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Advances in statistical distribution theory inspired by communications systems

by

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Declaration

I, Johannes Theodorus Ferreira, declare that this thesis, which I hereby submit for the degree *Philosophiae Doctor* (Mathematical Statistics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE: ....

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Summary

This thesis contributes to statistical distribution theory by developing bivariate- and matrix variate distributions with their origins in the complex elliptical class. These contributions are inspired by the communications systems domain, where the underlying distribution is often assumed to be normal. By proposing the underlying distribution to be from the complex elliptical class allows the practitioner to assume different underlying distributions to alleviate the restriction of normality. The building blocks of the contributions in this thesis are systematically described and motivated. Through these advances and contributions within statistical distribution theory, proposed application within the communications systems field is presented. Key performance metrics are investigated under this complex elliptical assumption, and comparatively explored between two members, namely the normal and t.

Supervisor: Prof. Andriëtte Bekker
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Research outputs

A list of research outputs related to this thesis is given below:

Articles

- **J.T. Ferreira**, A. Bekker, and M. Arashi. 2016. "Bivariate noncentral distributions: an approach via the compounding method". South African Statistical Journal, **50**, pp. 103-122.
- **J.T. Ferreira**, A. Bekker, and M. Arashi. "Advances in Wishart type modelling for channel capacity". Accepted at *REVSTAT Statistical Journal*.
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Conference presentations

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Notation and abbreviations

For the convenience of the reader, this section contains a list of notation and abbreviation used throughout this thesis.

$\mathbb{R}^+$	Positive real line
$\sim$	Distributed as
$\equiv$	Identical to
l	Complex unitary, $\sqrt{-1}$
$\in$	Element of
$\lvert c\rvert$	Absolute value of scalar c
$\sum$	Summation operator
$\int$	Integral operator
!	Factorial
log, lim	Natural logarithm, limit
min, max	Minimum, maximum
$\exp(\mathbf{A})$	Euler's constant e with complex square matrix $\mathbf{A}$ as argument
$\mathrm{tr}(\mathbf{A})$	Trace of the complex square matrix $\mathbf{A}$
$\mathrm{etr}(\mathbf{A})$	$\exp(tr(\mathbf{A}))$ if $\mathbf{A}$ is a complex square matrix
$\mathbb{C}_1^{n\times p}$	Space of all complex matrices of dimension $n \times p$
$\mathbb{C}_2^{p\times p}$	Space of all Hermitian positive definite complex matrices of dimension $p \times p$
$f(\cdot)$	Probability density function
$f_{normal}(\cdot)$	Probability density function
$f_t(\cdot)$	Probability density function
$f^{nc}(\cdot)$	Probability density function of noncentral variable
$F(\cdot)$	Cumulative distribution function
$E(\cdot)$	Expected value operator
$\mathcal{L}$	Laplace transform operator
v	Degrees of freedom
$\det \mathbf{A}$	Determinant of the complex square matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	Inverse of a complex square matrix $\mathbf{A}$
$\mathbf{A}^H$	Conjugate transpose of complex matrix $\mathbf{A}$
$\mathbf{\Lambda}$	Square diagonal matrix with $(\lambda_1, ..., \lambda_p)$ on the diagonal, also indicated by $diag(\lambda_1, ..., \lambda_p)$
$\mathbf{\Delta}$	Noncentral matrix
$\mathbf{I}_p$	Identity matrix of dimension p
$\otimes$	Kronecker product
$\{x\}_{i,j}$	Matrix $\mathbf{X}$ with $(i,j)_{th}$ element $x_{i,j}$
$\mathcal{C}V_{p,n}$	Stiefel manifold of orthonormal p-frames in an n-dimensional space
$U(p)$	Unitary space of size $p \times p$

$\delta(x)$	Dirac delta function
$\Gamma(\cdot)$	Gamma function
$\gamma(\cdot,\cdot)$	Lower incomplete gamma function
$\Gamma(\cdot,\cdot)$	Upper incomplete gamma function
$\mathcal{C}\Gamma_p(\alpha)$	Complex p-dimensional multivariate gamma function
$\mathcal{C}\Gamma_p(\alpha,\kappa)$	Complex p-dimensional multivariate gamma function pertaining to partition κ
$(\alpha)_t$	Pochhammer coefficient
$[\alpha]_\kappa$	Generalised hypergeometric coefficient pertaining to partition κ
$B(\alpha,\beta)$	Beta function
$\Phi_3(a,b;x,y)$	Humbert hypergeometric series of two variables x and y with parameters a,b
$\mathfrak{C}_n^v(\cdot)$	Gegenbauer polynomial of degree n and parameter v
${}_pF_q(\cdot)$	Hypergeometric series with p upper parameters and q lower parameters of real scalar argument
$C_\kappa(\cdot)$	Zonal polynomial of Hermitian matrix argument corresponding to partition κ
${}_r\mathcal{C}F_s(\cdot)$	Hypergeometric series with r upper parameters and s lower parameters of Hermitian matrix argument
${}_r\mathcal{C}F_s^{(p)}(\cdot,\cdot)$	Hypergeometric series with r upper parameters and s lower parameters of double Hermitian matrix arguments of dimension p
$G_{r,s}^{m,n}(\cdot)$	Meijer's G-function

pdf	Probability density function
cdf	Cumulative distribution function
ecdf	Empirical cumulative distribution function
ISCW	Integral series of complex Wishart type
ISSCW	Integral series of singular complex Wishart type
MIMO	Multiple input, multiple output
LOS	Line-of-sight
SNR	Signal-to-noise ratio

Contents

Chapter 1

Introduction

Contents

1.1 The realm of statistical distribution theory

The normal distribution forms the cornerstone of multivariate statistical methods. It exists in many forms - univariate, multivariate, matrix variate - and in most cases, the theoretical results in literature relating thereto are relatively simple; along with (mostly) straightforward computational implementation. Furthermore, in several cases within multivariate statistical methods the limiting distribution of several statistics is approximately normal. In essence, together with the central limit theorem, the normal distribution in general acts as a reasonable approximation to the distribution of these statistics. If the approximation of the normal is not sufficient for the practitioner, the exact distributions of these statistics in many cases remain known and implementable, albeit generally with greater computational challenges than that of the normal counterpart. In the univariate, multivariate, and matrix variate cases, the t- , beta-, gamma-, Wishart-, and Dirichlet distributions take up important positions alongside the normal within the realm of statistical distribution theory.

However, many times data display characteristics that do not align with the notion of normality; such as exhibiting heavier tails, whilst still maintaining other attractive characteristics; such as symmetry. To account for such characteristic differences, elliptical distributions have been proposed as viable candidates that constitute a broader class of distributions which still contains the normal distribution as a special case. Much research has been done during recent decades to provide alternatives to the well-studied normal model in both real- and complex contexts, and is elaborated on in subsequent sections in this chapter. It is this frame of mind of elliptical distributions which motivates further study under this assumption, aiming to alleviate the restriction of normality.

In this thesis, particular focus will be on the class of complex elliptical distributions. From this complex elliptical origin, new distributions within the univariate-, bivariate-, and matrix variate domain are derived. The statistical distribution theory literature is advanced with these contributions of complex elliptical results (with emphasis on bivariate- and matrix variate distributions). These advances are inspired from a communications systems point of view, where the assumption of underlying normality is known and well-studied. This assumption however has been questioned, but little literature addresses a change from underlying normal to that of another reasonable underlying distribution within this domain. This thesis aims to relieve the restriction of underlying normality to that of any member of the complex elliptical class - specifically to allow for an underlying t distribution. In this thesis, particular performance measures in the communications systems environment are derived under a complex elliptical assumption, and comparatively investigated between the well-studied normal case as well as for an underlying t distribution.

1.2 Rationale for study

This section describes the overarching motivation and inspiration for this study in statistical distribution theory that emanates from the communications systems domain.

In the communications systems discipline, fading channels are characterized as statistical distributions used to model and describe the signal degradation from the transmitter to the receiver of wireless signals. Certain assumptions such as geographical area, type of transmitters or receivers etc., give rise to certain distributions being preferred to describe the fading of signals. Some of these include the Rayleigh and Rician distributions (see Miller (1974)), Hoyt distribution (see Hoyt (1947)), and Nakagami (see Nakagami (1960)) distribution. de Souza and Yacoub (2008) mentioned that the basis for assuming Rayleigh fading, for example, in communications systems, is derived based on the assumption that from the central limit theorem for large number of partial waves, the resultant process can be decomposed into two orthogonal zero mean and equal standard deviation normal random processes. This usual complex matrix variate normal assumption, say for some matrix variable $\mathbf{X} \in \mathbb{C}_1^{n\times p}$, is essential as it forms the basis of mathematical reasoning and structure in the design of communications systems. Not only does this assumption link to the type of fading model assumed to be in place, but this distributional assumption is vital for the mathematical derivation of performance measures of communications systems. To quote Bury (1999):

> Engineers face numerous uncertainties in the design and development of products and processes. To deal with the uncertainties inherent in measured information, they make use of a variety of statistical techniques.

These performance measures are derived via functions of $\mathbf{X}$; such as $\mathbf{S} = \mathbf{X}^H\mathbf{X} \in \mathbb{C}_2^{p\times p}$, the complex Wishart distribution, or their multivariate counterparts, such as the joint diagonal distribution of the elements of $\mathbf{S}$ which is multivariate gamma distributed. However, it is crucial to note that this assumption of complex normal is an approximation and the restriction of the complex normal distribution is unnecessary - it is not always a large number of interfering signals. A more general assumption than normal may not be that far from reality (see also Ollila et. al. (2011)).

The question thus arises of how to possibly alleviate this restriction of normality, and then, what other underlying distribution would be a reasonable contender for the well known normal model? In mathematical statistics, the t distribution has since its inception been known to approximate the normal distribution within reason. The t distribution, in principle, has particular relevance within multivariate statistical methods- and other applications, even though results pertaining to its use is not as widely derived or available as the normal counterpart. Kotz and Nadarajah (2004) and Ahsanullah et. al. (2014) discusses the importance of the t distribution as realistic alternative to that of the normal, due to its retention of attractive properties such as symmetry but with tail behaviour suitable for modeling real-world data with improved precision.

Table 1.1 contrasts the percentiles of the standard normal distribution with that of the t distribution for varying degrees of freedom.

CDF	t_5	t_{10}	t_{15}	t_{30}	$N(0,1)$
0.90	3.3649	2.7638	2.6025	2.4570	2.3263
0.95	2.0150	1.8125	1.7531	1.6970	1.6449
0.99	1.4759	1.3722	1.3406	1.3100	1.2816

Table 1.1 Percentiles of $N(0,1)$ and t distribution for varying degrees of freedom

Both the normal- and t distribution are members of a broader class of distributions that will be the focus of this thesis as underlying model: the elliptical class of distributions (Arashi et. al. (2012)). To this end, generalisation of the underlying complex normal distribution to that of the complex elliptical distribution may prove powerful, as it provides an opportunity to the practitioner to change the underlying distribution to that of any member of the complex elliptical class. To achieve this generalisation and subsequent proposition of new models, the weighted representation of the complex elliptical distribution as introduced by Chu (1973) and studied by Provost and Cheong (2002) is utilised. However, specific focus will be on the complex t distribution, as reasonable and meaningful candidate versus the well-studied complex normal distribution. Notably, Choi et. al. (2007) introduced the complex matrix variate t distribution into the communications systems environment to model severely fading multiple input, multiple output (MIMO) channels. Thus, by considering the t distribution as underlying distribution in communications systems is of interest.

By this development, the generalisation of the known- and published complex normal results to the complex elliptical distribution contributes to the discipline of statistical distribution theory, whilst proposing new results under a complex t distribution assists the development and stimulates further research within the communications systems domain.

1.3 Statistical origins

In this section, the main statistical tools- and distributions are described which act as fundamental building blocks of this thesis. In addition, a brief review on select published literature within statistical distribution theory and communications systems is given to provide a holistic overview of concepts in this thesis.

1.3.1 The elliptical class

Gupta et. al. (2013) describes the elliptical distributions as a suitable alternative to the usual normal model; especially in the context of multivariate statistical analysis. The matrix variate component of the elliptical distributions is particularly useful: to still have results within the matrix variate domain pertaining to studying the sample covariance matrix; describing repeated measurements on multivariate variables, whilst having the option to depart from the normal model's comparatively stringent assumptions of tail behaviour.

Krishnaiah and Lin (1986) studied the complex elliptical class of distributions, with particular mention of the normal- and t distributions as members. Some important properties are studied; such as the characteristic function and the stochastic representation of a complex elliptical random variable. Sutradhar and Ali (1989) considered the real matrix variate elliptical distribution as a generalisation of its normal counterpart, together with the t distribution as an important subclass. A useful reference and bibliographic overview of elliptical distributions' early development and research can be found in Chmielewski (1981). Diaz-Garcia and Gutierrez-Jaimez (2011) focussed on compound and scale mixtures from an elliptical viewpoint with particular focus on vector- and spherical distributions of hypergeometric type. Furthermore, Diaz-Garcia et. al. (2002), Diaz-Garcia and Leiva-Sanchez (2005), Diaz-Garcia and Gutierrez Jaimez (2006), and Caro-Lopera et. al. (2010) contributed to the theory of matrix variate elliptical models, although the computational use many of their proposed models remain limited.

Chu (1973) provided a representation of an elliptical probability density function (pdf) as an integral series of normal pdfs. What is especially useful of this representation is that the elliptical pdf can be viewed as a weighted representation with a scale change in variance of the normal pdf. Provost and Cheong (2002) studied the same but in particular for the complex matrix variate case with specific focus on Hermitian quadratic forms. The advantage of this representation of a distribution with its origin in the elliptical class, is the convenience for the practitioner to change the underlying distribution to any member of the elliptical class via a weight function, denoted by $\mathcal{W}(t)$.

This thesis focus on the complex matrix variate elliptical class, represented as a scale mixture of complex matrix variate normal pdfs (analogous to Chu (1973), and similarly to Provost and Cheong (2002)). Specific attention is given to the complex matrix variate normal distribution, and the complex matrix variate t distribution. To assist with the description of the complex matrix variate elliptical class, the complex matrix variate normal distribution and the complex matrix variate t distribution is presented first.

Firstly, $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ follows the complex matrix variate normal distribution, denoted by $\mathbf{X} \sim \mathcal{C}N_{n\times p}(\mathbf{M}, \mathbf{\Phi} \otimes \mathbf{\Sigma})$ with matrix parameters $\mathbf{M} \in \mathbb{C}_1^{n\times p}$, $\mathbf{\Phi} \in \mathbb{C}_2^{n\times n}$, $\mathbf{\Sigma} \in \mathbb{C}_2^{p\times p}$, if it has the following pdf (see James (1964)):

$$f(\mathbf{X}) = \frac{1}{\pi^{pn} \det(\mathbf{\Phi})^p \det(\mathbf{\Sigma})^n} \operatorname{etr}\left[-\left(\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})^H \mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\right)\right] \tag{1.1}$$

with mean $E(\mathbf{X}) = \mathbf{M}$ and covariance $cov(\mathbf{X}) = \mathbf{\Phi}\otimes\mathbf{\Sigma}$.

Secondly, $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ follows the complex matrix variate t distribution, denoted by $\mathbf{X} \sim \mathcal{C}t_{n\times p}(\mathbf{M}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, v)$, with matrix parameters $\mathbf{M} \in \mathbb{C}_1^{n\times p}$, $\mathbf{\Phi} \in \mathbb{C}_2^{n\times n}$, $\mathbf{\Sigma} \in \mathbb{C}_2^{p\times p}$ and degrees of freedom $v > 0$, if it has the following pdf (see Arashi et. al. (2012)):

$$f(\mathbf{X}) = \frac{v^{np}\mathcal{C}\Gamma(np+v)}{\pi^{np}\mathcal{C}\Gamma_p(v)} \left\{1 + \frac{1}{v}\operatorname{tr}\left(\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})^H \mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\right)\right\}^{-(np+v)} \tag{1.2}$$

where $\mathcal{C}\Gamma_p(v)$ denotes the complex p-dimensional multivariate gamma function (see Result D.47).

Finally, $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ follows the complex matrix variate elliptical distribution (whose distribution is absolutely continuous), denoted by $\mathbf{X} \sim \mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, h)$, with matrix parameters $\mathbf{M} \in \mathbb{C}_1^{n\times p}$, $\mathbf{\Phi} \in \mathbb{C}_2^{n\times n}$, $\mathbf{\Sigma} \in \mathbb{C}_2^{p\times p}$, if it has the following pdf (see Arashi et. al. (2012)):

$$f(\mathbf{X}) = \frac{1}{\det(\mathbf{\Phi})^p \det(\mathbf{\Sigma})^n} h\left[-\operatorname{tr}\left(\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})^H \mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\right)\right] \tag{1.3}$$

where $h(\cdot)$ is the corresponding generator function.

The following lemma is due to Provost and Cheong (2002) (resulting from Chu (1973)), and forms the cornerstone of distributional representation of the complex matrix variate elliptical distribution in this thesis.

Lemma 1.3.1 *If* $\mathbf{X} \sim \mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, h)$ *with pdf* $f(\mathbf{X})$ *(see (1.3)), then there exists a scalar weight function* $\mathcal{W}(\cdot)$ *on* $\mathbb{R}^+$ *such that:*

$$f(\mathbf{X}) = \int_{\mathbb{R}^+} f_{\mathcal{C}N_{n\times p}(\mathbf{M},\mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})}(\mathbf{X}|t)\mathcal{W}(t)\,dt \tag{1.4}$$

where

$$f_{\mathcal{C}N_{n\times p}(\mathbf{M},\mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})}(\mathbf{X}|t) = \frac{1}{\pi^{pn}\det(\mathbf{\Phi})^p \det(t^{-1}\mathbf{\Sigma})^n}\ \operatorname{etr}\left[-\left(t\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\right)\right]$$

is the pdf of $\mathbf{X}|t \sim \mathcal{C}N_{n\times p}(\mathbf{M}, \mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})$ *(see (1.1)), and the weight function* $\mathcal{W}(\cdot)$ *is given by:*

$$\mathcal{W}(t) = \pi^{np}t^{-np}\mathcal{L}^{-1}\left\{h\left[-\operatorname{tr}\left(\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\right)\right]\right\}$$

where $\mathcal{L}$ *is the Laplace transform operator.*

Proof. Let $s = \operatorname{tr}\left(\mathbf{\Phi}^{-1}(\mathbf{X}-\mathbf{M})^H\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\right)$. Using (1.3) we have

$$\begin{aligned} f(\mathbf{X}) &= \det(\mathbf{\Phi})^{-p}\det(\mathbf{\Sigma})^{-n}\,h[-s] \\ &= \det(\mathbf{\Phi})^{-p}\det(\mathbf{\Sigma})^{-n}\,\mathcal{L}\left[\mathcal{W}(t)\,\pi^{-np}t^{np}\right] \\ &= \det(\mathbf{\Phi})^{-p}\det(\mathbf{\Sigma})^{-n}\int_{\mathbb{R}^+}\pi^{-np}t^{np}\exp(-ts)\,\mathcal{W}(t)\,dt \\ &= \int_{\mathbb{R}^+}\pi^{-np}\det(\mathbf{\Phi})^{-p}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\exp(-ts)\,\mathcal{W}(t)\,dt \end{aligned}$$

and the result follows. ∎

Remark 1.1 *Under the assumptions of Lemma 1.3.1, using Fubbini's Theorem (see Arashi et. al. (2012)), it follows that:*

$$1 = \int_{\mathbb{C}_1^{n\times p}} f(\mathbf{X})d\mathbf{X} = \int_{\mathbb{R}^+}\mathcal{W}(t)\left(\int_{\mathbb{C}_1^{n\times p}} f_{\mathcal{C}N_{n\times p}(\mathbf{M},\mathbf{\Phi}\otimes t^{-1}\mathbf{\Sigma})}(\mathbf{X}|t)d\mathbf{X}\right)dt = \int_{\mathbb{R}^+}\mathcal{W}(t)\,dt.$$

Thus for a non-negative weight function $\mathcal{W}(\cdot)$*, the function* $\mathcal{W}(\cdot)$ *is a pdf of a non-negative random variable. Therefore Lemma 1.3.1 can only be interpreted as a representation of a scale mixture of complex matrix variate normal distributions. However,* $\mathcal{W}(\cdot)$ *is not always positive and can be negative on some domains (see Provost and Cheong (2002) for some examples). The only limitation of Lemma 1.3.1 is that it defines those complex matrix variate elliptical distributions whose inverse Laplace transform exist. There are some mild sufficient conditions that ensure the inverse Laplace transform exists for most of the well-known complex matrix variate elliptical distributions (see Chu (1973)).*

Remark 1.2 ***Complex matrix variate normal distribution*** *(Arashi et. al. (2012))*
If $\mathbf{X} \sim \mathcal{C}N_{n\times p}(\mathbf{M}, \mathbf{\Phi}\otimes\mathbf{\Sigma})$ *with pdf (1.1), the weight function* $\mathcal{W}(\cdot)$ *in Lemma 1.3.1 is given by:*

$$\mathcal{W}(t) = \delta(t-1) \tag{1.5}$$

where $\delta(\cdot)$ is the dirac delta or impulse function having the property $\int_{\mathbb{R}^+} f(x)\delta(x)dx = f(0)$, for every Borel-measurable function $f(\cdot)$.

Remark 1.3 ***Complex matrix variate t distribution*** *(Arashi et. al. (2012))*
If $\mathbf{X} \sim \mathcal{C}t_{n\times p}(\mathbf{M}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, v)$ with pdf (1.2), the weight function $\mathcal{W}(\cdot)$ in Lemma 1.3.1 is given by:

$$\mathcal{W}(t) = \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) \tag{1.6}$$

where $v > 0$ denotes the degrees of freedom and $\Gamma(\cdot)$ denotes the gamma function (see Result C.5).

For some select results, the univariate real elliptical distribution is also of interest. The result here is analogous to the main result from Chu (1973). If X follows an elliptical distribution with mean m, variance σ^2 and with generator function h, and has pdf:

$$f(x) = \frac{1}{\sigma} h\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

then there exists a scalar weight function $\mathcal{W}(\cdot)$ on $\mathbb{R}^+$ such that:

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^+} \frac{1}{\sqrt{2\pi}\sigma t^{-\frac{1}{2}}} \exp\left(-\frac{(x-m)^2}{2\sigma^2 t^{-1}}\right) \mathcal{W}(t)dt \\ &= \int_{\mathbb{R}^+} f(x|t)\,\mathcal{W}(t)dt \end{aligned} \tag{1.7}$$

where $f_{N(m,\sigma^2 t^{-1})}(x|t)$ is the pdf of a normal distribution with mean m and variance $\sigma^2 t^{-1}$. In this case, the distribution of X is denoted by $E(m, \sigma^2, h)$.

1.3.2 Bivariate- and matrix variate distributions

In statistics, particularly multivariate analysis, the majority of results rely on the assumption of underlying normality. Some of these examples includes the derivation of the Wishart distribution which plays a vital role in inference regarding the sample covariance matrix in multivariate normal settings (Gupta and Nagar (2000)); the study of the behaviour of the joint eigenvalues of such Wishart distributed matrix random variables (James (1964)); repeated measurements on multivariate variables (Gupta and Nagar (2000) and references therein); amongst others. A brief review on literature that is needed as background for this study within statistical distribution theory and communications systems is now discussed.

Bivariate distributions have been extensively explored in the literature (see for example, Balakrishnan and Lai (2009), Chen et. al. (2014), and Mahdavi et. al. (2017)). Within the context of communications systems, bivariate distributions have received significant attention (see for example, Nakagami (1960), Reig et. al. (2002), Pibongungon (2005), Mendes and Yacoub (2007), de Souza and Yacoub (2008), de Souza et. al. (2012), Lopez-Martinez et. al. (2013), Reig et. al. (2014), Ermelova and Tirkkonen (2014), and Villavicencio et. al. (2016)). In particular, Nakagami (1960) is considered a benchmark paper within the communications systems discipline as it provided a platform for a multitude of research regarding fading- and shadowing distributions emanating from it. The bivariate Nakagami distribution is particularly useful as fading distribution in a wireless environment (Tan and Beaulieu (1997)). Importantly, the bivariate Nakagami distribution is related to the bivariate gamma distribution - this is illustrated later on in this chapter. Bivariate gamma distributions have also been of special interest due to their pliable- and computable mathematical nature, exhibiting satisfactory fits to measured data subjected to multipath/shadowing fading (Pibongungon (2005)). The bivariate noncentral gamma distribution has also been investigated, see for example Knusel and Bablok (1996), Chen (2005), and de Oliveira and Ferreira (2013).

In the past few decades, various monographs have been published which aims to report on the vast amount of literature available in the 20^{th} and 21^{st} century on matrix variate distributions. In particular, published works by Gupta and Nagar (2000) and Gupta et. al. (2013) provide a comprehensive survey of literature and results useful within the matrix variate statistical domain. Gupta and Nagar (2000) covers, amongst others, matrix variate normal distribution and development from a real perspective; and Gupta et. al. (2013) approaches the elliptically contoured matrix variate distribution. However, groundbreaking work in multivariate analysis, and subsequently matrix variate distribution theory has been undertaken by pioneers including, but not limited to: Constantine (1963), James (1964), Hayakawa (1969), Khatri (1969), Hsu (1981), Muirhead (1982), Gross and Richards (1989), Gupta and Varga (1995), Anderson (2003), Ratnarajah (2005), Koev and Edelman (2006), and Dubbs and Edelman (2014). Most of this body of work focuses on the central real or complex Wishart distribution and studies properties thereof. The noncentral Wishart distribution is studied by James (1964), Jayaweera and Poor (2003), McKay and Collings (2005), McKay (2006), and Ordonez et. al. (2009), amongst others.

Much of this literature also focuses on the distribution of the eigenvalues of the Wishart distributed random variables under consideration; in some cases also the distributions of the smallest- and largest eigenvalues. The stochastic behaviour of the eigenvalues from a random matrix almost surely represents the stochastic nature of the entire matrix, based on a relatively smaller number of random variables as compared to the ensemble of a large number of the random variables. James (1964) provided the genesis of the study of eigenvalues from Wishart distributions for both real- and complex cases; the proposed methodology has been widely used and applied in multivariate statistical theory, see for example Muirhead (1982). In a communications systems context, Ordonez et. al. (2009) states that the distributions of the eigenvalues of the adopted channel model is necessary in order to derive and evaluate expressions for performance measures. Several papers have investigated the distribution of the joint eigenvalues of Wishart matrices, including Ratnarajah and Vaillancourt (2003), Ratnarajah (2005), Ratnarajah and Vaillancourt (2005), Jin et. al. (2006), Rui et. al. (2007), and Zhou et. al. (2015).

Of further interest, are the distributions of quadratic forms of real- and complex matrix variates. James (1964), Gupta and Varga (1995), Ratnarajah (2005), and McKay and Collings (2005) have been productive in the research of such quadratic forms, with Al-Naffouri et. al. (2016) having a relatively recent refreshing contribution on the topic. Al-Naffouri et. al. (2016) discussed the importance to determine the distributions of quantities relating to normal random variables, particularly that of the quadratic form. The quadratic form pertaining to normal random variables is indispensable in many applications of multivariate statistical methods, communications systems, and signal processing. Therefore, the study of quadratic forms originating from the complex elliptical class is vital for the advancement of not only statistical distribution theory, but also to that of other fields, such as communications systems. A challenge within distributions of quadratic forms remains the Hayakawa polynomial component when the underlying distribution of the quadratic form is noncentral, which to this day eludes convenient computation and tractable analytic solutions.

1.4 Communications systems

In the following section, a brief overview of the MIMO model specification is given relevant to this thesis.

1.4.1 Rician- and Rayleigh type fading

In this thesis, particular interest lies in Rician- and Rayleigh fading models. These models are inherently statistical distributions in nature, where the behaviour of signals and their degradation can be described by the Rician- and Rayleigh distributions respectively - in a communications system setting, this behaviour is called fading. By substituting the underlying normal assumption to that of elliptical implies that both Rician- and Rayleigh distributions have to be reconsidered from the elliptical viewpoint as well. Rician fading is of interest when there is a direct line-of-sight (LOS) component between transmitters and receivers (see Figure 1.1); mathematically this translates to assuming a nonzero mean for the underlying process; with Rayleigh fading corresponding to a zero mean for the underlying process (see Kang and Alouini (2006)a and Kang and Alouini (2006)b). Figure 1.1 visualises typical signal paths between transmitters and receivers. The assumed type of fading is dependent on the expected path of a signal.

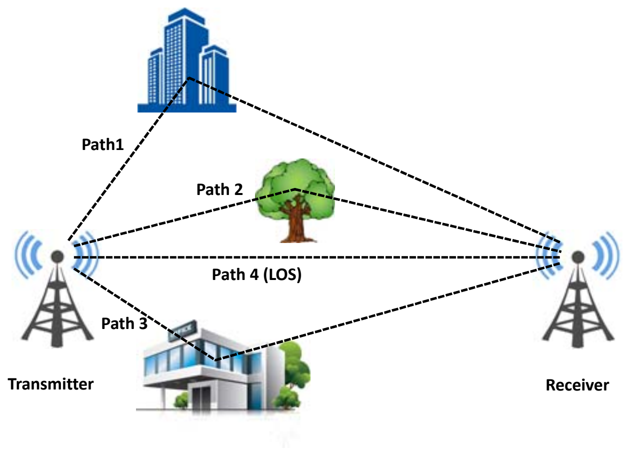


Figure 1.1 MIMO antenna system

The following lemma states the Rician distribution, and is due to Miller (1974). The subsequent results defines these fading distributions with their origin in the complex elliptical class, and is called Rician- and Rayleigh *type* fading respectively.

Lemma 1.4.1 *Consider* $R = \sqrt{X^2 + Y^2}$ *where* X, Y *are independent normally distributed random variables with mean* m_1 *and* m_2 *respectively and common covariance* $t^{-1}\sigma^2$. *The pdf of* R *is given by:*

$$f(r|t) = \frac{r}{\sigma^2 t^{-1}} \exp\left(-\frac{r^2+s^2}{2\sigma^2 t^{-1}}\right) I_0\left(\frac{rs}{\sigma^2 t^{-1}}\right) \tag{1.8}$$

where $r > 0$, $s^2 = m_1^2 + m_2^2$, $\sigma^2 > 0$, *and* $I_0(\cdot)$ *denotes the modified Bessel function of the first kind (see Result C.17). This distribution is called a Rician distribution, or just Rician fading with pdf (1.8).*

Corollary 1.1 *Consider* $R = \sqrt{X^2 + Y^2}$ *where* X, Y *are independent elliptically distributed random variables with mean* m_1 *and* m_2 *respectively, common covariance* σ^2, *and with generator function* h *with pdf (1.7). The* *pdf of R is given by:*

$$\begin{aligned} f(r) &= \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 t^{-1}} \exp\left(-\frac{r^2+s^2}{2\sigma^2 t^{-1}}\right) I_0\left(\frac{rs}{\sigma^2 t^{-1}}\right) \mathcal{W}(t)dt \qquad (1.9) \\ &= \int\limits_{\mathbb{R}^+} f(r|t)\mathcal{W}(t)dt \end{aligned}$$

where $r > 0$, $s^2 = m_1^2 + m_2^2$, $\sigma^2 > 0$, and $f(r|t)$ denotes the Rician distribution with pdf (1.8). This distribution is called a Rician type distribution, or just Rician type fading with pdf (1.9).

Remark 1.4 *When $m_1 = m_2 = 0$, (1.9) simplifies to:*

$$\begin{aligned} f(r) &= \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 t^{-1}} \exp\left(-\frac{r^2}{2\sigma^2 t^{-1}}\right) I_0\left(\frac{r\times 0}{\sigma^2 t^{-1}}\right) \mathcal{W}(t)dt \\ &= \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 t^{-1}} \exp\left(-\frac{r^2}{2\sigma^2 t^{-1}}\right) \mathcal{W}(t)dt \qquad (1.10) \end{aligned}$$

where $r > 0$. This distribution is called a Rayleigh type distribution, or just Rayleigh type fading with pdf (1.10).

Remark 1.5 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (1.9) simplifies to:*

$$f(r) = \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 t^{-1}} \exp\left(-\frac{r^2+s^2}{2\sigma^2 t^{-1}}\right) I_0\left(\frac{rs}{\sigma^2 t^{-1}}\right) \delta(t-1)\, dt.$$

Let $x = t-1$, then $t = x+1$ and $dx = dt$:

$$\begin{aligned} f(r) &= \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 (x+1)^{-1}} \exp\left(-\frac{r^2+s^2}{2\sigma^2 (x+1)^{-1}}\right) I_0\left(\frac{rs}{\sigma^2 (x+1)^{-1}}\right) \delta(x)\, dx \\ &= \frac{r}{\sigma^2} \exp\left(-\frac{r^2+s^2}{2\sigma^2}\right) I_0\left(\frac{rs}{\sigma^2}\right) \qquad (1.11) \end{aligned}$$

for $r > 0$. This distribution is known as the Rician distribution (see Shankar (2012), p. 201).

Remark 1.6 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (1.10) simplifies to:*

$$f(r) = \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 t^{-1}} \exp\left(-\frac{r^2}{2\sigma^2 t^{-1}}\right) \delta(t-1)\, dt.$$

Let $x = t-1$, then $t = x+1$ and $dx = dt$:

$$\begin{aligned} f(r) &= \int\limits_{\mathbb{R}^+} \frac{r}{\sigma^2 (x+1)^{-1}} \exp\left(-\frac{r^2}{2\sigma^2 (x+1)^{-1}}\right) \delta(x)\, dx \\ &= \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad (1.12) \end{aligned}$$

for $r > 0$. This distribution is known as the Rayleigh distribution (see Shankar (2012), p. 198).

1.4.2 Model specification

If a MIMO system is designed with n_t number of transmitters and n_r number of receivers, the MIMO channel is characterised mathematically by an $n_r \times n_t$ channel matrix, $\mathbf{H}$. The $(i,j)^{th}$ entry of $\mathbf{H}$, denotes by $\{h\}_{i,j}$

describes the path between the j^{th} transmitter and the i^{th} receiver. When $\mathbf{H}$ operates over some fading channel, $\mathbf{H}$ is assumed to be a random matrix depending on a variety of factors describing these signal paths. $\mathbf{H}$ is thus distributed according to some probability distribution, most often assumed to be complex matrix variate normal distributed (Ordonez et. al. (2009)). The following table contrasts the usual statistical notation of matrix dimensions with that of a random matrix $\mathbf{H}$ with the notation common in the communications system domain.

Statistical notation	**Communications systems notation**	**Interpretation**
n	n_r	number of receivers
p	n_t	number of transmitters

Table 1.2 Notation of communications system, particular to MIMO

A communications system can be characterised by the following relation:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v} \tag{1.13}$$

where $\mathbf{y}, \mathbf{v} \in \mathbb{C}_1^{n_r \times 1}$, $\mathbf{x} \in \mathbb{C}_1^{n_t \times 1}$ and $\mathbf{H} \in \mathbb{C}_1^{n_r \times n_t}$. In this case, $\mathbf{x}$ denotes the transmitted vector of signals, $\mathbf{y}$ denotes the received vector of signals, $\mathbf{H}$ denotes the channel matrix, and $\mathbf{v}$ denotes a noise component vector. In particular, the complex vector signal of $y_j \in \mathbf{y}$ is the complex j^{th} output signal given by:

$$y_j = \sum_{i=1}^{n_t} h_{ij} x_i + v_j \tag{1.14}$$

where $\{h\}_{i,j} \in \mathbf{H}$ describes the complex channel coefficient between input i and output j, $x_i \in \mathbf{x}$ is the complex i^{th} input signal, and $v_j \in \mathbf{v}$ is a complex noise component corresponding to the complex j^{th} output signal. The following figures visualise (1.13) and (1.14):

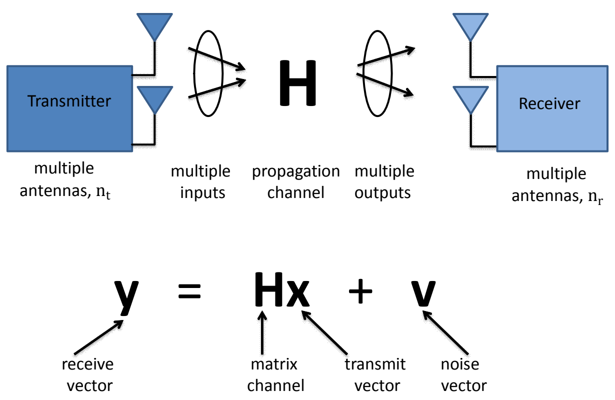


Figure 1.2 MIMO communications system

Usually, the $n_r \times n_t$ channel matrix $\mathbf{H}$ is assumed to follow complex matrix variate normal distribution, thus $\mathbf{H} \sim \mathcal{C}N_{n_r \times n_t}(\mathbf{M}, \mathbf{I}_{n_r} \otimes \boldsymbol{\Sigma})$, and in certain scenarios with the assumption of $\mathbf{M} = \mathbf{0}$. If the fading assumption is challenged from Rician- or Rayleigh fading (see (1.11) and (1.12)) to a more general assumption, such as Rician- or Rayleigh *type* fading (see (1.9) and (1.10)), this infers that the distribution of $\mathbf{H}$ has to be substituted with that of $\mathbf{H} \sim \mathcal{C}E_{n_r \times n_t}(\mathbf{M}, \mathbf{I}_{n_r} \otimes \boldsymbol{\Sigma}, h)$ or $\mathbf{H} \sim \mathcal{C}E_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \boldsymbol{\Sigma}, h)$ respectively. Thus, the channel matrix $\mathbf{H}$ and noise vector $\mathbf{v}$ are independently distributed according the complex matrix variate elliptical distributions, respectively, in other words, $\mathbf{H} \sim \mathcal{C}E_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \boldsymbol{\Sigma}, h)$, and $\mathbf{v} \sim \mathcal{C}E_{n_r}(\mathbf{0}, \sigma^2 \mathbf{I}_{n_r})$.

