strongly in } C([0,T]; \mathbf{W}).$$

To this aim, we first observe that z_n^δ solves

$$\begin{aligned} \partial_t z_n^\delta + (1-\delta)[\mathbf{u}_n^\delta \cdot \nabla z_n^\delta + y_n^\delta \cdot \nabla z^\delta] + \delta[\mathbf{u} \cdot \nabla z_n^\delta + y_n^\delta \cdot \nabla z] \\ = \tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}_n^\delta)h_n - \tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}^\delta)h, \end{aligned} \tag{5.7}$$

where we have set

$$z^\delta = \mathbf{rot}(\mathbf{u}^\delta + \alpha \mathrm{A}\mathbf{u}^\delta), \;\; z = \mathbf{rot}(\mathbf{u} + \alpha \mathrm{A}\mathbf{u}) \text{ and } \tilde{G} = \mathbf{rot} \circ G.$$

We have the following identity

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|z_n^\delta|^2 = (\tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}_n^\delta)h_n - \tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}^\delta)h, z_n^\delta) \\ +(\delta - 1)(\mathbf{y}_n^\delta \cdot \nabla z^\delta, z_n^\delta) - \delta(\mathbf{y}_n^\delta \cdot \nabla z, z_n^\delta). \end{aligned} \tag{5.8}$$

The proof of this identity uses the same regularization argument as in the proof of (3.17) and is omitted. We now invoke (2.12) and (2.4) to infer that there exists a constant $R_{11} > 0$ such that for any $n \in \mathbb{N}$

$$\begin{aligned} |z_n^\delta(t)|^2 \leqslant R_7 \int_0^t \big((1-\delta)|\nabla z^\delta(r)| + \delta |\nabla z(r)|\big) |z_n^\delta(r)|^2 dr + |\mathscr{T}_n^\delta(h_n - h)(t)| \\ + \int_0^t |[\tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}_n^\delta) - \tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}^\delta)]h_n(r)||z_n^\delta(r)|dr, \end{aligned} \tag{5.9}$$

where for each $n \in \mathbb{N}$ the linear map $\mathscr{T}_n^\delta : \mathrm{L}^2(0,T;\mathscr{H}_0) \to C([0,T];\mathbb{R})$ is defined by

$$\mathscr{T}_n^\delta \psi(\cdot) = \int_0^{\cdot} (\tilde{G}(\delta \mathbf{u} + (1-\delta)\mathbf{u}^\delta)\psi(r), z_n^\delta(r))dr, \quad \psi \in \mathrm{L}^2(0,T;\mathscr{H}_0). \tag{5.10}$$

From Assumption ($\mathbf{G}s$) and Remark 3.1(a) and the continuous embedding $\mathbf{V} \subset \mathbf{W}$ we infer that there exists a constant $R_{12} > 0$ such that

$$\int_0^t |[\tilde{G}(\delta\mathbf{u} + (1-\delta)\mathbf{u}_n^\delta) - \tilde{G}(\delta\mathbf{u} + (1-\delta)\mathbf{u}^\delta)]h_n(r)||z_n^\delta(r)|dr \leqslant R_8(1-\delta)\int_0^t |z_n^\delta(r)|^2 \|h_n(r)\|_{\mathscr{H}_0} dr. \tag{5.11}$$

Plugging this inequality in (5.9), using Gronwall's and the assumption $h_n \in S_M$ yield

$$|z_n^\delta(t)|^2 \leqslant |\mathscr{T}_n^\delta(h_n - h)(t)| \exp\left(R_7 \int_0^t \left((1-\delta)|\nabla z^\delta(r)| + \delta|\nabla z(r)|\right) dr + R_8 T^{\frac{1}{2}} M^{\frac{1}{2}}\right). \tag{5.12}$$

Now we claim that as $n \to \infty$

$$\sup_{t\in[0,T]} |\mathscr{T}_n^\delta(h_n - h)(t)| \to 0, \tag{5.13}$$

from which altogether with (3.7) completes the proof Proposition 3.14 for the case $s = 1$.

In order to complete the proof of the whole proposition, it remains to prove (5.13). To this end, we notice that thanks to the Assumption ($\mathbf{G}s$), Remark 3.1(a) and the estimate (5.5), the family $(\tilde{G}(\delta\mathbf{u} + (1-\delta)\mathbf{u}^\delta)\psi(\cdot), z_n^\delta(\cdot))$, $\psi \in S_M$, is uniformly bounded in $\mathrm{L}^2(0,T;\mathscr{H}_0)$. Thus, for each $n \in \mathbb{N}$ the linear map $\mathscr{T}_n^\delta$ is bounded and compact. Next, owing to the estimate (5.5) we can and will assume that there exists $z_\infty^\delta \in \mathrm{L}^\infty(0,T,\mathbf{L}^2)$ such that as $n \to \infty$

$$z_n^\delta \to z_\infty^\delta \text{ weak-}* \text{ in } \mathrm{L}^\infty(0,T;\mathrm{L}^2),$$

which implies that as $n \to \infty$

$$|||\, \mathscr{T}_n^\delta - \mathscr{T}_\infty^\delta ||| \to 0, \tag{5.14}$$

where $|||\cdot|||$ denotes the operator norm. Thus, the compactness of $\mathscr{T}_n$, $n \in \mathbb{N}$, and the weak convergence of h_n to h implies

$$\sup_{t\in[0,T]} |\mathscr{T}_n^\delta(h_n - h)(t)| \leqslant |||\mathscr{T}_n^\delta - \mathscr{T}_\infty^\delta\, |||\; \|h_n - h\|_{\mathrm{L}^2(0,T;\mathscr{H}_0)} + \|\mathscr{T}_\infty^\delta(h_n - h)\|_{C([0,T];\mathbb{R})} \to 0, \tag{5.15}$$

as $n \to \infty$. This completes the proof of Proposition 3.14.

5.2 Proof of Proposition 3.16

This section is devoted to the proof of Proposition 3.16. For the time being let us assume that the following two lemmata hold.

Lemma 5.1. *If all assumptions of Proposition 3.16 are satisfied, then for* $s \in \{0,1\}$

$$\lim_{n\to\infty} \mathbb{P}\biggl(\sup_{0\leqslant r\leqslant T} \|\mathbf{u}^{\varepsilon_n,\delta}_{h_n} - \mathbf{u}^{\delta}_{h_n}(r)\|_{D(\mathrm{A}^{1+\frac{s}{2}})} \geqslant \kappa\biggr) = 0, \tag{5.16}$$

for any $\kappa > 0$.

Lemma 5.2. *Under the assumptions of Proposition 3.16, the process* $\Gamma^{0,\delta}_{\xi}\left(\int_0^\cdot h_n(r)dr\right)$ *converges in distribution to* $\Gamma^{0,\delta}_{\xi}\left(\int_0^\cdot h(r)dr\right)$ *as* $C([0,T];D(\mathrm{A}^{1+\frac{s}{2}}))$*-valued random variables, where* $s \in \{0,1\}$.

We now give the promised proof of Proposition 3.16.

Proof of Proposition 3.16. Proposition 3.16 readily follows from [20, Theorem 11.3.3], Lemmata 5.1 and 5.2. □

We now proceed to the proofs of Lemmata 5.1 and 5.2.

