AN ANALYSIS OF GRADE 11 LEARNERS' LEVELS OF UNDERSTANDING OF FUNCTIONS IN TERMS OF APOS THEORY

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ABSTRACT

This article reports on a study of six Grade 11 learners' levels of understanding of concepts related to the function definition and representation. Task-based clinical interviews were used to elicit the learners' interpretations and reasoning when working with these function-related concepts. Indicators for Action-Process-Object-Schema (APOS) theory conception levels were formulated and used to categorise learners' written and interview responses into conception levels of understanding of the function concept. According to the South African Curriculum Assessment Policy Statement (CAPS) curriculum, Grade 11 learners are expected to be operating at the object and schema levels after instruction. However, the results indicated that learners were operating at the action and process levels as their understanding was characterised by vague definitions of function-related concepts and memorisation of procedures for translating between symbolic and graphical representations.

Keywords: Action-Process-Object-Schema (APOS) theory; clinical interview; conception level; function concept; representation; level of understanding

INTRODUCTION

The function concept is one of the most important concepts in the learning of mathematics (Dubinsky and Harel 1992). Yet, it is considered by many researchers to be one of the least understood and most difficult concepts to master (Eisenberg 1992). To this end, problems concerning its teaching and learning are often confronted (Mann 2000). In this study, we sought to "see" functions through the cognitive lenses of Grade 11 learners by eliciting and analysing their levels of understanding of concepts related to the function definition and representation. The participants in the study were taught the function-related concepts by their teacher who at that time was unaware that he was going to be involved in the study. The study was guided by the following research question: "What are Grade 11 learners' current levels of understandings of the function concept?"

THEORETICAL FRAMEWORK

Action-Process-Object-Schema (APOS) theory was chosen for the study mainly because it provides a framework for analysing the internal mental structures and mechanisms constructed and used by individuals as they are thinking about a mathematical concept (Dubinsky and Wilson 2013). These mental constructions are called actions, processes, objects and schemas. APOS theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept (Maharaj 2010). APOS theory has been used effectively to detect and evaluate learners' levels of understanding of the function concept (Dubinsky and Wilson 2013; Nyikahadzoyi 2006; Weller, Arnon and Dubinsky 2011; Weyer 2010). In the study, APOS theory was used to describe learners' mental constructions that characterise their understanding of the function concept at different conceptual levels as observed in clinical interviews. The four conception levels and their indicators are explained below with particular reference to the function concept.

Action level

At the action level, the individual has an external perception of the mathematical concept and can only carry out transformations via specific external cues and detailed step-by-step procedures (Dubinsky and McDonald 2002; Tziritas 2011). The following are indicators of learners operating at the action level:

- see a function as a relationship between two sets (Dubinsky and Harel 1992);
- can substitute numbers into a function expressed algebraically and then calculate to obtain an answer (Dubinsky and Harel 1992);
- think about a problem in a step-by-step manner and look at one step at a time (Dubinsky and McDonald 2002; Tziritas 2011); and
- exhibit strong tendencies to recall verbatim the definitions (Breidenbach, Dubinsky, Hawks and Nichols 1992).

Process level

At the process level, the individual begins to reflect upon the action which he or she is performing. The individual can also "reflect on, describe, or even reverse the steps" of a transformation on previously learned objects without actually performing those steps. Here, the individual may view the process as an input-output process (Dubinsky and McDonald 2002; Tziritas 2011). Learners operating at the process level exhibit the following indicators:

- look at the word "function" as a verb and see a function as doing something (Dubinsky and Harel 1992);
- see a function as an operation that accepts a given value and returns a corresponding value (Breidenbach *et al*. 1992);
- give a definition of a function that looks at the procedure as a whole with inputs, a process, and outputs (Weyer 2010); and
- look at an equation and see the procedure as a whole without having to plug in all the specific values (Dubinsky and Harel 1992).