Remark 1.7 *In practice, $\boldsymbol{\Sigma} \in \mathbb{C}_2^{n_t \times n_t}$ is often assumed to be $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_{n_t}$ when the geographical- or physical spacing of the transmit antennas is sufficiently far from each other to assume no correlation (and hence zero covariance). This assumption is often referred to as "semi correlated" in the literature (Kang and Alouini (2006)a). However, assuming nonzero covariance remains useful to study for correlated scenarios, especially when antennas are not sufficiently spatially separated, or lack scattering (see Kang and Alouini (2006)a). Note that if $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_{n_t} \in \mathbb{C}_2^{n_t \times n_t}$, then:*

$$\det(\boldsymbol{\Sigma})^{n_r} = \det\left(\sigma^2 \mathbf{I}_{n_t}\right)^{n_r} = \sigma^{2n_t \times n_r}. \tag{1.15}$$

By considering the complex matrix variate elliptical representation as given in Lemma 1.3.1, some key results pertaining to communications systems are derived under this complex matrix variate elliptical assumption for $\mathbf{H}$. In particular, the complex matrix variate normal distribution (see (1.5)) and the complex matrix variate t distribution (see (1.6)) may then act as underlying distributions for $\mathbf{H}$. Under this assumption, performance measures relating to this complex matrix variate elliptical assumption for $\mathbf{H}$ can be evaluated between these two candidate distributions. To this effect, key performance measures of communications systems are of interest in this study, and are defined in the following section.

1.4.3 Elements of interest

In this section, some elements of interest relating to communications systems relevant in this thesis is described.

1.4.3.1 Construction of Weibullised gamma type (Nakagami type) distributions

The Nakagami distribution has many applications in wireless communications (see de Souza and Yacoub (2008)). This distribution is derived from a gamma distribution: if $U \sim Gamma(\Omega, m)$ (see Result C.2), then $W = \left(\frac{U}{m}\right)^{\frac{1}{2}} \sim Nakagami(\Omega)$. Note that, in mathematical statistics literature, a transformation such as $\left(\frac{X}{m}\right)^{\frac{1}{\beta}}$ can be described as a Weibullised gamma distribution (see Malik (1967), McDonald and Xu (1995), Bekker et. al. (2000), Gupta and Nadarajah (2004), Chen et. al. (2014)). Thus, the Nakagami distribution would be a special case of such a Weibullised gamma distribution when $\beta = 2$. In this section, a systematic description of deriving such a Weibullised gamma distribution with its foundation in the elliptical class is described and motivated. This distribution is called a Weibullised gamma *type* distribution. From this Weibullised gamma type distribution, a Nakagami *type* distribution follows as a special case. It is worthwhile to note that in the literature, the terms power- and envelope distribution are used often. Within the context of this thesis, these would constitute the gamma type models as power distributions, and the Weibullised gamma type models as envelope distributions (see Simon and Alouini (2005), Shankar (2012)). The systematic construction of the Weibullised gamma type distribution is described in Figure 1.3. The subsequent lemma is useful for gaining access to the exponential distribution (and subsequently, gamma distribution) via the Rayleigh distribution (see (1.12)), from where the construction of the Weibullised gamma type distribution follows when considering the Rayleigh type distribution (see (1.10)).

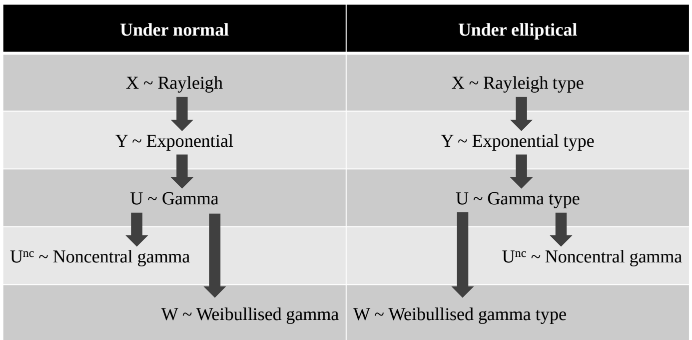


Figure 1.3 Schematic diagram of the construction of Weibullised gamma type distributions

Note that the noncentral gamma type distribution is also of interest in this study, and is elaborated on in Chapter 4, based on methodology proposed by Ferreira et. al. (2016).

Lemma 1.4.2 *Suppose that X follows a Rayleigh distribution with pdf (1.12). The transformation $Y = \frac{X^2\Omega}{2\sigma^2}$ with $\frac{dx}{dy} = \sigma\sqrt{\frac{2}{\Omega}\frac{1}{2}y^{-\frac{1}{2}}}$ results in Y following an exponential distribution with parameter Ω and pdf:*

$$
\begin{aligned}
f(y) &= f\left(\sqrt{\frac{2y\sigma^2}{\Omega}}\right)\left|\frac{d}{dy}\right| \\
&= \frac{\sqrt{\frac{2y\sigma^2}{\Omega}}}{\sigma^2}\exp\left(-\frac{\left(\sqrt{\frac{2y\sigma^2}{\Omega}}\right)^2}{2\sigma^2}\right)\sigma\sqrt{\frac{2}{\Omega}\frac{1}{2}}y^{-\frac{1}{2}} \\
&= \frac{1}{\Omega}\exp\left(-\frac{y}{\Omega}\right)
\end{aligned}
$$

with $y > 0$, denoted $Y \sim Exp(\Omega)$ where $\Omega > 0$.

By using Result C.22, consider the Laplace transform of Y:

$$
\begin{aligned}
\mathcal{L}(z) &= E(\exp(-zY)) \\
&= \int_0^\infty \exp(-zy)\, f(y)\, dy \\
&= \int_0^\infty \exp(-zy)\frac{1}{\Omega}\exp\left(-\frac{y}{\Omega}\right) dy \\
&= \frac{1}{\Omega}\int_0^\infty \exp\left(-y\left(z+\frac{1}{\Omega}\right)\right) dy \\
&= \frac{1}{\Omega\left(z+\frac{1}{\Omega}\right)} \\
&= \frac{1}{1+\Omega z}.
\end{aligned}
\tag{1.16}
$$

This Laplace transform (1.16) acts as a useful thinking tool as part of this motivation to eventually arrive at a Weibullised gamma type distribution. Consider now (1.10). As in Lemma 1.4.2, by applying the transformation $Y = \frac{X^2\Omega}{2\sigma^2}$ on variable X with pdf (1.10) leaves:

$$\begin{aligned} f(y) &= \int_0^\infty \left(\frac{1}{\Omega t^{-1}} \exp\left(-\frac{1}{\Omega t^{-1}} y \right) \right) \mathcal{W}(t)\, dt \\ &= \int_0^\infty f(y|t)\, \mathcal{W}(t)\, dt \end{aligned} \tag{1.17}$$

where $f(y|t)$ is the pdf of an $Exp\left(\Omega t^{-1}\right)$ distribution. This distribution with pdf (1.17) is called an exponential *type* distribution and is denoted by $Exp(\Omega, \mathcal{W}(\cdot))$.

The following theorems and subsequent results are useful in deriving the point of departure for Weibullised gamma type distributions.

Theorem 1.1 *Suppose $Y_1, Y_2, ..., Y_m$ are i.i.d. random variables which is distributed as $Exp\left(\Omega t^{-1}\right)$. Then $U = \sum_{i=1}^m Y_i$ is distributed as $Gamma\left(\Omega t^{-1}, m\right)$.*

Proof. *Consider from (1.16) the Laplace transform of U:*

$$\begin{aligned} \mathcal{L}(z) &= E(\exp(-zU)) \\ &= E\left(\exp\left(-z \sum_{i=1}^m Y_i \right) \right) \\ &= E(\exp(-(Y_1 z + ... + Y_m z))) \\ &= E(\exp(-Y_1 z)) ... E(\exp(-Y_m z)) \\ &= \prod_{i=1}^m \mathcal{L}_{Y_i}(z) \\ &= \frac{1}{(1+\Omega t^{-1} z)^m} \end{aligned}$$

which is the Laplace transform of a gamma random variable with parameters Ωt^{-1} and m (see Result C.2). ■

Corollary 1.2 *Suppose $Y_1, Y_2, ..., Y_m$ are i.i.d. random variables which is distributed as $Exp(\Omega, \mathcal{W}(\cdot))$ with pdf (1.17). Then $U = \sum_{i=1}^m Y_i$ is distributed as a gamma type distribution, denoted as $Gamma(\Omega, m, \mathcal{W}(\cdot))$, with Laplace transform:*

$$\begin{aligned} \mathcal{L}(z) &= \int_0^\infty \left(1+\Omega t^{-1} z\right)^{-m} \mathcal{W}(t)\, dt \\ &= \int_0^\infty \mathcal{L}(z|t)\, \mathcal{W}(t)\, dt. \end{aligned} \tag{1.18}$$

Corollary 1.3 *Suppose U is distributed as gamma type distribution with Laplace transform (1.18). Then the pdf of U is given by:*

$$\begin{aligned} f(u) &= \int_0^\infty \left(\frac{1}{(\Omega t^{-1})^m \Gamma(m)} u^{m-1} \exp\left(-\frac{1}{\Omega t^{-1}} u \right) \right) \mathcal{W}(t)\, dt \\ &= \int_0^\infty f(u|t)\, \mathcal{W}(t)\, dt \end{aligned} \tag{1.19}$$

where $u > 0$, $m, \Omega > 0$.

Remark 1.8 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (1.19) simplifies to:*

$$f(u) = \int_0^\infty \left(\frac{1}{(\Omega t^{-1})^m \Gamma(m)} u^{m-1} \exp\left(-\frac{1}{\Omega t^{-1}} u\right)\right) \delta(t-1)\, dt.$$

Let $x = t-1$, then $t = x+1$ and $dx = dt$:

$$\begin{aligned} f(u) &= \int_0^\infty \left(\frac{1}{\left(\Omega (x+1)^{-1}\right)^m \Gamma(m)} u^{m-1} \exp\left(-\frac{1}{\Omega (x+1)^{-1}} u\right)\right) \delta(x)\, dx \\ &= \frac{1}{\Omega^m \Gamma(m)} u^{m-1} \exp\left(-\frac{1}{\Omega} u\right) \end{aligned} \tag{1.20}$$

for $u > 0$. Thus (1.20) is identified as the well known gamma distribution with parameters Ω and m (see Result C.2).

Theorem 1.2 *Suppose U is distributed as gamma type distribution with pdf (1.19). Let $W = \left(\frac{U}{m}\right)^{\frac{1}{\beta}}$. Then the pdf of W is given by:*

$$f(w) = \frac{\beta m^m}{\Omega^m \Gamma(m)} w^{\beta m - 1} \int_0^\infty t^m \exp\left(-\frac{m}{\Omega t^{-1}} w^\beta\right) \mathcal{W}(t)\, dt \tag{1.21}$$

where $w > 0$ and $m, \Omega, \beta > 0$. This distribution is called a Weibullised gamma type distribution with the Nakagami type distribution as a special case when $\beta = 2$.

Proof. *Let $W = \left(\frac{U}{m}\right)^{\frac{1}{\beta}}$, then $U = mW^\beta$ with $\frac{du}{dw} = \beta m w^{\beta-1}$. Then from (1.19) it follows that:*

$$\begin{aligned} f(w|t) &= f\left(mw^\beta | t\right) \beta m w^{\beta-1} \\ &= \frac{1}{(\Omega t^{-1})^m \Gamma(m)} \left(mw^\beta\right)^{m-1} \exp\left(-\frac{1}{\Omega t^{-1}} m w^\beta\right) \beta m w^{\beta-1} \\ &= \frac{\beta m^m}{(\Omega t^{-1})^m \Gamma(m)} w^{\beta m-1} \exp\left(-\frac{m}{\Omega t^{-1}} w^\beta\right) \end{aligned}$$

and thus:

$$\begin{aligned} f(w) &= \int_0^\infty f(w|t) \mathcal{W}(t)\, dt \\ &= \int_0^\infty \frac{\beta m^m}{(\Omega t^{-1})^m \Gamma(m)} w^{\beta m-1} \exp\left(-\frac{m}{\Omega t^{-1}} w^\beta\right) \mathcal{W}(t)\, dt \end{aligned}$$

which leaves the final result. ■

Corollary 1.4 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (1.21) simplifies to:*

$$f(w) = \frac{\beta m^m}{\Omega^m \Gamma(m)} w^{\beta m-1} \int_0^\infty t^m \exp\left(-\frac{m}{\Omega t^{-1}} w^\beta\right) \delta(t-1)\, dt.$$

Let $x = t - 1$, then $t = x + 1$ and $dx = dt$:

$$
\begin{aligned}
f(w) &= \frac{\beta m^m}{\Omega^m \Gamma(m)} w^{\beta m-1} \int_0^\infty (x+1)^m \exp\left(-\frac{m w^\beta}{\Omega}(x+1)\right) \delta(x)\, dx \\
&= \frac{\beta \left(\frac{m}{\Omega}\right)^m}{\Gamma(m)} w^{\beta m-1} \exp\left(-\frac{m w^\beta}{\Omega}\right)
\end{aligned}
$$

where $w > 0$ and $m, \Omega, \beta > 0$. This resembles the generalised gamma distribution of Patil et. al. (1984), p. 69.

Corollary 1.5 *By choosing $\mathcal{W}(t)$ as (1.6), using Result C.22, (1.21) simplifies to:*

$$
\begin{aligned}
f(w) &= \frac{\beta m^m}{\Omega^m \Gamma(m)} w^{\beta m-1} \int_0^\infty t^m \exp\left(-\frac{m}{\Omega t^{-1}} w^\beta\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-\frac{vt}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\beta m^m}{\Omega^m \Gamma(m)} w^{\beta m-1} \int_0^\infty t^{\frac{v}{2}+m-1} \exp\left(-t\left(\frac{m w^\beta}{\Omega} + \frac{v}{2}\right)\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\beta m^m}{\Omega^m \Gamma(m)} w^{\beta m-1} \frac{\Gamma\left(\frac{v}{2}+m\right)}{\left(\frac{m w^\beta}{\Omega} + \frac{v}{2}\right)^{\frac{v}{2}+m}} \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}} \beta m^m}{\Omega^m B\left(m, \frac{v}{2}\right)} \frac{w^{\beta m-1}}{\left(\frac{m}{\Omega} w^\beta + \frac{v}{2}\right)^{\frac{v}{2}+m}} \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}} \beta m^m}{\Omega^m B\left(m, \frac{v}{2}\right)} \frac{w^{\beta m-1}}{\left(\frac{v}{2}\right)^{\frac{v}{2}+m} \left(\frac{\frac{m}{\Omega} w^\beta}{\frac{v}{2}} + 1\right)^{\frac{v}{2}+m}} \\
&= \frac{\beta \left(\frac{m}{\Omega \frac{v}{2}}\right)^m}{B\left(m, \frac{v}{2}\right)} \frac{w^{\beta m-1}}{\left(1 + \frac{m}{\Omega \frac{v}{2}} w^\beta\right)^{\frac{v}{2}+m}} \qquad (1.22)
\end{aligned}
$$

where $w > 0$, $m, \Omega, \beta, v > 0$, and $B(\cdot,\cdot)$ denotes the beta function (see Result C.8). This distribution resembles the generalised beta-prime distribution of Patil et. al. (1984), p. 26.

Remark 1.9 *Consider (1.21) when $\beta = 2$:*

$$
f(w) = \frac{2m^m}{\Omega^m \Gamma(m)} w^{2m-1} \int_0^\infty t^m \exp\left(-\frac{m}{\Omega t^{-1}} w^2\right) \mathcal{W}(t)\, dt \qquad (1.23)
$$

where $w > 0$ and m, Ω. This distribution is called a Nakagami type distribution.

Remark 1.10 *Consider the distributions with pdfs (1.19), (1.20), (1.21), (1.22), and (1.23). The parameter m in these distributions is called the fading parameter, and within a statistical context acts as a shape parameter within these distributions. Within the communications systems domain, this m determines the severity of fading of the transmitted signal between transmitter and receiver (see Matthaiou and Laurenson (2007)).*

1.4.3.2 Performance measures

By design, communications systems are constantly exposed to different external factors (for example, geographical area, architectural infrastructure) which may affect the optimal performance of such a system. Being able to evaluate such a system for its performance is thus of interest, to ensure the system maintains an acceptable level of communication between transmitters and receivers. In this thesis, three performance measures are of interest, which are briefly described below.

Capacity Teletar (1999) proposed channel capacity as a useful tool to measure the performance of MIMO communications systems, and motivated the study of capacity in MIMO settings for many researchers thereafter (see for example, Ratnarajah and Vaillancourt (2003), Ratnarajah (2005), Ratnarajah and Vaillancourt (2005), McKay and Collings (2005)). The capacity of a system is directly related to the distribution of the channel matrix $\mathbf{H}$, and can be represented and analysed in terms of the eigenvalues of $\mathbf{H}$ via its spectral decomposition.

Formally, the ergodic capacity C of the random MIMO channel $\mathbf{H}$ is given by (see Teletar (1999)):

$$C = E_{\mathbf{H}}\left(\log\det\left(\mathbf{I}_{n_t} + \frac{\rho}{n_t}\mathbf{H}^H\mathbf{H}\right)\right) \tag{1.24}$$

where ρ denotes the signal-to-noise ratio (SNR). The SNR is the quantity relating to the channel fading distribution used to describe the ratio of signal to noise evident in signal transmission between transmitters and receivers. By the singular value decomposition, construct $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^H$, $\mathbf{U} \in U(n_t)$, $\mathbf{V} \in U(n_r)$ (see Result D.39), and where $\mathbf{\Lambda} = diag(\lambda_1, ..., \lambda_{n_t})$ and $\lambda_1 > \lambda_2 > ... > \lambda_{n_t} > 0$ denotes the ordered eigenvalues of $\mathbf{H}^H\mathbf{H}$:

$$\begin{aligned}
C &= E_{\mathbf{H}}\left(\log\det\left(\mathbf{I}_{n_t} + \frac{\rho}{n_t}\left(\mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^H\right)^H\mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^H\right)\right) \\
&= E_{\mathbf{H}}\left(\log\prod_{i=1}^{n_t}\left(1 + \frac{\rho}{n_t}\lambda_i\right)\right) \\
&= E_{\mathbf{H}}\left(\sum_{i=1}^{n_t}\log\left(1 + \frac{\rho}{n_t}\lambda_i\right)\right) \\
&= \sum_{i=1}^{n_t}E_{\mathbf{H}}\left(\log\left(1 + \frac{\rho}{n_t}\lambda_i\right)\right) \\
&= n_t E_{\mathbf{H}}\left(\log\left(1 + \frac{\rho}{n_t}\lambda_1\right)\right).
\end{aligned}$$

This latter expectation, and hence the capacity C, depends only on the distribution of one of the eigenvalues, and therefore leaves C to be relatively simple to evaluate regardless of the assumed underlying model; given tractable marginal eigenvalue pdfs. Note that, in (1.24) if $\log_e$ is used then the measurement unit for capacity is termed "nats" (Ratnarajah and Vaillancourt (2003)).

Outage probability The outage probability of a fading channel is defined by Simon and Alouini (2005) as:

$$P_{out}(x_{th}) = F(x_{th}) = \int_0^{x_{th}} f(x)\,dx \tag{1.25}$$

where x_{th} is termed the output threshold, and $f(x)$ is the pdf of X, the SNR of the channel fading distribution. In statistical terms, (1.25) acts as the cdf of the SNR of the channel fading distribution. Outage probability relates to the probability of the SNR of the received signal of a fading channel performing below an acceptable, specified threshold x_{th}. If the SNR of the signal falls below this threshold, the communications system does not operate, which lends to the name of this performance measure - the probability of an outage. Note that x_{th} is referred to as the threshold SNR, and in literature X denotes either the instantaneous SNR $X = \frac{R^2 E_s}{N_0}$, or normalised power, $X = R^2$, where R denotes the channel envelope (see Simon and Alouini (2005)). Here, E_s denotes the energy per symbol and N_0 denotes the noise power.

For a fading channel subject to a bivariate distribution, say with variable (X_1, X_2), the outage probability in (1.25) can be evaluated as:

$$F(x_{th}) = P(\max(X_1, X_2) < x_{th})$$

since, by MIMO design, the receiving antenna with the highest SNR is evaluated (Xu et. al. (2009), Ermelova and Tirkkonen (2014)). Thus, the cdf of $\max(X_1, X_2)$ is of interest.

Equal gain combiner diversity Diversity combining is a well-known technique used to improve received signal strength at receiver channels (Karagiannidis (2004)). A popular performance measure following this technique is called equal gain combiner diversity (EGC diversity). In a multivariate fading channel with n_r receivers operating under a multivariate distribution with pdf $f(r_1, r_2, ..., r_{n_r})$, the EGC diversity is given by:

$$\gamma_{out} = \frac{E_s}{nN_o}(R_1 + R_2 + ... + R_{n_r})^2$$

where E_s denotes the energy per symbol of the fading channel, N_o denotes the noise component of the channel, and R_i represents the random variables with joint pdf $f(r_1, r_2, ..., r_{n_r})$. Note that γ_{out} denotes the EGC diversity output SNR. In this thesis, the focus is on fading channels operating under a bivariate gamma type distribution. Thus $n_r = 2$, and:

$$\gamma_{out} = \frac{E_s}{2N_o}(R_1 + R_2)^2 \tag{1.26}$$

is of interest. Particularly, the d^{th} moment of the EGC diversity output SNR receives special attention, and is given by:

$$\begin{aligned} \mu_d &= E\left(\left(\frac{E_s}{2N_o}(R_1 + R_2)^2\right)^d\right) \\ &= \left(\frac{E_s}{2N_o}\right)^d E\left((R_1 + R_2)^{2d}\right). \end{aligned}$$

The first moment, μ_1, represents an important performance measure of communications systems subject to (in this thesis) bivariate gamma type distributions. This μ_1 is usually easy to evaluate and serves as a suitable indicator of the communications system's reliability, and is derived and investigated for the bivariate gamma type I distribution later in this thesis.

1.5 Objective and outline of study

Inspired by communications systems, the focus of this thesis is to advance the statistical distribution theory literature with results emanating from the complex elliptical class, using the representation as in (1.4). In particular, this thesis aims to:

- Propose the use of the complex elliptical class with motivation from practical problems emanating from communications systems;
- Systematically derive distributions of complex Wishart type and its accompanied joint eigenvalues with its origins in the complex elliptical class;
- Systematically derive bivariate gamma type distributions and subsequently bivariate Weibullised gamma type distributions originating from the complex elliptical class, with bivariate noncentral contributions as well; and
- Explore the candidacy of the t distribution within the complex elliptical context as an assumed underlying model within communications systems.

This thesis is constructed in the following way.

In Chapter 2, the distribution of the quadratic form, $\mathbf{S} = \mathbf{X}^H \mathbf{A} \mathbf{X}$ where $\mathbf{A} \in \mathbb{C}_2^{n \times n}$, when $\mathbf{X}$ is assumed to be complex matrix variate elliptically distributed, is derived and studied, particular for the case when $E(\mathbf{X}) = \mathbf{0}$. Cases when $\mathbf{X}$ is nonsingular and singular are investigated. The distributions of the eigenvalues of $\mathbf{S}$ are also studied. These results are applied in the communication systems domain for the performance measure of channel capacity and comparatively investigated for different members of the complex elliptical class.

In Chapter 3, the complex matrix variate inverse Wishart type distribution, emanating from Chapter 2, is proposed. This contribution is used as the platform to derive a bivariate gamma type I distribution from the inverse of the diagonal elements of the complex matrix variate inverse Wishart type distribution. Key characteristics of the new bivariate gamma type I distribution are studied. Subsequently, a bivariate Weibullised gamma type I distribution emanating from the proposed bivariate gamma type I distribution is derived. The obtained results are applied in the communications systems domain for the outage probability and the EGC diversity of a MIMO communications system and investigated. The validity and accuracy of the derived analytical results are illustrated with a simulation study.

Chapter 4 continues on a bivariate gamma type distribution path; however, the genesis of this bivariate gamma type distribution is different from that of Chapter 3. The Rayleigh type distribution (see (1.10)) is used as a foundation from which a bivariate gamma type II distribution is proposed. The systematic construction of this bivariate gamma type II distribution is motivated and described, and key characteristics are studied. In particular, a bivariate noncentral gamma type II distribution is also proposed and derived. Subsequently, a bivariate Weibullised gamma type II distribution emanating from the proposed bivariate gamma type II distribution is derived and some of its characteristics are studied. The obtained results are applied in the communications systems domain for the outage probability of a MIMO communications system and investigated. Also, certain percentiles of the derived results are calculated to illustrate the computational and tractable nature of these models.

In Chapter 5, the distribution of the Wishart form, $\mathbf{S} = \mathbf{X}^H \mathbf{X}$, when $\mathbf{X}$ is assumed to be complex matrix variate elliptically distributed, is derived and studied, particular for the case when $E(\mathbf{X}) = \mathbf{M}$. The pdf of the joint eigenvalues of $\mathbf{S}$ are studied, and special cases highlighted. Valuable results pertaining to the assumed behaviour of the noncentrality matrix parameter are given and studied. In particular, the assumption of the noncentrality matrix parameter having rank 1 is of practical (and hence theoretical) interest. The distribution of the minimum eigenvalue of $\mathbf{S}$ is investigated for some special cases. The validity and accuracy of the derived analytical results are illustrated with a simulation study.

Chapter 6 contains conclusions relating to this thesis, as well as future directions of areas where research emanating from this thesis may take place.

Finally, the Appendices at the end of this thesis form a collection of fundamental results relevant to this study.

Chapter 2

Complex central Wishart type distributions

Contents

2.1 Introduction

In this chapter the distribution of $\mathbf{S} = \mathbf{X}^H\mathbf{A}\mathbf{X} \in \mathbb{C}_2^{p\times p}$ is investigated. This is commonly known as the quadratic form of $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ ($\mathbf{A} \in \mathbb{C}_2^{n\times n}$), and in this case, $\mathbf{X}$ is assumed to follow the complex matrix variate elliptical distribution with pdf (1.3). Specifically, cases when $\mathbf{X}$ is nonsingular and singular is of interest. The distributions of the eigenvalues of $\mathbf{S}$ are also derived. These results are applied in the communication systems domain for the performance measure of channel capacity (see (1.24)) and investigated comparatively for the underlying complex matrix variate normal- and t distributions.

2.2 Pdfs of quadratic forms and joint eigenvalues

In this section, the pdfs of the nonsingular- and singular quadratic forms of complex matrix variate elliptical random matrices are derived and certain special cases of them are highlighted. In addition, the joint pdfs of the eigenvalues of these quadratic forms are also derived.

2.2.1 Nonsingular case ($n \geq p$)

2.2.1.1 Pdf of the quadratic form

The following theorem gives the pdf of the quadratic form of the nonsingular complex matrix variate elliptical distribution.

Theorem 2.1 *Suppose that $n \geq p$ and $\mathbf{X} \sim \mathcal{C}E_{n\times p}(\mathbf{0}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, h)$, $\mathbf{\Phi} \in \mathbb{C}_2^{n\times n}$, $\mathbf{A} \in \mathbb{C}_2^{n\times n}$, and $\mathbf{\Sigma} \in \mathbb{C}_2^{p\times p}$. The quadratic form $\mathbf{S} = \mathbf{X}^H\mathbf{A}\mathbf{X} \in \mathbb{C}_2^{p\times p}$ has the integral series complex Wishart type (ISCW) distribution with pdf:*

$$f_{\mathbf{S}}(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-p}\,\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n} \tag{2.1}$$

where

$$\mathcal{G}(\mathbf{S}) = \int_{\mathbb{R}^+} t^{np}\ {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt$$

with $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$, ${}_0\mathcal{C}F_0^{(p)}(\cdot,\cdot)$ is the hypergeometric function of two Hermitian matrix arguments (see Result D.54), and $\mathcal{C}\Gamma_p(\cdot)$ denotes the complex multivariate gamma function (see Result D.47). This distribution is denoted as $\mathbf{S} \sim ISCW_p(n, \mathbf{\Phi}, \mathbf{\Sigma}, \mathcal{G}(\cdot))$.

Proof. From Lemma 1.3.1, consider a matrix variate $\mathbf{Y} \sim \mathcal{C}E_{n\times p}(\mathbf{0}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, h)$ (see (1.4)) with pdf:

$$\begin{aligned} f(\mathbf{Y}) &= \int_{\mathbb{R}^+} \pi^{-np}\det(\mathbf{\Phi})^{-p}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-t\mathbf{\Phi}^{-1}\mathbf{Y}\mathbf{\Sigma}^{-1}\mathbf{Y}^H\right)\mathcal{W}(t)dt \\ &= \int_{\mathbb{R}^+} t^{np}\det(\mathbf{\Phi})^{-p}\det(\mathbf{\Sigma})^{-n}\pi^{-np}\operatorname{etr}\left(-t\mathbf{\Phi}^{-1}\mathbf{Y}\mathbf{\Sigma}^{-1}\mathbf{Y}^H\right)\mathcal{W}(t)dt \end{aligned}$$

since $\mathbf{Y}|t \sim \mathcal{C}N_{n\times p}\left(\mathbf{0}, \mathbf{\Phi} \otimes t^{-1}\mathbf{\Sigma}\right)$ (see (1.1)). Let $\mathbf{Y} = \mathbf{A}^{-\frac{1}{2}}\mathbf{X}$, with Jacobian $J(\mathbf{Y} \to \mathbf{X}) = \det(\mathbf{A})^{-p}$ (see Result D.43). Then:

$$\begin{aligned} f(\mathbf{X}) &= \int_{\mathbb{R}^+} t^{np}\pi^{-np}\det(\mathbf{A})^{-p}\det(\mathbf{\Phi})^{-p}\det(\mathbf{\Sigma})^{-n}\operatorname{etr}\left(-t\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\mathbf{A}^{-\frac{1}{2}}\right)\mathcal{W}(t)dt \\ &= \int_{\mathbb{R}^+} t^{np}\pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}\operatorname{etr}\left(-t\mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)\mathcal{W}(t)dt \\ &= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)\mathcal{W}(t)dt \\ &= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}\int_{\mathbb{R}^+} t^{np}\ {}_0\mathcal{C}F_0\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)\mathcal{W}(t)dt \end{aligned} \tag{2.2}$$

where $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$. Consider now $\mathbf{S} = \mathbf{X}^H\mathbf{X} = \mathbf{Y}^H\mathbf{A}\mathbf{Y}$. The latter integral in (2.2) is invariant under a unitary transformation. Using Result D.52, Result D.61, Result D.41, and Result D.62:

$$\begin{aligned} f(\mathbf{S}) &= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}\int_{\mathbb{R}^+} t^{np}\int_{\mathbf{X}^H\mathbf{X}=\mathbf{S}} {}_0\mathcal{C}F_0\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)d\mathbf{X}\mathcal{W}(t)dt \\ &= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}\int_{\mathbb{R}^+} t^{np}\det(\mathbf{S})^{n-p}\ {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{X}^H\mathbf{X}\right)\frac{\pi^{np}}{\mathcal{C}\Gamma_p(n)}\mathcal{W}(t)dt \\ &= \frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\ {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{X}^H\mathbf{X}\right)\mathcal{W}(t)dt \end{aligned}$$

leaving the final result. ■

Some special cases of the pdf (2.1) are discussed next.

Remark 2.1 *1. If $\mathbf{A} = \mathbf{I}_n$, and $\mathbf{\Phi} = \mathbf{I}_n$, then $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ has the complex Wishart type distribution with the* *following pdf:*

$$f(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}. \tag{2.3}$$

Since $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}} = \mathbf{I}_n$ *and using Result D.54 and Result D.52, it follows that:*

$$\begin{aligned}
\mathcal{G}(\mathbf{S}) &= \int_{\mathbb{R}^+} t^{np}\ {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{I}_n, -t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt \\
&= \int_{\mathbb{R}^+} t^{np} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{I}_n)C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{k!C_\kappa(\mathbf{I}_n)}\mathcal{W}(t)dt \\
&= \int_{\mathbb{R}^+} t^{np} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{k!}\mathcal{W}(t)dt \\
&= \int_{\mathbb{R}^+} t^{np}\ {}_0\mathcal{C}F_0\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt \\
&= \int_{\mathbb{R}^+} t^{np}\ \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt.
\end{aligned} \tag{2.4}$$

2. If $\mathbf{A} = \mathbf{I}_n$, $\mathbf{\Phi} = \mathbf{I}_n$, *and* $\mathbf{\Sigma} = \sigma^2\mathbf{I}_p$, *then by using (1.15)* $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ *has the following pdf:*

$$f(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_p(n)\sigma^{2np}}$$

and

$$\begin{aligned}
\mathcal{G}(\mathbf{S}) &= \int_{\mathbb{R}^+} t^{np}\ \operatorname{etr}\left(-t\left(\sigma^2\mathbf{I}_p\right)^{-1}\mathbf{S}\right)\mathcal{W}(t)dt \\
&= \int_{\mathbb{R}^+} t^{np}\ \operatorname{etr}\left(-t\sigma^{-2}\mathbf{S}\right)\mathcal{W}(t)dt.
\end{aligned}$$

Corollary 2.1 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), see that (2.4) simplifies to:*

$$\mathcal{G}(\mathbf{S}) = \operatorname{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{S}\right)$$

which leaves the pdf of the complex Wishart distribution (see James (1964)):

$$f_{normal}(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-p}\operatorname{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}. \tag{2.5}$$

Corollary 2.2 *By choosing* $\mathcal{W}(t)$ *as the t distribution weight (1.6) and using Result C.22, see that (2.4) in (2.3) simplifies to:*

$$\begin{aligned}
\mathcal{G}(\mathbf{S}) &= \int_{\mathbb{R}^+} t^{np}\ \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\ \exp\left(-t\left(\frac{v}{2}+tr\left(\mathbf{\Sigma}^{-1}\mathbf{S}\right)\right)\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\Gamma\left(np+\frac{v}{2}\right)\left(\frac{v}{2}+tr\left(\mathbf{\Sigma}^{-1}\mathbf{S}\right)\right)^{-\left(np+\frac{v}{2}\right)}
\end{aligned}$$

where $\Gamma(\cdot)$ *denotes the gamma function (see Result C.5), which leaves the pdf of the complex Wishart distribution emanating from the complex matrix variate t distribution:*

$$f_t(\mathbf{S}) = \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\det(\mathbf{S})^{n-p}\Gamma\left(np+\frac{v}{2}\right)}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n\left(\frac{v}{2}+tr\left(\mathbf{\Sigma}^{-1}\mathbf{S}\right)\right)^{-\left(np+\frac{v}{2}\right)}}. \tag{2.6}$$

2.2.1.2 Pdf of the joint eigenvalues

Next, an expression for the pdf of the joint eigenvalues, $\mathbf{\Lambda}$, of $\mathbf{S} = \mathbf{X}^H\mathbf{A}\mathbf{X}$ is given where $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ has pdf (2.1).