Proof of Lemma 5.1. To lighten notation we set

$$\mathbf{u}^{\varepsilon}_n := \mathbf{u}^{\varepsilon_n,\delta}_{h_n}, \quad \mathbf{y}^{\varepsilon}_n := \mathbf{u}^{\varepsilon}_n + \alpha \mathrm{A}\mathbf{u}^{\varepsilon}_n, \quad z^{\varepsilon}_n := \mathbf{rot}\,\mathbf{y}^{\varepsilon}_n,$$

$$\mathbf{u}_n := \mathbf{u}^{\delta}_n, \quad \mathbf{v}_n := \mathbf{u}_n + \alpha \mathrm{A}\mathbf{u}_n, \quad z_n := \mathbf{rot}\,\mathbf{y}_n,$$

$$\eta^{\varepsilon}_n := \mathbf{u}^{\varepsilon}_n - \mathbf{u}_n, \quad \boldsymbol{\varphi}^{\varepsilon}_n = \eta^{\varepsilon}_n + \alpha\mathrm{A}\eta^{\varepsilon}_n \quad \omega^{\varepsilon}_n := \mathbf{rot}\,\boldsymbol{\varphi}^{\varepsilon}_n.$$

We also recall that $\mathbf{u}$ is the solution to (1.5), $\mathbf{v} = \mathbf{u} + \alpha\mathrm{A}\mathbf{u}$ and $z = \mathbf{rot}\,\mathbf{v}$.

We will first establish the lemma for $s = 0$. More precisely, we will show that

$$\lim_{n\to\infty} \mathbb{P}\biggl(\sup_{0\leqslant r\leqslant T} |\boldsymbol{\varphi}^{\varepsilon}_n(r)| \geqslant \kappa\biggr) = 0, \tag{5.17}$$

for any $\kappa > 0$. For this purpose, we first observe that $\boldsymbol{\varphi}^{\varepsilon}_n$ satisfies

$$d\boldsymbol{\varphi}^{\varepsilon}_n + (\varepsilon\mathrm{A}\mathbf{u}^{\varepsilon}_n + \lambda_\delta(\varepsilon)\mathcal{C}(\mathbf{rot}\,\boldsymbol{\varphi}^{\varepsilon}_n, \mathbf{u}^{\varepsilon}_n) + [\varrho_\delta(\varepsilon)]\mathcal{C}(\mathbf{rot}\,\mathbf{v}_n, \eta^{\varepsilon}_n) + \delta(\mathcal{C}(\mathbf{rot}\,\boldsymbol{\varphi}^{\varepsilon}_n, \mathbf{u}) + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \eta^{\varepsilon}_n))\,dt$$
$$= \left(\delta\varepsilon\lambda^{-1}_\delta(\varepsilon)\mathrm{A}\mathbf{u} + [G(Y^{\varepsilon,\delta}_n) - G(Y^{\delta}_n)]h_n\right)dt + \varepsilon^{\frac{1}{2}}\lambda^{-1}_\delta(\varepsilon)G(Y^{\varepsilon,\delta}_n)dW,$$

where

$$\varrho_\delta(\varepsilon) = \lambda_\delta(\varepsilon) - (1-\delta), \quad Y_n^{\varepsilon,\delta} := \delta \mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}_n^\varepsilon, \quad \text{and } Y_n^\delta := \delta\mathbf{u} + (1-\delta)\mathbf{u}_n.$$

Applying Itô's formula to $|\boldsymbol{\varphi}_n^\varepsilon|^2$ and $\boldsymbol{\varphi}_n^\varepsilon$, and using Cauchy-Schwarz's inequality and Assumption $(\mathbf{G}s)$ we infer that there exists a constant $R_9 > 0$

$$\begin{aligned} &d|\boldsymbol{\varphi}_n^\varepsilon|^2 - 2|(\varepsilon \mathrm{A}\mathbf{u}_n^\varepsilon + \lambda_\delta(\varepsilon)\mathcal{C}(\mathbf{rot}\,\boldsymbol{\varphi}_n^\varepsilon, \mathbf{u}_n^\varepsilon) + \delta(\mathcal{C}(\mathbf{rot}\,\boldsymbol{\varphi}_n^\varepsilon, \mathbf{u}) + \mathcal{C}(\mathbf{rot}\,\mathbf{v}, \eta_n^\varepsilon)), \boldsymbol{\varphi}_n^\varepsilon)|dt \\ &\leqslant 2|\boldsymbol{\varphi}_n^\varepsilon|[[\varrho_\delta(\varepsilon)]|\mathcal{C}(\mathbf{rot}\,\mathbf{v}_n, \eta_n^\varepsilon)| + \delta\varepsilon\lambda_\delta^{-1}(\varepsilon)|\mathrm{A}\mathbf{u}| + R_9\|\lambda_\delta(\varepsilon)\mathbf{u}_n^\varepsilon - (1-\delta)\mathbf{u}_n\|_\alpha \|h_n\|_{\mathscr{H}_0}]dt \\ &\quad + R_9\varepsilon\lambda_\delta^{-2}(\varepsilon)\|\delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}_n^\varepsilon\|_\alpha^2 + 2\varepsilon^{\frac{1}{2}}\lambda_\delta^{-1}(\varepsilon)(\boldsymbol{\varphi}_n^\varepsilon, G(Y_n^{\varepsilon,\delta})dW). \end{aligned} \tag{5.18}$$

Now, we infer from (2.13), the properties of $b(\cdot,\cdot,\cdot)$ (mainly (2.10), (2.11) and (2.12)), Hölder's inequality and the Sobolev embedding $\mathbf{L}^\infty \subset D(\mathrm{A})$ that there exists a constant $R_{10} > 0$

$$\begin{aligned} &|(\mathcal{C}(\mathbf{rot}\,\boldsymbol{\varphi}_n^\varepsilon, \mathbf{u}_n^\varepsilon), \boldsymbol{\varphi}_n^\varepsilon)| = |b(\boldsymbol{\varphi}_n^\varepsilon, \mathbf{u}_n^\varepsilon, \boldsymbol{\varphi}_n^\varepsilon)| \leqslant R_{10}|\boldsymbol{\varphi}_n^\varepsilon|^2|z_n^\varepsilon|, \\ &|(\mathcal{C}(\mathbf{rot}\,\mathbf{v}_n, \eta_n^\varepsilon), \boldsymbol{\varphi}_n^\varepsilon)| \leqslant R_{10}|z_n||\boldsymbol{\varphi}_n^\varepsilon|^2, \\ &|(\mathcal{C}(\mathbf{rot}\,\boldsymbol{\varphi}_n^\varepsilon, \mathbf{u}), \boldsymbol{\varphi}_n^\varepsilon)| \leqslant R_{10}|\boldsymbol{\varphi}_n^\varepsilon|^2|z|, \end{aligned}$$

and

$$|(\mathcal{C}(\mathbf{rot}\,\mathbf{v}, \eta_n^\varepsilon), \boldsymbol{\varphi}_n^\varepsilon)| \leqslant R_{10}|z||\boldsymbol{\varphi}_n^\varepsilon|^2.$$