Object level

At the object level, the individual reflects on a particular set of processes until he or she can construct transformations on the mathematical concept. Once this is achieved the individual is said to be at the object level (Dubinsky and McDonald 2002; Tziritas 2011). Learners operating at the object level exhibit the following indicators:

- regard the word "function" as a noun (Dubinsky and Harel 1992) and see a function as something that is being acted on (Dubinsky and McDonald 2002); and
- carry out actions, resulting in some kind of transformation on a function (Dubinsky and Harel 1992).

Schema level

At the schema level, objects and processes are interconnected in the individual's mind to construct schemas. The schema level is the highest level of understanding and schemas vary from person to person since the various connections can differ (Dubinsky and McDonald 2002; Tziritas 2011).

These four conception levels of APOS theory were used to investigate what learners understand of concepts related to functions and helped to determine the level at which a particular learner was operating. As an individual's conception changes from one level to the other, his/her understanding deepens and the lower level is encompassed within the higher level.

RESEARCH DESIGN

Since learners' understanding varies, it means that, to explore such understanding requires an in-depth study of a few learners. As such, the case study research method was deemed appropriate for the study. The participants were six Grade 11 learners of mixed ability who were purposively selected using the teacher's record of marks and from those who were willing to participate in the study. The sample may not be representative and their interpretations may not be generalisable, because this is not the primary concern of such sampling, rather the concern is to acquire in-depth information. Gender balance was also purposively sought from the group of volunteers. Ethical clearance was applied and granted from the university concerned.

Data collection instruments and methods

The individual in-depth, open-ended, task-based qualitative clinical interviews conducted with the learners in the sample were the primary data collection method. Hunting and Doig (1997) state that centering the dialogue on a task or problem gives the subject every opportunity to display behavior from which to infer which mental processes are being used when thinking about a task or solving a problem. A prepared interview guide was used to give structure to the interviews and also made it easy to organize and analyze the interview data. The interview questions were categorized into APOS theory conception levels which were justified by matching the demands of each question with the description and indicators of each conception level. The interviews were audio recorded and transcribed. The interview questions related to the function definition were as follows:

- What do you think of when you hear the word function in mathematics? (action level – the learner would have to recall the memorised definition of a function).
- Using your own words and any diagrams you may need to express your ideas, explain the meaning of the word function (process level – the learner would have to think about a relationship between an input and output variable).
- List and explain any special properties of a function that you can recall and explain how you would illustrate them (action-process level – the learner would have to recall verbatim the properties of a function and think about the transformation process which is used in changing input into output).
- Give me two examples of functions and two examples of non-functions (action-process level – the learner would have to think of an example of data from an input and an output variable).
- How do you distinguish a function from a non-function? (process-object level – the learner would have to think of using representations of a function like an equation or graph to distinguish a function from a non-function).
- Explain in your own words what you understand by an independent variable and a dependent variable? (process level – the learner would have to think about a relationship between input – independent variable and output – dependent variable).
- How do you identify the independent and dependent variables in a given functional relationship? (process-object level – the learner would have to identify these without an external cue and to reflect on a set of possible inputs and outputs and explain how they are connected).
- A function has a domain and a range (co-domain). Explain in your own words the meaning of a domain and a range (process level – the learner would have to explain the connection between inputs and outputs in a functional relationship).

- Where do you use functions in real life? You can use an example to explain the application of functions in real life (object level - the learner would have to identify and reflect on a set of possible inputs and outputs and explain how they are connected).

The learners had also written the June 2011 mathematics examination which had two questions that tested concepts related to functions. Thus, the learners were solving the same problems for the second time. Question 7 in Figure 1 tested the learners' understanding of the calculation of intercepts, determination of asymptotes from a given equation; and switching from the symbolic representation (equation) to the graphical representation. Learners at different levels of thinking about functions will approach this question differently.