Theorem 2.2 *Suppose that* $n \geq p$ *and* $\mathbf{S} \sim ISCW_p(n, \mathbf{\Phi}\otimes\mathbf{\Sigma}, \mathcal{G}(\cdot))$, *and let* $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ *represent the ordered eigenvalues of* $\mathbf{S} \in \mathbb{C}_2^{p\times p}$. *The joint distribution of the eigenvalues of* $\mathbf{S}$, $\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, ..., \lambda_p)$, *has pdf:*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\mathcal{G}(\mathbf{\Lambda})\mathcal{W}(t)dt \\ &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{B})C_\kappa(\mathbf{\Lambda})}{k!C_\kappa(\mathbf{I}_n)C_\kappa(\mathbf{I}_p)} \\ &\quad\times\int_{\mathbb{R}^+} t^{np}\ C_\kappa\left(-t\mathbf{\Sigma}_2^{-1}\right)\mathcal{W}(t)dt \end{aligned} \tag{2.7}$$

where $\mathcal{G}(\mathbf{\Lambda}) = \int_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}$, *and* $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$.

Proof. Using Result D.59 and substituting $f(\cdot)$ with pdf (2.1):

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(p)}\int_{\mathbf{E}\in U(p)}\frac{\det\left(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)^{n-p}\mathcal{G}(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H)}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}d\mathbf{E} \\ &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\int_{\mathbf{E}\in U(p)}\mathcal{G}(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H)d\mathbf{E} \\ &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\int_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}\mathcal{W}(t)dt \\ &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\mathcal{G}(\mathbf{\Lambda}) \end{aligned}$$

where

$$\mathcal{G}(\mathbf{\Lambda}) = \int_{\mathbb{R}^+} t^{np}\int_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}\mathcal{W}(t)dt. \tag{2.8}$$

By using Result D.54 and Result D.60, see that (2.8) can be written as:

$$\begin{aligned}\int\limits_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E} &= \int\limits_{\mathbf{E}\in U(p)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(\mathbf{B}\right)C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)}{k!C_\kappa\left(\mathbf{I}_n\right)}d\mathbf{E} \\ &= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(\mathbf{B}\right)}{k!C_\kappa\left(\mathbf{I}_n\right)} \int\limits_{\mathbf{E}\in U(p)} C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E} \\ &= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(\mathbf{B}\right)}{k!C_\kappa\left(\mathbf{I}_n\right)} \frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa\left(\mathbf{\Lambda}\right)}{C_\kappa\left(\mathbf{I}_p\right)}\end{aligned}$$

which leaves the final result. ∎

Some special cases of the pdf in (2.7) are discussed next.

Remark 2.2 *If $\mathbf{A} = \mathbf{I}_n$ and $\mathbf{\Phi} = \mathbf{I}_n$, then the pdf (2.7) of the joint eigenvalues, $\mathbf{\Lambda}$, of the complex Wishart type distribution, simplifies to*

$$f(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)}\det\left(\mathbf{\Lambda}\right)^{n-p}\left(\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det\left(\mathbf{\Sigma}\right)^n}\int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right)\mathcal{W}(t)dt. \tag{2.9}$$

Using Result D.54 and Result D.52, see that (2.8) simplifies to:

$$\begin{aligned}\mathcal{G}\left(\mathbf{\Lambda}\right) &= \int\limits_{\mathbb{R}^+} t^{np}\int\limits_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{I}_n, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}\mathcal{W}(t)dt \\ &= \int\limits_{\mathbb{R}^+} t^{np}\int\limits_{\mathbf{E}\in U(p)} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(\mathbf{I}_n\right)C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)}{k!C_\kappa\left(\mathbf{I}_n\right)}d\mathbf{E}\mathcal{W}(t)dt \\ &= \int\limits_{\mathbb{R}^+} t^{np}\int\limits_{\mathbf{E}\in U(p)} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)}{k!}d\mathbf{E}\mathcal{W}(t)dt \\ &= \int\limits_{\mathbb{R}^+} t^{np}\int\limits_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}\mathcal{W}(t)dt \\ &= \int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -\,t\mathbf{\Sigma}^{-1}\right)\mathcal{W}(t)dt.\end{aligned} \tag{2.10}$$

However, it can be observed that (2.10) depends on $\mathbf{\Sigma}$ *only through the eigenvalues of* $\mathbf{\Sigma}$*. Consider the eigenvalue decomposition of* $\mathbf{\Sigma}$ *as* $\mathbf{\Sigma} = \mathbf{F}\mathbf{\Upsilon}\mathbf{F}^H$ *where* $\mathbf{F} \in U(p)$*, and* $\mathbf{\Upsilon} = diag(a_1, \ldots, a_p)$*,* $a_1 > \ldots > a_p > 0$*, where* a_i $(i = 1, ..., p)$ *denotes the eigenvalues of the matrix* $\mathbf{\Sigma}$*. Let* $\hat{\mathbf{E}} = \mathbf{F}^H\mathbf{E} \in U(p)$*. Using Result D.62 and Result D.52, see that:*

$$\begin{aligned} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right) &= \int\limits_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E} \\ &= \int\limits_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\mathbf{F}\mathbf{\Upsilon}^{-1}\mathbf{F}^H\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E} \\ &= \int\limits_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\mathbf{\Upsilon}^{-1}\mathbf{F}^H\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\mathbf{F}\right) d\mathbf{E} \\ &= \int\limits_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\mathbf{\Upsilon}^{-1}\hat{\mathbf{E}}\mathbf{\Lambda}\hat{\mathbf{E}}^H\right) d\hat{\mathbf{E}} \\ &= \int\limits_{\mathbf{E}\in U(p)} {}_0\mathcal{C}F_0^{(p)}\left(-t\mathbf{\Upsilon}^{-1}\hat{\mathbf{E}}\mathbf{\Lambda}\hat{\mathbf{E}}^H\right) d\hat{\mathbf{E}} \\ &= {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Upsilon}^{-1}\right). \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.9) leaves:

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{i=1}^{p}\lambda_i^{n-p}\right)\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Sigma})^n} \\ &\quad\times \int\limits_{\mathbb{R}^+} t^{np}\ {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Upsilon}^{-1}\right)\mathcal{W}(t)dt. \end{aligned} \tag{2.12}$$

Using Result D.58 and Result D.57, (2.12) simplifies to:

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Sigma})^n}\int\limits_{\mathbb{R}^+} t^{np}\frac{\mathcal{C}\Gamma_p(p)\det\left(\exp\left(-ta_i\lambda_j\right)\right)}{\pi^{\frac{p(p-1)}{2}}\prod_{k<l}^{p}(\lambda_k-\lambda_l)\prod_{k<l}^{p}(ta_k-ta_l)}\mathcal{W}(t)dt \\ &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det(\mathbf{\Sigma})^n}\int\limits_{\mathbb{R}^+} t^{np}\frac{\mathcal{C}\Gamma_p(p)\det\left(\exp\left(-ta_i\lambda_j\right)\right)}{\pi^{\frac{p(p-1)}{2}}t^{\frac{p(p-1)}{2}}\prod_{k<l}^{p}(\lambda_k-\lambda_l)\prod_{k<l}^{p}(a_k-a_l)}\mathcal{W}(t)dt \\ &= \frac{\pi^{\frac{p(p-1)}{2}}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right)}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n\prod_{k<l}^{p}(a_k-a_l)}\int\limits_{\mathbb{R}^+} t^{np-\frac{p(p-1)}{2}}\det\left(\exp\left(-ta_i\lambda_j\right)\right)\mathcal{W}(t)dt. \end{aligned} \tag{2.13}$$

Remark 2.3 *If* $\mathbf{A} = \mathbf{I}_n$*,* $\mathbf{\Phi} = \mathbf{I}_n$*, and* $\mathbf{\Sigma} = \sigma^2\mathbf{I}_p$*, then by using (1.15), (2.7) simplifies to:*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\sigma^{2np}} \\ &\quad\times \int\limits_{\mathbb{R}^+} t^{np}\exp\left(-t\sigma^{-2}\sum_{i=1}^{p}\lambda_i\right)\mathcal{W}(t)dt \end{aligned} \tag{2.14}$$

From (2.9) using Result D.54, Result D.56, Result D.52, and Result D.62:

$$\begin{aligned}
{}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right) &= {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\sigma^{-2}\mathbf{I}_p\right) \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(\mathbf{\Lambda}\right) C_\kappa\left(-t\sigma^{-2}\mathbf{I}_p\right)}{k! C_\kappa\left(\mathbf{I}_p\right)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\left(-t\sigma^{-2}\right)^k C_\kappa\left(\mathbf{\Lambda}\right) C_\kappa\left(\mathbf{I}_p\right)}{k! C_\kappa\left(\mathbf{I}_p\right)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \left(-t\sigma^{-2}\right)^k \frac{C_\kappa\left(\mathbf{\Lambda}\right)}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(-t\sigma^{-2}\mathbf{\Lambda}\right)}{k!} \\
&= \operatorname{etr}\left(-t\sigma^{-2}\mathbf{\Lambda}\right) \\
&= \exp\left(-t\sigma^{-2}\sum_{i=1}^{p}\lambda_i\right)
\end{aligned}$$

which leaves the final result.

Corollary 2.3 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), from (2.9) see that:*

$$\mathcal{G}(\mathbf{\Lambda}) = {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -\mathbf{\Sigma}^{-1}\right)$$

which leaves the pdf of the joint eigenvalues of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ having the complex Wishart distribution (see James (1964)):

$$f_{normal}(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)} \det\left(\mathbf{\Lambda}\right)^{n-p} \left(\prod_{k<l}^{p}\left(\lambda_k - \lambda_l\right)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p)\det\left(\mathbf{\Sigma}\right)^n} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -\mathbf{\Sigma}^{-1}\right). \tag{2.15}$$

Corollary 2.4 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6) and using Result C.22 and Result D.56, from (2.9) see that:*

$$\begin{aligned}
\mathcal{G}(\mathbf{\Lambda}) &= \int_{\mathbb{R}^+} t^{np}\, {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right) C_\kappa\left(\mathbf{\Lambda}\right)}{k! C_\kappa\left(\mathbf{I}_p\right)} dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right) C_\kappa\left(\mathbf{\Lambda}\right)}{k! C_\kappa\left(\mathbf{I}_p\right)} \int_{\mathbb{R}^+} t^{np+k+\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right) C_\kappa\left(\mathbf{\Lambda}\right)}{k! C_\kappa\left(\mathbf{I}_p\right)} \Gamma\left(np + k + \frac{v}{2}\right) \left(\frac{v}{2}\right)^{-\left(np+k+\frac{v}{2}\right)} \\
&= \frac{1}{\Gamma(\frac{v}{2})\left(\frac{v}{2}\right)^{np}} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right) C_\kappa\left(\mathbf{\Lambda}\right)}{\left(\frac{v}{2}\right)^k k! C_\kappa\left(\mathbf{I}_p\right)} \Gamma\left(np + k + \frac{v}{2}\right)
\end{aligned}$$

which leaves the pdf of the joint eigenvalues of the complex Wishart distribution emanating from the complex *matrix variate t distribution:*

$$f_t(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)} \det(\mathbf{\Lambda})^{n-p} \left(\prod_{k<l}^{p} (\lambda_k - \lambda_l)^2\right)}{\mathcal{C}\Gamma_p(n)\mathcal{C}\Gamma_p(p) \det(\mathbf{\Sigma})^n \Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{np}} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa(-\mathbf{\Sigma}^{-1}) C_\kappa(\mathbf{\Lambda})}{\left(\frac{v}{2}\right)^k k! C_\kappa(\mathbf{I}_p)} \Gamma\left(np + \frac{v}{2} + k\right) \tag{2.16}$$

where $v > 0$ is the degrees of freedom.

2.2.2 Singular case ($n < p$)

2.2.2.1 Pdf of the quadratic form

In this section the singular case of the quadratic form of the matrix variate elliptical distribution is considered, this is when $n < p$. The pdf of the quadratic form of the complex singular matrix variate elliptical distribution is derived and some special cases illustrated, and subsequently the corresponding joint eigenvalue pdf is considered.

Theorem 2.3 *Suppose that $n < p$ and $\mathbf{X} \sim \mathcal{C}E_{n\times p}(\mathbf{0}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, h)$, and let $\mathbf{A} \in \mathbb{C}_1^{n\times n}$. Let $\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, ..., \lambda_p)$, where $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ represent the ordered eigenvalues of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$. The quadratic form $\mathbf{S} = \mathbf{X}^H\mathbf{A}\mathbf{X}$ has the integral series singular complex Wishart type (ISSCW) distribution with pdf:*

$$f(\mathbf{S}) = \frac{\pi^{n(n-p)} \det(\mathbf{\Lambda})^{n-p} \mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_n(n) \det(\mathbf{\Phi}\mathbf{A})^p \det(\mathbf{\Sigma})^n} \tag{2.17}$$

where

$$\mathcal{G}(\mathbf{S}) = \int_{\mathbb{R}^+} t^{np} \, {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{S}\right) \mathcal{W}(t)dt$$

with $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$ and ${}_0\mathcal{C}F_0^{(n)}(\cdot,\cdot)$ is the complex hypergeometric function of two matrix arguments (see Result D.55). This distribution is denoted as $\mathbf{S} \sim ISSCW_n(p, \mathbf{\Phi}, \mathbf{\Sigma}, \mathcal{G}(\cdot))$.

Proof. From Lemma 1.3.1, consider a singular complex matrix variate $\mathbf{Y} \sim \mathcal{C}E_{n\times p}(\mathbf{0}, \mathbf{\Phi} \otimes \mathbf{\Sigma}, h)$ with pdf:

$$\begin{aligned} f(\mathbf{Y}) &= \int_{\mathbb{R}^+} \pi^{-np} \det(\mathbf{\Phi})^{-p} \det\left(t^{-1}\mathbf{\Sigma}\right)^{-n} \operatorname{etr}\left(-t\mathbf{\Phi}^{-1}\mathbf{Y}\mathbf{\Sigma}^{-1}\mathbf{Y}^H\right) \mathcal{W}(t)dt \\ &= \int_{\mathbb{R}^+} \pi^{-np} t^{np} \det(\mathbf{\Phi})^{-p} \det(\mathbf{\Sigma})^{-n} \operatorname{etr}\left(-t\mathbf{\Phi}^{-1}\mathbf{Y}\mathbf{\Sigma}^{-1}\mathbf{Y}^H\right) \mathcal{W}(t)dt \end{aligned}$$

since $\mathbf{Y}|t \sim \mathcal{C}N_{n\times p}\left(\mathbf{0}, \mathbf{\Phi} \otimes t^{-1}\mathbf{\Sigma}\right)$ (see (1.1)). Let $\mathbf{Y} = \mathbf{A}^{-\frac{1}{2}}\mathbf{X}$, with Jacobian $J(\mathbf{Y} \to \mathbf{X}) = \det(\mathbf{A})^{-p}$ (see Result D.43). Using Result D.52:

$$\begin{aligned} f(\mathbf{X}) &= \int_{\mathbb{R}^+} t^{np}\pi^{-np} \det(\mathbf{A})^{-p} \det(\mathbf{\Phi})^{-p} \det(\mathbf{\Sigma})^{-n} \operatorname{etr}\left(-t\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\mathbf{A}^{-\frac{1}{2}}\right) \mathcal{W}(t)dt \\ &= \int_{\mathbb{R}^+} t^{np}\pi^{-np} \det(\mathbf{\Phi}\mathbf{A})^{-p} \det(\mathbf{\Sigma})^{-n} \operatorname{etr}\left(-t\mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right) \mathcal{W}(t)dt \\ &= \pi^{-np} \det(\mathbf{\Phi}\mathbf{A})^{-p} \det(\mathbf{\Sigma})^{-n} \int_{\mathbb{R}^+} t^{np} \operatorname{etr}\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right) \mathcal{W}(t)dt \\ &= \pi^{-np} \det(\mathbf{\Phi}\mathbf{A})^{-p} \det(\mathbf{\Sigma})^{-n} \int_{\mathbb{R}^+} t^{np} \, {}_0\mathcal{C}F_0\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right) \mathcal{W}(t)dt \end{aligned}$$

where $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$. Consider a spectral decomposition by letting $\mathbf{X}^H = \mathbf{E}_1\mathbf{\Upsilon}\mathbf{E}$, where $\mathbf{E}_1 \in \mathcal{C}V_{p,n}$, $\mathbf{\Upsilon} \in \mathbb{C}_2^{p\times p}$, and $\mathbf{E} \in \mathcal{C}V_{n,n}$, and see that

$$\mathbf{S} = \mathbf{Y}^H\mathbf{A}\mathbf{Y} = \mathbf{X}^H\mathbf{X} = \mathbf{E}_1\mathbf{\Upsilon}\mathbf{E}\mathbf{E}^H\mathbf{\Upsilon}\mathbf{E}_1^H = \mathbf{E}_1\mathbf{\Upsilon}^2\mathbf{E}_1^H = \mathbf{E}_1\mathbf{\Lambda}\mathbf{E}_1^H$$

where $\mathbf{\Upsilon}^2 = \mathbf{\Lambda}$. Using Result D.42:

$$f(\mathbf{S}) = \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}2^{-n}\det(\mathbf{\Lambda})^{n-p}\int\limits_{\mathbb{R}^+}\int\limits_{\mathbf{X}^H\mathbf{X}=\mathbf{S}} t^{np}\,{}_0\mathcal{C}F_0\left(-t\mathbf{B}\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)d\mathbf{X}\mathcal{W}(t)dt.$$

The latter integral in (2.2) is invariant under a unitary transformation. Using Result D.61 and Result D.62:

$$\begin{aligned}
f(\mathbf{S}) &= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}2^{-n}\det(\mathbf{\Lambda})^{n-p} \\
&\quad\times\int\limits_{\mathbb{R}^+}\int\limits_{\mathbf{X}^H\mathbf{X}=\mathbf{S}}\int\limits_{\hat{\mathbf{E}}\in U(n)} t^{np}\,{}_0\mathcal{C}F_0\left(-t\hat{\mathbf{E}}\mathbf{B}\hat{\mathbf{E}}^H\mathbf{X}\mathbf{\Sigma}^{-1}\mathbf{X}^H\right)d\hat{\mathbf{E}}d\mathbf{X}\mathcal{W}(t)dt \\
&= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}2^{-n}\det(\mathbf{\Lambda})^{n-p}\int\limits_{\mathbb{R}^+}\int\limits_{\mathbf{X}^H\mathbf{X}=\mathbf{S}} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{X}^H\mathbf{X}\right)d\mathbf{X}\mathcal{W}(t)dt.
\end{aligned}$$

Finally, by using Result D.37:

$$\begin{aligned}
f(\mathbf{S}) &= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}2^{-n}\det(\mathbf{\Lambda})^{n-p}\int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\int\limits_{U(n)} d\mathbf{E}\mathcal{W}(t)dt \\
&= \pi^{-np}\det(\mathbf{\Phi}\mathbf{A})^{-p}\det(\mathbf{\Sigma})^{-n}2^{-n}\det(\mathbf{\Lambda})^{n-p}\int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt\frac{2^n\pi^{n^2}}{\mathcal{C}\Gamma_n(n)} \\
&= \frac{\pi^{-np+n^2}\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_n(n)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n}\int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B},-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt
\end{aligned}$$

which leaves the final result. ■

Some special cases of the pdf (2.17) are discussed next.

Remark 2.4 *1. If $\mathbf{A} = \mathbf{I}_n$ and $\mathbf{\Phi} = \mathbf{I}_n$, then $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ has the complex singular Wishart type distribution with the following pdf:*

$$f(\mathbf{S}) = \frac{\pi^{n(n-p)}\det(\mathbf{\Lambda})^{n-p}\mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_n(n)\det(\mathbf{\Sigma})^n} \tag{2.18}$$

and using Result D.55 and Result D.52:

$$\begin{aligned}
\mathcal{G}(\mathbf{S}) &= \int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{I}_n,-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt \\
&= \int\limits_{\mathbb{R}^+} t^{np}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{I}_n)\,C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{k!C_\kappa(\mathbf{I}_n)}\mathcal{W}(t)dt \\
&= \int\limits_{\mathbb{R}^+} t^{np}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{k!}\mathcal{W}(t)dt \\
&= \int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt \\
&= \int\limits_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)dt.
\end{aligned} \tag{2.19}$$

2. *If* $\mathbf{A} = \mathbf{I}_n$, $\mathbf{\Phi} = \mathbf{I}_n$, *and* $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_p$, *then by using (1.15),* $\mathbf{S} \in \mathbb{C}_2^{p \times p}$ *has the following pdf:*

$$f(\mathbf{S}) = \frac{\pi^{n(n-p)} \det(\mathbf{\Lambda})^{n-p} \mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_n(n)\sigma^{2np}}$$

and

$$\begin{aligned} \mathcal{G}(\mathbf{S}) &= \int_{\mathbb{R}^+} t^{np} \ \operatorname{etr}\left(-t\left(\sigma^2 \mathbf{I}_p\right)^{-1} \mathbf{S}\right) \mathcal{W}(t) dt \\ &= \int_{\mathbb{R}^+} t^{np} \ \operatorname{etr}\left(-t\sigma^{-2}\mathbf{S}\right) \mathcal{W}(t) dt. \end{aligned}$$

Corollary 2.5 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), see that (2.19) in (2.18) simplifies to:*

$$\mathcal{G}(\mathbf{S}) = \operatorname{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{S}\right)$$

which leaves the pdf of the complex singular Wishart distribution (see Ratnarajah and Vaillancourt (2005)):

$$f_{normal}(\mathbf{S}) = \frac{\pi^{n(n-p)} \det(\mathbf{\Lambda})^{n-p} \operatorname{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{S}\right)}{\mathcal{C}\Gamma_n(n) \det(\mathbf{\Sigma})^n}. \tag{2.20}$$

Corollary 2.6 *By choosing* $\mathcal{W}(t)$ *as the* t *distribution weight (1.6) and using Result C.22, see that (2.19) in (2.18) simplifies to:*

$$\begin{aligned} \mathcal{G}(\mathbf{S}) &= \int_{\mathbb{R}^+} t^{-np} \ \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1} \exp\left(-t\left(\frac{v}{2} + tr\mathbf{\Sigma}^{-1}\mathbf{S}\right)\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(np + \frac{v}{2}\right) \left(\frac{v}{2} + \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{S}\right)^{-\left(np+\frac{v}{2}\right)} \end{aligned}$$

which leaves the pdf of $\mathbf{S} \in \mathbb{C}_2^{p \times p}$ *as:*

$$f_t(\mathbf{S}) = \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\pi^{n(n-p)} \det(\mathbf{\Lambda})^{n-p} \Gamma\left(np + \frac{v}{2}\right)}{\mathcal{C}\Gamma_n(n) \det(\mathbf{\Sigma})^n \left(tr\mathbf{\Sigma}^{-1}\mathbf{S} + \frac{v}{2}\right)^{np+\frac{v}{2}}}. \tag{2.21}$$

2.2.2.2 Pdf of the joint eigenvalues

Next, expressions for the joint pdf of the eigenvalues of the quadratic form $\mathbf{S}$ under the complex singular matrix variate elliptical distribution are derived (see (2.17)).

Theorem 2.4 *Suppose that* $n < p$ *and* $\mathbf{S} \sim ISCW_n\left(p, \mathbf{\Phi} \otimes \mathbf{\Sigma}, \mathcal{G}(\cdot)\right)$ *(see (2.17)), and let* $\lambda_1 > \lambda_2 > ... > \lambda_n > 0$ *represent the ordered eigenvalues of* $\mathbf{S} \in \mathbb{C}_2^{p \times p}$*. Then the joint distribution of the eigenvalues of* $\mathbf{S}$*,* $\mathbf{\Lambda} = diag(\lambda_1, \lambda_2, ..., \lambda_p)$*, has pdf*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{n(n-1)} \det(\mathbf{\Lambda})^{p-n} \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p) \det(\mathbf{\Phi}\mathbf{A})^p \det(\mathbf{\Sigma})^n} \\ &\quad \times \int_{\mathbb{R}^+} t^{np} \int_{\mathbf{E} \in \mathcal{C}V_{n,p}} {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E} \mathcal{W}(t) dt \end{aligned} \tag{2.22}$$

where $\mathbf{B} = \mathbf{A}^{-\frac{1}{2}}\mathbf{\Phi}^{-1}\mathbf{A}^{-\frac{1}{2}}$ *and where* $\mathcal{C}V_{n,p}$ *denotes the Stiefel manifold (see Result D.37).*

Proof. Consider a partial spectral decomposition where $\mathbf{S} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^H$. The transformation from $\mathbf{S}$ to $\mathbf{E}, \mathbf{\Lambda}$ has Jacobian

$$J(\mathbf{S} \to \mathbf{E}, \mathbf{\Lambda}) = (2\pi)^{-n} \det\left(\mathbf{\Lambda}^{n-p}\right)^{-2} \prod_{k<l}^{n} (\lambda_k - \lambda_l)^2,$$

see Ratnarajah and Vaillancourt (2005). Therefore, by using (2.17) and Result D.37:

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \int\limits_{\mathbb{R}^+} t^{np} \int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} f(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H) J(\mathbf{S} \to \mathbf{E}, \mathbf{\Lambda})\, d\mathbf{E}\mathcal{W}(t)dt \\
&= \frac{\pi^{n(n-p)}}{\mathcal{C}\Gamma_n(n)\det(\mathbf{\Phi}\mathbf{A})^p \det(\mathbf{\Sigma})^n} (2\pi)^{-n} \det\left(\mathbf{\Lambda}^{n-p}\right)^{-2} \det(\mathbf{\Lambda})^{n-p} \left(\prod_{k<l}^{n}(\lambda_k - \lambda_l)\right)^2 \\
&\quad \times \int\limits_{\mathbb{R}^+} t^{np} \int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) \mathbf{E}^H d\mathbf{E}\mathcal{W}(t)dt \\
&= \frac{\pi^{n^2-np-n}2^{-n}\det(\mathbf{\Lambda})^{p-n}\left(\prod_{k<l}^{n}(\lambda_k - \lambda_l)\right)^2}{\mathcal{C}\Gamma_n(n)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n} \frac{2^n\pi^{np}}{\mathcal{C}\Gamma_n(p)} \\
&\quad \times \int\limits_{\mathbb{R}^+} t^{np} \int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E}\mathcal{W}(t)dt \\
&= \frac{\pi^{n(n-1)}\det(\mathbf{\Lambda})^{p-n}\left(\prod_{k<l}^{n}(\lambda_k - \lambda_l)\right)^2}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det(\mathbf{\Phi}\mathbf{A})^p\det(\mathbf{\Sigma})^n} \int\limits_{\mathbb{R}^+} t^{np} \int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{B}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E}\mathcal{W}(t)dt
\end{aligned}$$

leaving the final result. ■

Some special cases of the pdf (2.22) are discussed next.

Remark 2.5 *1. If* $\mathbf{A} = \mathbf{I}_n$ *and* $\mathbf{\Phi} = \mathbf{I}_n$, *the joint pdf of the eigenvalues of the complex singular Wishart type distribution,* $f(\mathbf{\Lambda})$, *simplifies to:*

$$f(\mathbf{\Lambda}) = \frac{\pi^{n(n-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{n}(\lambda_k - \lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det(\mathbf{\Sigma})^n} \int\limits_{\mathbb{R}^+} t^{np}\, {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right) \mathcal{W}(t)dt. \qquad (2.23)$$

Similarly to Ratnarajah and Vaillancourt (2005), from (2.22), using Result D.55 see that:

$$\begin{aligned}
\int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{I}_n, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E} &= \int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{I}_n)C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)}{k!C_\kappa(\mathbf{I}_p)}d\mathbf{E} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{I}_n)}{k!C_\kappa(\mathbf{I}_p)}\int\limits_{\mathbf{E}\in\mathcal{C}V_{n,p}} C_\kappa\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{I}_n)}{k!C_\kappa(\mathbf{I}_p)}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa(\mathbf{\Lambda})}{C_\kappa(\mathbf{I}_n)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{I}_n)}{k!C_\kappa(\mathbf{I}_p)}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa(\mathbf{\Lambda})}{C_\kappa(\mathbf{I}_n)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa(\mathbf{\Lambda})}{k!C_\kappa(\mathbf{I}_p)}. \qquad (2.24)
\end{aligned}$$

Substituting (2.24) into (2.22) and using Result D.55 leaves:

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \frac{\pi^{n(n-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{n}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det(\mathbf{\Sigma})^n} \\
&\quad\times\int\limits_{\mathbb{R}^+} t^{np}\int\limits_{E\in\mathbb{C}V_{n,p}} {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{I}, -t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}\mathcal{W}(t)dt \\
&= \frac{\pi^{n(n-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{n}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det(\mathbf{\Sigma})^n} \\
&\quad\times\int\limits_{\mathbb{R}^+} t^{np}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa(\mathbf{\Lambda})}{k!C_\kappa(\mathbf{I}_p)}\mathcal{W}(t)dt \\
&= \frac{\pi^{n(n-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{n}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det(\mathbf{\Sigma})^n} \\
&\quad\times\int\limits_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right)\mathcal{W}(t)dt
\end{aligned}$$

leaving the final result.

2. *If $\mathbf{A} = \mathbf{I}_n$, $\mathbf{\Phi} = \mathbf{I}_n$, and $\mathbf{\Sigma} = \sigma^2\mathbf{I}_p$, then by using (1.15), (2.22) simplifies to:*

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \frac{\pi^{n(n-1)}\det(\mathbf{\Lambda})^{n-p}\left(\prod_{k<l}^{n}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\sigma^{2np}} \qquad (2.25) \\
&\quad\times\int\limits_{\mathbb{R}^+} t^{np}\exp\left(-t\sigma^{-2}\sum_{i=1}^{n}\lambda_i\right)\mathcal{W}(t)dt
\end{aligned}$$

From (2.23) and by using Result D.55, see that:

$$\begin{aligned}
{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right) &= {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -t\sigma^{-2}\mathbf{I}_p\right) \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(\mathbf{\Lambda}\right)C_\kappa\left(-t\sigma^{-2}\mathbf{I}_p\right)}{k!C_\kappa\left(\mathbf{I}_p\right)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(-t\sigma^{-2}\right)^k C_\kappa\left(\mathbf{\Lambda}\right)C_\kappa\left(\mathbf{I}_p\right)}{k!C_\kappa\left(\mathbf{I}_p\right)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\left(-t\sigma^{-2}\right)^k\frac{C_\kappa\left(\mathbf{\Lambda}\right)}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\sigma^{-2}\mathbf{\Lambda}\right)}{k!} \\
&= \mathrm{etr}\left(-t\sigma^{-2}\mathbf{\Lambda}\right) \\
&= \exp\left(-t\sigma^{-2}\sum_{i=1}^{n}\lambda_i\right).
\end{aligned}$$

Corollary 2.7 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), see that (2.24) in (2.23) simplifies to:*

$$\mathcal{G}(\mathbf{\Lambda}) = {}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -\mathbf{\Sigma}^{-1}\right)$$

which leaves the pdf of the joint eigenvalues of the complex singular Wishart distribution (see Ratnarajah and Vaillancourt (2005)):

$$f_{normal}(\mathbf{\Lambda}) = \frac{\pi^{n(n-1)}\det\left(\mathbf{\Lambda}\right)^{n-p}\left(\prod_{k<l}^{n}\left(\lambda_k-\lambda_l\right)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det\left(\mathbf{\Sigma}\right)^n}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -\mathbf{\Sigma}^{-1}\right). \tag{2.26}$$

Corollary 2.8 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6) and by using Result C.22, Result D.55 and Result D.56, see that (2.24) in (2.23) simplifies to:*

$$\begin{aligned}
&\int_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda}, -t\mathbf{\Sigma}^{-1}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa\left(\mathbf{\Lambda}\right)}{k!C_\kappa\left(\mathbf{I}_p\right)}dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-t\mathbf{\Sigma}^{-1}\right)C_\kappa\left(\mathbf{\Lambda}\right)}{k!C_\kappa\left(\mathbf{I}_p\right)}dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right)C_\kappa\left(\mathbf{\Lambda}\right)}{k!C_\kappa\left(\mathbf{I}_p\right)}\int_{\mathbb{R}^+} t^{np+k+\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right)C_\kappa\left(\mathbf{\Lambda}\right)}{k!C_\kappa\left(\mathbf{I}_p\right)}\Gamma\left(np+k+\frac{v}{2}\right)\left(\frac{v}{2}\right)^{-\left(np+k+\frac{v}{2}\right)} \\
&= \frac{1}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{np}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right)C_\kappa\left(\mathbf{\Lambda}\right)}{\left(\frac{v}{2}\right)^k k!C_\kappa\left(\mathbf{I}_p\right)}\Gamma\left(np+k+\frac{v}{2}\right)
\end{aligned}$$

which leaves the joint pdf of the eigenvalues of $\mathbf{S} \in \mathbb{C}_2^{p \times p}$ *as:*

$$f_t(\mathbf{\Lambda}) = \frac{\pi^{n(n-1)} \det(\mathbf{\Lambda})^{p-n} \left(\prod_{k<l}^{n} (\lambda_k - \lambda_l)^2 \right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p) \det(\mathbf{\Sigma})^n \Gamma\left(\frac{v}{2}\right) \left(\frac{v}{2}\right)^{np}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa\left(-\mathbf{\Sigma}^{-1}\right) C_\kappa(\mathbf{\Lambda})}{\left(\frac{v}{2}\right)^k k! C_\kappa(\mathbf{I}_p)} \Gamma\left(np + k + \frac{v}{2}\right). \tag{2.27}$$

2.3 Illustrative application

The channel capacity (see (1.24)) for an $n_r \times n_t$ communications systems can be investigated, using the pdf of the joint eigenvalues of the quadratic form of $\mathbf{S} \in \mathbb{C}_2^{n_t \times n_t}$. Particularly, the case when $\mathbf{A} = \mathbf{I}_{n_r}$ and $\mathbf{\Phi} = \mathbf{I}_{n_r}$ is of interest for both nonsingular (see (2.13)) and singular (see (2.23)) cases.

2.3.1 Nonsingular case ($n_r \geq n_t$)

In this section, the expressions for the capacity of a correlated- and uncorrelated Rayleigh-type $n_r \times 2$ channel environment (see (1.10)) is derived when the underlying distribution is complex matrix variate elliptical. It is specifically derived here for a two-input ($n_t = 2$), n_r output communication system to be able to graphically illustrate the capacity of the system. This section investigates both cases when $\mathbf{\Sigma}$ has off-diagonal entries (i.e. correlated) as well as when $\mathbf{\Sigma} = \sigma^2\mathbf{I}_2$, thus being uncorrelated. Both the correlated- and uncorrelated cases are compared to the normal assumption as in Ratnarajah and Vaillancourt (2003).

Theorem 2.5 *Consider a two-input,* n_r *output matrix* $\mathbf{H} \sim \mathcal{C}E_{n_r \times 2}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma}, h)$ *subject to a Rayleigh type fading channel (see (1.10)), with* $n_r \geq 2$*. The capacity* C *(see (1.24)) is given by:*

$$\begin{aligned} C \;=\; & \frac{(a_1 a_2)^{n_r}}{\Gamma(n_r)\Gamma(n_r-1)(a_1-a_2)} \int_0^\infty \log\left(1 + \frac{\rho}{2}\lambda_1\right) \\ & \times \Big\{ \lambda_1^{n_r-1} \Gamma(n_r-1) a_2^{-(n_r-1)} \int_{\mathbb{R}^+} t^{n_r} \exp(-t a_1 \lambda_1) \mathcal{W}(t) dt \\ & - \lambda_1^{n_r-1} \Gamma(n_r-1) a_1^{-(n_r-1)} \int_{\mathbb{R}^+} t^{n_r} \exp(-t a_2 \lambda_1) \mathcal{W}(t) dt \\ & - \lambda_1^{n_r-2} \Gamma(n_r) a_2^{-n_r} \int_{\mathbb{R}^+} t^{n_r-1} \exp(-t a_1 \lambda_1) \mathcal{W}(t) dt \\ & + \lambda_1^{n_r-2} \Gamma(n_r) a_1^{-n_r} \int_{\mathbb{R}^+} t^{n_r-1} \exp(-t a_2 \lambda_1) \mathcal{W}(t) dt \Big\} d\lambda_1 \end{aligned} \tag{2.28}$$

where $a_1 > a_2$ *are the ordered eigenvalues of* $\mathbf{\Sigma}$.