Collecting these inequalities together, plugging them in (5.18) and using Young's inequality yield

$$\begin{aligned} d|\boldsymbol{\varphi}_n^\varepsilon|^2 &\leqslant 2\varepsilon^{\frac{1}{2}}\lambda_\delta^{-1}(\varepsilon)(\boldsymbol{\varphi}_n^\varepsilon, G(Y_n^{\varepsilon,\delta})dW) + \varepsilon^2|z_n|^2 + \delta^2\varepsilon^2\lambda_\delta^{-2}(\varepsilon)|z|^2 + |\varrho_\delta(\varepsilon)|\|\mathbf{u}_n\|_\alpha^2\|h_n\|_{\mathscr{H}_0}^2 \\ &\quad + R_{10}|\boldsymbol{\varphi}_n^\varepsilon|^2\big[1 + 2\lambda_\delta(\varepsilon)|z_n^\varepsilon| + \varrho_\delta(\varepsilon)|z_n| + \delta|z| + \lambda_\delta(\varepsilon)\|h_n\|_{\mathscr{H}_0}\big], \end{aligned}$$

which along with Proposition 3.5 implies that

$$|\boldsymbol{\varphi}_n^\varepsilon(t\wedge\tau_n)|^2 \leqslant T\varSigma_N^{\varepsilon,\delta} + \int_0^{t\wedge\tau_N} \varTheta_N^{\varepsilon,\delta}(r)|\boldsymbol{\varphi}_n^\varepsilon(r)|^2 dr + 2\varepsilon\frac{1}{2}\lambda_\delta^{-1}(\varepsilon)\tilde{\mathscr{M}}_n^\varepsilon(t\wedge\tau_n),$$

where

$$\varSigma_N^{\varepsilon,\delta} := \varepsilon^2 N + \delta^2\varepsilon^2\lambda_\delta^{-2}(\varepsilon)R_0^2 + \varrho_\delta(\varepsilon)NR_{10}, \tag{5.19}$$

$$\varTheta_N^{\varepsilon,\delta}(\cdot) = N\lambda_\delta(\varepsilon) + \varrho(\varepsilon) + 2\delta R_0 + \lambda_\delta(\varepsilon)\|h_n(\cdot)\|_{\mathscr{H}_0}, \tag{5.20}$$

$$\tilde{\mathscr{M}}_n^\varepsilon(t) := \int_0^t (\boldsymbol{\varphi}_n^\varepsilon(r), G(Y_n^{\varepsilon,\delta})(r)dW(r)), \tag{5.21}$$

and for each number $N>0$ the family of stopping times $\tau_n:=\tau_{n,N}$ is defined by

$$\tau_{n,N}=\inf\{t\geqslant 0; |z_n(t)|>N\}\wedge\{t\geqslant 0; |z^\varepsilon_n(t)|>N\}\wedge T. \tag{5.22}$$

Now, from an application of BDG's inequality and Young's inequality we infer that for any constant $\tilde\beta>0$ there exists a constant $R_{11}>0$ such that for any $n\geqslant 1$ and $N\geqslant 1$

$$\begin{aligned}
&2\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)\mathbb{E}\sup_{r\in[0,t\wedge\tau_n]}|\tilde{\mathscr{M}}^\varepsilon_n(r)|\\
&\quad\leqslant 2\varepsilon^{\frac12}\lambda_\delta^{-1}(\varepsilon)R_{11}\mathbb{E}\left(\sup_{r\in[0,t]}|\boldsymbol{\varphi}^\varepsilon_n(r\wedge\tau_n)|^2\int_0^{t\wedge\tau_n}\|G(Y^{\varepsilon,\delta}_n)(r)\|^2_{\mathscr{L}_2(\mathscr{H}_0,\mathbf{V})}dr\right)^{\frac12}\\
&\quad\leqslant \varepsilon\lambda_\delta^{-2}(\varepsilon)R_{11}\int_0^{t\wedge\tau_n}(1+\delta^2\|\mathbf{u}(r)\|^2_\alpha+\lambda^2_\delta(\varepsilon)\|\mathbf{u}_n(r)\|^2_\alpha)dr\\
&\qquad\quad+\tilde\beta\mathbb{E}\sup_{r\in[0,t]}|\boldsymbol{\varphi}^\varepsilon_n(r\wedge\tau_n)|^2+\varepsilon R_{11}\int_0^{t\wedge\tau_n}|\boldsymbol{\varphi}^\varepsilon_n(r)|^2dr\\
&\quad\leqslant\tilde\beta\mathbb{E}\sup_{r\in[0,t]}|\boldsymbol{\varphi}^\varepsilon_n(r\wedge\tau_n)|^2+\varepsilon\lambda_\delta^{-2}(\varepsilon)R_{11}T(1+\delta^2R_0^2+\lambda^2_\delta(\varepsilon)N)+\varepsilon R_{11}\int_0^{t\wedge\tau_n}|\boldsymbol{\varphi}^\varepsilon_n(r)|^2dr.
\end{aligned} \tag{5.23}$$

Notice that since $(h_n)_{n\in\mathbb{N}}\subset\mathscr{S}_M$ and by the definitions of $\varrho_\delta(\varepsilon)$ and $\lambda_\delta(\varepsilon)$, there exists a constant $R_{12}>1$ such that

$$\sup_{n\in\mathbb{N}}e^{\int_0^T\Theta^{\varepsilon,\delta}_N(r)dr}\leqslant R_{12}.$$

Thus, choosing $\tilde\beta$ so that $2\tilde\beta R_{12}<1$ and applying the version of Gronwall's lemma given in [12, Lemma A.1] we obtain

$$\mathbb{E}\sup_{r\in[0,t]}|\boldsymbol{\varphi}^\varepsilon_n(r\wedge\tau_n)|^2\leqslant\varSigma^{\varepsilon,\delta}_N R_{12}. \tag{5.24}$$

Now, since $\varrho_\delta(\varepsilon)\to 0$ and $\varepsilon\lambda_\delta^{-\ell}(\varepsilon)\to 0$, $\ell\in\{1,2\}$, as $\varepsilon\to 0$ we infer that

$$\mathbb{E}\sup_{r\in[0,t]}|\boldsymbol{\varphi}^\varepsilon_n(r\wedge\tau_n)|^2\to 0 \text{ as } n\to\infty. \tag{5.25}$$

Next, let $\gamma>0$ and $\kappa>0$ be arbitrary numbers. It is not difficult to check that

$$\begin{aligned}
\mathbb{P}(\sup_{r\in[0,T]}|\boldsymbol{\varphi}^\varepsilon_n(r)|^2\geqslant\kappa)\leqslant&\mathbb{P}(\sup_{r\in[0,T]}|\boldsymbol{\varphi}^\varepsilon_n(r)|^2,\tau_n=T)+\mathbb{P}(\sup_{r\in[0,T]}|z_n(r)|\geqslant N)\\
&+\mathbb{P}(\sup_{r\in[0,T]}|z^\varepsilon_n(r)|\geqslant N)\\
&\leqslant\frac1\kappa\mathbb{E}\sup_{r\in[0,t]}|\boldsymbol{\varphi}^\varepsilon_n(r)|^2+\frac1N\mathbb{E}\sup_{r\in[0,T]}(|z_n(r)|+|z^\varepsilon_n(r)|).
\end{aligned} \tag{5.26}$$

Owing to estimate (3.12) and (3.17) one can find $N > 0$ such that

$$\frac{1}{N}\mathbb{E}\sup_{r\in[0,T]}(|z_n(r)| + |z_n^\varepsilon(r)|) < \frac{\gamma}{2}.$$

Thus, thanks to (5.25) and (5.26) we infer that for all n large enough

$$\mathbb{P}(\sup_{r\in[0,T]}|\boldsymbol{\varphi}_n^\varepsilon(r)|^2 \geqslant \kappa) < \gamma,$$

which completes the proof of Lemma 5.1 for $s = 0$.

