A class of nilpotent Lie algebras admitting a compact subgroup of automorphisms*

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Abstract

The realification of the $(2n+1)$-dimensional complex Heisenberg Lie algebra is a $(4n+2)$-dimensional real nilpotent Lie algebra with a 2-dimensional commutator ideal coinciding with the centre, and admitting the compact algebra $\mathfrak{sp}(n)$ of derivations. We investigate, in general, whether a real nilpotent Lie algebra with 2-dimensional commutator ideal coinciding with the centre admits a compact Lie algebra of derivations. This also gives us the occasion to revisit a series of classic results, with the expressed aim of attracting the interest of a broader audience.

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1 Introduction

Metabelian Lie algebras $\mathfrak{h} = V \oplus \langle x, y\rangle$ of dimension $(n+2)$ defined by a pair of alternating forms F_1, F_2 on the n-dimensional vector space V, putting, for any $v, w \in V$, $[v, w] = F_1(v, w)x + F_2(v, w)y$, are called nilpotent Lie algebras of type $\{n, 2\}$. The *type* $\{d_1, \ldots, d_c\}$ of a nilpotent Lie algebra $\mathfrak{g}$ with descending central series $\mathfrak{g}^{(i)} = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$ is defined, according to the literature beginning with Vergne [20], by the integers $d_i = \dim \frac{\mathfrak{g}^{(i-1)}}{\mathfrak{g}^{(i)}}$. Nilpotent (real or complex) Lie algebras of type $\{n, 2\}$ have been classified firstly by Gauger [11], applying the canonical reduction of the pair F_1, F_2, but, according to results of Belitskii, Lipyanski, and Sergeichuk [3], it is not possible to carry this argument further. On the contrary, it seems possible to broaden these families of Lie algebras by considering their derivations, this has been done in [9] extending a real Lie algebras of type $\{n, 2\}$ by a single compact derivation. We mention that also nilpotent Lie algebras of type $\{n, 1, 1\}$ can be explicitly described (cf. [2]), and derivations of a nilpotent Lie algebra of type $\{2n, 1, 1\}$ are determined in [1].

In the present paper, we are interested in the following problem. The realification of the $(2n+1)$-dimensional complex Heisenberg Lie algebra $\widehat{\mathfrak{h}}$ is a $(4n+2)$-dimensional real Lie algebra $\mathfrak{h}$ of type $\{4n, 2\}$,

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admitting the compact algebra $\mathfrak{sp}(n)$ of derivations. We ask, in general, whether a real Lie algebra $\mathfrak{h}$ of type $\{2n, 2\}$ admits a compact Lie algebra of derivations, that is, the Lie algebra of a compact Lie group, a question arising in the study of the isometry groups of homogeneous nilmanifolds. In fact, the realifications $\mathfrak{h} = V \oplus \langle x, y\rangle$ of the complex Heisenberg algebras $\widehat{\mathfrak{h}}$ are the only H-type algebras with two-dimensional centre (cf. [19], Section 5, p. 3252), where an H-type algebra is a nilpotent Lie algebra of class 2 with an inner product such that the operator $J : \mathfrak{h} \longrightarrow \mathrm{End}(V)$, $z \mapsto J_z$, defined by $\langle J_z v, w\rangle = \langle z, [v, w]\rangle$, fulfills $J_z^2 = -\langle z, z\rangle \mathrm{id}_V$.

Notice that a non-commutative simple compact Lie algebra of derivations of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{2n, 2\}$ must induce the null map on the two-dimensional commutator ideal $\mathfrak{h}'$, because a non-commutative simple compact Lie algebra cannot have a two-dimensional representation.

Generally speaking, for the compact Lie algebra $\mathfrak{g}$ of a compact Lie group G, the opposite of the Killing form induces on $\mathfrak{g}$ an $\mathrm{Ad}(G)$-invariant inner product, and, up to scalar multiplication, this is the unique $\mathrm{Ad}(G)$-invariant inner product. With respect to this inner product, $\mathrm{Ad}(G)$ acts by orthogonal transformations of $\mathrm{SO}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})$ acts by skew-symmetric matrices of $\mathfrak{so}(\mathfrak{g})$. Therefore, a compact Lie algebra can be embedded into $\mathfrak{so}(\mathfrak{g})$, a simpler and stronger version of Ado's theorem.

In the context of (real) nilpotent Lie algebras with low-dimensional commutator ideals, the situation is as follows. Let $\mathfrak{g}$ be a nilpotent Lie algebra with commutator ideal $\mathfrak{g}'$ and centre $\mathfrak{z}$.

- If $\dim \mathfrak{g}' = 0$, then $\mathfrak{g}$ is Abelian and the maximal compact subgroup is $\mathrm{O}(n)$.
- If $\dim \mathfrak{g}' = 1$ and $\dim \mathfrak{z} = 1$, then $\mathfrak{g}$ is the $(2n + 1)$-dimensional Heisenberg algebra $\mathfrak{h}_{2n+1}$ and the maximal compact subgroup is $\mathrm{U}(n) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{O}(2n)$ (cf. [4]). Note that if instead $\dim \mathfrak{z} = \ell + 1$ then $\mathfrak{g} \cong \mathfrak{h}_{2n+1} \oplus \mathbb{R}^\ell$.
- If $\dim \mathfrak{g}' = 2$ and $\dim \mathfrak{z} = 1$, then $\mathfrak{g}$ is a uniquely determined by its dimension (cf. [2]); the $(2n + 4)$- and $(2n + 5)$-dimensional groups turn out to have maximal compact subgroup $\mathrm{U}(n) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{O}(2n)$. Note again that if instead $\dim \mathfrak{z} = \ell + 1$, then we simply have a trivial Abelian extension of the case already discussed.
- Finally, if $\mathfrak{g}' = \mathfrak{z}$ and $\dim \mathfrak{g}' = 2$, we have the case considered in this paper. This is arguably the first interesting case (in this list) as there exist Lie algebras of the same dimension of this kind with different maximal compact subgroups of automorphisms (see Examples 1, 2).