Proof. The expression for the capacity in (1.24) is in terms of the pdf of an unordered eigenvalue; whereas (2.13) is the pdf of an ordered eigenvalue. Substituting $p \equiv n_t = 2$ and dividing by $n_t! = 2! = 2$ to obtain the pdf of the unordered eigenvalues:

$$f(\lambda_1, \lambda_2) = \frac{(\lambda_1\lambda_2)^{n_r-2} (\lambda_1 - \lambda_2)(a_1 a_2)^{n_r}}{2\Gamma(n_r)\Gamma(n_r-1)(a_2-a_1)} \int_{\mathbb{R}^+} t^{2n_r-1} \det(\exp(-t a_i \lambda_j)) \mathcal{W}(t) dt \tag{2.29}$$

where (see Result D.47):

$$\begin{aligned} \mathcal{C}\Gamma_2(2) &= \pi^{\frac{2(2-1)}{2}} \prod_{k=1}^{2} \Gamma(2-k+1) &= \pi\Gamma(2)\Gamma(1), \text{ and} \\ \mathcal{C}\Gamma_2(n_r) &= \pi^{\frac{2(2-1)}{2}} \prod_{k=1}^{2} \Gamma(n_r-k+1) &= \pi\Gamma(n_r)\Gamma(n_r-1). \end{aligned} \tag{2.30}$$

From (2.29), see that:

$$\begin{aligned} \det(\exp(-ta_i\lambda_j)) &\equiv \det \begin{bmatrix} \exp(-ta_1\lambda_1) & \exp(-ta_1\lambda_2) \\ \exp(-ta_2\lambda_1) & \exp(-ta_2\lambda_2) \end{bmatrix} \\ &= \exp(-t(a_1\lambda_1 + a_2\lambda_2)) - \exp(-t(a_1\lambda_2 + a_2\lambda_1)). \end{aligned} \tag{2.31}$$

By substituting (2.31) into (2.29), the capacity (see (1.24)) is obtained as:

$$\begin{aligned} C &= 2\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_0^\infty f(\lambda_1,\lambda_2)d\lambda_2 d\lambda_1 \\ &= \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_0^\infty \frac{(\lambda_1\lambda_2)^{n_r-2}(\lambda_1-\lambda_2)(a_1a_2)^{n_r}}{\Gamma(n_r)\Gamma(n_r-1)(a_2-a_1)} \\ &\quad \times \int_{\mathbb{R}^+} t^{2n_r-1}\left(\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \exp(-t(a_1\lambda_2+a_2\lambda_1))\right)\mathcal{W}(t)dtd\lambda_2 d\lambda_1 \\ &= K\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_0^\infty \left(\lambda_1^{n_r-1}\lambda_2^{n_r-2} - \lambda_1^{n_r-2}\lambda_2^{n_r-1}\right) \\ &\quad \times \int_{\mathbb{R}^+} t^{2n_r-1}\left(\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \exp(-t(a_1\lambda_2+a_2\lambda_1))\right)\mathcal{W}(t)dtd\lambda_2 d\lambda_1 \\ &= K\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\int_{\mathbb{R}^+} t^{2n_r-1}\int_0^\infty \left(\lambda_1^{n_r-1}\lambda_2^{n_r-2} - \lambda_1^{n_r-2}\lambda_2^{n_r-1}\right) \\ &\quad \times \left(\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \exp(-t(a_1\lambda_2+a_2\lambda_1))\right)d\lambda_2\mathcal{W}(t)dtd\lambda_1 \end{aligned} \tag{2.32}$$

where $K = \frac{(a_1a_2)^{n_r}}{\Gamma(n_r)\Gamma(n_r-1)(a_2-a_1)}$. From (2.32) and by using Result C.22, consider:

$$\begin{aligned} &\int_0^\infty \left(\lambda_1^{n_r-1}\lambda_2^{n_r-2} - \lambda_1^{n_r-2}\lambda_2^{n_r-1}\right)\left(\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \exp(-t(a_1\lambda_2+a_2\lambda_1))\right)d\lambda_2 \\ &= \int_0^\infty (\lambda_1^{n_r-1}\lambda_2^{n_r-2}\exp(-t(a_1\lambda_1+a_2\lambda_2)) - \lambda_1^{n_r-1}\lambda_2^{n_r-2}\exp(-t(a_1\lambda_2+a_2\lambda_1)) \\ &\quad -\lambda_1^{n_r-2}\lambda_2^{n_r-1}\exp(-t(a_1\lambda_1+a_2\lambda_2)) + \lambda_1^{n_r-2}\lambda_2^{n_r-1}\exp(-t(a_1\lambda_2+a_2\lambda_1)))d\lambda_2 \\ &= \lambda_1^{n_r-1}\exp(-ta_1\lambda_1)\int_0^\infty \lambda_2^{n_r-2}\exp(-ta_2\lambda_2)d\lambda_2 - \lambda_1^{n_r-1}\exp(-ta_2\lambda_1)\int_0^\infty \lambda_2^{n_r-2}\exp(-ta_1\lambda_2)d\lambda_2 \\ &\quad -\lambda_1^{n_r-2}\exp(-ta_1\lambda_1)\int_0^\infty \lambda_2^{n_r-1}\exp(-ta_2\lambda_2)d\lambda_2 + \lambda_1^{n_r-2}\exp(-ta_2\lambda_1)\int_0^\infty \lambda_2^{n_r-1}\exp(-ta_1\lambda_2)d\lambda_2 \\ &= \lambda_1^{n_r-1}\exp(-ta_1\lambda_1)\Gamma(n_r-1)(ta_2)^{-(n_r-1)} - \lambda_1^{n_r-1}\exp(-ta_2\lambda_1)\Gamma(n_r-1)(ta_1)^{-(n_r-1)} \\ &\quad -\lambda_1^{n_r-2}\exp(-ta_1\lambda_1)\Gamma(n_r)(ta_2)^{-n_r} + \lambda_1^{n_r-2}\exp(-ta_2\lambda_1)\Gamma(n_r)(ta_1)^{-n_r}. \end{aligned} \tag{2.33}$$

Substituting (2.33) into (2.32):

$$
\begin{aligned}
C &= K\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\left\{\int_{\mathbb{R}^+} t^{2n_r-1}\lambda_1^{n_r-1}\exp\left(-ta_1\lambda_1\right)\Gamma\left(n_r-1\right)\left(ta_2\right)^{-(n_r-1)}\mathcal{W}(t)dt\right.\\
&\quad -\int_{\mathbb{R}^+} t^{2n_r-1}\lambda_1^{n_r-1}\exp\left(-ta_2\lambda_1\right)\Gamma\left(n_r-1\right)\left(ta_1\right)^{-(n_r-1)}\mathcal{W}(t)dt\\
&\quad -\int_{\mathbb{R}^+} t^{2n_r-1}\lambda_1^{n_r-2}\exp\left(-ta_1\lambda_1\right)\Gamma\left(n_r\right)\left(ta_2\right)^{-n_r}\mathcal{W}(t)dt\\
&\quad \left.+\int_{\mathbb{R}^+} t^{2n_r-1}\lambda_1^{n_r-2}\exp\left(-ta_2\lambda_1\right)\Gamma\left(n_r\right)\left(ta_1\right)^{-n_r}\mathcal{W}(t)dt\right\}\\
&= \frac{\left(a_1a_2\right)^{n_r}}{\Gamma\left(n_r\right)\Gamma\left(n_r-1\right)\left(a_2-a_1\right)}\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\\
&\quad \times\left\{\lambda_1^{n_r-1}\Gamma\left(n_r-1\right)a_2^{-(n_r-1)}\int_{\mathbb{R}^+} t^{n_r}\exp\left(-ta_1\lambda_1\right)\mathcal{W}(t)dt\right.\\
&\quad -\lambda_1^{n_r-1}\Gamma\left(n_r-1\right)a_1^{-(n_r-1)}\int_{\mathbb{R}^+} t^{n_r}\exp\left(-ta_2\lambda_1\right)\mathcal{W}(t)dt\\
&\quad -\lambda_1^{n_r-2}\Gamma\left(n_r\right)a_2^{-n_r}\int_{\mathbb{R}^+} t^{n_r-1}\exp\left(-ta_1\lambda_1\right)\mathcal{W}(t)dt\\
&\quad \left.+\lambda_1^{n_r-2}\Gamma\left(n_r\right)a_1^{-n_r}\int_{\mathbb{R}^+} t^{n_r-1}\exp\left(-ta_2\lambda_1\right)\mathcal{W}(t)dt\right\}d\lambda_1
\end{aligned}
$$

which leaves the final result. ■

Corollary 2.9 *Consider a two-input, n_r output matrix $\mathbf{H}\sim\mathcal{C}t_{n_r\times 2}(\mathbf{0},\mathbf{I}_{n_r}\otimes\mathbf{\Sigma},h)$ subject to a Rayleigh type fading channel (see (1.10)) with weight function (1.6), with $n_r\geq 2$. The capacity C (see (1.24)) is given by:*

$$
\begin{aligned}
C &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{a_1^{n_r}a_2\Gamma\left(n_r+\frac{v}{2}\right)}{\left(a_1-a_2\right)\Gamma\left(n_r\right)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}\left(a_1\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)}d\lambda_1 \\
&\quad -\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{a_1a_2^{n_r}\Gamma\left(n_r+\frac{v}{2}\right)}{\left(a_1-a_2\right)\Gamma\left(n_r\right)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}\left(a_2\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)}d\lambda_1\\
&\quad -\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{a_1^{n_r}\Gamma\left(n_r+\frac{v}{2}-1\right)}{\left(a_1-a_2\right)\Gamma\left(n_r-1\right)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}\left(a_1\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)}d\lambda_1\\
&\quad +\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{a_2^{n_r}\Gamma\left(n_r+\frac{v}{2}-1\right)}{\left(a_1-a_2\right)\Gamma\left(n_r-1\right)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}\left(a_2\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)}d\lambda_1
\end{aligned}
\tag{2.34}
$$

where $v>0$ is the degrees of freedom.

Proof. Consider from (2.28) the following integrals, by substituting the weight function for the t distribution (see (1.6)), and by using Result C.22:

$$\begin{aligned}
\int_{\mathbb{R}^+} t^{n_r} \exp(-ta_1\lambda_1)\mathcal{W}(t)dt &= \int_{\mathbb{R}^+} t^{n_r} \exp(-ta_1\lambda_1) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_r+\frac{v}{2}-1} \exp\left(-t\left(a_1\lambda_1 + \frac{v}{2}\right)\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_r + \frac{v}{2}\right)\left(a_1\lambda_1 + \frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} \qquad (2.35)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^+} t^{n_r} \exp(-ta_2\lambda_1)\mathcal{W}(t)dt &= \int_{\mathbb{R}^+} t^{n_r} \exp(-ta_2\lambda_1) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_r+\frac{v}{2}-1} \exp\left(-t\left(a_2\lambda_1 + \frac{v}{2}\right)\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_r + \frac{v}{2}\right)\left(a_2\lambda_1 + \frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} \qquad (2.36)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^+} t^{n_r-1} \exp(-ta_1\lambda_1)\mathcal{W}(t)dt &= \int_{\mathbb{R}^+} t^{n_r-1} \exp(-ta_1\lambda_1) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_r+\frac{v}{2}-1-1} \exp\left(-t\left(a_1\lambda_1 + \frac{v}{2}\right)\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_r + \frac{v}{2} - 1\right)(a_1\lambda_1 + v)^{-\left(n_r+\frac{v}{2}-1\right)} \qquad (2.37)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^+} t^{n_r-1} \exp(-ta_2\lambda_1)\mathcal{W}(t)dt &= \int_{\mathbb{R}^+} t^{n_r-1} \exp(-ta_2\lambda_1) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_r+\frac{v}{2}-1-1} \exp\left(-t\left(a_2\lambda_1 + \frac{v}{2}\right)\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_r + \frac{v}{2} - 1\right)\left(a_2\lambda_1 + \frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)}. \qquad (2.38)
\end{aligned}$$

Substituting (2.35), (2.36), (2.37), and (2.38) into (2.28), leaves the capacity as:

$$
\begin{aligned}
C &= \frac{(a_1 a_2)^{nr}}{\Gamma(n_r)\Gamma(n_r-1)(a_2-a_1)} \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) \\
&\quad \times \Big\{ \lambda_1^{n_r-1}\Gamma(n_r-1)\, a_2^{-(n_r-1)}\Gamma\left(n_r+\frac{v}{2}\right)\left(a_1\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} \\
&\quad -\lambda_1^{n_r-1}\Gamma(n_r-1)\, a_1^{-(n_r-1)}\Gamma\left(n_r+\frac{v}{2}\right)\left(a_2\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} \\
&\quad -\lambda_1^{n_r-2}\Gamma(n_r)\, a_2^{-n_r}\Gamma\left(n_r+\frac{v}{2}-1\right)\left(a_1\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)} \\
&\quad +\lambda_1^{n_r-2}\Gamma(n_r)\, a_1^{-n_r}\Gamma\left(n_r+\frac{v}{2}-1\right)\left(a_2\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)} \Big\} d\lambda_1 \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{a_1^{n_r} a_2 \Gamma\left(n_r+\frac{v}{2}\right)}{(a_2-a_1)\Gamma(n_r)} \int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}\left(a_1\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} d\lambda_1 \\
&\quad - \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{a_1 a_2^{n_r} \Gamma\left(n_r+\frac{v}{2}\right)}{(a_2-a_1)\Gamma(n_r)} \int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}\left(a_2\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} d\lambda_1 \\
&\quad - \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{a_1^{n_r} \Gamma\left(n_r+\frac{v}{2}-1\right)}{(a_2-a_1)\Gamma(n_r-1)} \int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}\left(a_1\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)} d\lambda_1 \\
&\quad + \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{a_2^{n_r} \Gamma\left(n_r+\frac{v}{2}-1\right)}{(a_2-a_1)\Gamma(n_r-1)} \int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}\left(a_2\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)} d\lambda_1
\end{aligned}
$$

which leaves the final result. ■

In the next theorem, the capacity is derived under the assumption of no correlation between transmitters, or $\mathbf{\Sigma} = \sigma^2\mathbf{I}_2$.

Theorem 2.6 *Consider a two-input, n_r output matrix $\mathbf{H} \sim \mathcal{C}E_{n_r\times 2}(\mathbf{0}, \mathbf{I}_{n_r}\otimes\sigma^2\mathbf{I}_2, h)$ subject to a Rayleigh type fading channel (see (1.10)), with $n_r \geq 2$. The capacity C (see (1.24)) is given by:*

$$
\begin{aligned}
C &= \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\Big\{ \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r} t^{n_r+1}\exp\left(-t\sigma^{-2}\lambda_1\right)}{2\Gamma(n_r)\sigma^2}\mathcal{W}(t)dt \\
&\quad - \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r-1} t^{n_r}\exp\left(-t\sigma^{-2}\lambda_1\right)}{\Gamma(n_r-1)}\mathcal{W}(t)dt \\
&\quad + \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r-2} t^{n_r-1}\Gamma(n_r+1)\exp\left(-t\sigma^{-2}\lambda_1\right)}{2\Gamma(n_r-1)\sigma^{-2}}\mathcal{W}(t)dt \Big\} d\lambda_1.
\end{aligned} \tag{2.39}
$$

Proof. The expression for the capacity in (1.24) is in terms of the pdf of an unordered eigenvalue; whereas (2.14) is the pdf of an ordered eigenvalue. Substituting $p \equiv n_t = 2$ and dividing by 2! to obtain the pdf of the unordered eigenvalues:

$$\begin{aligned} f(\lambda_1,\lambda_2) &= \frac{\pi^2(\lambda_1\lambda_2)^{n_r-2}(\lambda_1-\lambda_2)^2}{2!\mathcal{C}\Gamma_2(n_r)\mathcal{C}\Gamma_2(2)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}\sum_{i=1}^{2}\lambda_i\right)\mathcal{W}(t)dt \\ &= \frac{\pi^2(\lambda_1\lambda_2)^{n_r-2}(\lambda_1-\lambda_2)^2}{2\pi^2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right)\mathcal{W}(t)dt \\ &= \frac{\lambda_1^{n_r}\lambda_2^{n_r-2}-2\lambda_1^{n_r-1}\lambda_2^{n_r-1}+\lambda_1^{n_r-2}\lambda_2^{n_r}}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right)\mathcal{W}(t)dt. \end{aligned} \tag{2.40}$$

From (2.40), the marginal pdf of λ_1 is given by:

$$\begin{aligned} f(\lambda_1) &= \int_0^\infty f(\lambda_1,\lambda_2)d\lambda_2 \\ &= \int_0^\infty \frac{\left(\lambda_1^{n_r}\lambda_2^{n_r-2}-2\lambda_1^{n_r-1}\lambda_2^{n_r-1}+\lambda_1^{n_r-2}\lambda_2^{n_r}\right)}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right)\mathcal{W}(t)dtd\lambda_2 \\ &= \int_0^\infty \frac{\lambda_1^{n_r}\lambda_2^{n_r-2}}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right)\mathcal{W}(t)dtd\lambda_2 \\ &\quad -\int_0^\infty \frac{2\lambda_1^{n_r-1}\lambda_2^{n_r-1}}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right)\mathcal{W}(t)dtd\lambda_2 \\ &\quad +\int_0^\infty \frac{\lambda_1^{n_r-2}\lambda_2^{n_r}}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right)\mathcal{W}(t)dtd\lambda_2 \\ &= \frac{\lambda_1^{n_r}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\int_0^\infty \lambda_2^{n_r-2}\exp\left(-t\sigma^{-2}\lambda_2\right)d\lambda_2\mathcal{W}(t)dt \\ &\quad -\frac{\lambda_1^{n_r-1}\exp\left(-t\sigma^2\lambda_1\right)}{\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\int_0^\infty \lambda_2^{n_r-1}\exp\left(-t\sigma^{-2}\lambda_2\right)d\lambda_2\mathcal{W}(t)dt \\ &\quad +\frac{\lambda_1^{n_r-2}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\int_0^\infty \lambda_2^{n_r}\exp\left(-t\sigma^{-2}\lambda_2\right)d\lambda_2\mathcal{W}(t)dt \\ &= \frac{\lambda_1^{n_r}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\Gamma(n_r-1)\left(t\sigma^{-2}\right)^{-(n_r-1)}\mathcal{W}(t)dt \\ &\quad -\frac{\lambda_1^{n_r-1}\exp\left(-t\sigma^2\lambda_1\right)}{\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\lambda_2^{n_r-1}\Gamma(n_r)\left(t\sigma^{-2}\right)^{-n_r}\mathcal{W}(t)dt \\ &\quad +\frac{\lambda_1^{n_r-2}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_r)\Gamma(n_r-1)\sigma^{2n_r}}\int\limits_{\mathbb{R}^+} t^{2n_r}\lambda_2^{n_r}\Gamma(n_r+1)\left(t\sigma^{-2}\right)^{-(n_r+1)}\mathcal{W}(t)dt. \end{aligned} \tag{2.41}$$

By subsituting (2.41) into (1.24), the capacity is given by:

$$\begin{aligned} C &= 2\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right] f(\lambda_1)d\lambda_1 \\ &= \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\left\{\int_{\mathbb{R}^+} \frac{\lambda_1^{n_r} t^{n_r+1}\exp\left(-t\sigma^{-2}\lambda_1\right)}{2\Gamma(n_r)\sigma^2}\mathcal{W}(t)dt \right. \\ &\quad - \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r-1} t^{n_r}\exp\left(-t\sigma^{-2}\lambda_1\right)}{\Gamma(n_r-1)}\mathcal{W}(t)dt \\ &\quad \left. + \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r-2} t^{n_r-1}\Gamma(n_r+1)\exp\left(-t\sigma^{-2}\lambda_1\right)}{2\Gamma(n_r-1)\sigma^{-2}}\mathcal{W}(t)dt\right\} d\lambda_1 \end{aligned}$$

leaving the final result. ■

Corollary 2.10 *Consider a two-input, n_r output matrix $\mathbf{H} \sim \mathcal{C}t_{n_r \times 2}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \sigma^2\mathbf{I}_2, h)$ subject to a Rayleigh type fading channel (see (1.10)) with weight function (1.6), with $n_r \geq 2$. The capacity C (see (1.24)) is given by:*

$$\begin{aligned} C &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\Gamma\left(n_r+\frac{v}{2}+1\right)}{\sigma^2\Gamma(n_r)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r}\left(\frac{\lambda_1}{\sigma^2}+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}+1\right)} d\lambda_1 \\ &\quad -\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{2\Gamma\left(n_r+\frac{v}{2}\right)}{\Gamma(n_r-1)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-1}\left(\frac{\lambda_1}{\sigma^2}+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} d\lambda_1 \\ &\quad +\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\Gamma\left(n_r+\frac{v}{2}-1\right)\Gamma(n_r+1)}{\sigma^{-2}\Gamma(n_r-1)}\int_0^\infty \log\left[1+\frac{\rho}{2}\lambda_1\right]\lambda_1^{n_r-2}\left(\frac{\lambda_1}{\sigma^2}+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)} d\lambda_1 \end{aligned} \tag{2.42}$$

where $v > 0$ is the degrees of freedom.

Proof. By substituting the weight function for the t distribution (see (1.6)), and by Result C.22, the integrals from (2.39) simplifies to:

$$\begin{aligned} \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r} t^{n_r+1}}{2\Gamma(n_r)\sigma^2}\exp\left(-t\sigma^2\lambda_1\right)\mathcal{W}(t)dt &= \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r} t^{n_r+1}\exp\left(-t\sigma^{-2}\lambda_1\right)}{2\Gamma(n_r)\sigma^2}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\lambda_1^{n_r}}{2\Gamma(n_r)\sigma^2}\int_{\mathbb{R}^+} t^{n_r+\frac{v}{2}}\exp\left(-t\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\lambda_1^{n_r}}{2\Gamma(n_r)\sigma^2}\Gamma\left(n_r+\frac{v}{2}+1\right)\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}+1\right)} \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r-1} t^{n_r}}{\Gamma(n_r-1)}\exp\left(-t\sigma^2\lambda_1\right)\mathcal{W}(t)dt &= \int_{\mathbb{R}^+} \frac{\lambda_1^{n_r-1} t^{n_r}\exp\left(-t\sigma^{-2}\lambda_1\right)}{\Gamma(n_r-1)}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\lambda_1^{n_r-1}}{\Gamma(n_r-1)}\int_{\mathbb{R}^+} t^{n_r+\frac{v}{2}-1}\exp\left(-t\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\lambda_1^{n_r-1}}{\Gamma(n_r-1)}\Gamma\left(n_r+\frac{v}{2}\right)\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} \end{aligned} \tag{2.44}$$

and

$$\begin{aligned}
&\int\limits_{\mathbb{R}^+} \frac{\lambda_1^{n_r-2} t^{n_r-1} \Gamma(n_r+1)}{2\Gamma(n_r-1)\sigma^{-2}} \exp\left(-t\sigma^2\lambda_1\right) \mathcal{W}(t)dt \\
&= \int\limits_{\mathbb{R}^+} \frac{\lambda_1^{n_r-2} t^{n_r-1} \Gamma(n_r+1) \exp\left(-t\sigma^{-2}\lambda_1\right)}{2\Gamma(n_r-1)\sigma^{-2}} \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\lambda_1^{n_r-2}\Gamma(n_r+1)}{2\Gamma(n_r-1)\sigma^{-2}} \int\limits_{\mathbb{R}^+} t^{n_r+\frac{v}{2}-2} \exp\left(-t\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\lambda_1^{n_r-2}\Gamma(n_r+1)}{2\Gamma(n_r-1)\sigma^{-2}} \Gamma\left(n_r+\frac{v}{2}-1\right) \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)}. \qquad (2.45)
\end{aligned}$$

Substituting (2.43), (2.44), and (2.45) into (2.39), the capacity as in (1.24) is given by:

$$\begin{aligned}
C &= 2\int\limits_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) f(\lambda_1)d\lambda_1 \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_r+\frac{v}{2}+1\right)}{2\Gamma(n_r)\sigma^2} \int\limits_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) \lambda_1^{n_r} \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}+1\right)} d\lambda_1 \\
&\quad - \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_r+\frac{v}{2}\right)}{\Gamma(n_r-1)} \int\limits_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) \lambda_1^{n_r-1} \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}\right)} d\lambda_1 \\
&\quad + \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma(n_r+1)\Gamma\left(n_r+\frac{v}{2}-1\right)}{2\Gamma(n_r-1)\sigma^{-2}} \int\limits_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) \lambda_1^{n_r-2} \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_r+\frac{v}{2}-1\right)} d\lambda_1
\end{aligned}$$

which leaves the final result. ■

These expressions for the capacity under a complex matrix variate t distribution is coded in SAS; this code is contained in the Appendix. Figure 2.1 illustrates the channel capacity (2.34) versus n_r for different values of ρ (the signal to noise ratio), and for $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$ and $v = 10$. Table 2.1 shows the capacity (2.34) in nats for different values of n_r. Figure 2.2 illustrates the channel capacity (2.42) versus n_r for different values of ρ , and for $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $v = 10$. Table 2.2 shows the capacity (2.42) in nats for different values of n_r.

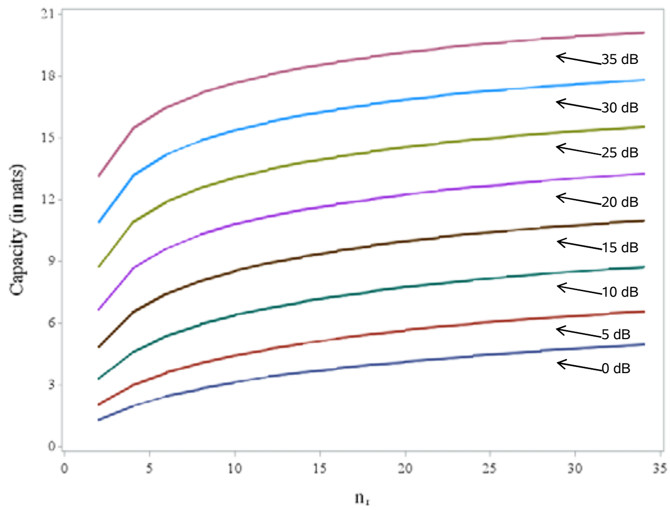


Figure 2.1 (2.34) against n_r for different values of ρ

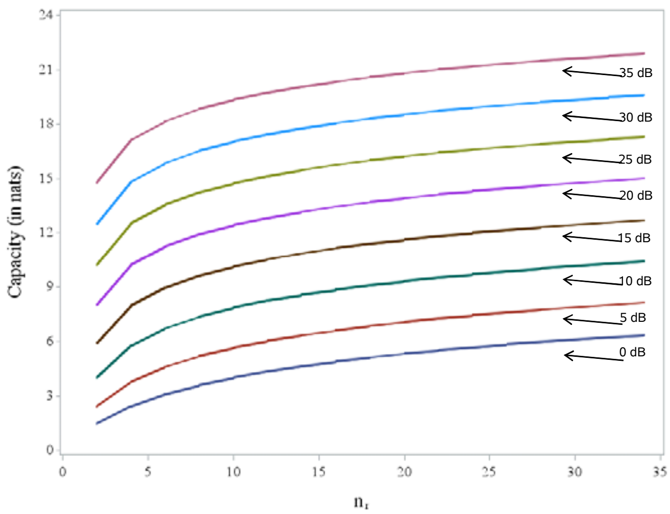


Figure 2.2 (2.42) against n_r for different values of ρ

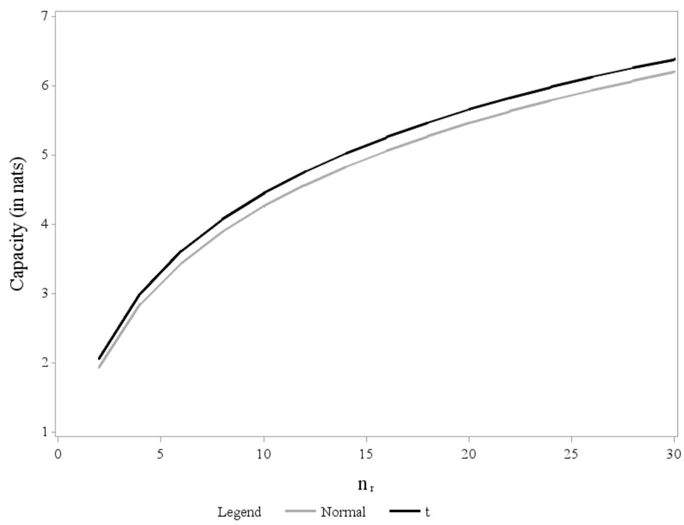


Figure 2.3 (2.42) and eq. 32 of Ratnarajah and Vaillancourt (2003) against n_r, for $\rho = 20$

n_r	0 dB	5 dB	10 dB	15 dB	20 dB	25 dB	30 dB	35 dB
2	1.2916	2.0609	3.3057	4.8558	6.6852	8.7821	10.9059	13.1656
4	1.9816	2.9984	4.5956	6.5129	8.6450	10.8836	13.1643	15.4598
6	2.4582	3.6126	5.3811	7.4294	9.6327	11.9010	14.1924	16.4914
8	2.8266	4.0737	5.9455	8.0592	10.2922	12.5715	14.8665	17.1666
10	3.1289	4.4445	6.3856	8.5381	10.7872	13.0721	15.368	17.6696
12	3.3862	4.7550	6.7460	8.9240	11.1831	13.4713	15.7691	18.0700
14	3.6105	5.0222	7.0506	9.2467	11.5125	13.8028	16.1012	18.4021
16	3.8095	5.2564	7.3141	9.5234	11.7939	14.0856	16.3842	18.6850
18	3.9882	5.4646	7.5456	9.7650	12.0988	14.3313	16.6298	18.9303
20	4.1502	5.6515	7.7414	9.9786	12.2547	14.5476	16.8458	19.1458

Table 2.1 Capacity (2.34) for a $n_r \times 2$ system for different values of ρ and $v = 10$

n_r	0 dB	5 dB	10 dB	15 dB	20 dB	25 dB	30 dB	35 dB
2	1.4843	2.4498	4.0298	5.9281	8.0291	10.2403	12.5045	14.7941
4	2.4402	3.7860	5.7830	7.9676	10.2292	12.5184	14.8167	17.1179
6	3.1083	4.6148	6.7334	8.9714	11.2528	13.5486	15.8490	18.1509
8	3.6156	5.2064	7.3788	9.6373	11.9256	14.2237	16.5284	18.8269
10	4.0228	5.6647	7.8668	10.1360	12.4279	14.7270	17.0285	19.3307
12	4.3622	6.0382	8.2591	10.5948	12.8287	15.1285	17.4302	19.7325
14	4.6583	6.3532	8.5869	10.8670	13.1623	15.4625	17.7643	20.0668
16	4.9069	6.6253	8.8684	11.1516	13.4479	14.7484	18.0503	20.3525
18	5.1324	6.8648	9.1149	11.4004	13.6974	15.9981	18.3000	20.6022
20	5.3350	7.0785	9.3342	11.6214	13.9189	16.2197	18.5215	20.8237

Table 2.2 Capacity (2.42) in nats for a $n_r \times 2$ system for different values of ρ and $v = 10$

From Table 2.1 and Table 2.2 it can be observed that the existence of nonzero correlation between the receiving antennas degrade the system capacity as the capacity is lower than the case of zero correlation. This same phenomenon has been observed under the complex matrix variate normal assumption as well (see Ratnarajah and Vaillancourt (2003)). However, the capacity under the assumption of an underlying complex matrix variate t distribution is seen to be higher (see Table 2.1 and Table 2.2) than that under the complex matrix variate normal assumption (see Ratnarajah and Vaillancourt (2003), Table 1 and Table 2).

2.3.2 Singular case ($n_t > n_r$)

For the singular case, the correlated Rayleigh $2 \times n_t$ channel is considered next, and its capacity derived. Subsequently the uncorrelated Rayleigh $2 \times n_t$ channel's capacity is derived and studied. As in the previous section, this thesis examines both cases when $\mathbf{\Sigma}$ has off diagonal entries (i.e. correlated) as well as when $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_2$, thus being uncorrelated. The investigation of the complex matrix variate t distribution is approached in this thesis similar to the approach of Ratnarajah and Vaillancourt (2005).

Theorem 2.7 *Consider a two output, n_t input matrix $\mathbf{H} \sim \mathcal{C}E_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \mathbf{\Sigma}, h)$ subject to a Rayleigh type* *fading channel (see (1.10)), with $n_t \geq 2$. The capacity C (see (1.24)) is given by:*

$$\begin{aligned} C &= K\int_0^\infty\int_0^{\lambda_1}\left(\log\left(1+\frac{\rho}{n_t}\lambda_1\right)+\log\left(1+\frac{\rho}{n_t}\lambda_2\right)\right)(\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2) \\ &\quad\times\int_{\mathbb{R}^+} t^{n_t+1}\det\left(\exp\left(-ta_i\lambda_j\right)\right)\mathcal{W}(t)dtd\lambda_2 d\lambda_1 \end{aligned} \tag{2.46}$$

where $K = \left(\Gamma(n_t)\Gamma(n_t-1)\prod_{i=1}^{n_t}a_i^2\prod_{k<l}^{n_t}(a_k-a_l)\right)^{-1}$, *and* $a_1 > a_2 > ... > a_{n_t} > 0$ *are the eigenvalues of* $\mathbf{\Sigma}$.