In order to prove the lemma for $s = 1$, we first recall that $\omega_n^\varepsilon := \mathbf{rot}\,\boldsymbol{\varphi}_n^\varepsilon$. It is not difficult to prove that ω_n^ε satisfies

$$d\omega_n^\varepsilon + [\lambda_\delta(\varepsilon)(\mathbf{u}_n^\varepsilon\cdot\nabla\omega_n^\varepsilon) + \varrho_\delta(\varepsilon)\eta_n^\varepsilon\cdot\nabla z_n]dt + \delta[\mathbf{u}\cdot\nabla\omega_n^\varepsilon + \eta_n^\varepsilon\cdot\nabla z]dt$$
$$= [\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\mathrm{A}\,\mathbf{rot}\,\mathbf{u} + \tilde{G}(Y_n^\delta)h_n - \tilde{G}(Y_n^{\varepsilon,\delta})h_n - \varepsilon\mathrm{A}\,\mathbf{rot}\,\mathbf{u}_n^\varepsilon]dt + \varepsilon^{\frac{1}{2}}\lambda_\delta^{-1}(\varepsilon)\tilde{G}(Y_n^{\varepsilon,\delta})dW$$

where

$$\varrho_\delta(\varepsilon) = \lambda_\delta(\varepsilon) - (1-\delta),\ \ Y_n^{\varepsilon,\delta} := \delta\mathbf{u} + \lambda_\delta(\varepsilon)\mathbf{u}_n^\varepsilon,\ \ Y_n^\delta := \delta\mathbf{u} + (1-\delta)\mathbf{u}_n.$$

Now, for each number $N > 0$ we define a family of stopping times $\tilde{\tau}_{n,N}$ by

$$\tilde{\tau}_{n,N} = \inf\{t \geqslant 0; |\nabla z_n(t)| > N\} \wedge T. \tag{5.27}$$

For the sake of simplicity we just write $\tilde{\tau}_n$ in place of $\tilde{\tau}_{n,N}$. Using a regularization technique as in the proof of Lemma and Itô's formula we can derive the following identity

$$\begin{aligned}|\omega_n^\varepsilon(t\wedge\tilde{\tau}_n)|^2 = -2&\int_0^{t\wedge\tilde{\tau}_n}([\varepsilon\mathrm{A}\,\mathbf{rot}\,\mathbf{u}_n^\varepsilon + \varrho_\delta(\varepsilon)\eta_n^\varepsilon\cdot\nabla z_n + \delta\eta_n^\varepsilon\cdot\nabla z](r), \omega_n^\varepsilon(r))\,dr\\ &+2\int_0^{t\wedge\tilde{\tau}_n}\Big(\delta\varepsilon\lambda_\delta^{-1}(\varepsilon)\mathrm{A}\,\mathbf{rot}\,\mathbf{u}(r) + [\tilde{G}(Y_n^{\varepsilon,\delta}) - \tilde{G}(Y_n^\delta)]h_n(r), \omega_n^\varepsilon(r)\Big)\,dr\\ +\varepsilon\lambda_\delta^{-2}(\varepsilon)&\int_0^{t\wedge\tilde{\tau}_n}\|\tilde{G}(Y_n^{\varepsilon,\delta})\|^2_{\mathscr{L}(\mathscr{H}_0,\mathbf{L}^2)}dr + 2\varepsilon^{\frac{1}{2}}\lambda_\delta^{-1}(\varepsilon)\int_0^{t\wedge\tilde{\tau}_n}(\omega_n^\varepsilon(r), \tilde{G}(Y_n^{\varepsilon,\delta})dW(r))\\ &= J_{1,n}(t) + J_{2,n}(t) + J_{3,n}(t) + \mathscr{M}_n^\varepsilon(t).\end{aligned} \tag{5.28}$$

From (2.12), the Sobolev embedding $\mathbf{W}\subset\mathbf{L}^\infty$ and (2.4) we infer that there exist constants $R_{13}, R_{14}, R_{15} > 0$ such that for any $n\in\mathbb{N}$

$$\begin{aligned}J_{1,n}(t) &\leqslant \int_0^{t\wedge\tilde{\tau}_n}|(\varepsilon\mathrm{A}\,\mathbf{rot}\,\mathbf{u}_n^\varepsilon(r), \omega_n^\varepsilon(r))|dr + R_{13}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon|^2\left(\varrho_\delta(\varepsilon)|\nabla z_n(r)| + \delta R_1\right)dr\\ &\leqslant \frac{\varepsilon^2}{2}R_{14}\int_0^{t\wedge\tilde{\tau}_n}|z_n^\varepsilon|^2dr + R_{15}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon(r)|^2\left(1 + \varrho_\delta(\varepsilon)|\nabla z_n(r)| + \delta R_1\right)dr,\end{aligned}$$

where we used Cauchy-Schwarz's and Young's inequalities to obtain the last line. From Assumption ($\mathbf{G}s$) and Remark 3.1(a), the Sobolev embedding $\mathbf{V} \subset \mathbf{W}$, Cauchy-Schwarz's and Young's inequalities we derive that there exist 2 constants $R_{16}, R_{17} > 0$ such that for any $n \in \mathbb{N}$

$$\begin{aligned}J_{2,n}(t) &\leqslant \delta^2\varepsilon^2\lambda_\delta^{-2}(\varepsilon)R_{16}\int_0^{t\wedge\tilde{\tau}_n}|z(r)|^2dr + R_{16}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon(r)|^2(1+\lambda_\delta(\varepsilon)\|h_n(r)\|_{\mathscr{H}_0})dr\\&\quad+R_{16}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon(r)|\varrho_\delta(\varepsilon)\|\mathbf{u}_n(r)\|_\alpha\|h_n(r)\|_{\mathscr{H}_0}dr\\&\leqslant \delta^2\varepsilon^2\lambda_\delta^{-2}(\varepsilon)R_{16}R_0T + R_{17}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon(r)|^2(1+\lambda_\delta(\varepsilon)\|h_n(r)\|_{\mathscr{H}_0})dr\\&\quad+\varrho_\delta^2(\varepsilon)R_{17}\int_0^{t\wedge\tilde{\tau}_n}|z_n(r)|^2\|h_n(r)\|^2_{\mathscr{H}_0}dr.\end{aligned}$$

For the term $J_{3,n}$ we have the following estimate which easily follows from Assumption ($\mathbf{G}s$) with $s=1$

$$J_{3,n}(t) \leqslant \varepsilon\lambda_\delta^{-2}(\varepsilon)R_{18}\int_0^{t\wedge\tilde{\tau}_n}(1+\delta^2\|\mathbf{u}(r)\|_\alpha^2+\lambda_\delta^2(\varepsilon)|z_n(r)|^2)dr + \varepsilon R_{18}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon(r)|^2dr.$$

Now, BDG's inequality and Young's inequality yield

$$\mathbb{E}\sup_{r\in[0,t]}|\mathscr{M}_n^\varepsilon(r)| \leqslant R_{19}\mathbb{E}\left(\sup_{r\in[0,t]}|\omega_n^\varepsilon(r\wedge\tilde{\tau}_n)|^2J_{3,n}(t)\right)^{\frac{1}{2}}$$

$$\leqslant \beta\mathbb{E}\sup_{r\in[0,t]}|\omega_n^\varepsilon(r\wedge\tilde{\tau}_n)|^2 + \varepsilon\lambda_\delta^{-2}(\varepsilon)R_{20}\int_0^{t\wedge\tilde{\tau}_n}(1+\delta^2\|\mathbf{u}(r)\|_\alpha^2+\lambda_\delta^2(\varepsilon)|z_n(r)|^2)dr \tag{5.29}$$

$$+\varepsilon R_{20}\int_0^{t\wedge\tilde{\tau}_n}|\omega_n^\varepsilon(r)|^2dr, \tag{5.30}$$

for any $\beta > 0$ and a certain constant $R_{20} > 0$ independent of n.