QUESTION 7

Given $f(x)=\dfrac{1}{x-4}+2$

7.1	Calculate the co-ordinates of the *x* and *y* intercepts of *f*.	(4)
7.2	Determine the equations of the asymptotes of $f(x)$.	(2)
7.3	Sketch the graph of $f(x)$ showing all the critical points.	(4)

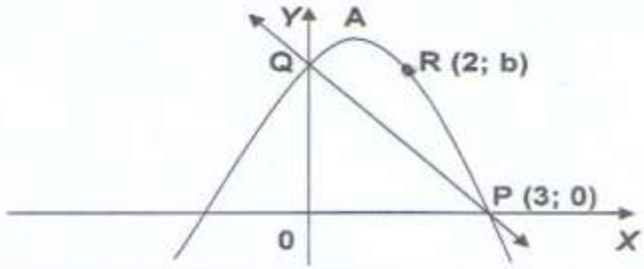
Figure 1: Extract from the Mathematics Paper 1 (June 2011)

On the other hand, the purpose of question 8 (see Figure 2) was to elicit the learners' understanding of the coordinates of the intercept at Q; the gradient of a straight line given two points on the line; the coordinates of a turning point from a given equation, in this case the action conception; and being able to switch from the graphical representation of a function to the symbolic representation, namely, process, object and schema conceptions.

QUESTION 8

The sketch below, not drawn to scale, shows the graphs of the functions defined by: $h(x) = -2(x-3)(x+1)$ and $g(x) = mx+c$.

where A is the turning point of $h(x)$ and R(2;b) is a point on $h(x)$



8.1	Calculate the coordinates of the turning point A.	(3)
8.2	Calculate the coordinates of Q.	(1)
8.3	Determine the numerical values of m and b.	(4)
8.4	Write down the equation of $g(x)$.	(1)

Figure 2: Extract from the Mathematics Paper 1 (June 2011)

The nature of questions 7 and 8 compelled the researcher to generate specific indicators for the action, process, object and schema level of APOS theory using the descriptions and indicators of these conception levels in the theoretical framework. The following are indicators of APOS theory created specifically for the function-related concepts:
Learners operating at the action level can:

- repeat the explanation of intercepts, asymptotes and turning points (critical points) just as they were given in class;
- use rules without reason, for instance, they say "this is what we do or what we were told"; and
- repeat the only examples and explanations of the procedures as they were given in class.

Learners at the action level cannot:

- formulate their own example of a function or non-function;
- distinguish a function from a non-function;
- explain how and why the procedure they use works;
- explain why at the *x*-intercept $y = 0$ and at the *y*-intercept $x = 0$; and
- explain why at the turning point the *x*-coordinate = $\frac{-b}{2a}$, nor they do know how this formula came about.

Learners operating at the process level can:

- explain the meaning of critical points;
- easily calculate the critical points;
- give partial explanations of their procedures;
- sketch the graph of a given equation by first calculating the critical points and plotting them; and
- think of graph or curve sketching as an entire activity and internalise the procedure.

Learners at the process level cannot:

- use the set-theoretical definition of a function to distinguish a function from a non-function;
- create their own example of a function or non-function;
- link the vertical line test with the one-to-one property of the function concept;
- explain why they take the steps of the procedures that they use, such as finding the critical points;
- attempt questions which require the use of a sketched graph;
- use the critical points they identify on a sketched graph to determine the equation of that graph; and
- see that critical points connect the graph and its equation.

Learners operating at the object level can:

- look at the graphical representation of a function and verify whether it is a function or not by using the vertical line test;
- explain why a graph which passes the vertical line test represents a function;
- interpret and relate parts of algebraic expressions or equations representing functions;
- describe how they transform functions and predict how functions are transformed by looking at the graphs of transformed functions and arriving at conclusions of the properties of graphs relating to different equations;
- identify critical points on a drawn graph and write down their coordinates;
- readily relate symbolic and graphical representations of some basic functions; and
- relate functions with transformations of functions.