We give remarkable examples for nilmanifolds M such that the group of isometries of M contains the compact group $\mathrm{SO}_2(\mathbb{R})$.

In the present Introduction, we introduce the Heisenberg Lie algebra in the contexts of algebra, complex analysis and quantum mechanics. Although not directly connected, these results appear in all introductory books in these fields. Nevertheless, they serve us to emphasize the non-trivial rôle of compact derivations of nilpotent Lie algebras.

1.1 Heisenberg algebra, Wirtinger derivatives, and Weyl algebra

Heisenberg Lie algebras are the most elementary non-Abelian Lie algebras. Such a Lie algebra $\mathfrak{h} = V \oplus \langle u\rangle$ has dimension $(2n + 1)$ and is defined by a non-degenerate alternating form F on the $2n$-dimensional subspace V, putting $[v, w] = F(v, w)u$, for any $v, w \in V$. The choice of a symplectic basis $\{p_1, q_1, \ldots, p_n, q_n\}$ of V allows one to write

$$[p_i, q_j] = \delta_{ij} u, \quad (i, j = 1, \ldots, n),$$

where δ_{ij} is the Kronecker symbol. This basic definition underpins the reason why one meets Heisenberg Lie algebras very frequently in the scientific literature.

The same Cauchy-Riemann conditions $\partial u/\partial q = \partial v/\partial p$, $\partial u/\partial p = -\partial v/\partial q$, describing the behaviour of the real form of a complex holomorphic function $h = u + iv$ of a complex variable $z = q + ip$, can be formulated in terms of the Heisenberg Lie algebra, as corresponding to the Wirtinger derivatives $\partial/\partial z = \frac{1}{2}(\partial/\partial q - i\partial/\partial p)$ and $\partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial q + i\partial/\partial p)$, whose product is the Laplacian operator $\frac{1}{4}\nabla^2 = \frac{1}{4}\ (\partial^2/\partial p^2 + \partial^2/\partial^2 q)$. The realification of the 3-dimensional complex Heisenberg Lie algebra

$\widehat{\mathfrak{h}} = \langle \partial/\partial z, z, 1\rangle$ induced by the derivation rule $[\partial/\partial z, z] = 1$ is therefore a 6-dimensional real Lie algebra $\mathfrak{h} = \langle \partial/\partial q, i\,\partial/\partial p, q, p, 1, i\rangle$, and in the case of multivariate holomorphic functions

$$h(z_1, \dots, z_n) = u(q_1 + ip_1, \dots, q_n + ip_n) + iv(q_1 + ip_1, \dots, q_n + ip_n),$$

the realification of the $(2n+1)$-dimensional complex Heisenberg Lie algebra $\mathfrak{h} = \langle \partial/\partial z_1, \dots, \partial/\partial z_n, z_1, \dots, z_n, 1\rangle$ is a $(4n+2)$-dimensional real Lie algebra, where the commutator ideal is the Gauss plane $\mathbb{C} = \langle 1, i\rangle$.

Finally, we denote by $\mathfrak{W}$ the Lie algebra generated by the differential operators ${x_j}^h \partial^k/\partial {x_j}^k$ (with ${x_j}^h\big(f(x)\big) := {x_j}^h f(x)$) (it is actually a Lie subalgebra of the n-th Weyl algebra of differential operators with polynomial coefficients in n variables). If we map $\mathfrak{h}$ into $\mathfrak{W}$ via

$$p_h \mapsto x_h, \quad q_h \mapsto -\partial/\partial x_h, \quad u \mapsto \mathrm{id},$$

we obtain an isomorphic copy, because

$$[x_h, -\partial/\partial x_k]\big(f(x)\big) = -x_h \partial f(x)/\partial x_k + \partial/\partial x_k\big(x_h f(x)\big) = \delta_{hk} f(x).$$

1.2 Oscillator Lie algebra

A classic subject in Mathematical Physics is the description of n uncoupled harmonic oscillators via an extension of the Heisenberg Lie algebra called, in fact, oscillator Lie algebra, which is defined as follows: let $\mathfrak{h}$ be the $(2n+1)$-dimensional Heisenberg Lie algebra, generated by $\{p_1, q_1, \dots, p_n, q_n; u\}$ and defined by the brackets

$$[p_h, q_k] = \delta_{hk} u, \quad (h, k = 1, \dots, n),$$

and let $\mathfrak{g}$ (the oscillator algebra) be the extension of $\mathfrak{h}$ by the outer derivation H, acting as

$$[H, p_h] = q_h, \quad [H, q_h] = -p_h.$$

We notice that the derivation H is *compact*, that is, its exponentiation $\exp(H)$ generates the compact one-parameter group of automorphisms of $\mathfrak{h}$, which rotate each pair (p_h, q_h) onto $(\cos(t)p_h + \sin(t)q_h, -\sin(t)p_h + \cos(t)q_h)$.

The corresponding (simply connected) oscillator groups admit a rich family of bi-invariant Lorentzian metrics (cf. [6, 10]); indeed, they are the only (indecomposable) connected simply connected non-Abelian solvable Lie groups which admit a bi-invariant Lorentzian metric [15, 17]. Beyond their significance to Lorentzian geometry, these groups also have some interesting applications in mathematical physics (see, e.g., [5, 8, 14]).