Collecting all these inequalities we obtain that $\mathbb{P}$-a.s.

$$\sup_{r\in[0,t]}|\omega_n^\varepsilon(r\wedge\tilde{\tau}_n)|^2 \leqslant \Phi_N^{\varepsilon,\delta} + \int_0^{t\wedge\tilde{\tau}_n}\Psi_N^{\varepsilon,\delta}(r)|\omega_n^\varepsilon(r)|^2dr + \sup_{r\in[0,t\wedge\tilde{\tau}_n]}|\mathscr{M}_n^\varepsilon(r)|, \tag{5.31}$$

with

$$\Phi_N^{\varepsilon,\delta} := R_{21}\Big(N^2([\varepsilon^2+\varepsilon]T+\varrho_\delta(\varepsilon)M)+\varepsilon\lambda_\delta^{-2}(\varepsilon)(\delta^2(\varepsilon+1)R_0^2+1)T\Big),$$

$$\Psi_N^{\varepsilon,\delta}(r) := 1+\varepsilon+\varrho_\delta(\varepsilon)N+\delta R_1+\lambda_\delta(\varepsilon)\|h_n(r)\|_{\mathscr{H}_0},$$

and the process $\mathscr{M}_n^\varepsilon$ satisfies the estimate (5.30). Notice that by the fact that $(h_n)_{n\in\mathbb{N}} \subset \mathscr{S}_M$ and by the definitions of $\varrho_\delta(\varepsilon)$ and $\lambda_\delta(\varepsilon)$, there exists a constant $R_{22} > 1$ such that

$$\sup_{n\in\mathbb{N}} e^{\int_0^T\Psi_N^{\varepsilon,\delta}(r)dr} \leqslant R_{22}.$$

Thus, choosing $\beta > 0$ in such a way that $2\beta R_{22} < 1$ and applying the version of Gronwall's lemma given in [12, Lemma A.1] we obtain

$$\mathbb{E} \sup_{r\in[0,t]} |\omega_n^\varepsilon(r)|^2 \leqslant \Phi_N^{\varepsilon,\delta} R_{22}. \tag{5.32}$$

Now, since $\varrho_\delta(\varepsilon) \to 0$ and $\varepsilon\lambda_\delta^{-\ell}(\varepsilon) \to 0$, $\ell \in \{1,2\}$, as $\varepsilon \to 0$ we infer that $\Psi_N^{\varepsilon,\delta} \to 0$ as $\varepsilon \to 0$. Thus,

$$\mathbb{E} \sup_{r\in[0,t]} |\omega_n^\varepsilon(r)|^2 \to 0 \text{ as } n \to \infty. \tag{5.33}$$

Next, let $\gamma > 0$ and $\kappa > 0$ be arbitrary numbers. It is not difficult to check that

$$\mathbb{P}(\sup_{r\in[0,T]} |\omega_n^\varepsilon(r)|^2 \geqslant \kappa) \leqslant \mathbb{P}(\sup_{r\in[0,T]} |\omega_n^\varepsilon(r)|^2, \tilde{\tau}_n = T) + \mathbb{P}(\sup_{r\in[0,T]} |\nabla z_n(r)| \geqslant N) \tag{5.34}$$

$$\leqslant \frac{1}{\kappa}\mathbb{E} \sup_{r\in[0,t]} |\omega_n^\varepsilon(r)|^2 + \frac{1}{N}\mathbb{E} \sup_{r\in[0,T]} |\nabla z_n(r)|. \tag{5.35}$$

Owing to estimate (3.12) one can find $N > 0$ such that $\frac{1}{N}\mathbb{E}\sup_{r\in[0,T]}|\nabla z_n(r)| < \frac{\gamma}{2}$. Thus, thanks to (5.33) and (5.35) we infer that for all n large enough

$$\mathbb{P}(\sup_{r\in[0,T]} |\omega_n^\varepsilon(r)|^2 \geqslant \kappa) < \gamma,$$

which completes the proof of Lemma 5.1. □

Proof of Lemma 5.2. Before diving into the depth of the proof we recall that S_M is a Polish space when endowed with the metric defined in (3.9). Now, since, by assumption, $h_n \to h$ in law as S_M-valued random variables, we can infer from the Skorokhod's theorem that one can find a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ on which there exist S_M-valued random variables $\bar{h}_n$, $\bar{h}$ having the same laws as h_n and h, respectively, and satisfying

$$\bar{h}_n \to h \text{ in } S_M, \bar{\mathbb{P}} - \text{a.s.}. \tag{5.36}$$

From the last property and Proposition 3.14 we derive that for $s \in \{0,1\}$

$$\varGamma_\xi^{0,\delta}\left(\int_0^\cdot \bar{h}_n(r)dr\right) \to \varGamma_\xi^{0,\delta}\left(\int_0^\cdot \bar{h}(r)dr\right) \text{ in } C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}})) \;\; \bar{\mathbb{P}} - \text{a.s.}. \tag{5.37}$$

Observe that Proposition 3.14 implies in particular that $\varGamma_\xi^{0,\delta} : S_M \to C([0,T]; D(\mathrm{A}^{1+\frac{s}{2}}))$ is continuous. Hence, from the equality of the laws of h_n (resp. h) and $\bar{h}_n$ (resp. $\bar{h}$) we infer that the laws of $\varGamma_\xi^{0,\delta}\left(\int_0^\cdot \bar{h}_n(r)dr\right)$ and $\varGamma_\xi^{0,\delta}\left(\int_0^\cdot \bar{h}(r)dr\right)$ are equal to the laws of $\varGamma_\xi^{0,\delta}\left(\int_0^\cdot h_n(r)dr\right)$ and $\varGamma_\xi^{0,\delta}\left(\int_0^\cdot h(r)dr\right)$, respectively. This observation and the convergence (5.37) complete the proof of Lemma 5.2. □

A Budhiraja-Dupuis' theorem

In this appendix we formulate a LDP result which follows from [5, Theorem 3.6 and Theorem 4.4]. Let $\mathscr{H}$, $\mathscr{H}_0$ be two separable Hilbert spaces and W a Wiener process as in Subsection 3.1. We recall that $\mathscr{S}$ is the set of all $\mathscr{H}_0$-valued predictable process h such that

$$\mathbb{P}\left(\int_0^T \|h(r)\|^2_{\mathscr{H}_0} dr < \infty\right) = 1. \tag{A.1}$$

We now recall the following result which is exactly [5, Theorem 3.6].