Duval (2006) argues that it is not possible to access mathematical objects directly; hence, they can only be accessed by means of a specific representation, and as a consequence, learners may confuse a mathematical object with the specific representation of the object used to access it. The notion of object in APOS may be thought to be consistent with this view. This means that learners having an object conception of a specific mathematical notion do so in a way that is independent of representation. The function "object" being represented is the same be it given either by a formula or by a graph. While it is true that generally learners will find it easier to convert from a symbolic to a graphical representation than the other way around, it would be expected that learners with an "object" conception of function should be able to consider both its symbolic and geometric representations as different ways of looking at the same mathematical "object". Thus, learners may conceivably have an object conception of the general notion of function but may not have related this object conception to other processes for specific classes of functions. These learners would have an object conception of function which they have yet to organize into a coherent schema for functions.

Learners operating at the object level cannot:

- use the logical definition of the function concept in formulating examples and non-examples;
- use the logical definition to determine whether a given relation is a function or non-function;
- easily switch from graph to equation;
- link the critical points located on the drawn graph and the ones they calculated using the equation; and
- see that the critical points connect the graph and its equation.

Learners at the schema level can:

- use the logical definition of the function concept in formulating examples and non-examples;
- use the logical definition to determine whether a given relation is a function or a non-function;
- switch from graph to equation and from equation back to graph;
- link the critical points located on the drawn graph and the ones calculated using the equation; and
- see that the critical points connect the graph and its equation.

Learners having an object conception of the general notion of function will progress to relating this notion and organizing it with new function-related processes and objects being studied (such as organizing functions in classes including linear, quadratic and rational functions, and relating the general notion of function to the specific properties of functions in each of these classes) into a function schema. Hence, it is reasonable to state that learners limited to an object understanding of the general notion of function might not "link the critical points located on the drawn graph and the ones they calculated using the equation and see that the critical points connect the graph and its equation". This is so because critical points of quadratic functions (the only critical points considered in problems 7 and 8) are obtained using processes and properties specific to this class of functions and do not immediately follow from the general notion of function. Hence, the learners would have needed to have expanded their understanding of the general notion of function to include such processes and properties, that is, would be referring to the schema of functions rather than to solely the general notion of function.

Each sub-question of questions 7 and 8 was allocated to an APOS theory conception level depending on the thinking structures required to answer it. Table 1 summarises the APOS theory conception levels for all the sub-questions of questions 7 and 8 of the June 2011 examination.

Table 1: Content analysis of instrument 1 (June examination)

Question	What concepts are covered?	APOS level/s
7.1	Calculation of the coordinates of the x- and y- intercepts	A and P
7.2	Determination of equations of asymptotes	A and P
7.3	Sketching of graph	A, P and O
8.1	Calculation of coordinates of turning point	A and P
8.2	Determination of the coordinates of the y-intercept	A
8.3	Calculation of gradient of straight line	A
8.4	Determination of equation of line from drawn graph	P

Key: A- Action, P- Process, O-Object, S-Schema

The learners were under no time constraints to complete the tasks. Each learner was given a copy of question 7 and instructed to answer the questions in their own way. They were provided with only a pencil and paper. Once it was estimated that the learners had finished their solutions, they were interviewed individually regarding the manner in which they approached the questions. The learners were also asked to think about other ways to answer the same questions. The procedure for question 7 was repeated for question 8.

Validity

In the research process it is important to provide checks and balances to maintain acceptable standards of scientific inquiry by addressing the need for rigorous data collection and methods of analysis (Bowen 2005). In the current study, rigor and trustworthiness were enhanced by using triangulation of three methods in collecting data, namely: task-based clinical interviews; participant observation; and document reviews in the form of learners' written work.

Internal validity refers to the quality of data collection and soundness of reasoning leading to the conclusions (Gravemeijer and Cobb 2001). To ensure the internal validity of the study, important episodes were analysed with multiple theoretical instruments of analysis, in other words, theoretical triangulation. External validity is mostly interpreted as the transferability of results (Confrey 2003). The question is how the results can be transferred from specific contexts so as to be useful for other contexts. The challenge is to present the results in such a way that others can adjust them to their local contexts (Barab and Kirshner 2002). This implies that the transferability and viability of the study results can better be judged in the future if applied in other situations.