Note that the universal enveloping algebra of $\mathfrak{h}$ contains the element $\widehat{H} = -\frac{1}{2}\sum(q_h^2 + p_h^2)$ such that

$$[\widehat{H}, p_h] = -\frac{1}{2}[q_h^2, p_h] = -\frac{1}{2}(q_h^2 p_h - p_h q_h^2) = q_h u$$

and, similarly,

$$[\widehat{H}, q_h] = -p_h u,$$

thus one has to factor through $\langle u - 1\rangle$, in order to find an isomorphic image of the oscillator algebra.

This induces the physical interpretation of the compact derivation H as the Hamiltonian of a dynamical system obtained by identifying the n vectors p_i with the coordinate operators $x_j : f(x) \mapsto x_j f(x)$, the n vectors q_j with the momentum operators $\dot{x}_j$ (of unitary mass), the vector u with the identity map $u : \xi \mapsto \xi$, subjected to the potential $V = -\frac{1}{2}\sum x_l^2$.

1.3 Compact derivations and Hamiltonians

Alternatively, the representation into the subalgebra $\mathfrak{W}$ of the n-th Weyl algebra of differential operators gives one the advantage that, putting $H \mapsto \sum a_{jhk}\, {x_j}^h \partial^k/\partial {x_j}^k \in \mathfrak{W}$, from $[H, x_l] = -\partial/\partial x_l$ and from $[H, \partial/\partial x_l] = x_l$, it follows directly that, up to scalar multiplication $a\,\mathrm{id}$,

$$H \mapsto -\frac{1}{2}\sum\left(\partial^2/\partial x_l^2 + x_l^2\right).$$

This results in another evocative physical interpretation, given by changing of basis $\{q, p\}$ into the basis $\{a, a^\dagger\}$ consisting of the vectors $a = \frac{\sqrt{2}}{2}(q + ip)$ and $a^\dagger = \frac{\sqrt{2}}{2}(q - ip)$ (notice the affinity to Wirtinger derivations). Since the quadratic form $H = \frac{1}{2}\sum_i(q_i^2 + p_i^2)$ splits in the present non-commutative case as

$$H = \frac{1}{2}\sum_i(q_i^2 + p_i^2) = \frac{1}{2}(aa^\dagger + a^\dagger a) = aa^\dagger - \frac{1}{2}iu,$$

inductively we find $[H, a^n] = in\,a^n$ and, similarly, $[H, {a^\dagger}^n] = -in\,{a^\dagger}^n$. This produces the following discrete families of eigenvectors for the Hamiltonian H which underlies the quantistic interpretation of the harmonic oscillator (see [22], §22; [12], §9): for $a^\dagger v = 0$, we get $Hv = -\frac{1}{2}iv$, and more generally, putting $\psi_n = a^n v$ for any integer n, we get a discrete set of eigenspaces

$$H\psi_n = Ha^n v = a^n(ni + H)v = i\left(n - \frac{1}{2}\right)\psi_n.$$

Similarly, for $aw = 0$, we get $Hw = \frac{1}{2}iw$, and putting $\varphi_n = {a^\dagger}^n w$, we get $H\varphi_n = i\left(-n - \frac{1}{2}\right)\varphi_n$. Turning to the model where $p \mapsto x$ and $q \mapsto -\partial/\partial x$, a solution to $a^\dagger v = 0$, resp. to $aw = 0$, is given by

$$a^\dagger v = \frac{\sqrt{2}}{2}(q - ip)v = \frac{\sqrt{2}}{2}(-\partial/\partial x - ix)v(x) = 0,$$

that is, $v'/v = -ix$, hence $v(x) = \exp(-\frac{1}{2}ix^2)$, resp. $w(x) = \exp(\frac{1}{2}ix^2)$.

1.4 Automorphisms and Riemannian isometries

Let $\mathbf{g}$ be a left-invariant Riemannian metric on a (real, connected) n-dimensional simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, i.e., $(G, \mathbf{g})$ is a homogeneous nilmanifold. The group of isometries $\mathrm{Iso}(G, \mathbf{g})$ consists of diffeomorphisms $\phi : G \to G$ such that $\phi^*\mathbf{g} = \mathbf{g}$. The isotropy subgroup of the identity $\mathrm{Iso}_{\mathbf{1}}(G, \mathbf{g}) = \{\phi \in \mathrm{Iso}(G, \mathbf{g}) \,:\, \phi(\mathbf{1}) = \mathbf{1}\}$ is a subgroup of the automorphism group $\mathrm{Aut}(G)$ ([21]). Accordingly, $\mathrm{Iso}(G, \mathbf{g})$ decomposes as a semidirect product of the group of left translations and $\mathrm{Iso}_1(G, \mathbf{g})$. Moreover, $\mathrm{Iso}_{\mathbf{1}}(G, \mathbf{g})$ is a subgroup of $\mathrm{O}(n)$ and hence is compact. It therefore follows that the isometry group is at least n-dimensional and at most $\binom{n+1}{2}$-dimensional.