Proof. Note that (2.23) is the pdf of an ordered eigenvalue:

$$f(\mathbf{\Lambda}) = \frac{\pi^{n(n-1)}\left(\prod_{i=1}^{n}\lambda_i^{p-n}\right)\left(\prod_{k<l}^{n}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(p)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\,{}_0\mathcal{C}F_0^{(n)}\left(\mathbf{\Lambda},-t\mathbf{\Sigma}^{-1}\right)\mathcal{W}(t)dt.$$

Thus, for the $2\times n_t$ case:

$$f(\lambda_1,\lambda_2) = \frac{\pi^{2(2-1)}\left(\prod_{i=1}^{2}\lambda_i^{n_t-2}\right)\left(\prod_{k<l}^{2}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_2(2)\mathcal{C}\Gamma_2(n_t)\det(\mathbf{\Sigma})^2}\int_{\mathbb{R}^+} t^{2n_t}\,{}_0\mathcal{C}F_0^{(2)}\left(\mathbf{\Lambda},-t\mathbf{\Sigma}^{-1}\right)\mathcal{W}(t)dt. \tag{2.47}$$

Using Result D.58 and Result D.57 the complex hypergeometric function of two Hermitian matrix arguments in (2.47) can be simplified as:

$$\begin{aligned} {}_0\mathcal{C}F_0^{(2)}\left(\mathbf{\Lambda},-t\mathbf{\Sigma}^{-1}\right) &= \frac{\left(\prod_{j=1}^{2}(j-1)!\right)\det\left(\exp\left(-ta_i\lambda_j\right)\right)}{\prod_{k<l}^{2}(\lambda_k-\lambda_l)\prod_{k<l}^{n_t}(ta_k-ta_l)} \\ &= \frac{\det\left(\exp\left(-ta_i\lambda_j\right)\right)}{(\lambda_1-\lambda_2)\,t^{\frac{n_t(n_t-1)}{2}}\prod_{k<l}^{n_t}(a_k-a_l)}. \end{aligned} \tag{2.48}$$

Note that $\det(\mathbf{\Sigma})^2 = \left(\prod_{i=1}^{n_t}a_i\right)^2 = \prod_{i=1}^{n_t}a_i^2$. Substituting (2.48) into (2.47) leaves:

$$\begin{aligned} f(\lambda_1,\lambda_2) &= \frac{(\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2)}{\Gamma(n_t)\Gamma(n_t-1)\det(\mathbf{\Sigma})^2\prod_{k<l}^{n_t}(a_k-a_l)}\int_{\mathbb{R}^+} t^{2n_t-\frac{n_t(n_t-1)}{2}}\det\left(\exp\left(-ta_i\lambda_j\right)\right)\mathcal{W}(t)dt \\ &= \frac{(\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2)}{\Gamma(n_t)\Gamma(n_t-1)\prod_{i=1}^{n_t}a_i^2\prod_{k<l}^{n_t}(a_k-a_l)}\int_{\mathbb{R}^+} t^{2n_t-\frac{n_t^2}{2}-\frac{n_t}{2}}\det\left(\exp\left(-ta_i\lambda_j\right)\right)\mathcal{W}(t)dt \\ &= \frac{(\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2)}{\Gamma(n_t)\Gamma(n_t-1)\prod_{i=1}^{n_t}a_i^2\prod_{k<l}^{n_t}(a_k-a_l)}\int_{\mathbb{R}^+} t^{\frac{3n_t}{2}-\frac{n_t^2}{2}}\det\left(\exp\left(-ta_i\lambda_j\right)\right)\mathcal{W}(t)dt. \end{aligned} \tag{2.49}$$

Substituting (2.49) into (1.24):

$$\begin{aligned}
C &= \int_0^\infty \int_0^{\lambda_1} \left(\log\left(1+\frac{\rho}{n_t}\lambda_1\right) + \log\left(1+\frac{\rho}{n_t}\lambda_2\right)\right) f(\lambda_1,\lambda_2) d\lambda_2 d\lambda_1 \\
&= \int_0^\infty \int_0^{\lambda_1} \left(\log\left(1+\frac{\rho}{n_t}\lambda_1\right) + \log\left(1+\frac{\rho}{n_t}\lambda_2\right)\right) \frac{(\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2)}{\Gamma(n_t)\Gamma(n_t-1)\prod_{i=1}^{n_t} a_i^2 \prod_{k<l}^{n_t}(a_k-a_l)} \\
&\quad \times \int_{\mathbb{R}^+} t^{\frac{3n_t}{2}-\frac{n_t^2}{2}} \det(\exp(-ta_i\lambda_j)) \mathcal{W}(t) dt d\lambda_2 d\lambda_1 \\
&= K \int_0^\infty \int_0^{\lambda_1} \left(\log\left(1+\frac{\rho}{n_t}\lambda_1\right) + \log\left(1+\frac{\rho}{n_t}\lambda_2\right)\right) (\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2) \\
&\quad \times \int_{\mathbb{R}^+} t^{\frac{3n_t}{2}-\frac{n_t^2}{2}} \det(\exp(-ta_i\lambda_j)) \mathcal{W}(t) dt d\lambda_2 d\lambda_1
\end{aligned}$$

where

$$K = \left(\Gamma(n_t)\Gamma(n_t-1)\prod_{i=1}^{n_t} a_i^2 \prod_{k<l}^{n_t}(a_k-a_l)\right)^{-1} \tag{2.50}$$

which leaves the final result. ■

Corollary 2.11 *Consider a two output, n_t input matrix $\mathbf{H} \sim \mathcal{C}t_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \mathbf{\Sigma}, \nu)$ subject to a Rayleigh type fading channel (see (1.10)) with weight function (1.6), with $n_t \geq 2$. The capacity C (see (1.24)) is given by:*

$$\begin{aligned}
C &= K\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_0^\infty \int_0^{\lambda_1} \left(\log\left(1+\frac{\rho}{n_t}\lambda_1\right) + \log\left(1+\frac{\rho}{n_t}\lambda_2\right)\right) (\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2) \\
&\quad \times \int_{\mathbb{R}^+} t^{n_t+\frac{v}{2}} \exp\left(-t\frac{v}{2}\right) \det(\exp(-ta_i\lambda_j)) dt d\lambda_2 d\lambda_1
\end{aligned} \tag{2.51}$$

where K is as in (2.50), $v > 0$ is the degrees of freedom and $a_1 > a_2 > ... > a_{n_t} > 0$ are the eigenvalues of $\mathbf{\Sigma}$.

Proof. Consider from (2.46) the following by substituting the weight function for the t distribution (see (1.6)):

$$\begin{aligned}
\int_{\mathbb{R}^+} t^{n_t+1} \det(\exp(-ta_i\lambda_j)) \mathcal{W}(t) dt &= \int_{\mathbb{R}^+} t^{n_t+1} \det(\exp(-ta_i\lambda_j)) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_t+\frac{v}{2}} \exp\left(-t\frac{v}{2}\right) \det(\exp(-ta_i\lambda_j)) dt.
\end{aligned} \tag{2.52}$$

Using (2.46) and (2.52) the capacity (1.24) is then given by:

$$\begin{aligned}
C &= K\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_0^\infty \int_0^{\lambda_1} \left(\log\left(1+\frac{\rho}{n_t}\lambda_1\right) + \log\left(1+\frac{\rho}{n_t}\lambda_2\right)\right) (\lambda_1\lambda_2)^{n_t-2}(\lambda_1-\lambda_2) \\
&\quad \times \int_{\mathbb{R}^+} t^{n_t+\frac{v}{2}} \exp\left(-t\frac{v}{2}\right) \det(\exp(-ta_i\lambda_j)) dt d\lambda_2 d\lambda_1
\end{aligned}$$

which leaves the final result. ■

In the next theorem, the capacity is derived under the assumption of no correlation between transmitters, or $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_{n_t}$.

Theorem 2.8 *Consider a two output, n_t input matrix $\mathbf{H} \sim \mathcal{C}E_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \sigma^2 \mathbf{I}_{n_t}, h)$ subject to a Rayleigh type fading channel (see (1.10)), with $n_t \geq 2$. The capacity C (see (1.24)) is given by:*

$$
\begin{aligned}
C &= \frac{1}{\sigma^{2n_t+2}\Gamma(n_t)} \int_0^\infty \log\left[1 + \frac{\rho}{n_t}\lambda_1\right] \lambda_1^{n_t} \int_{\mathbb{R}^+} t^{n_t+1} \exp\left(-t\sigma^{-2}\lambda_1\right) \mathcal{W}(t) dt d\lambda_1 \\
&\quad - \frac{2}{\sigma^{2n_t}\Gamma(n_t-1)} \int_0^\infty \log\left[1 + \frac{\rho}{n_t}\lambda_1\right] \lambda_1^{n_t-1} \int_{\mathbb{R}^+} t^{n_t} \exp\left(-t\sigma^{-2}\lambda_1\right) \mathcal{W}(t) dt d\lambda_1 \\
&\quad + \frac{\Gamma(n_t+1)}{\sigma^{2n_t-2}\Gamma(n_t)\Gamma(n_t-1)} \int_0^\infty \log\left[1 + \frac{\rho}{n_t}\lambda_1\right] \lambda_1^{n_t-2} \int_{\mathbb{R}^+} t^{n_t-1} - t\sigma^{-2}\lambda_1 \mathcal{W}(t) dt d\lambda_1.
\end{aligned} \tag{2.53}
$$

Proof. The expression for the capacity in (1.24) is in terms of the pdf of an unordered eigenvalue; whereas (2.25) is the pdf of an ordered eigenvalue. Substituting $n \equiv n_r = 2$ and dividing by $n_r! = 2! = 2$ to obtain the pdf of the unordered eigenvalues:

$$
\begin{aligned}
f(\lambda_1, \lambda_2) &= \frac{\pi^{2(2-1)} \left(\prod_{i=1}^{2} \lambda_i^{n_t-2}\right) \left(\prod_{k<l}^{2} (\lambda_k - \lambda_l)\right)}{2! \mathcal{C}\Gamma_2(2) \mathcal{C}\Gamma_2(n_r) \sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2} \sum_{i=1}^{2} \lambda_i\right) \mathcal{W}(t) dt \\
&= \frac{\pi^2 (\lambda_1\lambda_2)^{n_t-2} (\lambda_1 - \lambda_2)^2}{2\pi^2 \Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right) \mathcal{W}(t) dt \\
&= \frac{\lambda_1^{n_t}\lambda_2^{n_t-2} - 2\lambda_1^{n_t-1}\lambda_2^{n_t-1} + \lambda_1^{n_t-2}\lambda_2^{n_t}}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right) \mathcal{W}(t) dt.
\end{aligned}
$$

The marginal pdf of λ_1 is given by:

$$\begin{aligned}
f(\lambda_1) &= \int_0^\infty f(\lambda_1,\lambda_2)d\lambda_2 \\
&= \int_0^\infty \frac{\lambda_1^{n_t}\lambda_2^{n_t-2} - 2\lambda_1^{n_t-1}\lambda_2^{n_t-1} + \lambda_1^{n_t-2}\lambda_2^{n_t}}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right) \mathcal{W}(t)dtd\lambda_2 \\
&= \int_0^\infty \frac{\lambda_1^{n_t}\lambda_2^{n_t-2}}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right) \mathcal{W}(t)dtd\lambda_2 \\
&\quad - \int_0^\infty \frac{2\lambda_1^{n_t-1}\lambda_2^{n_t-1}}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right) \mathcal{W}(t)dtd\lambda_2 \\
&\quad + \int_0^\infty \frac{\lambda_1^{n_t-2}\lambda_2^{n_t}}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \exp\left(-t\sigma^{-2}(\lambda_1+\lambda_2)\right) \mathcal{W}(t)dtd\lambda_2 \\
&= \frac{\lambda_1^{n_t}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \int_0^\infty \lambda_2^{n_t-2}\exp\left(-t\sigma^{-2}\lambda_2\right) d\lambda_2 \mathcal{W}(t)dt \\
&\quad - \frac{\lambda_1^{n_t-1}\exp\left(-t\sigma^2\lambda_1\right)}{\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \int_0^\infty \lambda_2^{n_t-1}\exp\left(-t\sigma^{-2}\lambda_2\right) d\lambda_2 \mathcal{W}(t)dt \\
&\quad + \frac{\lambda_1^{n_t-2}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t} \int_0^\infty \lambda_2^{n_t}\exp\left(-t\sigma^{-2}\lambda_2\right) d\lambda_2 \mathcal{W}(t)dt \\
&= \frac{\lambda_1^{n_t}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t}\Gamma(n_t-1)\left(t\sigma^{-2}\right)^{-(n_t-1)} \mathcal{W}(t)dt \\
&\quad - \frac{\lambda_1^{n_t-1}\exp\left(-t\sigma^2\lambda_1\right)}{\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t}\lambda_2^{n_t-1}\Gamma(n_t)\left(t\sigma^{-2}\right)^{-n_t} \mathcal{W}(t)dt \\
&\quad + \frac{\lambda_1^{n_t-2}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t}\lambda_2^{n_t}\Gamma(n_t+1)\left(t\sigma^{-2}\right)^{-(n_t+1)} \mathcal{W}(t)dt.
\end{aligned}$$

The capacity (see (1.24)) for an uncorrelated Rayleigh fading model of dimension $2 \times n_t$ under the singular complex elliptical class is given by:

$$\begin{aligned}
C &= 2\int_0^\infty \log\left(1+\frac{\rho}{n_t}\lambda_1\right)\int_0^\infty f(\lambda_1,\lambda_2)d\lambda_2 d\lambda_1 \\
&= \int_0^\infty \log\left(1+\frac{\rho}{n_t}\lambda_1\right)\left\{\frac{\lambda_1^{n_t}\exp\left(-t\sigma^2\lambda_1\right)}{\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t}\Gamma(n_t-1)\left(t\sigma^{-2}\right)^{-(n_t-1)} \mathcal{W}(t)dt\right. \\
&\quad - \frac{2\lambda_1^{n_t-1}\exp\left(-t\sigma^2\lambda_1\right)}{\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t}\lambda_2^{n_t-1}\Gamma(n_t)\left(t\sigma^{-2}\right)^{-n_t} \mathcal{W}(t)dt \\
&\quad \left. + \frac{\lambda_1^{n_t-2}\exp\left(-t\sigma^2\lambda_1\right)}{2\Gamma(n_t)\Gamma(n_t-1)\sigma^{4n_t}} \int_{\mathbb{R}^+} t^{2n_t}\lambda_2^{n_t}\Gamma(n_t+1)\left(t\sigma^{-2}\right)^{-(n_t+1)} \mathcal{W}(t)dt\right\}
\end{aligned}$$

leaving the final result. ■

Corollary 2.12 *Consider a two output, n_t input matrix $\mathbf{H} \sim \mathcal{C}E_{2\times n_t}(\mathbf{0}, \mathbf{I}_2 \otimes \sigma^2\mathbf{I}_{n_t}, h)$ subject to a Rayleigh* *type fading channel (see (1.10)) with weight function (1.6), with $n_t \geq 2$. The capacity C (see (1.24)) is given by:*

$$\begin{aligned} C &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_t+\frac{v}{2}+1\right)}{\sigma^{2n_t+2}\Gamma\left(n_t\right)} \int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right] \lambda_1^{n_t} \left(\frac{\lambda_1}{\sigma^2}+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}+1\right)} d\lambda_1 \\ &\quad -\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{2\Gamma\left(n_t+\frac{v}{2}\right)}{\sigma^{2n_t}\Gamma\left(n_t-1\right)} \int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right] \lambda_1^{n_t-1} \left(\frac{\lambda_1}{\sigma^2}+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}\right)} d\lambda_1 \\ &\quad +\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_t+\frac{v}{2}-1\right)\Gamma\left(n_t+1\right)}{\sigma^{2n_t-2}\Gamma\left(n_t\right)\Gamma\left(n_t-1\right)} \int_0^\infty \log\left[1+\frac{\rho}{n_t}\lambda_1\right] \lambda_1^{n_t-2} \left(\frac{\lambda_1}{\sigma^2}+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}-1\right)} d\lambda_1 \end{aligned} \tag{2.54}$$

where $v > 0$ is the degrees of freedom.

Proof. By substituting the weight function for the t distribution (see (1.6)), and by Result C.22, the integrals from (2.53) simplifies to:

$$\begin{aligned} \int_{\mathbb{R}^+} t^{n_t+1} \exp\left(-t\sigma^{-2}\lambda_1\right) \mathcal{W}(t) dt &= \int_{\mathbb{R}^+} t^{n_t+1} \exp\left(-t\sigma^{-2}\lambda_1\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_t+\frac{v}{2}} \exp\left(-t\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_t+\frac{v}{2}+1\right) \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}+1\right)} \end{aligned} \tag{2.55}$$

and

$$\begin{aligned} \int_{\mathbb{R}^+} t^{n_t} \exp\left(-t\sigma^{-2}\lambda_1\right) \mathcal{W}(t) dt &= \int_{\mathbb{R}^+} t^{n_t} \exp\left(-t\sigma^{-2}\lambda_1\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_t+\frac{v}{2}-1} \exp\left(-t\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_t+\frac{v}{2}\right) \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}\right)} \end{aligned} \tag{2.56}$$

and

$$\begin{aligned} \int_{\mathbb{R}^+} t^{n_t-1} \exp\left(-t\sigma^{-2}\lambda_1\right) \mathcal{W}(t) dt &= \int_{\mathbb{R}^+} t^{n_t-1} \exp\left(-t\sigma^{-2}\lambda_1\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{\mathbb{R}^+} t^{n_t+\frac{v}{2}-2} \exp\left(-t\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Gamma\left(n_t+\frac{v}{2}-1\right) \left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}-1\right)} \end{aligned} \tag{2.57}$$

Substituting (2.55), (2.56), and (2.57) into (2.53), leaves the capacity (see (1.24)) as:

$$\begin{aligned} C &= 2\int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right) f(\lambda_1)d\lambda_1 \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_t+\frac{v}{2}+1\right)}{2\Gamma\left(n_t\right)\sigma^{2n_t-2}} \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\lambda_1^{n_t}\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(n_t+\frac{v}{2}+1\right)} d\lambda_1 \\ &\quad -\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_t+\frac{v}{2}\right)}{\Gamma\left(n_t-1\right)\sigma^{2n_t}} \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\lambda_1^{n_t-1}\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(nt+\frac{v}{2}\right)} d\lambda_1 \\ &\quad +\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(n_t+1\right)\Gamma\left(n_t+\frac{v}{2}-1\right)}{2\Gamma\left(n_t\right)\Gamma\left(n_r-1\right)\sigma^{2n_t+2}} \int_0^\infty \log\left(1+\frac{\rho}{2}\lambda_1\right)\lambda_1^{n_t-1}\left(\sigma^{-2}\lambda_1+\frac{v}{2}\right)^{-\left(nt+\frac{v}{2}-1\right)} d\lambda_1 \end{aligned}$$

which leaves the final result. ■

Figure 2.4 illustrates the channel capacity (2.51) and (2.54) versus ρ for $n_t = 4$, for $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$ and $\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ respectively, and $v = 10$.

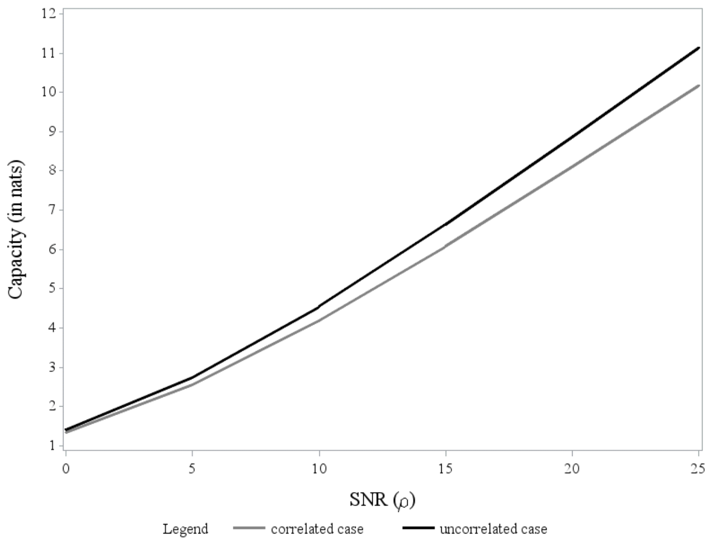


Figure 2.4 (2.51) and (2.54) against ρ, for $n_t = 4$

As in the nonsingular case (see Figure 2.1 and Figure 2.2), it is observed that existence of nonzero correlation between transmitters degrades the system capacity.

2.4 Summary of results and conclusion

A summary of theoretical results in this chapter is provided for the convenience of the reader in Table 2.3. Specifically, the results relating to the special cases of the quadratic form, $\mathbf{S}$, under consideration in this chapter are mentioned.

Distribution of X	Pdf of $\mathbf{S}$	Pdf of $\mathbf{\Lambda}$
Nonsingular		
Elliptical	(2.1)	(2.7)
Elliptical when $\mathbf{A}, \mathbf{\Phi} = \mathbf{I}_n$	(2.3)	(2.9)
Normal when $\mathbf{A}, \mathbf{\Phi} = \mathbf{I}_n$	(2.5)	(2.15)
t when $\mathbf{A}, \mathbf{\Phi} = \mathbf{I}_n$	(2.6)	(2.16)
Singular		
Elliptical	(2.17)	(2.22)
Elliptical when $\mathbf{A}, \mathbf{\Phi} = \mathbf{I}_n$	(2.18)	(2.23)
Normal when $\mathbf{A}, \mathbf{\Phi} = \mathbf{I}_n$	(2.20)	(2.26)
t when $\mathbf{A}, \mathbf{\Phi} = \mathbf{I}_n$	(2.21)	(2.27)

Table 2.3 Pdfs of nonsingular- and singular complex matrix variate distributions derived in this chapter

In this chapter the distribution of the quadratic form and its corresponding joint eigenvalues with an underlying complex matrix variate elliptical distribution was derived. Using (1.4), the form of the expressions for the pdfs of the quadratic form and its corresponding eigenvalues are computable and circumvents cumbersome computations. Some special cases were highlighted and shows the well-known Wishart distribution as a special case when the complex matrix variate normal distribution is under consideration. Another special case that is illustrated is the case of no correlation, when the covariance matrix only has nonzero entries on the diagonal and 0 entries on the off-diagonal elements. This case is of specific interest in the performance measure of channel capacity in the MIMO environment.

Specifically, the complex matrix variate t distribution was applied and the literature is enriched with its representation in this chapter. As an illustration of the obtained results the channel capacity within a MIMO environment is derived and studied. The channel capacity and is investigated for correlated and uncorrelated scenarios in the nonsingular and singular cases. It is observed that:

1. Correlation between transmitters/receivers degrade system capacity; and
2. The capacity of the system is higher in the case of underlying complex matrix variate complex t distribution than that compared to an underlying complex matrix variate normal distribution.

It is worthwhile to discuss the candidacy of the complex matrix variate t distribution as underlying choice for the practitioner for $\mathbf{H}$. When no correlation exists between receivers (i.e. independent observations), the well-known central limit theorem can be applied which results in $\mathbf{H} \sim \mathcal{C}N_{n_r \times n_t}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma})$. However, if the receivers- and transmitters are correlated simultaneously, i.e. $\mathbf{H} \sim \mathcal{C}N_{n_r \times n_t}(\mathbf{0}, \mathbf{\Sigma}_{n_r} \otimes \mathbf{\Sigma}_{n_t})$, then the well-known central limit theorem does not apply. In that case the complex matrix variate elliptical model may provide greater flexibility in this regard.

Although the results in this chapter are presented for the $\mathbf{I}_{n_r} \otimes \mathbf{\Sigma}$ and related cases, in the case of $\mathbf{\Sigma}_{n_r} \otimes \mathbf{\Sigma}_{n_t}$ the covariance structure can be simplified to $\mathbf{I}_{n_r} \otimes \mathbf{\Sigma}$ via a transformation. Even though the complex matrix variate normal distribution can then still be applied, these numerical examples of the capacity illustrate that the derived expressions under the complex matrix variate t distribution provide significant insights on the behaviour of performance measures when the assumption of the complex matrix variate normal distribution is challenged.

Chapter 3

Bivariate gamma type I distributions

Contents

3.1 Introduction

In this chapter, the main contribution is the derivation of the joint distribution of the diagonal elements of $(\mathbf{H}^H\mathbf{H})^{-1}$, where $\mathbf{H} \sim CE_{n,2}(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{\Sigma}, h)$ (see (1.3)); this distribution is called a bivariate gamma type I distribution. This bivariate gamma type I distribution is studied via its pdf, cdf, and product moment. As a particular contribution, the underlying complex matrix variate t distribution is studied, alongside the studied complex matrix variate normal model. From this bivariate gamma type I distribution, a bivariate Weibullised gamma type I distribution is derived which emanates from the bivariate gamma type I distribution. The derivation of these and subsequent bivariate distributions (see Chapter 4) provides a platform to gain valuable insight into the construction of such bivariate distributions, and also expands the knowledge base of candidates for modeling within the wireless communications domain. Application of the derived results are investigated with regards to the outage probability (see (1.25)) and the EGC diversity (see (1.26)) of a MIMO system operating under a bivariate gamma type I distribution; which is analyzed in a broad generality from an elliptical viewpoint, and comparatively investigated for the underlying complex matrix variate normal- and t cases.

The figure below visualises the main gist of theoretical derivations in this chapter.

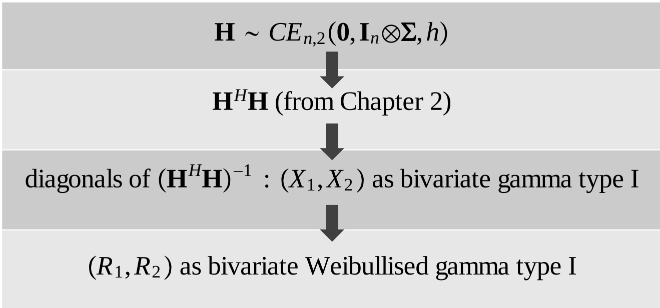


Figure 3.1 Outline of theoretical derivations in this chapter

3.2 Bivariate gamma type I distribution

In this section, a bivariate gamma type I distribution (extending the work of Xu et. al. (2009)) emanating from the complex matrix variate elliptical class is proposed. Xu et. al. (2009) considered the joint distribution of the diagonal elements of $\mathbf{W} = [w_{11}\ w_{12};\ w_{12}^*\ w_{22}]$, where $\mathbf{W} = (\mathbf{H}^H\mathbf{H})^{-1}$, with $\mathbf{H} \in \mathbb{C}_1^{n\times 2}$ distributed according to a complex matrix variate normal distribution $CN_{n,2}(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{\Sigma})$ (see (1.1)) with $\mathbf{\Sigma} = [1\ \xi;\ \xi^*\ 1]$ as the covariance matrix. Consider now $\mathbf{H} \sim CE_{n,2}(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{\Sigma}, h)$ (with pdf (1.3)). Then $\mathbf{S} = \mathbf{H}^H\mathbf{H} \in \mathbb{C}_2^{2\times 2}$ is complex Wishart type distributed, and the pdf is given by (see (2.3)):

$$f(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-2}\, \mathcal{G}(\mathbf{S})}{\mathcal{C}\Gamma_2(n) \det(\mathbf{\Sigma})^n}$$

where $\mathcal{C}\Gamma_2(n)$ denotes the complex multivariate gamma function (see Result D.47), and where

$$\mathcal{G}(\mathbf{S}) = \int_{\mathbb{R}^+} t^{2n} \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right) \mathcal{W}(t)\, dt.$$

3.2.1 Pdf

The following lemma derives a complex matrix variate *inverse* Wishart type distribution. Subsequently, a bivariate gamma type I emanating from the diagonal elements of this complex matrix variate inverse Wishart type distribution is derived.

Lemma 3.2.1 *Suppose* $\mathbf{S} \in \mathbb{C}_2^{2\times 2}$ *follows a complex matrix variate Wishart type distribution with pdf (2.3). Then* $\mathbf{W} = \mathbf{S}^{-1} \in \mathbb{C}_2^{2\times 2}$ *follows a complex matrix variate inverse Wishart type distribution with pdf:*

$$\begin{aligned} f(\mathbf{W}) &= \int_{\mathbb{R}^+} \frac{t^{2n}}{\mathcal{C}\Gamma_2(n) \det(\mathbf{W})^{n+2} \det(\mathbf{\Sigma})^n} \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{W}^{-1}\right) \mathcal{W}(t)\, dt \qquad (3.1) \\ &= \int_{\mathbb{R}^+} f(\mathbf{W}|t) \mathcal{W}(t)\, dt. \end{aligned}$$

Proof. *Consider* $\mathbf{W} = \mathbf{S}^{-1} \in \mathbb{C}_2^{2\times 2}$, *with Jacobian* $J(\mathbf{S} \to \mathbf{W}^{-1}) = \det(\mathbf{W})^{-2n}$ *(see Result D.44). From (2.3), the complex inverse Wishart pdf follows as:*

$$\begin{aligned} f(\mathbf{W}) &= f(\mathbf{W}^{-1}) \left| J(\mathbf{S} \to \mathbf{W}^{-1}) \right| \\ &= \frac{\det(\mathbf{W}^{-1})^{n-2}}{\mathcal{C}\Gamma_2(n) \det(\boldsymbol{\Sigma})^n} \int_{\mathbb{R}^+} t^{2n} \operatorname{etr}\left(-t\boldsymbol{\Sigma}^{-1}\mathbf{W}^{-1}\right) \mathcal{W}(t)\, dt \left| \det(\mathbf{W})^{-2n} \right| \\ &= \int_{\mathbb{R}^+} \frac{t^{2n}}{\mathcal{C}\Gamma_2(n) \det(\mathbf{W})^{n+2} \det(\boldsymbol{\Sigma})^n} \operatorname{etr}\left(-t\boldsymbol{\Sigma}^{-1}\mathbf{W}^{-1}\right) \mathcal{W}(t)\, dt \\ &= \int_{\mathbb{R}^+} f(\mathbf{W}|t) \mathcal{W}(t)\, dt, \end{aligned}$$

where

$$\begin{aligned} f(\mathbf{W}|t) &= \frac{t^{2n}}{\mathcal{C}\Gamma_2(n) \det(\mathbf{W})^{n+2} \det(\boldsymbol{\Sigma})^n} \operatorname{etr}\left(-t\boldsymbol{\Sigma}^{-1}\mathbf{W}^{-1}\right) \\ &= \frac{t^{2n}}{\mathcal{C}\Gamma_2(n)} \left(\frac{1}{w_{11}w_{22} - w_{12}w_{12}^*} \right)^{n+2} \left(\frac{1}{1 - \xi\xi^*} \right)^n \\ &\quad \times \exp\left(-\frac{t}{w_{11}w_{22} - w_{12}w_{12}^*(1-\xi\xi^*)} \left[w_{11} + w_{22} + w_{12}\xi^* + w_{12}^*\xi \right] \right) \end{aligned} \tag{3.2}$$

which concludes the proof. ■

Theorem 3.1 *Suppose* $\mathbf{W} \in \mathbb{C}_2^{2\times 2}$ *follows a complex matrix variate inverse Wishart type distribution with pdf (3.1). Then the pdf of* (X_1, X_2), *the inverse of the diagonal elements* W_{11} *and* W_{22} *of* $\mathbf{W}$, *is given by:*

$$\begin{aligned} f(x_1, x_2) &= \frac{2(1-a^2)}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{a^2}{1-a^2} \right)^k (x_1x_2)^{n+k-1} \\ &\quad \times \sum_{p=0}^{k} \binom{k}{p} (-1)^{k+p} \frac{(1-a^2)^p}{(x_1+x_2)^{n+k+p+1}} \\ &\quad \times \int_{\mathbb{R}^+} t^{n+k-p-1} \Gamma\left(n+k+p+1, \frac{t}{1-a^2}(x_1+x_2) \right) \mathcal{W}(t)\, dt, \end{aligned} \tag{3.3}$$

where $x_1, x_2 > 0$, $n > 0$, $0 \leq a^2 \leq 1$, $\Gamma(\cdot)$ *denotes the gamma function (see Result C.5) and* $\Gamma(\cdot,\cdot)$ *denotes the upper incomplete gamma function (see Result C.7). The distribution is called a bivariate gamma type I distribution.*

Proof. *Letting:*

$$\xi = ae^{-\iota b}, \quad a = |\xi|, \quad b = arg(\xi) \text{ and } 0 \leq a^2 \leq 1, w_{12} = \sqrt{w_{11}w_{22}}\rho e^{-\iota\theta}, \quad 0 \leq \theta < 2\pi, \ 0 \leq \rho < 1,$$

where $l^2 = -1$ is the complex unitary, then $J = 2\rho w_{11}w_{22}$. It follows from (3.2) that:

$$\begin{aligned} f(w_{11}, w_{22}, \rho, \theta|t) &= \frac{t^{2n}}{\mathcal{C}\Gamma_2(n)} \left(\frac{1}{w_{11}w_{22}(1-\rho^2)}\right)^{n+2} \left(\frac{1}{1-a^2}\right)^n \\ &\quad \times \exp\left(-\frac{t}{w_{11}w_{22}(1-\rho^2)(1-a^2)}\left[w_{11}+w_{22}+a\rho\sqrt{w_{11}w_{22}}\left(e^{l(\theta-b)}+e^{-l(\theta-b)}\right)\right]\right) \times |J| \\ &= \frac{t^{2n}}{\mathcal{C}\Gamma_2(n)} \left(\frac{1}{w_{11}w_{22}(1-\rho^2)}\right)^{n+2} \left(\frac{1}{1-a^2}\right)^n \\ &\quad \times \exp\left(-\frac{t}{w_{11}w_{22}(1-\rho^2)(1-a^2)}\left[w_{11}+w_{22}+2a\rho\sqrt{w_{11}w_{22}}\cos(\theta-b)\right]\right) \times 2\rho w_{11}w_{22}. \end{aligned}$$

To obtain the joint pdf of the inverted diagonal elements of $\mathbf{W}$*, the transformations $X_1 = W_{11}^{-1}$ and $X_2 = W_{22}^{-1}$ with Jacobian:*

$$\begin{aligned} & J \\ =\ & J\left(x_1, x_2 \to w_1, w_2\right) \\ =\ & \det\begin{pmatrix} \frac{dx_1}{dw_1} & \frac{dx_1}{dw_2} \\ \frac{dx_2}{dw_1} & \frac{dx_2}{dw_2} \end{pmatrix} \\ =\ & (x_1x_2)^{-2}. \end{aligned}$$

Using (2.30) arrives at:

$$\begin{aligned} f(x_1, x_2, \rho, \theta|t) &= \frac{2t^{2n}(x_1x_2)^{n-1}\rho(1-\rho^2)^{-(n+2)}}{\pi(1-a^2)^n\Gamma(n)\Gamma(n-1)} \\ &\quad \times \exp\left(-\frac{t}{(1-\rho^2)(1-a^2)}\left[x_1+x_2+2a\rho\sqrt{x_1x_2}\cos(\theta-b)\right]\right). \end{aligned}$$

The pdf of (X_1, X_2, ρ, θ) is given by:

$$\begin{aligned} f(x_1, x_2, \rho, \theta) &= \int_{\mathbb{R}^+} f(x_1, x_2, \rho, \theta|t)\mathcal{W}(t)\, dt \\ &= \frac{2(x_1x_2)^{n-1}\rho(1-\rho^2)^{-(n+2)}}{\pi(1-a^2)^n\Gamma(n)\Gamma(n-1)} \int_{\mathbb{R}^+} t^{2n}\mathcal{W}(t) \\ &\quad \times \exp\left(-\frac{t}{(1-\rho^2)(1-a^2)}\left[x_1+x_2+2a\rho\sqrt{x_1x_2}\cos(\theta-b)\right]\right) dt \\ &= \frac{2(x_1x_2)^{n-1}\rho(1-\rho^2)^{-(n+2)}}{\pi(1-a^2)^n\Gamma(n)\Gamma(n-1)} \int_{\mathbb{R}^+} t^{2n}\mathcal{W}(t) \\ &\quad \times \exp\left(-\frac{t\left(x_1+x_2\right)}{(1-\rho^2)(1-a^2)}\right)\exp\left(-\frac{2ta\rho\sqrt{x_1x_2}}{(1-\rho^2)(1-a^2)}\cos(\theta-b)\right) dt. \end{aligned} \tag{3.4}$$

Next, θ is eliminated by integrating $f(X_1, X_2, \rho, \theta)$ over θ on the domain $(0, 2\pi)$. Let $A(t) = \frac{-2ta\rho\sqrt{x_1x_2}}{(1-a^2)(1-\rho^2)}$. By *using Result C.11, see from (3.4):*

$$\begin{aligned} I(A(t)) &= \int_0^{2\pi} \exp\left(A(t)\cos\left(\theta - b\right)\right) d\theta \\ &= \int_0^{2\pi} \exp(A(t)\cos(\theta)) d\theta \\ &= \sum_{k=0}^{\infty} \frac{A(t)^{2k}}{(2k)!} \int_0^{2\pi} \cos^{2k}\theta d\theta \\ &= \sum_{k=0}^{\infty} \frac{A(t)^{2k}}{(2k)!} \frac{2\pi}{2^{2k}} \binom{2k}{b} \\ &= \sum_{k=0}^{\infty} \frac{2\pi}{(k!)^2} \left(\frac{A(t)^2}{4}\right)^k \end{aligned} \tag{3.5}$$

and $\int_0^{2\pi} \cos^{2k+1}\theta d\theta = 0 \ \ \forall\, k$. *Substituting (3.5) into (3.4) leaves:*

$$\begin{aligned} f(x_1, x_2, \rho) &= \frac{4(x_1x_2)^{n-1}\rho(1-\rho^2)^{-(n+2)}}{(1-a^2)^n\Gamma(n)\Gamma(n-1)} \int_{\mathbb{R}^+} t^{2n}\mathcal{W}(t) \\ &\quad \times \exp\left(-\frac{t(x_1+x_2)}{(1-\rho^2)(1-a^2)}\right) \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{A^2(t)}{4}\right)^k dt \\ &= \frac{4(x_1x_2)^{n-1}\rho(1-\rho^2)^{-(n+2)}}{(1-a^2)^n\Gamma(n)\Gamma(n-1)} \int_{\mathbb{R}^+} t^{2n}\mathcal{W}(t) \\ &\quad \times \exp\left(-\frac{t(x_1+x_2)}{(1-\rho^2)(1-a^2)}\right) \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{t^2a^2\rho^2x_1x_2}{(1-a^2)^2(1-\rho^2)^2}\right)^k dt \\ &= \frac{4(x_1x_2)^{n-1}\rho(1-\rho^2)^{-(n+2)}}{(1-a^2)^n\Gamma(n)\Gamma(n-1)} \int_{\mathbb{R}^+} t^{2n}\mathcal{W}(t) \exp\left(-t\left(\frac{x_1}{(1-\rho^2)(1-a^2)} + \frac{x_2}{(1-\rho^2)(1-a^2)}\right)\right) \\ &\quad \times \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{t^2a^2\rho^2x_1x_2}{(1-a^2)^2(1-\rho^2)^2}\right)^k dt. \end{aligned} \tag{3.6}$$

To obtain the pdf of (X_1, X_2), *let* $Z_i = \frac{X_i}{(1-a^2)}$, $i = 1, 2$, *and* $Y = \frac{1}{(1-\rho^2)}$, *with Jacobian:*

$$\begin{aligned} J\left(x_1, x_2, y \rightarrow z_1, z_2, y\right) &= \det \begin{pmatrix} \frac{dx_1}{dz_1} & \frac{dx_1}{dz_2} & \frac{dx_1}{dy} \\ \frac{dx_2}{dz_1} & \frac{dx_2}{dz_2} & \frac{dx_2}{dy} \\ \frac{dy}{dx_1} & \frac{dy}{dz_2} & \frac{dy}{dy} \end{pmatrix} \\ &= \det \begin{pmatrix} (1-a^2) & 0 & 0 \\ 0 & (1-a^2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \left(1-a^2\right)^2. \end{aligned} \tag{3.7}$$

Performing this transformation and Jacobian in (3.7) on (3.6) leads to:

$$\begin{aligned} f(z_1, z_2, y) &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} (z_1 z_2)^{n+k-1} y^{n+k} (y-1)^k \\ &\quad \times \int_{\mathbb{R}^+} t^{2(n+k)} \exp\left(-t(z_1+z_2)y\right) \mathcal{W}(t)\, dt. \end{aligned} \tag{3.8}$$

To obtain the pdf of (Z_1, Z_2) and using Result C.26 and Result C.29, it follows that:

$$\begin{aligned} f(z_1, z_2) &= \int_1^{\infty} f(z_1, z_2, y) dy \\ &= \int_1^{\infty} \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} (z_1 z_2)^{n+k-1} y^{n+k} (y-1)^k \\ &\quad \times \int_{\mathbb{R}^+} t^{2(n+k)} \exp\left(-t(z_1+z_2)y\right) \mathcal{W}(t)\, dtdy \\ &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} (z_1 z_2)^{n+k-1} \int_{\mathbb{R}^+} t^{2(n+k)} \\ &\quad \times \int_1^{\infty} y^{n+k} (y-1)^k \exp\left(-t(z_1+z_2)y\right) dy \mathcal{W}(t)\, dt \\ &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} (z_1 z_2)^{n+k-1} \sum_{p=0}^{k} \binom{k}{p} \frac{(-1)^{k+p}}{(z_1+z_2)^{n+k+p+1}} \\ &\quad \times \int_{\mathbb{R}^+} t^{n+k-p-1} \Gamma\left(n+k+p+1, t(z_1+z_2)\right) \mathcal{W}(t)\, dt. \end{aligned} \tag{3.9}$$

The pdf of (X_1, X_2) is given by:

$$f(x_1, x_2) = \left(\frac{1}{1-a^2}\right)^2 f\left(\frac{x_1}{1-a^2}, \frac{x_2}{1-a^2}\right). \tag{3.10}$$

Therefore from (3.10) and (3.9), (X_1, X_2) has the following bivariate gamma type I pdf:

$$\begin{aligned} f(x_1, x_2) &= \frac{2(1-a^2)}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{a^2}{1-a^2}\right)^k (x_1 x_2)^{n+k-1} \\ &\quad \times \sum_{p=0}^{k} \binom{k}{p} (-1)^{k+p} \frac{(1-a^2)^p}{(x_1+x_2)^{n+k+p+1}} \\ &\quad \times \int_{\mathbb{R}^+} t^{n+k-p-1} \Gamma\left(n+k+p+1, \frac{t}{1-a^2}(x_1+x_2)\right) \mathcal{W}(t)\, dt \end{aligned}$$

where $x_1, x_2 > 0$. ∎

Two special cases of the pdf for the bivariate gamma type I distribution (3.3) from the complex matrix variate elliptical distribution is of interest, namely the complex matrix variate normal (1.5) and the complex matrix variate t (1.6) distributions as underlying models for $\mathbf{H}$.