Theorem A.1. *Let $\Gamma : C([0,T];\mathscr{H}_0) \to \mathbb{R}$ be a bounded, Borel measurable function. Then*

$$-\log \mathbb{E}e^{-\Gamma(W)} = \inf_{h\in\mathscr{S}} \mathbb{E}\biggl\{\frac{1}{2}\int_0^T \|h(r)\|^2_{\mathscr{H}_0} + \Gamma\left(W + \int_0^{\cdot} h(r)dr\right)\biggr\}. \tag{A.2}$$

Now, let $\mathcal{E}$ be a Polish space, $(\Psi^\varepsilon)_{\varepsilon\in(0,1]}$ a family of Borel measurable maps from $C([0,T;\mathscr{H}_0])$ onto $\mathcal{E}$, and $(X^\varepsilon)_{\varepsilon\in(0,1]}$ a family of $\mathcal{E}$-valued random variables.We have the following result which can be proved by using Theorem A.1 and the idea in the proof of [5, Theorem 4.4].

Theorem A.2. *Let ϱ be a real-valued function defined on $(0,\infty)$ such that*

$$\varrho(\varepsilon) \to \infty \text{ as } \varepsilon \to 0.$$

Assume that there exists a Borel measurable map $\Psi^0 : C([0,T];\mathscr{H}_0) \to \mathcal{E}$ such that the following hold:

*(**A**1) if $(h_\varepsilon)_{\varepsilon\in(0,1]} \subset \mathscr{S}_M$, $M > 0$, converges in distribution to $h \in \mathscr{S}_M$ as $\mathscr{S}_M$-valued random variables, then $\Psi^\varepsilon(W + \varrho(\varepsilon)\int_0^{\cdot} h_\varepsilon(r)dr)$ converges in distribution to $\Psi^0(\int_0^{\cdot} h(r)dr)$.*

*(**A**2) For every $M > 0$ the set $K_M = \{\Psi^0(\int_0^{\cdot} h(r)dr) : h \in \mathscr{S}_M\}$ is a compact subset of $\mathcal{E}$.*

Then, the family $(X^\varepsilon)_{\varepsilon\in(0,1]}$ satisfies an LDP with speed $\varrho^2(\varepsilon)$ and rate function I given by

$$I(x) = \inf_{\{h\in \mathrm{L}^2(0,T;\mathscr{H}_0):\ x=\Psi^0(\int_0^{\cdot} h(r)dr)\}} \biggl\{\frac{1}{2}\int_0^T \|h(r)\|^2_{\mathscr{H}_0}\biggr\}. \tag{A.3}$$

B Some very important auxiliary results

Let $\mathscr{H}_0$ be as in Appendix A. We will prove the following theorem.

Theorem B.1. *Let $\delta \in \{0,1\}$, $\boldsymbol{\varphi} \in \mathrm{L}^\infty(0,T;D(\mathrm{A}^{\frac{3+\delta}{2}}))$, $\eta = \boldsymbol{\varphi} + \alpha\mathrm{A}\boldsymbol{\varphi}$, $h \in \mathrm{L}^2(0,T;\mathscr{H}_0)$ and $\xi \in \mathbf{W}$. If G satisfies Assumption (**G**s) with $s = 0$, then there exists a unique $\mathbf{u}^\delta_h \in C([0,T];\mathbf{W})$ satisfying: for any $\boldsymbol{\phi} \in \mathbf{W}$*

$$(\partial_t \mathbf{v}^\delta, \boldsymbol{\phi}) + (1-\delta)(\mathcal{C}(\mathbf{rot}\,\mathbf{v}^\delta, \mathbf{u}^\delta), \boldsymbol{\phi}) + \delta(\mathcal{C}(\mathbf{rot}\,\mathbf{v}_h, \boldsymbol{\varphi}) + \mathcal{C}(\mathbf{rot}\,\eta, \mathbf{u}^\delta_h), \boldsymbol{\phi}) = F^\delta_h, \tag{B.1a}$$

$$\mathbf{v}_h = \mathbf{u}^\delta_h + \alpha\mathrm{A}\mathbf{u}^\delta_h, \tag{B.1b}$$

$$\mathbf{u}^\delta_h(0) = \xi_\delta := (1-\delta)\xi, \tag{B.1c}$$

*where $F^\delta_h = G(\delta\boldsymbol{\varphi} + (1-\delta)\mathbf{u}^\delta_h)h$. If furthermore $\xi \in D(\mathrm{A}^2)$, $\boldsymbol{\varphi} \in \mathrm{L}^\infty(0,T;D(\mathrm{A}^{2+\frac{\delta}{2}}))$ and G satisfies Assumption (**G**s) with $s = 1$, then $\mathbf{u}^\delta_h \in C([0,T];\mathbf{W}) \cap \mathrm{L}^\infty(0,T;\mathbf{H}^4)$.*

Proof. The system we consider is a linear perturbation of the inviscid model for grade-two fluid or Lagrangian Averaged Euler (LAE) equations. The proof of the well-posedness result is very similar, and is even simpler, to the one given in [8] where the LAE equations with Navier-slip boundary conditions was analyzed. Thus, we will only outline the main lines of the proof of the firts part of the theorem and refer the reader to [8], see also [15] and [49], for the detail. The main idea is to use a Galerkin approximation by considering a special orthonormal basis $\{e_j; j \in \mathbb{N}\}$ of $\mathbf{V}$ whose elements are the eigenfunctions of the spectral problem

$$e_j \in \mathbf{W} \text{ and } (\mathbf{rot}(\psi + \alpha \mathrm{A}\psi), \mathbf{rot}(e_j + \alpha \mathrm{A}e_j)) = \tilde{\lambda}_j((\psi, e_j))_\alpha \text{ for all } \psi \in \mathbf{W}. \tag{B.2}$$

We should note that for each j $e_j \in \mathbf{H}^4$. The existence of the eigenfunctions and the proof of their regularity can be found in [13] and [8]. Now for the sake of simplicity we will write $\mathbf{u}^\delta$ in place of $\mathbf{u}^\delta_h$. For each $\ell \in \mathbb{N}$ we denote by

$$\mathbf{u}^\delta_\ell(t,x) := \sum_{i=1}^{\ell} \varphi^\delta_i(t) e_i(x) \text{ and } \mathbf{v}^\delta_\ell = \mathbf{u}^\delta_\ell + \alpha \mathrm{A}\mathbf{u}^\delta_\ell$$

the solutions of the system of ODEs

$$\Big(\partial_t \mathbf{v}^\delta_\ell + (1-\delta)\mathcal{C}(\mathbf{rot}\,\mathbf{v}^\delta_\ell, \mathbf{u}^\delta_\ell) + \delta(\mathcal{C}(\mathbf{rot}\,\mathbf{v}^\delta_\ell, \boldsymbol{\varphi}) + \mathcal{C}(\mathbf{rot}\,\eta, \mathbf{u}^\delta_\ell)), e_i\Big) = (F^\delta_h, e_i), \ \forall i \in \{1,\dots,\ell\}, \tag{B.3a}$$

$$\mathbf{u}^\delta_\ell(0) = \mathit{\Pi}_\ell \xi_\delta, \tag{B.3b}$$

where $\mathit{\Pi}_\ell$ is the orthogonal projection form $\mathbf{W}$ onto $\mathrm{X}_\ell := \operatorname{span}\{e_1,\dots,e_\ell\}$. We now derive uniform estimates for $\mathbf{u}^\delta_\ell$ in $\mathrm{L}^\infty(0,T;\mathbf{V})$. For this aim, we multiply (B.3) by φ^δ_i, sum over $i \in \{1,\dots,\ell\}$ and use (2.7) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^\delta_\ell\|^2_\alpha = (\delta \mathcal{C}(\mathbf{rot}\,\mathbf{v}^\delta_\ell, \boldsymbol{\varphi}) + F^\ell_h, \mathbf{u}^\delta_\ell).$$