Proposition 1. *Let G be a simply connected nilpotent Lie group and let $K \leq \mathrm{Aut}(G)$ be a compact subgroup of automorphisms.*

1. *There exists a left-invariant Riemannian metric $\mathbf{g}$ on G such that $K \leq \mathrm{Iso}_1(G, \mathbf{g})$.*
2. *If K is a maximal compact subgroup, then for any left-invariant Riemannian metric $\mathbf{g}$ on G we have that $\mathrm{Iso}_1(G, \mathbf{g}) \leq \phi K \phi^{-1}$ for some $\phi \in \mathrm{Aut}(G)$.*

Remark 1. G admits $\mathrm{O}(n)$ as a (maximal) compact subgroup of automorphisms only in the Abelian case. Accordingly, if G is nilpotent but non-Abelian, the dimension of $\mathrm{Iso}(G, \mathbf{g})$ is strictly less than $\frac{1}{2}n(n + 1)$. We note that there do exist non-nilpotent Lie groups that admit a left-invariant Riemannian metric whose isotropy subgroup could be the entire orthogonal group (e.g., SU(2) with Killing metric).

2 Examples of lower-dimensional nilpotent Lie algebras of type $\{n, 2\}$

Up to isomorphism, the smallest example of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ is the 5-dimensional Lie algebra of type $\{3, 2\}$ given by

$$[u_1, u_2] = x, \quad [u_1, u_3] = y$$

(cf. [7], the Lie algebra $L_{5,8}$, p. 646, and [18], the Lie algebra L_5^1, p. 162). This Lie algebra does not have a non-commutative compact Lie algebra of derivations, since its derivations inducing the null map on $\mathfrak{h}'$ are defined with respect to the basis $\{u_1, u_2, u_3, x, y\}$ by the matrices

$$\left(\begin{array}{c|cc|cc} a & 0 & 0 & 0 & 0 \\ \hline b & -a & 0 & 0 & 0 \\ c & 0 & -a & 0 & 0 \\ \hline d_1 & d_2 & d_3 & 0 & 0 \\ d_4 & d_5 & d_6 & 0 & 0 \end{array}\right).$$

Yet its group of automorphisms contains the group $\mathrm{SO}_2(\mathbb{R})$, acting as the group of automorphisms of the form $\exp(0 \oplus it \oplus it)$, $t \in \mathbb{R}$ (notice that the latter automorphism does operate non-trivially on $\mathfrak{h}'$).

As soon as $n = 4$, the algebra of derivations of a Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ can contain compact simple subalgebras, as the following example shows. On the other hand, an example of a Lie algebra of type $\{4, 2\}$ which does not contain any compact simple subalgebra is given in Example 2.

Example 1. Let $\mathcal{B} = \{u_1, u_2, u_3, u_4, x, y\}$ be a basis of the 6-dimensional Lie algebra $\mathfrak{h}$ of type $\{4, 2\}$ defined by

$$[u_1, u_3] = x, \quad [u_1, u_4] = -y, \quad [u_2, u_3] = y, \quad [u_2, u_4] = x$$

(cf. [7], the Lie algebra $L_{6,22}(\epsilon = -1)$, p. 647, and [18], the Lie algebra #5. ($\gamma = -1$), p. 168). A direct computation shows that the derivations of $\mathfrak{h}$ which induce the null map on $\mathfrak{h}'$ are represented, with respect to the basis $\mathcal{B}$, by matrices of the form

$$\left(\begin{array}{cccc|cc} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ -a_2 & a_1 & a_4 & -a_3 & 0 & 0 \\ -b_2 & c_2 & -a_1 & a_2 & 0 & 0 \\ c_2 & b_2 & -a_2 & -a_1 & 0 & 0 \\ \hline d_1 & d_2 & d_3 & d_4 & 0 & 0 \\ d_5 & d_6 & d_7 & d_8 & 0 & 0 \end{array}\right).$$

With $a_1 = 0$, $c_2 = -a_4$, $b_2 = a_3$, and all the entries d_i equal to zero, we get an algebra isomorphic to the compact real form $\mathfrak{su}(2)$. The 6-dimensional Lie algebra $\mathfrak{h}$ is manifestly the realification of the complex Lie algebra of the complex Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Example 2. The six-dimensional Lie algebra $\mathfrak{g}$ with ordered basis $(u_1, u_2, u_3, u_4, x, y)$ and nonzero commutators given by

$$[u_1, u_2] = x, \quad [u_1, u_3] = y, \quad [u_2, u_4] = y$$

has Lie algebra of derivations

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_1 & -a_2 & 0 & 0 & 0 & 0 \\ -a_3 & a_4 & 0 & 0 & 0 & 0 \\ a_5 & a_6 + a_7 & -a_1 + a_8 & a_3 & 0 & 0 \\ a_6 & a_9 & a_2 & -a_4 + a_8 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_1 + a_4 & 0 \\ a_{14} & a_{15} & a_{16} & a_{17} & a_7 & a_8 \end{pmatrix} : a_1, \ldots, a_{17} \in \mathbb{R} \right\}.$$

Since the subalgebra of matrices with $a_1 = a_2 = a_3 = a_4 = 0$

$$\mathfrak{s} = \left\{ \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a_5 & a_6 + a_7 & a_8 & 0 & 0 & 0 \\ a_6 & a_9 & 0 & a_8 & 0 & 0 \\ \hline a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{14} & a_{15} & a_{16} & a_{17} & a_7 & a_8 \end{array}\right) : a_5, \ldots, a_{17} \in \mathbb{R} \right\}.$$

is manifestly a solvable ideal, such that $\mathrm{Der}(\mathfrak{g})/\mathfrak{s}$ is isomorphic to $\mathfrak{gl}(2, \mathbb{R})$, we see that a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ must have dimension one, because $\mathfrak{s} \cap \mathfrak{k} = 0$.