Corollary 3.1 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (3.3) simplifies to:*

$$\begin{aligned} f_{normal}(x_1,x_2) &= \frac{2(1-a^2)}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{a^2}{1-a^2}\right)^k (x_1x_2)^{n+k-1} \\ &\times \sum_{p=0}^{k}\left(\begin{array}{c} k \\ p \end{array}\right)\frac{(-1)^{k+p}(1-a^2)^p}{(x_1+x_2)^{n+k+p+1}} \\ &\times\Gamma\left(n+k+p+1,\frac{1}{1-a^2}(x_1+x_2)\right) \end{aligned} \tag{3.11}$$

where $x_1, x_2 > 0$, which reflects the result of Xu et. al. (2009).

Corollary 3.2 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), by using Result C.28, (3.3) simplifies to:*

$$\begin{aligned} &\int_{\mathbb{R}^+} t^{n+k-p-1}\Gamma\left(n+k+p+1,\frac{t}{1-a^2}(x_1+x_2)\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\int_{\mathbb{R}^+} t^{\frac{\nu}{2}+n+k-p-2}\exp\left(-t\frac{v}{2}\right)\Gamma\left(n+k+p+1,\frac{t}{1-a^2}(x_1+x_2)\right)dt \\ &= \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\left(\frac{x_1+x_2}{1-a^2}\right)^{n+k+p+1}\Gamma\left(\frac{\nu}{2}+2n+2k\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}\right)^{\frac{\nu}{2}+2n+2k}} \\ &\times\ {}_2F_1\left(1,\frac{\nu}{2}+2n+2k;\frac{\nu}{2}+n+k-p;\frac{\frac{\nu}{2}}{\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}}\right). \end{aligned} \tag{3.12}$$

where ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function (see Result C.15). Substituting (3.12) into (3.3) leaves:

$$\begin{aligned} f_t(x_1,x_2) &= \frac{2(1-a^2)}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{a^2}{1-a^2}\right)^k(x_1x_2)^{n+k-1} \\ &\times\sum_{p=0}^{k}\left(\begin{array}{c} k \\ p \end{array}\right)(-1)^{k+p}\frac{(1-a^2)^p}{(x_1+x_2)^{n+k+p+1}} \\ &\times\frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\left(\frac{x_1+x_2}{1-a^2}\right)^{n+k+p+1}\Gamma\left(\frac{\nu}{2}+2m+2k\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}\right)^{\frac{\nu}{2}+2n+2k}} \\ &\times\ {}_2F_1\left(1,\frac{\nu}{2}+2n+2k;\frac{\nu}{2}+n+k-p;\frac{\frac{\nu}{2}}{\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}}\right) \\ &= \frac{2\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma(n)\Gamma(n-1)\Gamma\left(\frac{\nu}{2}\right)(1-a^2)^{n-2k}}\sum_{k=0}^{\infty}\frac{a^{2k}\Gamma\left(\frac{\nu}{2}+2n+2k\right)}{(k!)^2} \\ &\times(x_1x_2)^{n+k-1}\sum_{p=0}^{k}\left(\begin{array}{c} k \\ p \end{array}\right)(-1)^{k+p} \\ &\times\frac{{}_2F_1\left(1,\frac{\nu}{2}+2n+2k;\frac{\nu}{2}+m+k-p;\frac{\frac{\nu}{2}}{\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}}\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}\right)^{\frac{\nu}{2}+2n+2k}} \end{aligned} \tag{3.13}$$

where $x_1, x_2 > 0$ and $\left|\frac{\frac{\nu}{2}}{\frac{x_1+x_2}{1-a^2}+\frac{\nu}{2}}\right| < 1$.

In the figures below pdfs (3.11) and (3.13) are illustrated for arbitrary parameters.

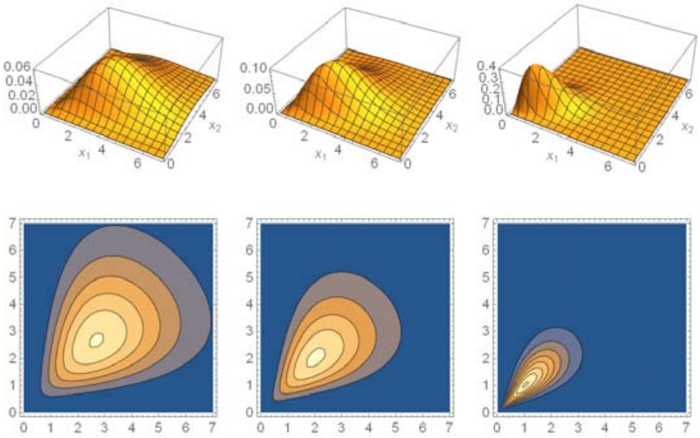


Figure 3.2 Pdf (3.11) with contourplots for $n = 5$ and $a = 0.25, 0.5, 0.75$ (fltr)

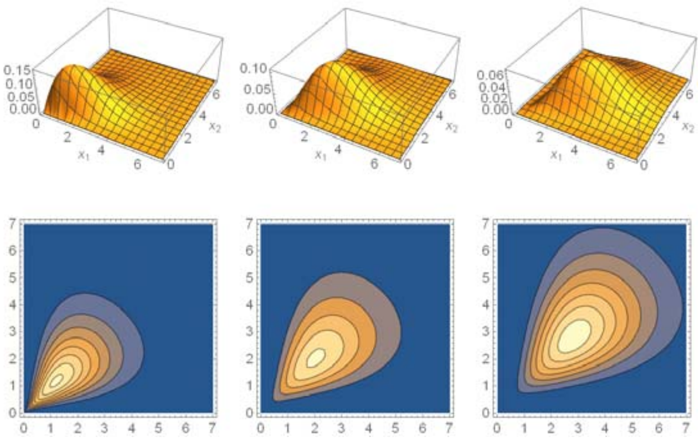


Figure 3.3 Pdf (3.11) with contourplots for $a = 0.5$ and $n = 4, 5, 6$ (fltr)

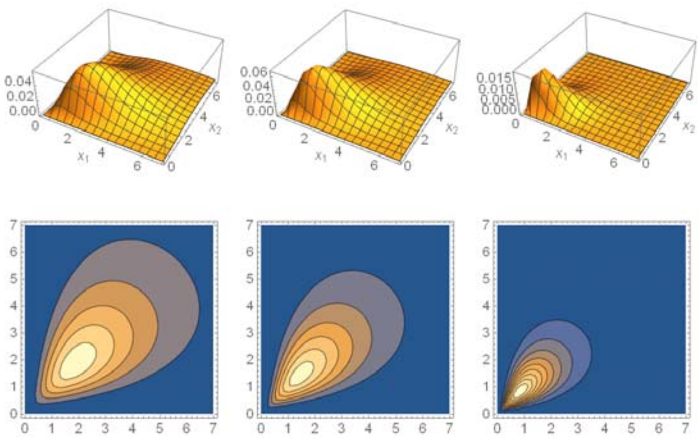


Figure 3.4 Pdf (3.13) with contourplots for $n = 5, v = 10$ and $a = 0.25, 0.5, 0.75$ (fltr)

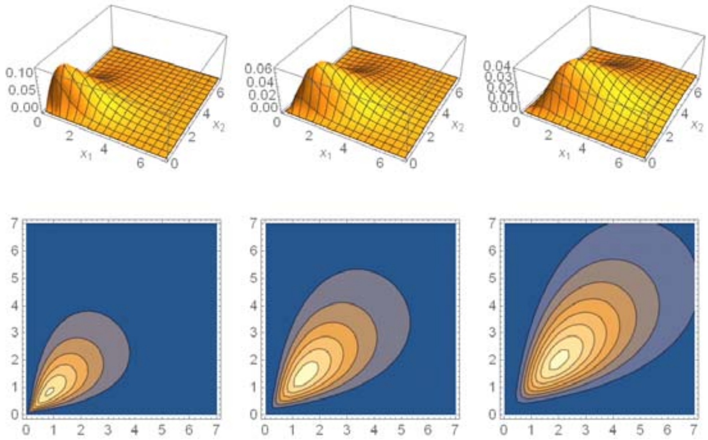


Figure 3.5 Pdf (3.13) with contourplots for $a = 0.5, v = 10$ and $n = 4, 5, 6$ (fltr)

From these figures, the following observations are made:

- It is observed from Figures 3.2 and 3.4 that when a approaches 1, the pdf becomes more dense; i.e. the variables become more concentrated.
- From Figure 3.3 and Figure 3.5, see that the underlying t case, in comparison to the normal distribution, exhibits fatter tails as expected.

3.2.2 Marginal distribution

The following theorem presents the pdf of the marginal distribution of X_1 and X_2 when (X_1, X_2) has pdf (3.3).

Theorem 3.2 *Suppose (X_1, X_2) follows a bivariate gamma type I distribution with pdf (3.3). The marginal distribution of X_1 is given by:*

$$\begin{aligned} f(x_1) &= \frac{2(1-a^2)}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{a^2}{1-a^2}\right)^k \sum_{p=0}^{k} \binom{k}{p} (-1)^{k+p} (1-a^2)^p \\ &\quad \times \int_{\mathbb{R}^+} t^{n+k-p-1} \int_0^{\infty} \frac{(x_1 x_2)^{n+k-1}}{(x_1+x_2)^{n+k+p+1}} \Gamma\left(n+k+p+1, \frac{t}{1-a^2}(x_1+x_2)\right) dx_2 \mathcal{W}(t)\, dt \quad (3.14) \end{aligned}$$

where $x_1 > 0$ and $n > 0$, $0 \leq a^2 \leq 1$.

Proof. *Since*

$$f(x_1) = \int_0^{\infty} f(x_1, x_2)\, dx_2$$

the expression (3.14) follows immediately. ■

Remark 3.1 *A similar expression as in (3.14) holds for X_2.*

3.2.3 Product moment

In this section the product moment of the bivariate gamma type I distribution with pdf (3.3) is derived. Subsequently the expression of the product moment is obtained for both underlying complex matrix variate normal- and t cases, and the correlation coefficient is investigated.

Remark 3.2 *The product moment of* (X_1, X_2) *with pdf (3.3) is given by:*

$$E\left(X_1^r X_2^d\right) = \left(1-a^2\right)^{r+d} E\left(Z_1^r Z_2^d\right) \tag{3.15}$$

since from (3.9):

$$\begin{aligned}
E\left(Z_1^r Z_2^d\right) &= \int\limits_{\mathbb{R}+}\int\limits_{\mathbb{R}+} z_1^r z_2^d f(z_1,z_2)dz_1dz_2 \\
&= \int\limits_{\mathbb{R}+}\int\limits_{\mathbb{R}+} \left(\frac{x_1}{1-a^2}\right)^r \left(\frac{x_2}{1-a^2}\right)^d f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \left(1-a^2\right)^{-(r+d)} \int\limits_{\mathbb{R}+}\int\limits_{\mathbb{R}+} x_1^r x_2^d f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \left(1-a^2\right)^{-(r+d)} E\left(X_1^r X_2^d\right).
\end{aligned}$$

From (3.8) and using Result C.22, see that:

$$\begin{aligned}
E(Z_1^r Z_2^d) &= \int\limits_{\mathbb{R}+}\int\limits_{\mathbb{R}+}\int\limits_1^\infty z_1^r z_2^d f(z_1,z_2,y)dz_1dz_2dy \\
&= \int\limits_{\mathbb{R}+}\int\limits_{\mathbb{R}+}\int\limits_1^\infty z_1^r z_2^d \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^\infty \frac{a^{2k}}{(k!)^2}(z_1z_2)^{n+k-1}y^{n+k}(y-1)^k \\
&\quad \times \int\limits_{\mathbb{R}+} t^{2(n+k)} \exp\left\{-ty(z_1+z_2)\right\} \mathcal{W}(t)\, dtdz_1dz_2dy \\
&= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^\infty \frac{a^{2k}}{(k!)^2} \int\limits_1^\infty\int\limits_{\mathbb{R}+}\int\limits_{\mathbb{R}+} z_1^{r+n+k-1} \exp\left\{-tyz_1\right\} dz_1 \\
&\quad \times \int\limits_{\mathbb{R}+} z_2^{d+n+k-1} \exp\left\{-tyz_2\right\} dz_2 y^{n+k}(y-1)^k t^{2(n+k)} \mathcal{W}(t)\, dtdy \\
&= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^\infty \frac{a^{2k}}{(k!)^2} \int\limits_1^\infty\int\limits_{\mathbb{R}+} \frac{\Gamma(r+n+k)}{(ty)^{r+n+k}} \frac{\Gamma(d+n+k)}{(ty)^{d+n+k}} \\
&\quad \times y^{n+k}(y-1)^k t^{2(n+k)} \mathcal{W}(t)\, dtdy \\
&= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^\infty \frac{a^{2k}}{(k!)^2} \int\limits_1^\infty\int\limits_{\mathbb{R}+} t^{-r-n-k-d-n-k+2n+2k} \\
&\quad \times \frac{\Gamma(r+n+k)}{y^{r+n+k}} \frac{\Gamma(d+n+k)}{y^{d+n+k}} y^{n+k}(y-1)^k \mathcal{W}(t)\, dtdy \\
&= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^\infty \frac{a^{2k}}{(k!)^2} \Gamma(r+n+k)\Gamma(d+n+k) \\
&\quad \times \int\limits_{\mathbb{R}+} t^{-(r+d)} \mathcal{W}(t)\, dt \int\limits_1^\infty y^{-(r+d+n+k)}(y-1)^k dy.
\end{aligned}$$

Further using Result C.27 and the definition of the Pochhammer symbol (see Result C.9), see that:

$$
\begin{aligned}
E(Z_1^r Z_2^d) &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} \Gamma(r+n+k)\Gamma(d+n+k) \\
&\quad \times \int_{\mathbb{R}^+} t^{-(r+d)} \mathcal{W}(t)\, dt B(r+d+n-1, k+1) \\
&= \frac{2\kappa(r+d)(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} \Gamma(r+n+k)\Gamma(d+n+k) \\
&\quad \times \frac{\Gamma(r+d+n-1)\Gamma(k+1)}{\Gamma(r+d+n+k)} \\
&= \frac{2\kappa(r+d)(1-a^2)^n \Gamma(r+n)\Gamma(d+n)}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} \\
&\quad \times \frac{\Gamma(r+n+k)\Gamma(d+n+k)}{\Gamma(r+n)\Gamma(d+n)} \frac{\Gamma(r+d+n)\, k!}{(r+d+n-1)\Gamma(r+d+n+k)} \\
&= \frac{2\kappa(r+d)(1-a^2)^n \Gamma(r+n)\Gamma(d+n)}{\Gamma(n)\Gamma(n-1)(r+d+n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{k!} \\
&\quad \times \frac{\Gamma(r+n+k)\Gamma(d+n+k)}{\Gamma(r+n)\Gamma(d+n)} \frac{\Gamma(r+d+n)}{\Gamma(r+d+n+k)} \\
&= \frac{2\kappa(r+d)(1-a^2)^m \Gamma(r+n)\Gamma(d+n)}{\Gamma(n)\Gamma(n-1)(r+d+n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{k!} \frac{(r+n)_k (d+n)_k}{(r+d+n)_k} \\
&= \frac{2\kappa(r+d)(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \frac{\Gamma(n+r)\Gamma(n+d)}{(n+r+d-1)} {}_2F_1(r,d;r+d+n;a^2)
\end{aligned} \tag{3.16}
$$

where $B(\cdot,\cdot)$ is the beta function (see Result C.8), and (provided it exists):

$$\kappa(u) = \int_{\mathbb{R}^+} t^{-u} \mathcal{W}(t)\, dt. \tag{3.17}$$

Next, the product moment for the underlying normal- and t cases are presented.

Corollary 3.3 *The product moment of (X_1, X_2) with pdf (3.11) is given by:*

$$E\left(X_1^r X_2^d\right) = \frac{2(1-a^2)^{r+d+n}}{\Gamma(n)\Gamma(n-1)} \frac{\Gamma(n+r)\Gamma(n+d)}{(n+r+d-1)} {}_2F_1(r,d;r+d+n;a^2) \tag{3.18}$$

since, using (1.5):

$$
\begin{aligned}
\kappa(r+d) &= \int_{\mathbb{R}^+} t^{-(r+d)} \delta(t-1) dt \\
&= \int_{\mathbb{R}^+} (x+1)^{-(r+d)} \delta(x) dx \\
&= 1.
\end{aligned}
$$

Corollary 3.4 *The product moment of (X_1, X_2) with pdf (3.13) is given by:*

$$
\begin{aligned}
E\left(X_1^r X_2^d\right) &= \frac{2(1-a^2)^{r+d+n} \left(\frac{v}{2}\right)^{r+d} \Gamma\left(\frac{v}{2}-(r+d)\right) \Gamma(n+r)\Gamma(n+d)}{\Gamma(n)\Gamma(n-1)\Gamma\left(\frac{v}{2}\right)(n+r+d-1)} \\
&\quad \times {}_2F_1(r,d;r+d+n;a^2)
\end{aligned} \tag{3.19}
$$

since using (1.6) and Result C.22:

$$\begin{aligned}\kappa(r+n) &= \int_{\mathbb{R}^+} t^{-(r+d)} \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_{\mathbb{R}^+} t^{\frac{v}{2}-(r+d)-1} \exp\left(-t\frac{v}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{v}{2}-(r+d)\right)}{\left(\frac{v}{2}\right)^{\frac{v}{2}-(r+d)}} \\ &= \left(\frac{v}{2}\right)^{r+d} \frac{\Gamma\left(\frac{v}{2}-(r+d)\right)}{\Gamma\left(\frac{\nu}{2}\right)}.\end{aligned}$$

By using the expression for the product moment in (3.18) and (3.19), the correlation coefficient can now be evaluated using Result C.30. In Figure 3.6 this correlation coefficient is graphed for arbitrary parameter values ($n = 3,\ v = 15,\ v = 30$) against a^2 to illustrate the effect of a^2 on the correlation between X_1 and X_2. It is observed that the correlation under the t model tends to that of the normal model when v increases.

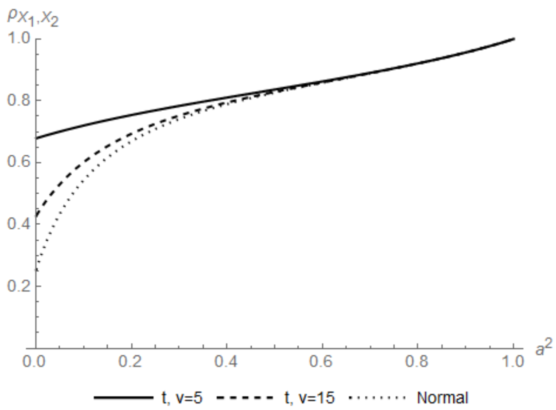


Figure 3.6 Correlation coefficient (C.30) for normal and t cases, for increasing a^2

Corollary 3.5 *The moment generating function (mgf) of (X_1, X_2) with pdf (3.3) is given by:*

$$M(q_1,q_2) = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} q_1^i q_2^{j-i} \frac{2\kappa(j)(1-a^2)^{n+j}}{\Gamma(n)\Gamma(n-1)} \frac{\Gamma(n+i)\Gamma(n+j-i)}{(n+j-1)} \, {}_2F_1(i, j-i; j+n; a^2)$$

where $n > 0$, $0 \leq a^2 \leq 1$, and with $\kappa(j)$ as defined in (3.17).

Proof. Substituting (3.15) and (3.16) into the definition of the mgf, together with Result C.29, yield:

$$\begin{aligned}
M(q_1,q_2) &= E\left(\exp\left(q_1X_1+q_2X_2\right)\right)\\
&= E\left(\sum_{j=0}^{\infty}\frac{1}{j!}(q_1X_1+q_2X_2)^j\right)\\
&= \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}q_1^iq_2^{j-i}E(X_1^iX_2^{j-i})\\
&= \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}q_1^iq_2^{j-i}\left(1-a^2\right)^{i+(j-i)}E_{Z_1,Z_2}\left(Z_1^iZ_2^{j-i}\right)\\
&= \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}q_1^iq_2^{j-i}\left(1-a^2\right)^{j}\frac{2\kappa(j)(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\\
&\quad\times\frac{\Gamma(n+i)\Gamma(n+j-i)}{(n+j-1)}\ {}_2F_1(i,j-i;j+n;a^2)\\
&= \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}q_1^iq_2^{j-i}\frac{2\kappa(j)(1-a^2)^{n+j}}{\Gamma(n)\Gamma(n-1)}\\
&\quad\times\frac{\Gamma(n+i)\Gamma(n+j-i)}{(n+j-1)}\ {}_2F_1(i,j-i;j+n;a^2)
\end{aligned}$$

which concludes the proof. ■

3.3 Bivariate Weibullised gamma type I distribution

The bivariate Nakagami distribution has been widely studied and has several applications in wireless communications (see de Souza and Yacoub (2008) and references therein). A bivariate Nakagami type I distribution that emanates from this bivariate gamma type I distribution is thus of particular interest. A bivariate Weibullised gamma type I distribution (see also Chen et. al. (2014)) is proposed, which originates from the bivariate gamma type I distribution (see (3.3)), of which the bivariate Nakagami type I distribution is a special case.

Theorem 3.3 *Suppose that (X_1, X_2) is bivariate gamma type I distributed with pdf (3.3). The pdf of (R_1, R_2), where $R_i = \left(\frac{X_i}{n}\right)^{\frac{1}{\beta_i}}$ is given by:*

$$\begin{aligned}
f\left(r_1,r_2\right) &= \frac{2\beta_1\beta_2}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\binom{k}{p}(-1)^{k+p}(1-a^2)^{p+1}}{(k!)^2}n^{n+k-p-1}\left(\frac{a^2}{1-a^2}\right)^k\frac{r_1^{\beta_1 n+\beta_1 k-1}r_2^{\beta_2 n+k\beta_2-1}}{(r_1^{\beta_1}+r_2^{\beta_2})^{n+k+p+1}}\\
&\quad\times\int_{\mathbb{R}^+}t^{n+k-p-1}\Gamma\left(n+k+p+1,\frac{t}{1-a^2}(nr_1^{\beta_1}+nr_2^{\beta_2})\right)\mathcal{W}\left(t\right)dt \qquad (3.20)
\end{aligned}$$

for $r_1, r_2 > 0$, $m, \beta_1, \beta_2 > 0$ and $0 \leq a^2 \leq 1$. This distribution is called a bivariate Weibullised gamma type I distribution.

Proof. *Consider the transformations* $R_1 = \left(\frac{X_1}{n}\right)^{\frac{1}{\beta_1}}$ *and* $R_2 = \left(\frac{X_2}{n}\right)^{\frac{1}{\beta_2}}$ *with Jacobian (see Result C.1):*

$$\begin{aligned} & J \\ = \; & J\left(x_1, x_2 \to r_1, r_2\right) \\ = \; & \det \begin{pmatrix} \frac{dx_1}{dr_1} & \frac{dx_1}{dr_2} \\ \frac{dx_2}{dr_1} & \frac{dx_2}{dr_2} \end{pmatrix} \\ = \; & \det \begin{pmatrix} \beta_1 n r_1^{\beta_1 - 1} & 0 \\ 0 & \beta_2 n r_2^{\beta_2 - 1} \end{pmatrix} \\ = \; & \beta_1 \beta_2 n^2 r_1^{\beta_1 - 1} r_2^{\beta_2 - 1}. \end{aligned}$$

The pdf of (R_1, R_2) *is obtained from (3.3):*

$$\begin{aligned} f(r_1, r_2) &= \frac{2(1-a^2)}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{a^2}{1-a^2}\right)^k \left(n r_1^{\beta_1} n r_2^{\beta_2}\right)^{n+k-1} \\ &\times \sum_{p=0}^{k} \binom{k}{p} (-1)^{k+p} \frac{(1-a^2)^p}{(n r_1^{\beta_1} + n r_2^{\beta_2})^{n+k+p+1}} \\ &\times \int_{\mathbb{R}^+} t^{n+k-p-1} \Gamma\left(n+k+p+1, \frac{t}{1-a^2}(n r_1^{\beta_1} + n r_2^{\beta_2})\right) \mathcal{W}(t)\, dt \beta_1 \beta_2 n^2 r_1^{\beta_1 - 1} r_2^{\beta_2 - 1} \\ &= \frac{2\beta_1\beta_2}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{a^2}{1-a^2}\right)^k r_1^{\beta_1 n + \beta_1 k - 1} r_2^{\beta_2 n + k\beta_2 - 1} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k+p} n^{n+k-p-1} \\ &\times \frac{(1-a^2)^{p+1}}{(r_1^{\beta_1} + r_2^{\beta_2})^{n+k+p+1}} \int_{\mathbb{R}^+} t^{n+k-p-1} \Gamma\left(n+k+p+1, \frac{t}{1-a^2}(n r_1^{\beta_1} + n r_2^{\beta_2})\right) \mathcal{W}(t)\, dt \end{aligned}$$

which leaves the final result. ■

Corollary 3.6 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), (3.20) simplifies to:*

$$\begin{aligned} f_{normal}(r_1, r_2) &= \frac{2\beta_1\beta_2}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{\binom{k}{p}(-1)^{k+p}(1-a^2)^{p+1}}{(k!)^2} n^{n+k-p-1} \left(\frac{a^2}{1-a^2}\right)^k \\ &\times \frac{r_1^{\beta_1 n + \beta_1 k - 1} r_2^{\beta_2 n + \beta_2 k - 1}}{(r_1^{\beta_1} + r_2^{\beta_2})^{n+k+p+1}} \Gamma\left(n+k+p+1, \frac{n r_1^{\beta_1} + n r_2^{\beta_2}}{1-a^2}\right) \end{aligned} \tag{3.21}$$

where $r_1, r_2 > 0$.

Corollary 3.7 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), and by using Result C.28, (3.20) simplifies to:*

$$\begin{aligned}
&\int_{\mathbb{R}^+} t^{n+k-p-1}\Gamma\left(n+k+p+1, \frac{t}{1-a^2}(nr_1^{\beta_1}+nr_2^{\beta_2})\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right) dt \\
= \; & \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_{\mathbb{R}^+} t^{\frac{\nu}{2}+n+k-p-2}\exp\left(-t\frac{v}{2}\right)\Gamma\left(n+k+p+1, \frac{t}{1-a^2}(nr_1^{\beta_1}+nr_2^{\beta_2})\right) dt \\
= \; & \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}\right)^{n+k+p+1}\Gamma\left(\frac{\nu}{2}+n+k-p-1+n+k+p+1\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{v}{2}\right)^{\frac{\nu}{2}+n+k-p-1+n+k+p+1}} \\
& \times {}_2F_1\left(1, \frac{\nu}{2}+n+k-p-1+n+k+p+1; \frac{\nu}{2}+n+k-p-1+1; \frac{\frac{v}{2}}{\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{v}{2}}\right) \\
= \; & \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}\right)^{n+k+p+1}\Gamma\left(\frac{\nu}{2}+2n+2k\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{v}{2}\right)^{\frac{\nu}{2}+2n+2k}} \\
& \times {}_2F_1\left(1, \frac{\nu}{2}+2n+2k; \frac{\nu}{2}+n+k-p; \frac{\frac{v}{2}}{\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{v}{2}}\right). \qquad (3.22)
\end{aligned}$$

Substituting (3.22) into (3.20) leaves:

$$\begin{aligned}
f_t(r_1,r_2) &= \frac{2\beta_1\beta_2}{\Gamma(n)\Gamma(n-1)}\frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\binom{k}{p}(-1)^{k+p}(1-a^2)^{p+1}}{(k!)^2}n^{n+k-p-1}\left(\frac{a^2}{1-a^2}\right)^k \\
&\times\frac{r_1^{\beta_1 n+\beta_1 k-1}r_2^{\beta_2 n+k\beta_2-1}}{(r_1^{\beta_1}+r_2^{\beta_2})^{n+k+p+1}}\frac{\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}\right)^{n+k+p+1}\Gamma\left(\frac{\nu}{2}+2n+2k\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}\right)^{\frac{\nu}{2}+2n+2k}} \\
&\times\ {}_2F_1\left(1,\frac{\nu}{2}+2n+2k;\frac{\nu}{2}+n+k-p;\frac{\frac{\upsilon}{2}}{\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}}\right) \\
&= \frac{2\beta_1\beta_2}{\Gamma(n)\Gamma(n-1)}\frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\binom{k}{p}(-1)^{k+p}(1-a^2)^{p+1}}{(k!)^2}n^{n+k-p-1}\left(\frac{a^2}{1-a^2}\right)^k \\
&\times\frac{r_1^{\beta_1 n+\beta_1 k-1}r_2^{\beta_2 n+k\beta_2-1}}{(r_1^{\beta_1}+r_2^{\beta_2})^{n+k+p+1}}\frac{\frac{n^{n+k+p+1}}{(1-a^2)^{n+k+p+1}}\left(r_1^{\beta_1}+r_2^{\beta_2}\right)^{n+k+p+1}\Gamma\left(\frac{\nu}{2}+2n+2k\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}\right)^{\frac{\nu}{2}+2n+2k}} \\
&\times\ {}_2F_1\left(1,\frac{\nu}{2}+2n+2k;\frac{\nu}{2}+n+k-p;\frac{\frac{\upsilon}{2}}{\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}}\right) \\
&= \frac{2\beta_1\beta_2}{\Gamma(n)\Gamma(n-1)\left(1-a^2\right)^n}\frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\binom{k}{p}(-1)^{k+p}}{(k!)^2}n^{2n+2k}\left(a^2\right)^k \\
&\times r_1^{\beta_1 n+\beta_1 k-1}r_2^{\beta_2 n+k\beta_2-1}\frac{\Gamma\left(\frac{\nu}{2}+2n+2k\right)}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}\right)^{\frac{\nu}{2}+2n+2k}} \\
&\times\ {}_2F_1\left(1,\frac{\nu}{2}+2n+2k;\frac{\nu}{2}+n+k-p;\frac{\frac{\upsilon}{2}}{\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}}\right) \qquad (3.23)
\end{aligned}$$

where $r_1, r_2 > 0$ *and* $\left|\frac{\frac{\upsilon}{2}}{\frac{(nr_1^{\beta_1}+nr_2^{\beta_2})}{1-a^2}+\frac{\upsilon}{2}}\right| < 1.$

Remark 3.3 *When* $\beta_1 = \beta_2 = 2$*, then (3.20) has the pdf of a bivariate Nakagami type I distribution:*

$$\begin{aligned}
f(r_1,r_2) &= \frac{8}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\binom{k}{p}(-1)^{k+p}}{(k!)^2}\left(\frac{a^2}{1-a^2}\right)^k n^{n+k-p-1}(1-a^2)^{p+1}\frac{(r_1r_2)^{2n+2k-1}}{(r_1^2+r_2^2)^{n+k+p+1}} \\
&\times\int_{\mathbb{R}^+} t^{n+k-p-1}\Gamma\left(n+k+p+1,\frac{t}{1-a^2}\left(nr_1^2+nr_2^2\right)\right)\mathcal{W}(t)\,dt \qquad (3.24)
\end{aligned}$$

for $r_1, r_2 > 0$*,* $n > 0$*,* $0 \le a^2 \le 1$*. For the special cases under consideration, observe the following:*

i) Substituting (1.5) into (3.24), a bivariate Nakagami distribution has pdf:

$$\begin{aligned}
f_{normal}(r_1,r_2) &= \frac{8}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\binom{k}{p}(-1)^{k+p}}{(k!)^2}\left(\frac{a^2}{1-a^2}\right)^k n^{n+k-p-1}(1-a^2)^{p+1} \\
&\times\frac{(r_1r_2)^{2n+2k-1}}{(r_1^2+r_2^2)^{n+k+p+1}}\Gamma\left(n+k+p+1,\frac{1}{1-a^2}\left(nr_1^2+nr_2^2\right)\right) \qquad (3.25)
\end{aligned}$$

where $r_1, r_2 > 0$.

ii) Substituting (1.6) into (3.24), see from (3.23) follows a bivariate Nakagami t distribution with pdf:

$$
\begin{aligned}
f_t(r_1, r_2) &= \frac{8}{\Gamma(n)\Gamma(n-1)\left(1-a^2\right)^n} \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \sum_{k=0}^{\infty}\sum_{p=0}^{k} \frac{\binom{k}{p}(-1)^{k+p}}{(k!)^2} \left(a^2\right)^k n^{2n+2k} \Gamma\left(\frac{\nu}{2}+2n+2k\right) \\
&\times \frac{(r_1 r_2)^{2n+2k-1}}{\left(\frac{\nu}{2}+n+k-p-1\right)\left(\frac{(nr_1^2+nr_2^2)}{1-a^2}+\frac{v}{2}\right)^{\frac{v}{2}+2n+2k}} \\
&\times {}_2F_1\left(1, \frac{\nu}{2}+2n+2k; \frac{\nu}{2}+n+k-p; \frac{\frac{v}{2}}{\frac{(nr_1^2+nr_2^2)}{1-a^2}+\frac{v}{2}}\right)
\end{aligned} \tag{3.26}
$$

where $r_1, r_2 > 0$ *and* $\left|\frac{\frac{v}{2}}{\frac{(nr_1^2+nr_2^2)}{1-a^2}+\frac{v}{2}}\right| < 1$.