Now, using (2.15), Cauchy-Schwarz's inequality, Assumption ($\mathbf{G}s$) and Remark 3.1(a) we easily derive that there exists a constant $R > 0$ such that for any $\ell \geqslant 1$

$$\frac{d}{dt}\|\mathbf{u}^\delta_\ell\|_\alpha \leqslant R(\delta\|\mathbf{u}^\delta_\ell\|_\alpha(\|\boldsymbol{\varphi}\|_{\mathbf{W}} + (1-\delta)\|h\|_{\mathscr{H}_0}) + R\|h\|_{\mathscr{H}_0}(1 + \delta\|\boldsymbol{\varphi}\|_\alpha),$$

which altogether with Gronwall's lemma imply that

$$\begin{aligned}\sup_{\ell\in\mathbb{N}}\sup_{0\leqslant r\leqslant T}\|\mathbf{u}^\delta_\ell(r)\|_\alpha \leqslant &\left(\|\xi_\delta\|_\alpha + R\int_0^T \|h(r)\|_X(1+\delta\|\boldsymbol{\varphi}(r)\|_\alpha)dr\right)\\ &\times \exp\left(\int_0^T [\|\boldsymbol{\varphi}(r)\|_\alpha + (1-\delta)\|h(r)\|_{\mathscr{H}_0}]dr\right).\end{aligned} \tag{B.4}$$

Next we shall derive an estimate for $\mathbf{rot}\,\mathbf{v}^\delta_\ell$ in $\mathbf{L}^2$. For this purpose we multiply (B.3) by $\tilde{\lambda}_i\varphi^\delta_i$, use (B.2) and argue as in [8, pages 1136] to infer that

$$\frac{1}{2}\frac{d}{dt}|\mathbf{rot}\,\mathbf{v}^\delta_\ell|^2 + ((1-\delta)\,\mathbf{rot}(\mathbf{rot}\,\mathbf{v}^\delta_\ell \times \mathbf{u}^\delta_\ell) + \delta(\mathbf{rot}(\mathbf{rot}\,\mathbf{v}^\delta_\ell \times \boldsymbol{\varphi}) + \mathbf{rot}(\mathbf{rot}\,\eta \times \mathbf{u}^\delta_\ell)) - \mathbf{rot}\,F^\delta_h, \mathbf{rot}\,\mathbf{v}^\delta_\ell) = 0. \tag{B.5}$$

Note that all the required steps to derive the above identity are rigorously justified thanks to the regularity of the e_i-s and the Assumption $\boldsymbol{\varphi} \in \mathrm{L}^\infty(0,T;D(\mathrm{A}^{\frac{3+\delta}{2}}))$. Using these regularity assumptions again and (2.11) we derive the following identity

$$\frac{1}{2}\frac{d}{dt}|\mathbf{rot}\,\mathbf{v}^\delta_\ell|^2 + \delta(\mathbf{u}^\delta_\ell \cdot \nabla\,\mathbf{rot}\,\eta, \mathbf{rot}\,\mathbf{v}^\delta_\ell) = (\mathbf{rot}\,F^\delta_h, \mathbf{rot}\,\mathbf{v}^\delta_\ell), \tag{B.6}$$

from which along with Cauchy-Schwarz's inequality, the Sobolev embedding $\mathbf{V} \cap \mathbf{L}^\infty \subset \mathbf{W}$, Assumption ($\mathbf{G}s$) and Remark 3.1(a) we derive that there exists a constant $R > 0$ such that for all $\ell \geqslant 1$

$$\frac{d}{dt}|\mathbf{rot}\, \mathbf{v}_\ell^\delta| \leqslant R(\delta|\mathbf{rot}\, \mathbf{v}_\ell^\delta| \|\mathbf{rot}\, \eta\|_{\mathbf{H}^1} + 1 + \delta\|\boldsymbol{\varphi}\|_\alpha + (1-\delta)\|\mathbf{u}_\ell^\delta\|_\alpha).$$

Gronwall's lemma and (B.4) now imply

$$\sup_{\ell\in\mathbb{N}} \sup_{0\leqslant r\leqslant T} |\mathbf{rot}\, \mathbf{v}_\ell^\delta(r)| \leqslant (\|\xi_\delta\|_{\mathbf{W}} + RT(1 + \delta \sup_{0\leqslant r\leqslant T} \|\boldsymbol{\varphi}\|_\alpha + R^\delta(\xi, h, \boldsymbol{\varphi})))e^{\delta R \int_0^T \|\boldsymbol{\varphi}(r)\|_{\mathbf{H}^4} dr}, \tag{B.7}$$

where $R^\delta(\xi, h, \boldsymbol{\varphi})$ denotes the term in the right-hand side of (B.4).

With these estimates at hand we can infer that the family $(\mathbf{u}_\ell^\delta)_{\ell\in\mathbb{N}}$ is uniformly bounded in $\mathrm{L}^\infty(0, T; \mathbf{W})$. Thanks to Assumption ($\mathbf{G}s$), (B.4) and (B.7) we can easily prove that $(\partial_t \mathbf{v}_\ell^\delta)_{\ell\in\mathbb{N}}$ is uniformly bounded in $\mathrm{L}^\infty(0, T; \mathbf{L}^2)$. Now we can pass to the limit to complete the existence of a solution $\mathbf{u}_h^0 \in C([0, T]; D(A)) \cap \mathrm{L}^\infty(0, T; \mathbf{W})$. The continuity of $\mathbf{u}_h^0 : [0, T] \to \mathbf{W}$ was established in [44] and in the recent paper [52]. The uniqueness of the solution can also be established as in [8, Section 4], see also [15, Theorem 3.6].

It now remains to prove the second part of the theorem. Firstly, using the same argument as in [15, Lemma 5.5] one can show that $z_h^\delta = \mathbf{rot}\, \mathbf{v}_h^\delta \in \mathrm{L}^\infty(0, T; \mathbf{L}^2)$ is the unique solution to

$$\partial_t z + [(1-\delta)\mathbf{u}^\delta + \delta\boldsymbol{\varphi}] \cdot \nabla z = g_h^\delta \tag{B.8a}$$

$$z(0) = \mathbf{rot}(\xi_\delta + \alpha \mathrm{A}\xi_\delta), \tag{B.8b}$$

where $g_h^\delta := \mathbf{rot}\, F_h^\delta - \delta\mathbf{u}^\delta \cdot \nabla\, \mathbf{rot}\, \eta \in \mathrm{L}^2(0, T; \mathbf{H}^1)$. Thanks to Assumption ($\mathbf{G}s$) with $s = 1$, (B.7) and the assumption $\boldsymbol{\varphi} \in \mathrm{L}^\infty(0, T; D(\mathrm{A}^{2+\frac{\delta}{2}}))$, we have $g_h^\delta \in \mathrm{L}^2(0, T; \mathbf{H}^1)$. Thus, from Theorem B.2, see below, we infer that $z_h^\delta \in \mathrm{L}^\infty(0, T; \mathbf{H}^1)$ or equivalently $\mathbf{u}^\delta \in \mathrm{L}^\infty(0, T; \mathbf{H}^4)$. This completes the proof of our theorem. ☐

We now state and prove the following result which was already used in the previous theorem.