Accordingly, the connected component of identity of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ has (maximal) compact subgroup

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong \mathbb{T}.$$

Example 3. The direct sum of the three-dimensional Heisenberg algebra with itself yields a two-step nilpotent Lie algebra with two-dimensional commutator; its automorphism group has a maximal compact subgroup isomorphic to $\mathbb{T}^2$.

3 Some notation

In order to study nilpotent Lie algebras of type $\{n, 2\}$ admitting a compact group of automorphisms, it is necessary to start from the description of the action of a compact one-dimensional group T of automorphisms, confining oneself to the T-*indecomposable* case. It turns out that the description depends on the reduction to canonical form of pairs of skew-Hermitian forms (cf. Theorem 1). Also, it is useful to represent $2h \times 2k$ real matrices as $h \times k$ matrices with coefficients in the algebra of split-quaternions.

3.1 Split-quaternions and pairs of skew-Hermitian forms

We introduce here the Clifford algebra of split-quaternions as the set

$$\mathbb{H}_- = \{z_1 + z_2\omega : z_i \in \mathbb{C}, \omega z = \bar{z}\omega, \omega^2 = 1\}.$$

We recall that, through the usual identification of the complex number $z = a + ib$ with the real matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and of the reflection $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with the split-quaternion ω, one obtains an isomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \Big(\frac{a+d}{2} + i\,\frac{b-c}{2}\Big) + \Big(\frac{b+c}{2} + i\,\frac{a-d}{2}\Big)\omega$$

of the algebra of real 2×2 matrices with the algebra $\mathbb{H}_-$ and, more generally, of the space of $\mathbb{R}^{2n\times 2m}$ matrices with the space of $\mathbb{H}_-^{n\times m}$ matrices. We denote furthermore:

i) by $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the real matrix corresponding to the imaginary unit, by 0 any $n \times m$ (real or complex) zero matrix, and by I_m the (real or complex) m-dimensional identity matrix;

ii) by M' the transpose of M, and by $M^\dagger$ the conjugate transpose of M;

iii) by $M_1 \oplus M_2$ the diagonal block matrix $\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$, and by $(\oplus M)$ the diagonal block matrix $M \oplus \cdots \oplus M$.

With the identification of the split-quaternion matrix ωI_n with the $2n \times 2n$ reflection $\Omega_{2n} = \Omega \oplus \cdots \oplus \Omega$, any matrix $M \in \mathbb{R}^{2n\times 2m}$ can be written in a unique way as $M = M_1 + M_2\Omega_{2m}$, where M_1 and M_2 are *realifications* of complex matrices $Z_1 = \widehat{M_1} = (z_{ij})$, $Z_2 = \widehat{M_2} = (u_{ij}) \in \mathbb{C}^{n\times m}$ such that, for $\overline{Z}_1 = (\bar{z}_{ij})$ and $\overline{Z}_2 = (\bar{u}_{ij})$, one has $\omega I_n\, Z_i = \overline{Z}_i\, \omega I_m$ $(i = 1, 2)$.

Since a bilinear form F is skew-Hermitian precisely when iF is Hermitian, the problem of the simultaneous reduction to canonical form of a pair of skew-Hermitian forms reduces to the one concerning a pair of Hermitian forms. Our reference for the following list is the popularizing paper by Lancaster and Rodman [13], which resumes a very long history.

A pair (H_1, H_2) of Hermitian matrices can be reduced, by simultaneous congruence $H_i \mapsto A^\dagger H_i A$, into the direct sum of diagonal blocks, which have one of the following four types (other than the null pair)

$$\begin{aligned}
&i)\, (\pm F_\epsilon, \pm G_\epsilon);\\
&ii)\, \big(\pm(\alpha F_\epsilon + G_\epsilon), \pm F_\epsilon\big);\\
&iii)\, \left(G_{2\epsilon+1}, \begin{pmatrix} 0 & 0 & F_\epsilon \\ 0 & 0 & 0 \\ F_\epsilon & 0 & 0 \end{pmatrix} \right);\\
&iv)\, \left(\begin{pmatrix} 0 & \beta F_\epsilon + G_\epsilon \\ \bar\beta F_\epsilon + G_\epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & F_\epsilon \\ F_\epsilon & 0 \end{pmatrix} \right),
\end{aligned} \tag{1}$$

where F_ϵ is the matrix that maps $(x_1, x_2, \dots, x_{\epsilon-1}, x_\epsilon)$ onto $(x_\epsilon, x_{\epsilon-1}, \dots, x_2, x_1)$, and G_ϵ is the matrix that maps $(x_1, x_2, \dots, x_{\epsilon-1}, x_\epsilon)$ onto $(x_{\epsilon-1}, x_{\epsilon-2}, \dots, x_1, 0)$, α is a real number and β is a complex non-real number. With the only exception of changing β with $\bar\beta$, different values of the parameters α and β, and of the sign $\pm$, correspond to pairs that are not congruent.