In the following figures the natures of pdfs (3.25) and (3.26) are illustrated for arbitrary parameters.

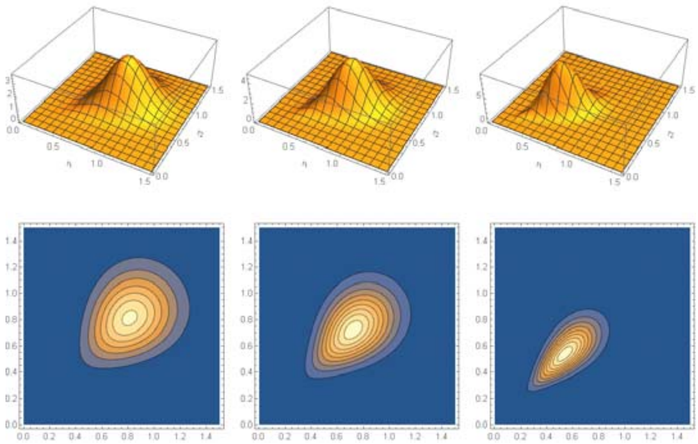


Figure 3.7 Pdf (3.25) with contourplots for $n = 5$ and $a = 0.25, 0.5, 0.75$ (fltr)

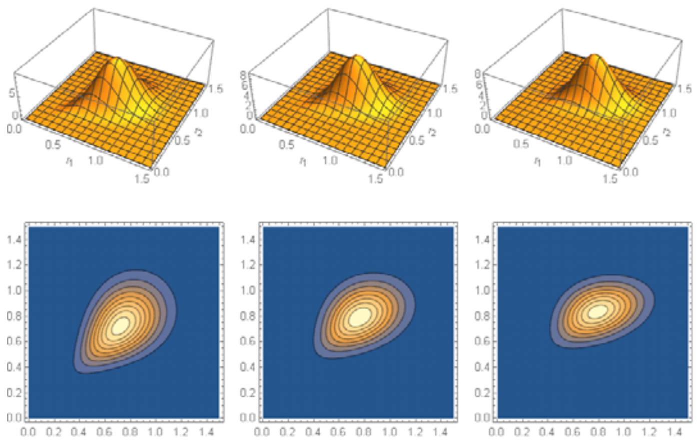


Figure 3.8 Pdf (3.21) with contourplots for $n = 5$, $\beta_1 = 2$, $a = 0.5$, and $\beta_2 = 2, 2.5, 3$ (fltr)

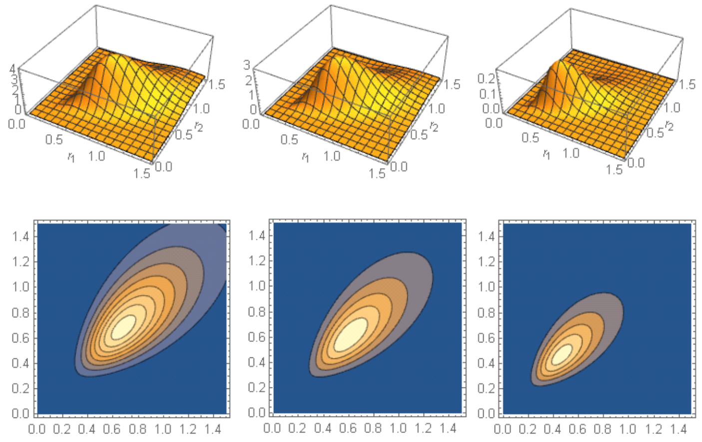


Figure 3.9 Pdf (3.26) with contourplots for $n = 5$, $v = 5$, and $a = 0.25, 0.5, 0.75$ (fltr)

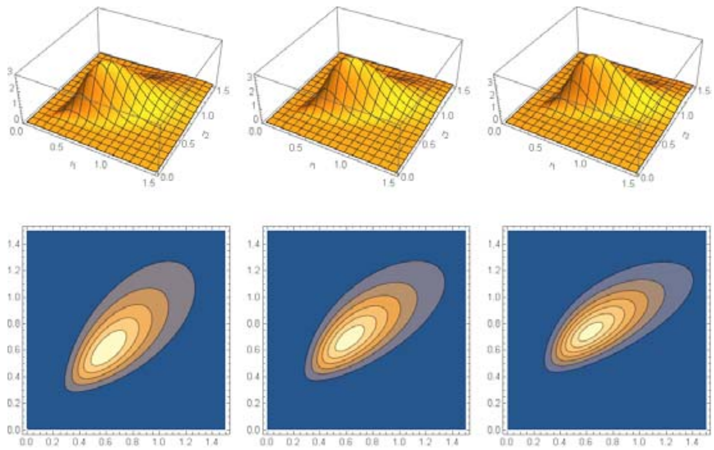


Figure 3.10 Pdf (3.23) with contourplots for $n = 5$, $v = 5$, $\beta_1 = 2$, $a = 0.5$, and $\beta_2 = 2, 2.5, 3$ (fltr)

From these figures, the following observations are made:

- It is observed from Figures 3.7 and 3.9 that when a approaches 1, the pdf becomes more dense; i.e. the variables become more concentrated.
- From Figure 3.8 and Figure 3.10, see that as β_2 increase, the corresponding variable R_2 becomes more dense.
- From Figures 3.7 and 3.9, the effect of the underlying t distribution is evident as the corresponding pdf (3.26) exhibits fatter tails than that of its normal counterpart (3.25).

3.4 Illustrative application

In this section, applications of results in this chapter are discussed. Let $\mathbf{H}$ be the matrix for MIMO system with two transmit antennas, where $\mathbf{H} \sim CE_{n,2}(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{\Sigma}, h)$ (see (1.3)) and is subject to Rayleigh type fading (see (1.10)).

3.4.1 Outage probability

To investigate the outage probability of a fading channel subject to the bivariate gamma type I distribution (3.3), the cdf of the maximum of (X_1, X_2) is of interest (see (1.25)). The cdf of $\max(X_1, X_2)$ is given by:

$$\begin{aligned}\hat{F}(z) &= F\left(\frac{z}{1-a^2}\right) \\ &= P\left(\max(z_1, z_2) < z\right).\end{aligned}$$

Using (3.8) and Result C.23, it follows that:

$$\begin{aligned}P(z_1 < z, z_2 < z, y) &= \int_0^z\int_0^z f(z_1, z_2, y)dz_1dz_2 \\ &= \int_0^z\int_0^z \Bigg\{\frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}(z_1z_2)^{n+k-1}y^{n+k}(y-1)^k \\ &\quad \times \int_{\mathbb{R}^+} t^{2(n+k)}\exp\left(-ty(z_1+z_2)\right)\mathcal{W}(t)\,dt\Bigg\}dz_1dz_2 \\ &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}y^{n+k}(y-1)^k \\ &\quad \times \int_{\mathbb{R}^+} t^{2(n+k)}\int_0^z z_1^{n+k-1}\exp\left(-tyz_1\right)dz_1\int_0^z z_2^{n+k-1}\exp\left(-tyz_2\right)dz_2\mathcal{W}(t)\,dt \\ &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}y^{n+k}(y-1)^k \\ &\quad \times \int_{\mathbb{R}^+} t^{2(n+k)}(ty)^{-2(n+k)}\gamma^2(n+k;tyz)\mathcal{W}(t)\,dt \\ &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}y^{-(n+k)}(y-1)^k\int_{\mathbb{R}^+}\gamma^2(n+k;tyz)\mathcal{W}(t)\,dt. \qquad (3.27)\end{aligned}$$

Subsequently, $F(z)$ can be obtained from (3.27) as:

$$\begin{aligned}F(z) &= \int_1^{\infty} P(z_1 < z, z_2 < z, y)dy \\ &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}\int_1^{\infty}\int_{\mathbb{R}^+} y^{-(n+k)}(y-1)^k\gamma^2(n+k;tyz)\mathcal{W}(t)\,dtdy. \qquad (3.28)\end{aligned}$$

Remark 3.4 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (3.28) simplifies to:*

$$F(z) = \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}\int_1^{\infty} y^{-(n+k)}(y-1)^k\gamma^2(n+k;yz)dy. \qquad (3.29)$$

Remark 3.5 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), (3.28) simplifies to:*

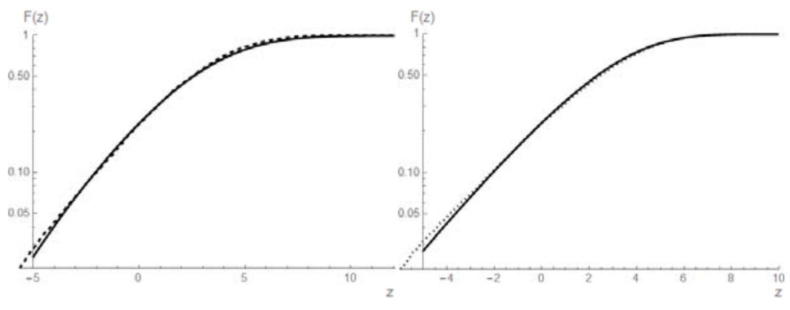
$$\begin{aligned} F(z) &= \frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} \int_1^{\infty} \int_{\mathbb{R}^+} y^{-(n+k)} (y-1)^k \gamma^2(n+k; tyz) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dtdy \\ &= \frac{2(1-a^2)^n \left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(n)\Gamma(n-1)\Gamma\left(\frac{v}{2}\right)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} \int_1^{\infty} y^{-(n+k)} (y-1)^k \int_{\mathbb{R}^+} \gamma^2(n+k; tyz) t^{\frac{v}{2}-1} \exp\left(-t\frac{v}{2}\right) dtdy. \quad (3.30) \end{aligned}$$

Remark 3.6 *Different expressions and methods are available (see Gradshteyn and Rhyzik (2007), eq. 3.383.3 and 8.351.2) for simplified expressions (3.29) and (3.30). However, due to cumbersome computational execution, the approach in this thesis is to utilise a direct integration of above expressions.*

For arbitrary parameters $n = 3$ and $a = 0.5$, a simulation is conducted to validate the analytical results. The results and accompanying figures are illustrated using Mathematica. Figure 3.11 illustrates the outage probability of the SNR for MIMO communications systems with 2 transmitters and $n = 3$ receivers with underlying complex matrix variate t distribution (see (3.30)). The code for this simulation can be found in the Appendix. The analytical results match the simulated results closely in Figure 3.11; this is supported by Table 3.1 indicating selected values of the outage probability for the analytical- and simulation cases. Particularly, $n = 5$ and $a = 0.5$.

Outage threshold	t, $v = 15$		Normal	
	Analytical	**Simulation**	**Analytical**	**Simulation**
-4	0.0432226	0.0434	0.0432296	0.0473
-2	0.105532	0.1003	0.102573	0.1045
0	0.230001	0.2247	0.228415	0.2271
2	0.433844	0.4381	0.453678	0.4395
4	0.678336	0.7035	0.740119	0.725
6	0.870862	0.9056	0.939678	0.937
8	0.963193	0.9881	0.994855	0.9955

Table 3.1 Analytical ((3.30) and (3.29)) for $v = 15$ and simulated values



t, v=15

Normal

Figure 3.11 Analytical ((3.30) and (3.29)) for $v = 15$, and simulated values

Figures 3.12 and 3.13 illustrate the outage probability results for the underlying complex matrix variate t distribution together with underlying complex matrix variate normal distribution for the case of 2 transmitters and 3 receivers with $a = 0.5$. The different behaviour for small and large outage thresholds is noteworthy. This observation provides significant insight to the theoretical contribution of the candidacy of the complex matrix variate t distribution in comparison to the complex matrix variate normal case. Note that Figure 3.13 is a magnified version of Figure 3.12 over a subset of the domain.

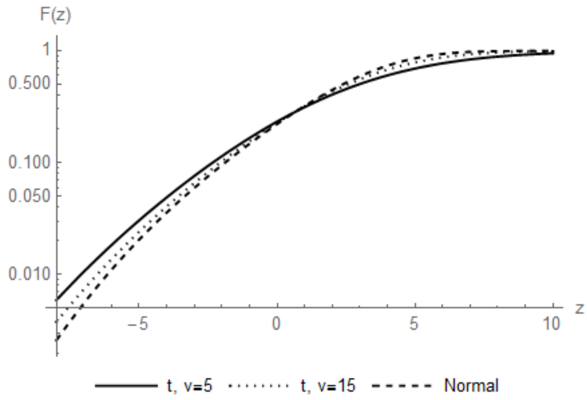


Figure 3.12 (3.30) and (3.29) against the output threshold, for $v = 5, 15$

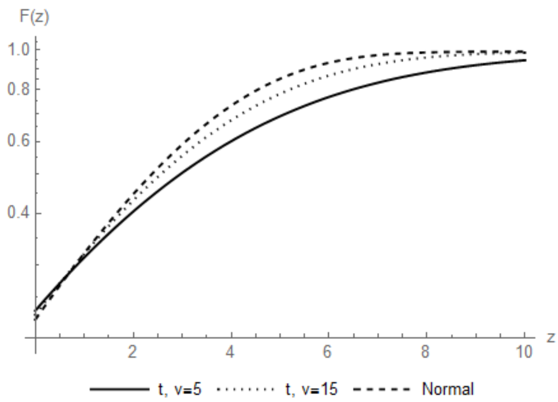


Figure 3.13 (3.30) and (3.29) against the output threshold, for $v = 5, 15$, for a subset of z

3.4.2 EGC diversity

The EGC diversity (see (1.26)) is a useful performance measure as well; in particular, that of the mean of the EGC diversity. In this section, an expression for the EGC diversity is derived for a communications system subject to a bivariate fading model with pdf (3.24). The d^{th} moment of the EGC diversity output SNR is evaluated using (3.9), (3.17), Result C.27, and Result C.29 as:

$$\begin{aligned} \mu_d &= E\left(\gamma_{out}\right)^d \\ &= \left(\frac{E_s}{2N_o}\right)^d E\left[(R_1+R_2)^{2d}\right] \\ &= \left(\frac{E_s}{2N_o}\right)^d \int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} (r_1+r_2)^{2d} f(r_1,r_2)dr_1dr_2 \\ &= \left(\frac{E_s}{2N_o}\right)^d \sum_{s=0}^{2d}\binom{2d}{s} \int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} r_1^s r_2^{2d-s} f(r_1,r_2)dr_1dr_2. \end{aligned} \tag{3.31}$$

Using (3.10), (3.9), and (3.8), (3.31) is solved using the following:

$$\begin{aligned} f(r_1,r_2) &= f\left(nr_1^2,nr_2^2\right)\left|J(x_1,x_2\to r_1,r_2)\right| \\ &= \left(\frac{1}{1-a^2}\right)^2 f\left(\frac{nr_1^2}{(1-a^2)},\frac{nr_2^2}{(1-a^2)}\right)\left|J(x_1,x_2\to r_1,r_2)\right| \\ &= 4n^2r_1r_2\left(\frac{1}{1-a^2}\right)^2 f\left(\frac{nr_1^2}{(1-a^2)},\frac{nr_2^2}{(1-a^2)}\right) \\ &= 4n^2r_1r_2\left(\frac{1}{1-a^2}\right)^2 \int_1^\infty f\left(\frac{nr_1^2}{(1-a^2)},\frac{nr_2^2}{(1-a^2)},y\right)dy. \end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.31) leaves:

$$\begin{aligned} \mu_d &= \left(\frac{E_s}{2N_o}\right)^d \sum_{s=0}^{2d}\binom{2d}{s} \int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} r_1^s r_2^{2d-s} f_{R_1,R_2}(r_1,r_2)dr_1dr_2 \\ &= 4n^2\left(\frac{1}{1-a^2}\right)^2\left(\frac{E_s}{2N_o}\right)^d \sum_{s=0}^{2d}\binom{2d}{s}\frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^\infty\frac{a^{2k}}{(k!)^2} \\ &\quad\times\int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} r_1^s r_2^{2d-s}\left(\frac{nr_1^2}{(1-a^2)}\times\frac{nr_2^2}{(1-a^2)}\right)^{n+k-1}\int_1^\infty y^{r+k}(y-1)^k \\ &\quad\times\int\limits_{\mathbb{R}^+} t^{2(n+k)}\exp\left(-ty\left(\frac{nr_1^2}{(1-a^2)}+\frac{nr_2^2}{(1-a^2)}\right)\right)\mathcal{W}\left(t\right)r_1r_2dtdr_1dr_2 \\ &= B(2d;s,k)\int\limits_{\mathbb{R}^+} t^{2(n+k)}\left(\int_1^\infty I(t,y)y^{n+k}(y-1)^k dy\right)\mathcal{W}\left(t\right)dt \end{aligned} \tag{3.33}$$

where

$$B(2d;s,k) = 4n^2\left(\frac{1}{1-a^2}\right)^2\left(\frac{E_s}{2N_o}\right)^d\sum_{s=0}^{2d}\binom{2d}{s}\frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^\infty\frac{a^{2k}}{(k!)^2} \tag{3.34}$$

and, using Result C.22:

$$\begin{aligned}
I(t,y) &= \int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} r_1^{s+1} r_2^{2d-s+1} \left(\frac{nr_1^2}{(1-a^2)} \times \frac{nr_2^2}{(1-a^2)}\right)^{n+k-1} \\
&\quad \times \exp\left(-ty\left(\frac{nr_1^2}{(1-a^2)} + \frac{nr_2^2}{(1-a^2)}\right)\right) dr_1 dr_2 \\
&= \int\limits_{\mathbb{R}^+} r_2^{2d-s+1} \left(\frac{nr_2^2}{(1-a^2)}\right)^{n+k-1} \exp\left(-\frac{tynr_2^2}{(1-a^2)}\right) dr_2 \\
&\quad \times \int\limits_{\mathbb{R}^+} r_1^{s+1} \left(\frac{nr_1^2}{(1-a^2)}\right)^{n+k-1} \exp\left(-\frac{tynr_1^2}{(1-a^2)}\right) dr_1 \\
&= \frac{1}{4}\left(\frac{(1-a^2)}{n}\right)^{d-\frac{s}{2}+1} \frac{1}{(ty)^{d-\frac{s}{2}+1+n+k-1}} \Gamma\left(d-\frac{s}{2}+n+k\right) \\
&\quad \times \left(\frac{(1-a^2)}{n}\right)^{\frac{s}{2}+1} \frac{1}{(ty)^{\frac{s}{2}+n+k}} \Gamma\left(\frac{s}{2}+n+k\right) \\
&= \frac{1}{4}(1-a^2)^{d+2} \left(\frac{1}{n}\right)^{d-\frac{s}{2}+1} \left(\frac{1}{n}\right)^{\frac{s}{2}+1} \frac{1}{(ty)^{d+2n+2k-1}} \\
&\quad \times \Gamma\left(d-\frac{s}{2}+n+k\right) \Gamma\left(\frac{s}{2}+n+k\right). \qquad (3.35)
\end{aligned}$$

Substituting (3.34) and (3.35) into (3.33), and using (3.17) and Result C.27:

$$\begin{aligned}
\mu_d &= \frac{1}{4}B(2d)(1-a^2)^{d+2} \left(\frac{1}{n}\right)^{d-\frac{s}{2}+1} \left(\frac{1}{n}\right)^{\frac{s}{2}+1} \\
&\quad \times \Gamma\left(d-\frac{s}{2}+n+k\right) \Gamma\left(\frac{s}{2}+n+k\right) \\
&\quad \times \int\limits_{\mathbb{R}^+} t^{2(n+k)} \left(\int\limits_1^\infty \frac{(y-1)^k}{(ty)^{d+n+k}} dy\right) \mathcal{W}(t)\, dt \\
&= \frac{1}{4}B(2d)(1-a^2)^{d+2} \left(\frac{1}{n}\right)^{d-\frac{s}{2}+1} \left(\frac{1}{n}\right)^{\frac{s}{2}+1} \\
&\quad \times \Gamma\left(d-\frac{s}{2}+n+k\right) \Gamma\left(\frac{s}{2}+n+k\right) \\
&\quad \times \int\limits_{\mathbb{R}^+} t^{-d} \mathcal{W}(t)\, dt \times \int\limits_1^\infty \frac{(y-1)^k}{y^{d+n+k}} dy \\
&= \frac{1}{4}B(2d)(1-a^2)^{d+2} \left(\frac{1}{n}\right)^{d-\frac{s}{2}+1} \left(\frac{1}{n}\right)^{\frac{s}{2}+1} \kappa(d) \\
&\quad \times \Gamma\left(d-\frac{s}{2}+n+k\right) \Gamma\left(\frac{s}{2}+n+k\right) \int_1^\infty \frac{(y-1)^k}{y^{d+n+k}} dy \\
&= \frac{1}{4}B(2d)(1-a^2)^{d+2} \left(\frac{1}{n}\right)^{d-\frac{s}{2}+1} \left(\frac{1}{n}\right)^{\frac{s}{2}+1} \kappa(d) \\
&\quad \times \Gamma\left(d-\frac{s}{2}+n+k\right) \Gamma\left(\frac{s}{2}+n+k\right) B(d+n-1,k+1). \qquad (3.36)
\end{aligned}$$

Finally, substituting (3.34) into (3.36) leaves:

$$\begin{aligned}
\mu_d &= \frac{1}{4}4n^2\left(\frac{1}{1-a^2}\right)^2\left(\frac{E_s}{2N_o}\right)^d\sum_{s=0}^{2d}\left(\begin{array}{c}2d\\s\end{array}\right)\frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\left(\frac{1}{n}\right)^{d-\frac{s}{2}+1}\left(\frac{1}{n}\right)^{\frac{s}{2}+1}\kappa(d)\\
&\quad\times\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}(1-a^2)^{d+2}\Gamma\left(d-\frac{s}{2}+n+k\right)\Gamma\left(\frac{s}{2}+n+k\right)B(d+n-1,k+1)\\
&= n^2\kappa(d)\left(\frac{1}{1-a^2}\right)^2\left(\frac{E_s}{2N_o}\right)^d\sum_{s=0}^{2d}\left(\begin{array}{c}2d\\s\end{array}\right)\frac{2(1-a^2)^n}{\Gamma(n)\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}\\
&\quad\times(1-a^2)^{d+2}\left(\frac{1}{n}\right)^{d-\frac{s}{2}+1}\left(\frac{1}{n}\right)^{\frac{s}{2}+1}\\
&\quad\times\Gamma\left(d-\frac{s}{2}+n+k\right)\Gamma\left(\frac{s}{2}+n+k\right)B(d+n-1,k+1)\\
&= \frac{2(1-a^2)^{n+d}\kappa(n)}{\Gamma(n)\Gamma(n-1)}\left(\frac{E_s}{2N_o}\right)^d\sum_{s=0}^{2d}\left(\begin{array}{c}2d\\s\end{array}\right)\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}\left(\frac{1}{n}\right)^{d-\frac{s}{2}}\left(\frac{1}{n}\right)^{\frac{s}{2}}\\
&\quad\times\Gamma\left(d-\frac{s}{2}+n+k\right)\Gamma\left(\frac{s}{2}+n+k\right)B(d+n-1,k+1) \qquad (3.37)
\end{aligned}$$

which is the final result for μ_d.

Remark 3.7 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (3.37) simplifies to:*

$$\begin{aligned}
\mu_{d,normal} &= \frac{2(1-a^2)^{n+d}}{\Gamma(n)\Gamma(n-1)n^d}\left(\frac{E_s}{2N_o}\right)^d\sum_{s=0}^{2d}\binom{2d}{s}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}\\
&\quad\times\Gamma\left(d-\frac{s}{2}+n+k\right)\Gamma\left(\frac{s}{2}+n+k\right)B(d+n-1,k+1). \qquad (3.38)
\end{aligned}$$

Remark 3.8 *By choosing $\mathcal{W}(t)$ as the t distribution weight (3.37), (3.37) simplifies to:*

$$\begin{aligned}
\mu_{d,t} &= \frac{2(1-a^2)^{n+d}\left(\frac{v}{2}\right)^d\Gamma\left(\frac{v}{2}-d\right)}{\Gamma(n)\Gamma(n-1)\Gamma\left(\frac{v}{2}\right)n^d}\left(\frac{E_s}{2N_o}\right)^d\sum_{s=0}^{2d}\binom{2d}{s}\sum_{k=0}^{\infty}\frac{a^{2k}}{(k!)^2}\\
&\quad\times\Gamma\left(d-\frac{s}{2}+n+k\right)\Gamma\left(\frac{s}{2}+n+k\right)B(d+n-1,k+1) \qquad (3.39)
\end{aligned}$$

since from (3.17) and (1.6):

$$\begin{aligned}
\kappa(d) &= \int_{\mathbb{R}^+} t^{-d}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt\\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\int_{\mathbb{R}^+} t^{\frac{v}{2}-d-1}\exp\left(-\frac{vt}{2}\right)dt\\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\Gamma\left(\frac{v}{2}-d\right)}{\left(\frac{v}{2}\right)^{\frac{v}{2}-d}}\\
&= \left(\frac{v}{2}\right)^d\frac{\Gamma\left(\frac{v}{2}-d\right)}{\Gamma\left(\frac{v}{2}\right)}.
\end{aligned}$$

In the following figures the behaviour of the EGC diversity ((3.38) and (3.39)) is illustrated for arbitrary parameters. In particular, the case $d = 1$, the mean EGC diversity, is of interest (see (1.26)).

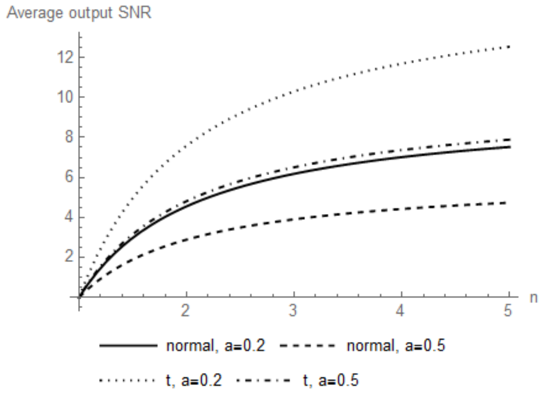


Figure 3.14 (3.38) and (3.39) against a, for different values of n, $v = 5$

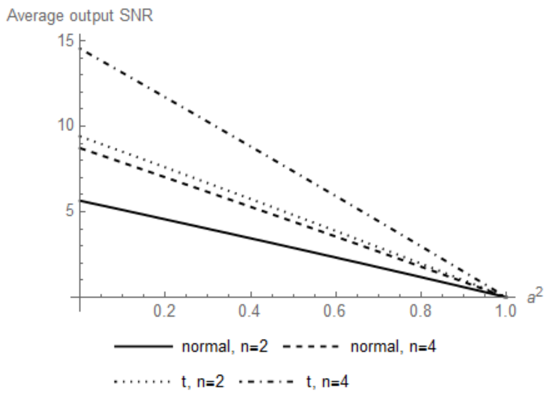


Figure 3.15 (3.38) and (3.39) against n, for different values of a, $v = 5$

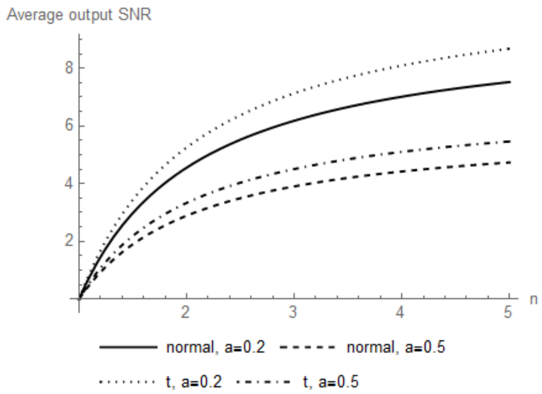


Figure 3.16 (3.38) and (3.39) against a, for different values of n, $v = 15$

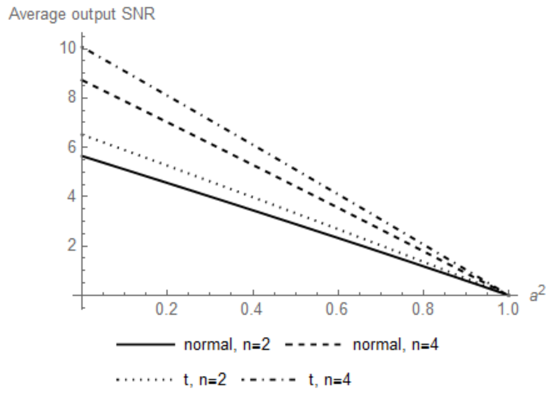


Figure 3.17 (3.38) and (3.39) against n, for different values of a, $v = 15$

It is observed that correlation between transmitters decreases the mean EGC diversity severely. Even so, for the underlying t distribution, the EGC diversity is higher than that of the normal case. Furthermore for both cases, as the number of receivers increase the mean EGC diversity increases correspondingly.

3.5 Summary of results and conclusion

In this section, a summary of theoretical results in this chapter is provided for the convenience of the reader. (X_1, X_2) denotes the bivariate gamma type I distribution which originates from the elliptical class (see (3.3)), and (R_1, R_2) denotes the corresponding bivariate Weibullised gamma type distribution emanating from (R_1, R_2).

Pdf			
	Elliptical	Normal	t
(X_1, X_2)	(3.3)	(3.11)	(3.13)
(R_1, R_2)	(3.20)	(3.21)	(3.23)
(R_1, R_2) for $\beta_1 = \beta_2 = 2$	(3.24)	(3.25)	(3.26)

Other results			
Outage probability of (X_1, X_2)	(3.28)	(3.29)	(3.30)
EGC diversity under (R_1, R_2)	(3.37)	(3.38)	(3.39)

Table 3.2 Summary of derived results relating to this chapter

In addition, the product moment of (X_1, X_2) with pdf (3.3) is also derived, see (3.15).

In this chapter new bivariate gamma- and bivariate Weibullised gamma type distributions have been presented which originated from the diagonal elements of a complex inverse Wishart type distribution. Specifically, the pdf, cdf, and product moments of the bivariate gamma type distribution have been derived. Since the complex elliptical class constitutes a flexible- and broad class of distributions, this chapter provides some insight in the possible usefulness for engineering applications by this assumption. The results have been applied to evaluate the outage probability of a MIMO system with two transmit antennas with underlying models the complex matrix variate normal- and t case.

Chapter 4

Bivariate gamma type II distributions

Contents

4.1 Introduction

In this chapter, a generalisation of a bivariate gamma distribution proposed by Nakagami (1960) is presented; this generalised bivariate gamma distribution emanates from exponential type random variables which in turn emanates from the complex elliptical class (see (1.4)). This generalised bivariate gamma distribution is called a bivariate gamma type II distribution, and contains the bivariate gamma distribution of Reig et. al. (2002) as a special case. This bivariate gamma type II distribution is studied via its pdf, product moment, and cdf. The normal- and t distribution are investigated as underlying models for this proposed bivariate gamma type I distribution. Notably, this bivariate gamma type II distribution has gamma type marginals, of which the gamma distribution is a special case of (see Result C.2). A bivariate noncentral gamma type II distribution is also proposed, stemming from this bivariate gamma type II distribution and based on the methodology of Ferreira et. al. (2016). From this bivariate gamma type II distribution, a bivariate Weibullised gamma type II distribution is proposed and studied. The pdf, Laplace transform, product moment, and cdf receives attention. Notably, under the normal assumption and for the case $\beta_1 = \beta_2 = 2$, this bivariate Weibullised gamma type II distribution reduces to the bivariate Nakagami distribution as studied by Reig et. al. (2002) and Pibongungon (2005). The bivariate Nakagami distribution is well known in the communiations systems domain and is popular to describe the signal behaviour of fading channels. Thus, to consider a bivariate Weibullised gamma type distribution which acts as an encompassive generalisation of this bivariate Nakagami distribution may prove useful within the communications systems framework. In particular, the bivariate gamma type II distribution is considered in an application context where outage probability (see (1.25)) of a fading channel operating under such a bivariate gamma type II distribution is evaluated.

This chapter's contributions can be summarised as follows:

1. A bivariate gamma type II distribution with gamma type marginals, and some of its statistical properties;
2. A bivariate noncentral gamma type II distribution, based on methodology proposed by Ferreira et. al. (2016);
3. A bivariate Weibullised gamma type II distribution and some of its statistical properties (containing the bivariate Nakagami distribution of Reig et. al. (2002) as a special case); and
4. Comparison of the new models in terms of the performance measure, outage probability.

The figure below visualise the main gist of theoretical derivations in this chapter.

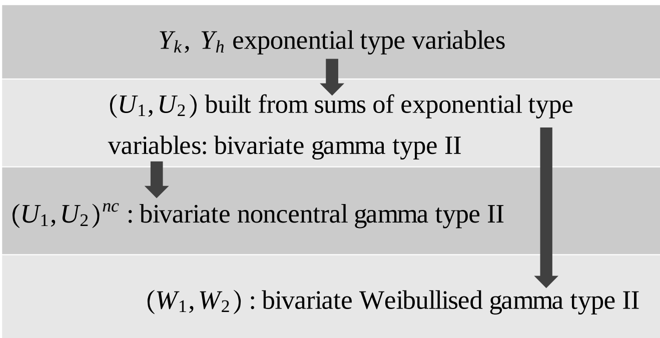


Figure 4.1 Outline of theoretical derivations in this chapter

4.2 Bivariate gamma type II distribution

In this section, a bivariate gamma type II distribution is constructed and studied, following the univariate description in Chapter 1. Following a similar approach as in Reig et. al. (2002), suppose:

$$
\begin{aligned}
U_1 &= \sum_{k=1}^{m_1} Y_k \\
U_2 &= U_{2a} + U_{2b} = \sum_{h=1}^{m_1} Y_h + \sum_{h=m_1+1}^{m_2} Y_h
\end{aligned}
$$

where Y_k $(k = 1, ..., m_1)$ and Y_h $(h = 1, ..., m_2)$ $(m_2 \geq m_1)$ are independent exponential type random variables (see (1.17)). Within the context of communications systems, m_1 and m_2 act as parameters describing the fading of the transmitted signal between transmitter and receiver (see Reig et. al. (2002), Pibongungon (2005), and references therein). Thus:

$$\begin{aligned} Y_k &\sim Exp(\Omega_1, \mathcal{W}(\cdot)) && \text{with } Y_k|t \sim Exp(\Omega_1); \\ Y_h &\sim Exp(\Omega_2, \mathcal{W}(\cdot)) && \text{with } Y_h|t \sim Exp(\Omega_2); \\ U_1 &\sim Gamma(\Omega_1, m_1, \mathcal{W}(\cdot)) && \text{with } U_1|t \sim Gamma(\Omega_1, m_1); \text{ and} \\ U_2 &\sim Gamma(\Omega_2, m_2, \mathcal{W}(\cdot)) && \text{with } U_2|t \sim Gamma(\Omega_2, m_2). \end{aligned}$$

The distribution of (U_1, U_2) is of particular interest.