Theorem B.2. *For $s \in [1, \infty)$ let*

$$\theta(s) = \begin{cases} 3 \text{ if } s = 1, \\ s + 1 \text{ if } s > 1. \end{cases}$$

Let $\boldsymbol{\psi} \in \mathrm{L}^\infty(0, T; D(\mathrm{A}^{\frac{\theta(s)}{2}}))$, $g \in \mathrm{L}^2(0, T; \mathbf{H}^s)$ and $z_0 \in \mathbf{H}^s$. Then, there exists a unique $z \in \mathrm{L}^\infty(0, T; \mathbf{H}^s)$ satisfying

$$\partial_t z + \boldsymbol{\psi} \cdot \nabla z = g, \tag{B.9a}$$

$$z(0) = z_0. \tag{B.9b}$$

Proof. We will follow the approach in [15, Lemma 5.2] and only outline the main idea of the proof. The existence is derived from Galerkin approximation based on an orthonormal basis $\{\psi_j ; j \in \mathbb{N}\}$ consisting of the eigenfunctions family of the spectral problem

$$\psi_j \in \mathbf{H}^s \text{ and } (\psi_j, \phi) + ((-\Delta)^{\frac{s}{2}}\psi_j, (-\Delta)^{\frac{s}{2}}\phi) = \alpha_j(\psi_j, \phi) \text{ for any } \phi \in \mathbf{H}^s.$$

Or equivalently

$$\psi_j \in \mathbf{H}^s \text{ and } \psi_j + (-\Delta)^s \psi_j = \alpha_j \psi_j,$$

which implies that each ψ_j is smooth as we want. Now, for each $\ell \in \mathbb{N}$ we denote by P_ℓ the orthogonal projection from $\mathbf{H}^s$ onto $Y_\ell := \operatorname{span}\{\psi_1, \dots, \psi_\ell\}$ and we let $z_\ell(t,x) = \sum_{j=1}^{\ell} r_j(t)\psi_j(x)$ be the solution to the following system of ODEs

$$\begin{aligned}
&(\partial_t z_\ell, \psi_i) + (\boldsymbol{\psi} \cdot \nabla z_\ell, \psi_i) = (g, \psi_i), \text{ for all } i \in \{1, \dots, \ell\},\\
&z_\ell(0) = P_\ell z_0.
\end{aligned}$$

Multiplying the above system by $\alpha_i r_i$ and summing over $i \in \{0, \dots, \ell\}$ yields

$$\frac{1}{2}\frac{d}{dt}|(I + \Lambda^s)z_\ell|^2 \leqslant |(\Lambda^s(\boldsymbol{\psi} \cdot \nabla z_\ell), \Lambda^s z_\ell)| + R\|g\|_{\mathbf{H}^s}|(I + \Lambda^s)z_\ell|, \tag{B.10}$$

where $\Lambda^s := (-\Delta)^{\frac{s}{2}}$ and $R > 0$ is a constant independent of ℓ. Observe that

$$|(\Lambda^s(\boldsymbol{\psi} \cdot \nabla z_\ell), \Lambda^s z_\ell)| \leqslant R \begin{cases} \|\nabla \boldsymbol{\psi}|_{\mathbf{L}^\infty}\|z_\ell\|^2_{\mathbf{H}^1} \text{ if } s = 1, \text{ see [15, page 333]},\\[2ex] \|\boldsymbol{\psi}\|_{h^{s+1}}\|z_\ell\|^2_{\mathbf{H}^s} \text{ if } s > 1, \text{ see [24, Corollary 2.1]}, \end{cases} \tag{B.11}$$

which along with the former identity implies that

$$|(I + \Lambda^s)z_\ell(t)| \leqslant \|z_0\|_{\mathbf{H}^s} + R\int_0^t \|g(r)\|_{\mathbf{H}^s}dr + R\int_0^t |(I + \Lambda^s)z_\ell(r)|\|\boldsymbol{\psi}(r)\|_{\mathbf{H}^{\theta(s)}}dr. \tag{B.12}$$

Now, Gronwall's inequality implies that

$$\sup_{\ell \in \mathbb{N}} \sup_{0 \leqslant s \leqslant T} \|z_\ell(t)\|_{\mathbf{H}^s} \leqslant \left(\|z_0\|_{\mathbf{H}^s} + R\int_0^T \|g(r)\|_{\mathbf{H}^s}dr\right) e^{R\int_0^T \|\boldsymbol{\psi}(r)\|_{\mathbf{H}^{\theta(s)}}dr}.$$

One can now pass to the limit to complete the proof of the existence result.

We omit the proof of the uniqueness because it is easy as we are dealing with a linear transport problem with regular coefficient. □

C Results on regularization by convolution

Let $\varrho \in C^\infty(\mathbb{T}^2)$ be an even, smooth function such that its support is compact and lies within a ball on the torus, and the ball lifts homeomorphically to the universal covering $\mathbf{R}^2$. We also assume that $\int_{\mathbb{T}^2} \varrho(x)dx = 1$. For each $k \in \mathbb{N}$ we set $\varrho_k(\cdot) = k^2\varrho(k\cdot)$ and define the convolution operator J_k by $J_k f = \varrho_k * f$.

We state the following result which is very crucial for the analysis in this paper.

Lemma C.1. *Let $\psi \in \mathrm{L}^2(0,T;\mathbf{W}^{1,\infty}(\mathbb{T}^2))$ and $\varphi \in \mathrm{L}^\infty(0,T;\mathbf{L}^2(\mathbb{T}^2))$. Then, as $k \to \infty$*

$$J_k(\psi \cdot \nabla\varphi) - \psi \cdot \nabla(J_k\varphi) \to 0 \text{ in } \mathrm{L}^2(0,T;\mathbf{L}^2(\mathbb{T}^2)).$$

Proof. This is a special case of [18, part ii) of Lemma II.1], thus we omit the proof. □

We also recall the following properties of J_k, see for instance [18] and also [52, Proposition 6.3 & Proposition 6.4].

Proposition C.2. *For all* $f \in \mathrm{L}^\gamma(0,T;\mathbf{L}^s(\mathbb{T}^2))$, $\gamma \in [1,\infty]$ *and* $s \in [1,\infty]$, *we have*

$$\lim_{k\to\infty} \|J_k f - f\|_{\mathrm{L}^\gamma(0,T;\mathbf{L}^s(\mathbb{T}^2))} = 0. \tag{C.1}$$

Now, let W be a Wiener process with covariance Q as introduced in Subsection 3.1. We state the following proposition.

Proposition C.3. *Let* $z, \zeta \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;\mathscr{L}(\mathscr{H},\mathbf{L}^2(\mathbb{T}^2))))$ *be predictable processes. Then, there exists a subsequence of* J_k *which is not relabeled such that*

$$\int_0^t (J_k z(s), J_k\xi(s)dW) \to \int_0^t (z(s),\zeta(s)dW) \text{ with probability } 1 \text{ for all } t\in[0,T], \tag{C.2}$$

as $k \to \infty$.

Proof. Thanks to Proposition C.2, the proposition is a corollary of [52, Proposition 6.7], so we omit the proof. □

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