RESULTS AND DISCUSSION

Research question: What are Grade 11 learners' current levels of understanding of the function concept?

This question was answered in two parts, first, under function definition and, second, under function representation.

Part 1: What do the learners understand of the function definition?

This research question was answered by analysing learners' responses with respect to what they said a function was; the examples and non-examples that they formulated; and the ways they used to identify the dependent and independent variables of a function. The examples of functions and non-functions that the learners gave could indicate the extent to which they understood the function concept. If they were able to formulate examples of functions and non-functions, it implied that they could recognize a function using its properties. This ability can in turn help to identify the dependent and independent variables in a given function and then distinguish a function from a non-function. The following excerpts provide the evidence of learners' level of understanding of the definition of the function concept. When the interviewer asked the learners to say what came to their mind when they heard the word function, or to explain the meaning of the word function in mathematics, they gave me the following responses:

Coco: Something having an input, the output must have a relationship, like, when you are working somewhere, when you get paid, there must be a correspondence between the money that you get paid and the hours that you work.
Diva: Function is a relationship between two things or variables.
Edy: If I hear the word function it is a relationship between two points.
Mat: Function means the relationship between two variables.
Monga: I think it is a relationship between two friends or intercepts.
Teko: Function is something that tells us about range and domain.

The learners gave the following examples of functions and non-functions:

Coco: when you are working somewhere, when you get paid, there must be a correspondence between the money that you get paid and the hours you work.
Diva: One cell phone has one number. You can't share one number with two phones... you can't use one number for the two cell phones.
Edy: I remember we were told that area of a rectangle is a function of its length and breadth.
Mat: Oh a number that is depending on two numbers like $6 = 2 \times 3$
Monga: The birth of a child is a function.
Teko: A shop owner depends on the customers for buying.

The learners distinguished a function from a non-function as follows:

Coco: A function like a domain can only sometimes correspond to the other element or two, but domain can correspond to one.
Diva: I will say here is a graph, then if it is a true function it has to cut in one, but if it is a non-function this would be like maybe into the circle, then it will cut in two points.
Edy: I am not sure but we were taught in class.
Mat: A function takes one, and then non-function takes many.
Monga: For a non-function one component appear maybe like more than once, then for the other one appear only once for a function, the first component to appear once.

Teko: A function has one domain but can share ranges. A non-function it can have a domain and two ranges.

The learners' gave the following explanations of dependent and independent variables:

Coco: An independent variable, I think is the variable that, even if you can change something, it does not affect it, but the dependent variable, if you affect the independent, also get affected because the dependent depend on the independent variables.

Diva: I think the …, dependent variable is the one that changes and the independent variable do not change.

Edy: Independent variable, okay ja let me talk of a tree, the independents in a tree are the water and the soil. The tree depends on the soil. That is why it is independent. Okay that is the independent. The tree cannot live without the water.

Mat: A dependent is something that cannot really work without maybe the help of something. It depends on something else in order for it to work.

Monga: An example will be like plants needed sunlight for growing so I will take plants as the independent value, it rely on the sun and the sun is the dependent because it rely on itself.

Teko: The first one that the student depends on his parents for money. The other one is independent or a wife does not depend on her husband for money.

The learners identified dependent and independent variables in a function as follows:

Coco: I will be like a function, someone like me, I only have the identity number, I am dependent on the identity number, which is how you can find me, with the identity number.

Edy: We as people we depend on water because you can't live without it and water can't depend on us as people.

Mat: Another example can be a tree. It can also depend on water or soil.

Monga: The apples are dependent on the tree because they can't live on their own but they need a tree.

Teko: A shop owner depends on the customers for buying.