3.2 T-indecomposable nilpotent Lie algebra of type $\{n, 2\}$

The following setting is the same as that of [9]. Let T be a compact one-dimensional group of automorphisms of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ (that is, with a 2-dimensional commutator ideal $\mathfrak{h}'$ coinciding with the centre $\mathfrak{z}$), let $\mathfrak{t}$ be the corresponding compact algebra of derivations of $\mathfrak{h}$, and let $n = 2m$ if n is even, and $n = 2m+1$ if n is odd, with $n \geq 3$. By the complete reducibility of T, we find a basis $\{e_1, \dots, e_n, x, y\}$ of $\mathfrak{h}$, such that $\{x, y\}$ is a basis of $\mathfrak{h}' = \mathfrak{z}$ and such that $\mathfrak{t}$ operates on $\mathfrak{h}$ as the algebra of matrices $\partial(t)$ with parameter $t \in \mathbb{R}$, defined as

$$\partial(t) := \begin{cases} (\alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) \oplus \beta t \cdot J & \text{for} \quad n = 2m, \\ (0 \oplus \alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) \oplus \beta t \cdot J & \text{for} \quad n = 2m+1, \end{cases} \tag{2}$$

where $\beta t \cdot J$ is the 2×2 matrix operating on $\mathfrak{h}' = \langle x, y \rangle$. Notice that, up to rescaling the parameter t, we can assume that either $\beta = 0$ or $\beta = 1$. Moreover, the case $\beta = 0$ occurs when T is contained in a non-Abelian compact group, as we noticed earlier, because a non-commutative simple compact Lie algebra cannot have a two-dimensional representation. Finally, up to interchanging the basis vector of each T-invariant plane in $\mathfrak{h}$, we can assume that α_i is non-negative, for all $i = 1, \dots, m$, and, up to interchanging the ordering of the planes in the basis, we can assume that $\alpha_i \leq \alpha_{i+1}$, for all $i = 1, \dots, m-1$.

In the case where $\mathfrak{h}$ contains two proper T-invariant ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ such that $[\mathfrak{i}_1, \mathfrak{i}_2] = 0$ and $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \mathfrak{h}'$, we say that $\mathfrak{h}$ is *T-decomposable* into the direct sum of $\mathfrak{i}_1$ and $\mathfrak{i}_2$ with amalgamated centre, and we restrict our interest on T-indecomposable Lie algebras $\mathfrak{h}$ of type $\{n, 2\}$. Namely, if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two T-indecomposable nilpotent Lie algebras of type $\{n, 2\}$ such that the action of the group T on the centre $\mathfrak{z}_1$ of $\mathfrak{h}_1$ and on $\mathfrak{z}_2$ of $\mathfrak{h}_2$ coincides, then the direct sum of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ with amalgamated centre is a T-decomposable nilpotent Lie algebra of type $\{n, 2\}$, and any T-decomposable nilpotent Lie algebra of type $\{n, 2\}$ is obtained in this way.

4 Compact derivations of irreducible Lie algebras of type $\{n, 2\}$

Once again, we recall that, if $\mathfrak{k}$ is a simple compact algebra of derivations of the nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$, then it induces on the 2-dimensional commutator subalgebra $\mathfrak{h}'$ the null map, the algebra $\mathfrak{k}$ having no two-dimensional representation. Any element in $\mathfrak{k}$ generates a 1-dimensional compact subalgebra of derivations of $\mathfrak{h}$, thus the coefficient β must be zero, in the case where T is contained in a non-commutative compact group of automorphisms of $\mathfrak{h}$.

4.1 One-dimensional compact subalgebra of derivations of $\mathfrak{h}$

Theorem 1. *Let $\mathfrak{h}$ be a T-indecomposable Lie algebra of type $\{n, 2\}$ and let $\{e_1, \dots, e_n, x, y\}$ be a basis such that $\mathfrak{t}$ operates on $\mathfrak{h}$ as in (2). If $\beta = 0$, then n is even, $\alpha_1 = \cdots = \alpha_m$, and the pair (A, B) is the realification of a pair of complex skew-Hermitian matrices $(\widehat{A} = iH_1, \widehat{B} = iH_2)$, where, up to a change of basis, (H_1, H_2) is one of the four pairs in (1).*

Proof. With respect to the chosen basis, we define the pair of alternating matrices $\Big(A = (a_{ij}), B = (b_{ij})\Big)$ by putting

$$[e_i, e_j] = a_{ij}x + b_{ij}y.$$

Write for $t \in \mathbb{R}$

$$\partial_0(t) := \begin{cases} (\alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) & \text{for } n \text{ even} \\ (0 \oplus \alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) & \text{for } n \text{ odd,} \end{cases}$$

hence $\partial(t) = \partial_0(t) \oplus \beta t \cdot J$. Since $\mathfrak{t}$ operates as an algebra of derivations of $\mathfrak{h}$, that is,

$$[e_i, e_j]^{\partial(t)} = [e_i^{\partial(t)}, e_j] + [e_i, e_j^{\partial(t)}],$$

for a generator of $\mathfrak{t}$, e. g. for $t = 1$, we get

$$\begin{aligned} \beta B &= \partial_0(1)' A + A\partial_0(1) \\ -\beta A &= \partial_0(1)' B + B\partial_0(1), \end{aligned}$$

and, since $\beta = 0$,

$$\begin{aligned} 0 &= \partial_0(1)' A + A\partial_0(1) \\ 0 &= \partial_0(1)' B + B\partial_0(1). \end{aligned} \tag{3}$$

We arrange the matrices A and B into 2×2 blocks A_{hk} and B_{hk} with $h, k = 1, \ldots, m$ (in the case where $n = 2m + 1$, we denote the 1×2 blocks of the first row with A_{0k} and B_{0k} and we put $\alpha_0 = 0$). Then (3) is equivalent to

$$\begin{aligned} 0 &= -\alpha_h J A_{hk} + \alpha_k A_{hk} J \\ 0 &= -\alpha_h J B_{hk} + \alpha_k B_{hk} J. \end{aligned} \tag{4}$$

(Notice that, since $\alpha_0 = 0$, the above equations still hold, with a slight abuse of notation, in the case where $h = 0$.)