Analysis of the learners' written and interview responses revealed that they understood a function as any relationship (Diva, Edy, Mat and Monga), correspondence (Coco) or connection between inputs and outputs (Teko). The weakness in the learners' understanding was that there was no mention of a dependence relationship between two sets of dependent and independent variables. This incomplete understanding of a function resulted in four of the learners (Teko, Monga, Mat and Diva) failing to formulate correct examples of functions and non-functions. Only two learners (Coco and Edy) managed to formulate correct examples of functions, but failed to give examples of non-functions. The learners' concept images of a function could not enable them to explain and identify the dependent and independent variables when given a function. Based on their responses, five out of the six learners were found to be operating at the action level, while only Coco was operating at the process level for the function definition. These concept images and reasoning of learners are also reported in the literature (Dubinsky and Harel

1992; Breidenbach *et al.* 1992; Hitt 1998; Polaki 2005). They indicate that learners do not understand the key idea (that of dependence relationship) of the function concept.

The learners' responses were categorised into APOS levels depending on the indicators they displayed in their oral and written responses and in terms of what they could do and what they could not do. Table 2 summarises the conception levels at which learners were operating. The interview excerpts complemented the analysis of written responses and gave credence to the allocation of operating levels to the learners.

Table 2: Learners' initial APOS theory conception levels on the function concept

Learner	A	P	O	S	Indicators of APOS theory conception levels
Monga	X				Has the basic idea of a relationship Cannot formulate an example and a non-example Cannot explain and identify dependent and independent variables in his own example
Coco		X			Can use the one-to-one correspondence property Can formulate reasonable examples of functions but not of non-functions Can explain and identify dependent and independent variables
Mat	X				Has the basic idea of a relationship Cannot formulate examples of functions and non-functions Cannot explain and identify dependent and independent variables
Diva	X				Has the basic idea of a relationship Gives a vague example of a function but used the vertical line test to give an example of a non-function Gives shallow explanations of dependent and independent variables
Edy	X				Has the basic idea of a relationship Recalls verbatim examples of functions and non-functions but cannot formulate his own examples Cannot clearly explain and identify dependent and independent variables
Teko	X				Recalls some basic definition aspects Gives vague examples and non-examples of functions Gives vague explanations of dependent and independent variables

Part 2: What do the learners understand by the representation of the function concept?

This question was answered by analysing the learners' responses with respect to what they said a representation of a function was; how they explained and calculated the aspects of a function representation (intercepts, asymptotes, turning points); and how they switched from one representation to the other. The learners were asked to explain how they represent a function and they gave the following responses:

Diva: I can do it by an expression.
Edy: Sort of an equation.
Mat: A function can be represented as an equation, as you can use the table to get the function, and then 'mina' (myself) I favour the one which we use equation, because when you use equation, it becomes more easier and then straight forward.
Teko: Draw a graph by using an equation of a function. You can also write a table to represent your values.

Monga: You can write an equation.
Coco: Okay when you are given a function you represent it in a table and on a graph by using an equation and it's easy to get these using an equation.

Similar to Eisenberg's (1992) findings, the study participants also preferred the equation as a representation of the function concept. This may be because their teacher had taught this representation in isolation without linking it to the other representations. This ability to repeat what was done in class verbatim indicates that the learners were operating at the action level of APOS theory. The following two excerpts from Diva and Teko were sampled as typical learners' responses for questions 7 and 8, respectively:

Diva's response to question 7:

Interviewer: How do you explain an intercept?
Diva: Intercepts are the points you are supposed to plot so you can get your graph.
Interviewer: How do you calculate these intercepts?
Diva: I want to find the y-intercept first by putting $x = 0$ and then I'm going to find the x-intercept by putting $y = 0$ and calculate.
Interviewer: Why put $x = 0$ on y-intercept and $y = 0$ on x-intercept?
Diva : Sir because our teacher said when you calculate intercepts you must always let y or x be equal to zero.
Interviewer: You managed to find the asymptotes; can you explain to me what these are?
Diva: An asymptote, I think is the line where …, which shows us that the graph can only approach, not mean to touch or cross.
Interviewer: Can you explain how you obtained these asymptotes?
Diva: I don't know how to explain to someone how to find it. I say zero is equated to the denominator umm..., I forgot how I calculate like that... $f(x) = \frac{1}{x-4} + 2$ ok, for x, I will take this one, and I say $x - 4 = 0$.
Interviewer: Why do you equate $x - 4$ to 0?
Diva: This is what we were told!
Interviewer: Tell me, how did you sketch this graph?
Diva: I want to show you my axis before writing a …, is like this as I have plotted, and then it will be q, this is y-asymptote. Here I put x, x - axis and the y - axis, then I look for $y = 0$ is here, and $x = 1.75$, I'm going to put it here and for $x = 0$, $y = 3.5$, I think is here, so I check my asymptote, for y is 2, I write a dotted line, for $x = 4$, and for y, so then I join my points.

Analysis of the above dialogue reveals that Diva could not explain what an intercept is, but she could explain the procedure of calculating the intercepts indicating procedural understanding. However, she failed to explain why $x = 0$ on the y-axis and $y = 0$ on the x-axis indicating that she had an incomplete understanding of the procedure of calculating the intercepts which might cause problems later when obtaining these intercepts from a drawn graph. Similarly, Diva determined the asymptotes correctly, but could not explain the procedure indicating that she has insufficient knowledge related to the concept of asymptote. Diva could explain how to sketch the graph using the critical points she had calculated. Her inability to explain concepts and why particular procedures work indicated that she was operating at the action level where she could just follow a procedure without understanding it as also documented in the literature (Polaki 2005).

Teko's response to question 8:

Interviewer: Looking at your solutions to question 8, I can see that you calculated the coordinates of the turning point correctly. What is your meaning of a turning point of a graph?
Teko: Turning point is where my graph turns, goes back where it comes from or same direction where it originates.
Interviewer: Explain to me how you calculated the coordinates of a turning point.
Teko: To calculate the turning point, I used the formula $x = \frac{-b}{2a}$.
Interviewer: Where does this formula come from and what does it say?
Teko: Yes, from the quadratic formula but I'm not sure but this is the *x*-coordinate.
Interviewer: What is Q on the diagram and how did you calculate its coordinates?
Teko: Q is a point on the *x*-axis; we know that on the *y*-axis, the value of *x* is 0, 'umm', so we substitute this value into one of these equations, on this, on coordinates of *x* to find the value of *y*, so the coordinate of Q is (0; 6).
Interviewer: Correct. In 8.3 what is this m and b? How did you find them?
Teko: 8.3 numerical value of m and b....m is the gradient of this line b is on this point and then… the formula of the gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$, the value of b which is 6, the value of m is -2.
Interviewer: Now how do you find the equation of $g(x)$?
Teko: I already calculated the gradient which is -2, so, usually this point b for *y*, it will be 2 for *x*, to find the value of c is 4, which means $g(x) = -2x + 4$.

Based on the above responses, we might conclude that Teko had an idea of what a turning point is, he just memorised that the *x*-coordinate of the turning point is $\frac{-b}{2a}$. Teko could calculate the turning point by substituting numbers into the formula but could not explain why $x = \frac{-b}{2a}$ works and how it came about, which are indicators of the action level of APOS theory (Dubinsky and Harel 1992).

The learners in the study were more familiar with two representations of the function concept, namely, the graph (Teko, Coco) and the equation or expression (Diva, Edy, Mat and Monga). However, from the task-based clinical interviews all six learners could use a given equation to calculate the critical points and to draw the graph, but found it difficult to use a drawn graph to determine its equation. The learners' concept images, reasoning and difficulties indicated that they did not understand the meanings, procedures of calculating the critical points of a graph and the process of translating from the graphical representation to the symbolic representation. These critical points are important to learners' ability to translate from one representation to another. Markovits, Eylon and Bruckheimer (1986) and Zaslavsky (1997) have indicated that translating functions from graphical to algebraic was more difficult than vice versa for learners, as was the case for the participants in the study. They could not use the definitions of the critical points to identify and extract them from the drawn graph. Table 3 summarises the learners' conception levels on the representation of the function concept based on their responses to clinical interview questions.

Table 3: Summary of learners' APOS theory conception levels on the representation of the function concept

	Explanations on aspects of a representation of a function									
Learner	**Preferred representation**	**Intercepts**	**Calculation of intercepts**	**Asymptotes**	**Determination of asymptotes**	**Turning point**	**Determination of coordinates of turning point**	**Procedure of sketching graph**	**Translating from graph to equation**	**APOS conception level in which a learner is generally operating**
Mat	Equation	A	A	A	P	A	P	P	A	Action
Teko	Table	A	A	A	A	A	P	A	A	Action
Coco	Table	P	P	A	P	A	P	P	A	Process
Diva	Expression	A	P	A	P	A	P	A	A	Action
Monga	Equation	A	A	A	A	A	P	A	A	Action
Edy	Equation	A	A	A	A	A	P	A	A	Action

Key: A- Action, P- Process, O- Object and S- Schema

CONCLUSION

Analysis of the learners' interview responses revealed the following weaknesses, difficulties and misconceptions:

- They could only mention that a function is a relationship without explaining the nature of the relationship. Understanding of a function is based on vague examples.
- They could not identify the variables (dependent and independent) in their examples of a function.
- They could not use their definition of the function concept to formulate examples and non-examples of functions.
- They could not use their function definition to determine whether given relationships were functions or non-functions.
- They confused the uniqueness condition of the function definition with the notion of one-to-one correspondence.

The literature also revealed that it is common for learners to have the weaknesses, difficulties and misconceptions listed above (Breidenbach et al. 1992; Dubinsky and Harel 1992; Hitt 1998; Sfard 1992), which indicates that there is a need for intervention to help learners overcome these obstacles to their understanding.

Learners also have a tendency of memorising and carrying out procedures or algorithms without understanding them, which limits their use of these procedures since they can only perform them in one direction. For example, the learners in the current study were able to successfully use the procedures of determining the intercepts, asymptotes and turning points from a given equation, but found it difficult to extract the same critical points from a drawn graph representing the same equation and then to formulate the equation. Schwarz and Hershkowitz (1999) attribute the difficulties in translating from one representational form to another to the fact that different representations of a function have different properties for mathematical work with functions. For example, the critical points can be read from a graph, but can only be calculated from a given equation. As such learners should be helped to read or extract these critical points from graphs and calculate them with understanding. Moreover, it is important for teachers to help learners understand the mathematical procedures before applying them in problem solving. The learners' weaknesses in understanding concepts related to function representation are summarised here:

- They had difficulties in answering questions that refer to a drawn graph. They could not deduce the critical points from the sketched graph and use them to determine the required equations.
- They had difficulties in determining the equation of the function represented by the drawn graph using the indicated critical points. This indicated that the learners did not understand the equivalence between the algebraic and graphical representation of the function concept.
- They could calculate the intercepts but did not know why, at the *x*-intercept $y = 0$ and at the *y*-intercept $x = 0$. They could just follow the procedure without understanding it.
- They memorised the formula $x = \frac{-b}{2a}$ for the turning point without knowing what it means and where it formula comes from.
- They could not tell when a function has an asymptote and what an asymptote means.
- They could easily move from equation to graph, but could not use a drawn graph to find the critical points and to determine the equation. This was probably because moving from an equation to a graph needs more of a procedural understanding, while moving from a graph to an equation requires more of a conceptual understanding.

If teachers could determine exactly what learners' weaknesses are, they could use these as a starting point to teach the function concept and its representations. The focus of successful teaching should be where the learners are using the APOS theory which would enable teachers to take learners to higher levels of understanding.

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