From the equations (4) we deduce that, if it were $\alpha_h = 0 \neq \alpha_k$, then A_{hk} and B_{hk} would be zero, but this would mean that $\mathfrak{h}$ is T-decomposable, a contradiction. It follows that α_h is positive for any $h = 1, \ldots, m$ and $n = 2m$ is even. Considering A and B as split-quaternion matrices, we write $A_{hk} = z_1 + z_2\omega$, $B_{hk} = z_3 + z_4\omega$ for suitable complex numbers z_1, z_2, z_3, z_4. Then the equations (4) give

$$\begin{aligned} (\alpha_h - \alpha_k)z_1 = 0, \ (\alpha_h - \alpha_k)z_3 = 0 \\ (\alpha_h + \alpha_k)z_2 = 0, \ (\alpha_h + \alpha_k)z_4 = 0. \end{aligned} \tag{5}$$

The latter two equations in (5) give $z_2 = 0 = z_4$, that is, A_{hk} and B_{hk} are the realification of two complex numbers. Moreover, from the former two equations we obtain that either A_{hk} and B_{hk} are zero, or $\alpha_h = \alpha_k$. As $\mathfrak{h}$ is T-indecomposable, we exclude the first case, hence we have that $\partial(t) = t \cdot \Big((\oplus \alpha J)\Big) \oplus 0$. Since A and B are the realification of complex $m \times m$ skew-Hermitian matrices $\widehat{A}$ and $\widehat{B}$, and T operates on them as the complex scalar matrix $\alpha i I_m$, up to a suitable change of basis in the m-dimensional complex space, which leaves T invariant, we can assume that $(\widehat{A}, \widehat{B})$ is in the canonical form given in the claim. Since the only T-invariant real planes are $\langle e_1, e_2 \rangle, \ldots, \langle e_{n-1}, e_n \rangle$, each of these pair defines a T-indecomposable nilpotent Lie algebra over the real numbers. □

Remark 2. Up to rescaling the parameter t, we can assume $\alpha_1 = 1$ in the above theorem, but we prefer to leave it, because, in the case where $\mathfrak{h}$ is T-decomposable, different values of α_k can occur (cf. [16], 3.3). For the same reason, we do not simplify the case where $(\widehat{A}, \widehat{B}) = \big(\pm(\alpha i F_\epsilon + i G_\epsilon), \pm i F_\epsilon\big)$, which, by the change of basis $\{x' = x, y' = \alpha x + y\}$ in $\mathfrak{h}'$ would transform in $(\pm i G_\epsilon, \pm i F_\epsilon)$.

Remark 3. For $n = 4$ and $(\widehat{A}, \widehat{B})$ of type $iv)$, with $\epsilon = 1$ and $\beta = -i$, we obtain

$$(\widehat{A}, \widehat{B}) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right),$$

which corresponds to Example 1. Pairs obtained by the direct sum of t copies of this pair correspond to the $(2t + 1)$-dimensional complex Heisenberg algebras. Besides being the only H-type algebra with two-dimensional centre as mentioned in the introduction, these are the only nilpotent real Lie algebras of type $\{n, 2\}$ which are complex Lie algebras (of type $\{\frac{n}{2}, 1\}$), because this is the only linearly dependent pair over $\mathbb{C}$.

4.2 Compact algebras of derivations of Lie algebras of type $\{n, 2\}$

Each of the above T-indecomposable Lie algebras allows a one-parameter compact algebra of derivations, acting, with respect to the given basis, as

$$\partial(t) = t \cdot (\oplus J) \oplus 0,$$

however, the following holds:

Theorem 2. *The only T-indecomposable Lie algebras having a compact non-Abelian Lie algebra of derivations are the ones where the pair (H_1, H_2) is of type iv), which admits $\mathfrak{sp}(1)$ as a maximal compact algebra of derivations.*

Proof. If $\mathfrak{h}$ is of type i) or ii), then applying (3) we see that each element in $\mathrm{Der}(\mathfrak{h})$ is the realification of

$$i \begin{pmatrix} x_{11} & 0 & \dots & 0 \\ x_{21} & x_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_{\epsilon 1} & x_{\epsilon 2} & \dots & x_{\epsilon\epsilon} \end{pmatrix} + \omega \begin{pmatrix} z_{11} & 0 & \dots & 0 \\ z_{21} & z_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ z_{\epsilon 1} & z_{\epsilon 2} & \dots & z_{\epsilon\epsilon} \end{pmatrix}$$

with $x_{ij} \in \mathbb{R}$ and $z_{ij} \in \mathbb{C}$. Since the subalgebra $\mathfrak{s}$ where $x_{ii} = z_{ii} = 0$ is manifestly a solvable ideal, such that $\mathrm{Der}(\mathfrak{h})/\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, a maximal compact subalgebra has dimension one.

Similarly, if $\mathfrak{h}$ is of type iii), then applying (3) we see that the $2(\epsilon+1)$-dimensional ideal $\mathfrak{k}/\mathfrak{h}'$, generated by the last $2(\epsilon+1)$ elements of the given real basis of $\mathfrak{h}/\mathfrak{h}'$, is invariant under each derivation $\partial \in \mathrm{Der}(\mathfrak{h})$ and such that its restriction $\bar{\partial}$ to $\mathfrak{k}/\mathfrak{h}'$, and, respectively, the derivation $\tilde{\partial}$ induced on $\mathfrak{h}/\mathfrak{k}$ are block matrices of the shape

$$\bar{\partial} = \underbrace{X \oplus \cdots \oplus X}_{\epsilon+1}, \qquad \tilde{\partial} = \underbrace{Y \oplus \cdots \oplus Y}_{\epsilon},$$

with $X \in \mathfrak{gl}(2, \mathbb{R})$ and $Y = JX'J$. Since the subalgebra $\mathfrak{s}$ where $X = 0$ is manifestly a solvable ideal, such that $\mathrm{Der}(\mathfrak{h})/\mathfrak{s}$ is isomorphic to $\mathfrak{gl}(2, \mathbb{R})$, a maximal compact subalgebra has dimension one.

Finally, if $\mathfrak{h}$ is of type iv), then applying (3) we see that each element in $\mathrm{Der}(\mathfrak{h})$ is the realification of a split-quaternion matrix of the shape

$$\left(\begin{array}{c|c} -\bar{A}_1 & A_2\omega \\ \hline A_3\omega & A_1 \end{array}\right)$$

where, for $h = 1, 2, 3$, the complex matrices A_h are

$$A_h = \sum_{k=1}^{\epsilon} z_{hk} J_\epsilon(0)^{k-1} = \begin{pmatrix} z_{h1} & 0 & 0 & \\ z_{h2} & z_{h1} & 0 & \ddots \\ z_{h3} & z_{h2} & z_{h1} & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{6}$$

with $J_\epsilon(0)$ the Jordan block of dimension ϵ and eigenvalue $\lambda = 0$. By a conjugation, we put them in the shape $\left(\begin{array}{c|c} -\bar{A}_1 & A_2 \\ \hline A_3 & A_1 \end{array}\right)$, and when we take A_h to be scalar complex matrices of the form

$$A_1 = ixI_\epsilon, \quad A_2 = -\bar{A}_3 = zI_\epsilon,$$

with $x \in \mathbb{R}$ and $z \in \mathbb{C}$, we get the compact subalgebra $\mathfrak{k}$ of the elements of the shape $\left(\begin{array}{c|c} -ixI_\epsilon & zI_\epsilon \\ \hline -\bar{z}I_\epsilon & ixI_\epsilon \end{array}\right)$, which is manifestly isomorphic to $\mathfrak{sp}(1)$.

Notice that the subalgebra defined by taking $z_{h1} = 0$ in (6), for $h = 1, 2, 3$, is indeed the nilpotent radical, and the quotient Lie algebra is manifestly isomorphic to the Lie algebra of matrices of the shape $\begin{pmatrix} z_{11}I_\epsilon & z_{21}I_\epsilon \\ z_{31}I_\epsilon & -z_{11}I_\epsilon \end{pmatrix}$, which in turn is manifestly isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Hence, the compact subalgebra $\mathfrak{k}$ is maximal. $\square$

Remark 4. The above Theorem 2 shows that nilpotent Lie algebras of type $\{n, 2\}$ given by a pair of skew-Hermitian forms (iH_1, iH_2) with (H_1, H_2) of type i), ii), and iii) do not admit a non-Abelian compact algebra of derivations. On the other hand, a *direct sum* of two algebras of type i), and consequently of type ii), admits $\mathfrak{sp}(1)$ as a maximal compact algebra of derivations, operating, with respect to the given basis, as the realification of the matrices:

$$\left(\begin{array}{c|c} -ixI_\epsilon & zI_\epsilon \\ \hline zI_\epsilon & ixI_\epsilon \end{array}\right),$$

with $c_0, c \in \mathbb{R}$, $z_1, \ldots, z_q \in \mathbb{C}$. Also, a direct sum of q ones of type iii) admits $\mathfrak{so}(2q)$ as a maximal compact algebra of derivations, acting on the $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}$, with $n = q(2\epsilon+1)$, acting with respect to the given basis, as block matrices (Δ_{hk}), with $1 \leq h, k \leq q$, where each block Δ_{hk} is in turn a diagonal block matrix of type $iaI_{2\epsilon+1}$, for $h = k$, and of type

$$z_1 I_{2\epsilon+1} + \left(\begin{array}{c|cccc} z_2 & & & & \\ \hline & -z_2 & & & \\ & & \ddots & & \\ & & & & -z_2 \end{array}\right) \omega I_{2\epsilon+1} \in \mathbb{H}_-^{(2\epsilon+1)\times(2\epsilon+1)}$$

for $h < k$. For instance, for $q = 2$ and $\epsilon = 1$, the 12×12 pair (A, B) defining the 14-dimensional real Lie algebra $\mathfrak{h}$ is the realification of the 6×6 pair of skew-Hermitian matrices

$$(\widehat{A}, \widehat{B}) = \left(\left(\begin{array}{ccc|ccc} 0 & i & 0 & & & \\ i & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & i & 0 \\ & & & i & 0 & 0 \\ & & & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|ccc} 0 & 0 & i & & & \\ 0 & 0 & 0 & & & \\ i & 0 & 0 & & & \\ \hline & & & 0 & 0 & i \\ & & & 0 & 0 & 0 \\ & & & i & 0 & 0 \end{array}\right) \right)$$

and each compact derivation is represented, with respect to the given basis, by a matrix of the form

$$\left(\begin{array}{ccc|ccc} ia_1 & & & z_1 & & \\ & ia_1 & & & z_1 & \\ & & ia_1 & & & z_1 \\ \hline -\bar{z}_1 & & & ia_2 & & \\ & -\bar{z}_1 & & & ia_2 & \\ & & -\bar{z}_1 & & & ia_2 \end{array}\right) + \left(\begin{array}{ccc|ccc} & & & z_2 & & \\ & & & & -z_2 & \\ & & & & & -z_2 \\ \hline -z_2 & & & & & \\ & z_2 & & & & \\ & & z_2 & & & \end{array}\right) \omega I_6,$$

with $a_1, a_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$.

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