4.2.1 Pdf

The Laplace transform of (U_1, U_2) is given by:

$$\begin{aligned} \mathcal{L}(s_1, s_2) &= \int_0^\infty \int_0^\infty \exp[-(s_1 u_2 + s_2 u_2)] f(u_1, u_2) du_1 du_2 \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \exp[-(s_1 u_2 + s_2 u_2)] f(u_1, u_2|t) \mathcal{W}(t) dt du_1 du_2 \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \exp[-(s_1 u_2 + s_2 u_2)] f(u_1, u_2|t) du_1 du_2 \mathcal{W}(t) dt \\ &= \int_0^\infty \mathcal{L}(s_1, s_2|t) \mathcal{W}(t) dt \end{aligned} \tag{4.1}$$

where $f(u_1, u_2)$ denotes the pdf of (U_1, U_2). The Laplace transform of $(U_1, U_2|t)$ is decomposed as follows:

$$\begin{aligned} \mathcal{L}(s_1, s_2|t) &= E(\exp(-s_1 U_1 - s_2 U_2)) \\ &= E(\exp(-s_1 U_1 - s_2 (U_{2a} + U_{2b}))) \\ &= E(\exp(-s_1 U_1 - s_2 U_{2a} - s_2 U_{2b})) \\ &= E(\exp(-s_1 U_1 - s_2 U_{2a}) \exp(-s_2 U_{2b})) \\ &= \mathcal{L}_{U_1, U_{2a}}(s_1, s_2|t) \mathcal{L}_{U_{2b}}(s_2|t) \end{aligned} \tag{4.2}$$

since U_1 and U_{2a} is independent of U_{2b}. From (1.18), see that:

$$\begin{aligned} \mathcal{L}_{U_{2b}}(s_2|t) &= \frac{1}{(1 + \Omega_2 t^{-1} s_2)^{m_2 - m_1}} \\ &= \frac{1}{(\Omega_2 t^{-1})^{m_2 - m_1} \left(s_2 + \frac{1}{\Omega_2 t^{-1}}\right)^{m_2 - m_1}} \end{aligned} \tag{4.3}$$

and the Laplace transform of $(U_1, U_{2a}|t)$ is given by (see Result C.16):

$$\begin{aligned} \mathcal{L}_{U_1, U_{2a}}(s_1, s_2|t) &= \left(\Omega_1 t^{-1}\right)^{-m_1} \left(\Omega_2 t^{-1}\right)^{-m_1} (1-\rho)^{-m_1} \\ &\times \left[\left(s_1 + \frac{1}{(\Omega_1 t^{-1})(1-\rho)} \right) \left(s_2 + \frac{1}{(\Omega_2 t^{-1})(1-\rho)} \right) - \frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)} \right]^{-m_1} \end{aligned} \tag{4.4}$$

where ρ represents the correlation coefficient between variables U_1 and U_{2a} (see Result C.30). Substituting (4.3) and (4.4) into (4.2) results in:

$$\begin{aligned}\mathcal{L}(s_1, s_2|t) &= \left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}(1-\rho)^{-m_1}\left[s_2+\frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} \\ &\quad\times\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)-\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right]^{-m_1}.\end{aligned} \tag{4.5}$$

Substituting (4.5) in (4.1) leaves:

$$\begin{aligned}\mathcal{L}(s_1, s_2) &= \int_0^\infty \left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}(1-\rho)^{-m_1}\left[s_2+\frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} \\ &\quad\times\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)-\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right]^{-m_1}\mathcal{W}(t)\,dt.\end{aligned} \tag{4.6}$$

This Laplace transform (4.6) resembles a bivariate gamma type II distribution with parameters $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$.

Remark 4.1 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.6) reflects the result of Reig et. al. (2002):*

$$\begin{aligned}\mathcal{L}(s_1, s_2) &= (\Omega_1)^{-m_1}(\Omega_2)^{-m_2}(1-\rho)^{-m_1}\left[s_2+\frac{1}{\Omega_2}\right]^{-(m_2-m_1)} \\ &\quad\times\left[\left(s_1+\frac{1}{\Omega_1(1-\rho)}\right)\left(s_2+\frac{1}{\Omega_2(1-\rho)}\right)-\frac{\rho}{\Omega_1\Omega_2(1-\rho^2)}\right]^{-m_1}\end{aligned}$$

where $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$.

The following lemma is useful for simplifying (4.6), in order to derive the pdf of (U_1, U_2).

Lemma 4.2.1 *In order to obtain the pdf of (U_1, U_2), consider the following term from (4.5) by using Result C.29:*

$$\begin{aligned}&\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)-\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right]^{-m_1} \\ &= \sum_{k=0}^\infty \frac{(m_1)_k}{k!}\left(\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right)^k \frac{1}{\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)\right]^k}\end{aligned}$$

where $(a)_k$ denotes the Pochhammer symbol (see Result C.9). Thus, the Laplace transform in (4.5) can be written in series form as:

$$\begin{aligned}\mathcal{L}(s_1, s_2|t) &= \left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}(1-\rho)^{-m_1}\left[s_2+\frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} \\ &\quad\times\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)-\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right]^{-m_1} \\ &= \left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}(1-\rho)^{-m_1}\left[s_2+\frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} \\ &\quad\times\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)\right]^{-m_1}\sum_{k=0}^\infty\frac{(m_1)_k}{k!} \\ &\quad\times\left(\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right)^k\frac{1}{\left[\left(s_1+\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)\right]^k}.\end{aligned} \tag{4.7}$$

By substituting (4.7) into (4.1), the pdf of the bivariate gamma type II distribution can be derived via an inverse Laplace approach. This is done in the next theorem.

Theorem 4.1 *If (U_1, U_2) is distributed as bivariate gamma type II with Laplace transform (4.6), then the pdf of (U_1, U_2) is given by:*

$$\begin{aligned} f(u_1,u_2) &= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!} \rho^k \left(\frac{1}{\Omega_1 (1-\rho)} \right)^{m_1+k} \left(\frac{1}{\Omega_2 (1-\rho)} \right)^{m_2+k} \\ &\quad \times \frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)} \frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)} \int_0^{\infty} \exp\left[-t \left(\frac{u_1}{\Omega_1(1-\rho)} + \frac{u_2}{\Omega_2(1-\rho)} \right) \right] \\ &\quad \times {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{t\rho}{\Omega_2(1-\rho)} u_2 \right) t^{m_1+m_2+2k} \mathcal{W}(t)\, dt \qquad (4.8) \end{aligned}$$

for $u_1, u_2 > 0$, where $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, $m_2 \geq m_1$, and where ${}_1F_1(\cdot)$ denotes the confluent hypergeometric function (see Result C.14). This distribution is called a bivariate gamma type II distribution.

Proof. *From (4.1), see that by applying an inverse Laplace transform results in:*

$$\begin{aligned} f(u_1,u_2) &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \exp(s_1u_2 + s_2u_2)\, \mathcal{L}(s_1,s_2|t)\, du_1 du_2 \mathcal{W}(t)\, dt \\ &= \int_0^{\infty} f(u_1,u_2|t)\, \mathcal{W}(t)\, dt. \qquad (4.9) \end{aligned}$$

By considering (4.7), the pdf of $(U_1, U_2|t)$ is obtained as:

$$\begin{aligned} f(u_1,u_2|t) &= \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_1u_1 + s_2u_2)\, \mathcal{L}(s_1,s_2|t)\, ds_1 ds_2 \\ &= \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!} \left(\Omega_1 t^{-1}\right)^{-m_1} \left(\Omega_2 t^{-1}\right)^{-m_2} (1-\rho)^{-m_1} \left(\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)} \right)^k \\ &\quad \times \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_1u_1) \left(s_1 + \frac{1}{(\Omega_1 t^{-1})(1-\rho)} \right)^{-m_1-k} ds_1 \\ &\quad \times \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_2u_2) \left(s_2 + \frac{1}{(\Omega_2 t^{-1})(1-\rho)} \right)^{-m_1-k} \left[s_2 + \frac{1}{\Omega_2 t^{-1}} \right]^{-(m_2-m_1)} ds_2 \\ &= \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!} \left(\Omega_1 t^{-1}\right)^{-m_1} \left(\Omega_2 t^{-1}\right)^{-m_2} (1-\rho)^{-m_1} \left(\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)} \right)^k \\ &\quad \times V_1 \cdot V_2. \qquad (4.10) \end{aligned}$$

Consider from (4.10):

$$\begin{aligned} V_1 &= \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_1u_1) \left(s_1 + \frac{1}{(\Omega_1 t^{-1})(1-\rho)} \right)^{-m_1-k} ds_1 \\ &= \frac{1}{\Gamma(m_1+k)} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_1u_1)\, \Gamma(m_1+k) \left(s_1 + \frac{1}{(\Omega_1 t^{-1})(1-\rho)} \right)^{-m_1-k} ds_1. \end{aligned}$$

Using Result C.33 and letting $y_1 = s_1 + \frac{1}{(\Omega_1 t^{-1})(1-\rho)}$*, thus* $\frac{dy_1}{ds_1} = 1$*,* V_1 *can be solved as:*

$$\begin{aligned} V_1 &= \frac{1}{\Gamma(m_1+k)} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_1 u_1) \Gamma(m_1+k) \left(s_1 + \frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)^{-m_1-k} ds_1 \\ &= \frac{1}{\Gamma(m_1+k)} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(y_1 u_1) \exp\left(-\frac{1}{(\Omega_1 t^{-1})(1-\rho)} u_1\right) \Gamma(m_1+k) y^{-m_1-k} dy_1 \\ &= \frac{1}{\Gamma(m_1+k)} \exp\left(-\frac{1}{(\Omega_1 t^{-1})(1-\rho)} u_1\right) \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(y_1 u_1) \Gamma(m_1+k) y^{-m_1-k} dy_1 \\ &= \frac{1}{\Gamma(m_1+k)} \exp\left(-\frac{1}{(\Omega_1 t^{-1})(1-\rho)} u_1\right) u_1^{m_1+k-1}. \end{aligned} \tag{4.11}$$

Also, consider from (4.10):

$$\begin{aligned} V_2 &= \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_2 u_2) \left(s_2 + \frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)^{-m_1-k} \left[s_2 + \frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} ds_2 \\ &= \frac{1}{\Gamma(m_2+k)} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_2 u_2) \Gamma(m_2+k) \left(s_2 + \frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)^{-m_1-k} \left[s_2 + \frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} ds_2. \end{aligned}$$

Using Result C.34 and letting $y_2 = s_2 + \frac{1}{(\Omega_2 t^{-1})(1-\rho)}$*, thus* $\frac{dy_2}{ds_2} = 1$*,* V_2 *can be solved as:*

$$\begin{aligned} V_2 &= \frac{1}{\Gamma(m_2+k)} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(s_2 u_2) \Gamma(m_2+k) \left(s_2 + \frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)^{-m_1-k} \left[s_2 + \frac{1}{\Omega_2 t^{-1}}\right]^{-(m_2-m_1)} ds_2 \\ &= \frac{1}{\Gamma(m_2+k)} \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(y_2 u_2) \exp\left(-\frac{1}{(\Omega_2 t^{-1})(1-\rho)} u_2\right) \Gamma(m_2+k) y_2^{-m_1-k} \left[y_2 - \frac{\rho}{(\Omega_2 t^{-1})(1-\rho)}\right]^{-(m_2-m_1)} dy_2 \\ &= \frac{1}{\Gamma(m_2+k)} \exp\left(-\frac{1}{(\Omega_2 t^{-1})(1-\rho)} u_2\right) \frac{1}{2\pi i} \int_{c-l\infty}^{c+l\infty} \exp(y_2 u_2) \Gamma(m_2+k) y_2^{-m_1-k} \left[y_2 - \frac{\rho}{(\Omega_2 t^{-1})(1-\rho)}\right]^{-(m_2-m_1)} dy_2 \\ &= \frac{1}{\Gamma(m_2+k)} \exp\left(-\frac{1}{(\Omega_2 t^{-1})(1-\rho)} u_2\right) u_2^{m_2+k-1} \, {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{\rho}{(\Omega_2 t^{-1})(1-\rho)} u_2\right). \end{aligned} \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10) leaves the pdf of $(U_1, U_2|t)$ *as:*

$$
\begin{aligned}
f(u_1,u_2|t) &= \sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}(1-\rho)^{-m_1}\left(\frac{\rho}{(\Omega_1 t^{-1})(\Omega_2 t^{-1})(1-\rho^2)}\right)^k \\
&\quad \times \frac{1}{\Gamma(m_1+k)}\exp\left(-\frac{1}{(\Omega_1 t^{-1})(1-\rho)}u_1\right)u_1^{m_1+k-1} \\
&\quad \times \frac{1}{\Gamma(m_2+k)}\exp\left(-\frac{1}{(\Omega_2 t^{-1})(1-\rho)}u_2\right)u_2^{m_2+k-1}\,{}_1F_1\left(m_2-m_1, m_2+k; \frac{\rho}{(\Omega_2 t^{-1})(1-\rho)}u_2\right) \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)^{m_1+k}\left(\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\exp\left[-\frac{u_1}{(\Omega_1 t^{-1})(1-\rho)}-\frac{u_2}{(\Omega_2 t^{-1})(1-\rho)}\right] \\
&\quad \times {}_1F_1\left(m_2-m_1, m_2+k; \frac{\rho}{(\Omega_2 t^{-1})(1-\rho)}u_2\right) \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}\right)\right] \\
&\quad \times {}_1F_1\left(m_2-m_1, m_2+k; \frac{t\rho}{\Omega_2(1-\rho)}u_2\right)t^{m_1+m_2+2k}. \qquad (4.13)
\end{aligned}
$$

Finally, because of the uniqueness of the inverse Laplace transform, substituting (4.13) into (4.9) leaves the final result. ■

Remark 4.2 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), (4.8) simplifies to:*

$$
\begin{aligned}
f_{normal}(u_1,u_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\exp\left[-\frac{u_1}{\Omega_1(1-\rho)}\right]\exp\left[-\frac{u_2}{\Omega_2(1-\rho)}\right] \\
&\quad \times {}_1F_1\left(m_2-m_1, m_2+k; \frac{\rho}{\Omega_2(1-\rho)}u_2\right) \qquad (4.14)
\end{aligned}
$$

for $u_1, u_2 > 0$ *and where* $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, *and* $m_2 \geq m_1$. *This result reflects the result by Reig et. al. (2002).*

Remark 4.3 *By choosing $\mathcal{W}(t)$ as (1.6) and using Result C.32, (4.8) simplifies to:*

$$\begin{aligned}
f_t(u_1,u_2) &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\\
&\quad\times\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\int_0^{\infty}t^{m_1+m_2+2k+\frac{v}{2}-1}\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right]\\
&\quad\times{}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)dt\\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\\
&\quad\times\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{-\left(m_1+m_2+2k+\frac{v}{2}\right)}\\
&\quad\times{}_2F_1\left(m_2-m_1,m_1+m_2+2k+\frac{v}{2};m_2+k;\frac{\rho}{\Omega_2(1-\rho)\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)}u_2\right) \qquad (4.15)
\end{aligned}$$

for $u_1,u_2>0$ and where $m_1,m_2,\Omega_1,\Omega_2>0$, $-1<\rho<1$, and $m_2\geq m_1$. ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function (see Result C.15), with restriction $\left|\frac{\rho}{\Omega_2(1-\rho)\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)}u_2\right|<1$.

Remark 4.4 *Note that (4.15) can be rewritten as follows:*

$$\begin{aligned}
f_t(u_1,u_2) &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\\
&\quad\times\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\int_0^{\infty}t^{m_1+m_2+2k+\frac{v}{2}-1}\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right]\\
&\quad\times{}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)dt\\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\\
&\quad\times\frac{\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)}{\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{m_1+m_2+2k+\frac{v}{2}}}\int_0^{\infty}\frac{\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{m_1+m_2+2k+\frac{v}{2}}}{\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)}t^{m_1+m_2+2k+\frac{v}{2}-1}\\
&\quad\times\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right]{}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)dt\\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)}\\
&\quad\times\frac{\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)}{\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{m_1+m_2+2k+\frac{v}{2}}}E_T\left({}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)\right)
\end{aligned}$$

where $T\sim Gamma\left(m_1+m_2+2k+\frac{v}{2},\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right)$ (see Result C.2).

In the following figures the natures of pdfs (4.14) and (4.15) are illustrated for arbitrary choices of parameters.

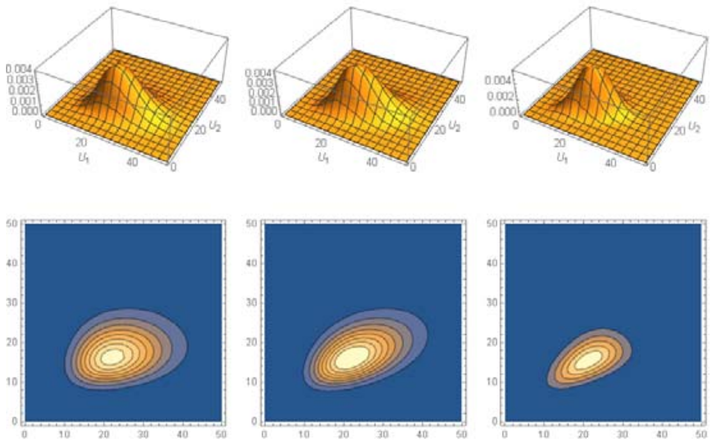


Figure 4.2 Pdf (4.14) for $m_1 = 10$, $m_2 = 12$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, and varying $\rho = 0.25, 0.5, 0.75$ (fltr)

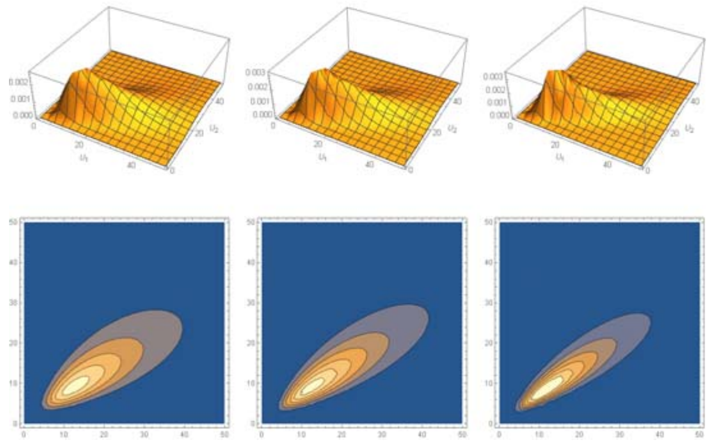


Figure 4.3 Pdf (4.15) for $v = 5$, $m_1 = 10$, $m_2 = 12$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, and varying $\rho = 0.25, 0.5, 0.75$ (fltr)

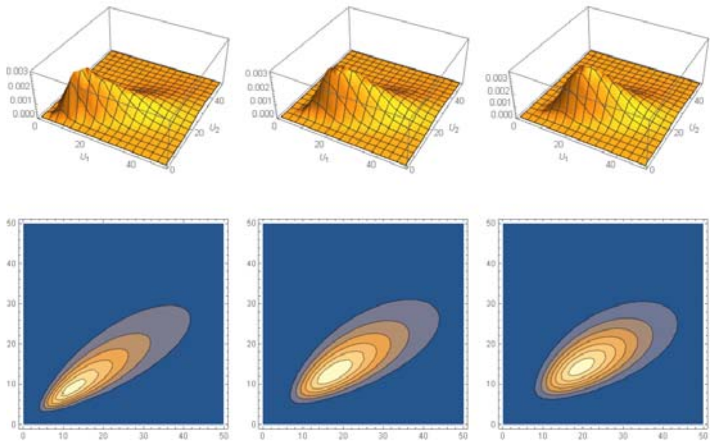


Figure 4.4 Pdf (4.15) for $m_1 = 10$, $m_2 = 12$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, $\rho = 0.5$, and varying $v = 5, 15, 30$ (fltr)

From the above figures, the following can be observed:

- In Figures 4.2 and 4.3 the effect of increasing ρ is observable, particularly indicating increased concentration (or correlation) between the variables.

- Figure 4.4 indicates the effect on the tails of (4.15) for increasing values of v. As v increases, (4.15) begins to illustrate the characteristics of (4.14) in Figure 4.2 (middle figure).

4.2.2 Marginal distributions

In this section the marginal distributions of the bivariate gamma type II distribution with pdf (4.8) are derived.

Theorem 4.2 *Suppose (U_1, U_2) is distributed with pdf (4.8). Then the marginal pdf of U_1 is given by*

$$f(u_1) = \frac{u_1^{m_1-1}}{\Omega_1^{m_1}\Gamma(m_1)} \int_0^\infty t^{m_1} \exp\left[-t\left(\frac{u_1}{\Omega_1}\right)\right] \mathcal{W}(t)\, dt \tag{4.16}$$

where $u_1 > 0$ for $m_1, \Omega_1 > 0$.

Proof. *From (4.8) the marginal distribution of U_1 is given by:*

$$\begin{aligned} f(u_1) &= \int_0^\infty f(u_1, u_2)\, du_2 \\ &= \int_0^\infty \int_0^\infty f(u_1, u_2|t)\, \mathcal{W}(t)\, dt du_2 \\ &= \int_0^\infty \int_0^\infty f(u_1, u_2|t)\, du_2 \mathcal{W}(t)\, dt. \end{aligned} \tag{4.17}$$

By using Result C.32, consider from (4.17):

$$\begin{aligned}
\int_0^\infty f(u_1,u_2|t)\,du_2 &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)\Gamma(m_2+k)}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]t^{m_1+m_2+2k} \\
&\quad\times\int_0^\infty u_2^{m_2+k-1}\exp\left[-\frac{t}{\Omega_2(1-\rho)}u_2\right]{}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)du_2 \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)\Gamma(m_2+k)}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]t^{m_1+m_2+2k} \\
&\quad\times\Gamma(m_2+k)\left(\frac{t}{\Omega_2(1-\rho)}\right)^{-(m_2+k)}{}_2F_1\left(m_2-m_1,m_2+k;m_2+k;\frac{\frac{t}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)}}\rho\right) \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)} \\
&\quad\times\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]{}_1F_0(m_2-m_1;\rho)\,t^{m_1+k} \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)} \\
&\quad\times\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right](1-\rho)^{m_2-m_1}t^{m_1+k} \\
&= (1-\rho)^{-m_1}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]t^{m_1+k} \qquad (4.18)
\end{aligned}$$

Substituting (4.18) into (4.17) leaves:

$$\begin{aligned}
f(u_1) &= \int_0^\infty f(u_1,u_2|t)\,du_2\,\mathcal{W}(t)\,dt \\
&= \int_0^\infty (1-\rho)^{-m_1}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\frac{u_1^{m_1+k-1}}{\Gamma(m_1+k)}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]t^{m_1+k}\mathcal{W}(t)\,dt \\
&= \int_0^\infty (1-\rho)^{-m_1}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]\left(\frac{t}{\Omega_1(1-\rho)}\right)^{m_1}u_1^{m_1-1}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!\Gamma(m_1+k)}\left(\frac{u_1\rho t}{\Omega_1(1-\rho)}\right)^k\mathcal{W}(t)\,dt \\
&= \int_0^\infty (1-\rho)^{-m_1}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]\left(\frac{t}{\Omega_1(1-\rho)}\right)^{m_1}\frac{u_1^{m_1-1}}{\Gamma(m_1)}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{u_1\rho t}{\Omega_1(1-\rho)}\right)^k\mathcal{W}(t)\,dt \\
&= \int_0^\infty (1-\rho)^{-m_1}\exp\left[-t\frac{u_1}{\Omega_1(1-\rho)}\right]\left(\frac{t}{\Omega_1(1-\rho)}\right)^{m_1}\frac{u_1^{m_1-1}}{\Gamma(m_1)}\exp\left(\frac{u_1\rho t}{\Omega_1(1-\rho)}\right)\mathcal{W}(t)\,dt \\
&= \frac{u_1^{m_1-1}}{(1-\rho)^{m_1}\Gamma(m_1)}\int_0^\infty \exp\left[-t\left(\frac{u_1(1+\rho)}{\Omega_1(1-\rho)}\right)\right]\left(\frac{t}{\Omega_1(1-\rho)}\right)^{m_1}\mathcal{W}(t)\,dt.
\end{aligned}$$

Finally setting $\rho = 0$*:*

$$\begin{aligned} f(u_1) &= \frac{u_1^{m_1-1}}{\Gamma(m_1)} \int_0^\infty \exp\left[-t\left(\frac{u_1}{\Omega_1}\right)\right]\left(\frac{t}{\Omega_1}\right)^{m_1} \mathcal{W}(t)\, dt \\ &= \frac{u_1^{m_1-1}}{\Omega_1^{m_1}\Gamma(m_1)} \int_0^\infty t^{m_1} \exp\left[-t\left(\frac{u_1}{\Omega_1}\right)\right] \mathcal{W}(t)\, dt \end{aligned}$$

which leaves the final result. ∎

Theorem 4.3 *Suppose* (U_1, U_2) *is distributed with pdf (4.8). Then the marginal pdf of* U_2 *is given by*

$$f(u_2) = \frac{u_2^{m_2-1}}{\Omega_2^{m_2}\Gamma(m_2)} \int_0^\infty t^{m_2} \exp\left[-t\left(\frac{u_2}{\Omega_2}\right)\right] \mathcal{W}(t)\, dt \tag{4.19}$$

where $u_2 > 0$ *for* $m_2, \Omega_2 > 0$.

Proof. *From (4.8) the marginal distribution of* U_2 *is given by:*

$$\begin{aligned} f(u_2) &= \int_0^\infty f(u_1, u_2)\, du_1 \\ &= \int_0^\infty \int_0^\infty f(u_1, u_2|t)\, \mathcal{W}(t)\, dt du_1 \\ &= \int_0^\infty \int_0^\infty f(u_1, u_2|t)\, du_1 \mathcal{W}(t)\, dt. \end{aligned} \tag{4.20}$$

By using Result C.22, consider from (4.20):

$$\begin{aligned} \int_0^\infty f(u_1, u_2|t)\, du_1 &= (1-\rho)^{m_2} \sum_{k=0}^\infty \frac{(m_1)_k}{k!}\rho^k \left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \frac{u_2^{m_2+k-1}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times t^{m_1+m_2+2k} \exp\left[-\frac{t}{\Omega_2(1-\rho)}u_2\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{t\rho}{\Omega_2(1-\rho)}u_2\right) \\ &\quad \times \int_0^\infty u_1^{m_1+k-1} \exp\left[-\frac{t}{\Omega_1(1-\rho)}u_1\right] du_1 \\ &= (1-\rho)^{m_2} \sum_{k=0}^\infty \frac{(m_1)_k}{k!}\rho^k \left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \frac{u_2^{m_2+k-1}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times t^{m_1+m_2+2k} \exp\left[-\frac{t}{\Omega_2(1-\rho)}u_2\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{t\rho}{\Omega_2(1-\rho)}u_2\right) \\ &\quad \times \frac{\Gamma(m_1+k)}{\left(\frac{t}{\Omega_1(1-\rho)}\right)^{m_1+k}} \\ &= (1-\rho)^{m_2} \sum_{k=0}^\infty \frac{(m_1)_k}{k!}\rho^k \left(\frac{t}{\Omega_2(1-\rho)}\right)^{m_2+k} \frac{u_2^{m_2+k-1}}{\Gamma(m_2+k)} \\ &\quad \times \exp\left[-\frac{t}{\Omega_2(1-\rho)}u_2\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{t\rho}{\Omega_2(1-\rho)}u_2\right) \end{aligned} \tag{4.21}$$

Substituting (4.21) into (4.20):

$$\begin{aligned} f\left(u_2\right) &= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!} \rho^k \frac{u_2^{m_2+k-1}}{\Gamma\left(m_2+k\right)} \left(\frac{1}{\Omega_2\left(1-\rho\right)}\right)^{m_2+k} \\ &\quad \times \int_0^{\infty} t^{m_2+k} \exp\left[-\frac{t}{\Omega_2\left(1-\rho\right)} u_2\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{t\rho}{\Omega_2\left(1-\rho\right)} u_2\right) \mathcal{W}\left(t\right) dt. \end{aligned}$$

Finally setting $\rho = 0$ *:*

$$\begin{aligned} f\left(u_2\right) &= \int_0^{\infty} u_2^{m_2-1} \left(\frac{1}{\Omega_2}\right)^{m_2} t^{m_2} \exp\left[-\frac{t}{\Omega_2} u_2\right] \frac{1}{\Gamma\left(m_2\right)} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!\left(m_2\right)_k} \left(\frac{t\rho u_2}{\Omega_2}\right)^k \mathcal{W}\left(t\right) dt \\ &= \frac{1}{\Gamma\left(m_2\right)} \int_0^{\infty} u_2^{m_2-1} \left(\frac{1}{\Omega_2}\right)^{m_2} t^{m_2} \exp\left[-\frac{t}{\Omega_2} u_2\right] {}_1F_1\left(m_1, m_2; \frac{t\rho u_2}{\Omega_2}\right) \mathcal{W}\left(t\right) dt \\ &= \frac{u_2^{m_2-1}}{\Omega_2^{m_2}\Gamma\left(m_2\right)} \int_0^{\infty} t^{m_2} \exp\left[-t\left(\frac{u_2}{\Omega_2}\right)\right] \mathcal{W}\left(t\right) dt \end{aligned}$$

which leaves the final result. ■

Remark 4.5 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), (4.16) and (4.19) simplifies to:*

$$f\left(u_i\right) = \frac{u_i^{m_i-1}}{\Omega_i^{m_i}\Gamma\left(m_i\right)} \exp\left[-\left(\frac{u_i}{\Omega_i}\right)\right] \tag{4.22}$$

for $u_i > 0$ *and* $m_i, \Omega_i > 0$ *where* $i = 1, 2$. *This distribution with pdf (4.22) is the gamma distribution (see Result C.2).*

Remark 4.6 *By choosing* $\mathcal{W}(t)$ *as (1.6) and using Result C.22, (4.16) and (4.19) simplifies to:*

$$\begin{aligned} f\left(u_i\right) &= \frac{u_i^{m_i-1}}{\Omega_i^{m_i}\Gamma\left(m_i\right)} \int_0^{\infty} t^{m_i} \exp\left[-t\left(\frac{u_i}{\Omega_i}\right)\right] \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-\frac{vt}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{u_i^{m_i-1}}{\Omega_i^{m_i}\Gamma\left(m_i\right)} \int_0^{\infty} t^{m_i+\frac{v}{2}-1} \exp\left[-t\left(\frac{u_i}{\Omega_i}+\frac{v}{2}\right)\right] dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \frac{u_i^{m_i-1}}{\Omega_i^{m_i}\Gamma\left(m_i\right)} \frac{\Gamma\left(m_i+\frac{v}{2}\right)}{\left(\frac{u_i}{\Omega_i}+\frac{v}{2}\right)^{m_i+\frac{v}{2}}} \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{B\left(\frac{v}{2}, m_i\right)\Omega_i^{m_i}} \frac{u_i^{m_i-1}}{\left(\frac{u_i}{\Omega_i}+\frac{v}{2}\right)^{m_i+\frac{v}{2}}} \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{B\left(\frac{v}{2}, m_i\right)\Omega_i^{m_i}} \frac{u_i^{m_i-1}}{\left(\frac{v}{2}\right)^{m_i+\frac{v}{2}}\left(\frac{u_i}{\frac{v}{2}\Omega_i}+1\right)^{m_i+\frac{v}{2}}} \\ &= \frac{1}{B\left(\frac{v}{2}, m_i\right)\left(\frac{v}{2}\Omega_i\right)^{m_i}} \frac{u_i^{m_i-1}}{\left(\frac{u_i}{\frac{v}{2}\Omega_i}+1\right)^{m_i+\frac{v}{2}}} \end{aligned} \tag{4.23}$$

for $u_i > 0$ *and* $m_i, \Omega_i, v > 0$ *where* $i = 1, 2$.

Remark 4.7 *It is observed that (4.16) and (4.19) are gamma type distributions, with the gamma distribution* *(see Result C.2) as special cases (see (4.22)). By this motivation it supports that (4.8) is called a bivariate gamma type I distribution with gamma type marginals.*

4.2.3 Product moment

In this section an expression for the product moment of this bivariate gamma type II distribution with pdf (4.8) is derived.

Theorem 4.4 *The product moment of (U_1, U_2) with pdf (4.8) is given by:*

$$\begin{aligned} E\left(U_1^r U_2^d\right) &= \Omega_1^r \Omega_2^d (1-\rho)^{r+d+m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!} \rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \int_0^\infty t^{-(r+d)} \mathcal{W}(t)\, dt \end{aligned} \tag{4.24}$$

where $r, d > 0$, and $m_2, m_1, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$.

Proof. *Consider from (4.8):*

$$\begin{aligned} E\left(U_1^r U_2^d\right) &= \int_0^\infty \int_0^\infty u_1^r u_2^d f(u_1, u_2)\, du_1 du_2 \\ &= \int_0^\infty \int_0^\infty u_1^r u_2^d \int_0^\infty f(u_1, u_2|t)\, \mathcal{W}(t)\, dt du_1 du_2 \\ &= \int_0^\infty \int_0^\infty \int_0^\infty u_1^r u_2^d f(u_1, u_2|t)\, du_1 du_2 \mathcal{W}(t)\, dt \\ &= \int_0^\infty E\left(U_1^r U_2^d|t\right) \mathcal{W}(t)\, dt. \end{aligned} \tag{4.25}$$

From (4.25) and (4.13):

$$\begin{aligned}
E\left(U_1^r U_2^d|t\right) &= \int_0^\infty\int_0^\infty u_1^r u_2^d f\left(u_1,u_2|t\right)du_1 du_2 \\
&= \int_0^\infty\int_0^\infty u_1^r u_2^d \left(1-\rho\right)^{m_2}\sum_{k=0}^\infty \frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1 t^{-1}\left(1-\rho\right)}\right)^{m_1+k}\left(\frac{1}{\Omega_2 t^{-1}\left(1-\rho\right)}\right)^{m_2+k} \\
&\quad\times\exp\left[-\frac{u_1}{\left(\Omega_1 t^{-1}\right)\left(1-\rho\right)}\right]\exp\left[-\frac{u_2}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}\right]\frac{u_1^{m_1+k-1}}{\Gamma\left(m_1+k\right)}\frac{u_2^{m_2+k-1}}{\Gamma\left(m_2+k\right)} \\
&\quad\times {}_1F_1\left(m_2-m_1,m_2+k;\frac{\rho}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}u_2\right)du_1 du_2 \\
&= \left(1-\rho\right)^{m_2}\sum_{k=0}^\infty \frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1 t^{-1}\left(1-\rho\right)}\right)^{m_1+k}\left(\frac{1}{\Omega_2 t^{-1}\left(1-\rho\right)}\right)^{m_2+k}\frac{1}{\Gamma\left(m_1+k\right)\Gamma\left(m_2+k\right)} \\
&\quad\int_0^\infty u_1^{r+m_1+k-1}\exp\left[-\frac{u_1}{\left(\Omega_1 t^{-1}\right)\left(1-\rho\right)}\right]du_1\int_0^\infty u_2^{d+m_2+k-1}\exp\left[-\frac{u_2}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}\right] \\
&\quad\times {}_1F_1\left(m_2-m_1,m_2+k;\frac{\rho}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}u_2\right)du_2 \\
&= \left(1-\rho\right)^{m_2}\sum_{k=0}^\infty \frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1 t^{-1}\left(1-\rho\right)}\right)^{m_1+k}\left(\frac{1}{\Omega_2 t^{-1}\left(1-\rho\right)}\right)^{m_2+k}\frac{1}{\Gamma\left(m_1+k\right)\Gamma\left(m_2+k\right)} \\
&\quad V_1\cdot V_2. \qquad (4.26)
\end{aligned}$$

Using Result C.22, consider from (4.26):

$$\begin{aligned}
V_1 &= \int_0^\infty u_1^{r+m_1+k-1}\exp\left[-\frac{1}{\left(\Omega_1 t^{-1}\right)\left(1-\rho\right)}u_1\right]du_1 \\
&= \frac{\Gamma\left(r+m_1+k\right)}{\left(\frac{1}{\left(\Omega_1 t^{-1}\right)\left(1-\rho\right)}\right)^{r+m_1+k}} \\
&= \Gamma\left(r+m_1+k\right)\left(\left(\Omega_1 t^{-1}\right)\left(1-\rho\right)\right)^{r+m_1+k} \\
&= t^{-(r+m_1+k)}\Gamma\left(n+m_1+k\right)\left(\Omega_1\left(1-\rho\right)\right)^{r+m_1+k} \qquad (4.27)
\end{aligned}$$

and using Result C.32:

$$\begin{aligned}
V_2 &= \int_0^\infty u_2^{d+m_2+k-1}\exp\left[-\frac{1}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}u_2\right]{}_1F_1\left(m_2-m_1,m_2+k;\frac{\rho}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}u_2\right)du_2 \\
&= \Gamma\left(d+m_2+k\right)\left(\frac{1}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}\right)^{-(d+m_2+k)}{}_2F_1\left(m_2-m_1,d+m_2+k;m_2+k;\frac{\frac{1}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}}{\frac{1}{\left(\Omega_2 t^{-1}\right)\left(1-\rho\right)}}\rho\right) \\
&= t^{-(d+m_2+k)}\Gamma\left(d+m_2+k\right)\left(\frac{1}{\Omega_2\left(1-\rho\right)}\right)^{-(d+m_2+k)}{}_2F_1\left(m_2-m_1,d+m_2+k;m_2+k;\rho\right). \qquad (4.28)
\end{aligned}$$

Substituting (4.27) and (4.28) into (4.26) leaves:

$$\begin{aligned} E\left(U_1^r U_2^d|t\right) &= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}\right)^{m_1+k} \left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}\right)^{m_2+k} \frac{1}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times t^{-(r+m_1+k)}\Gamma(r+m_1+k)\left(\Omega_1(1-\rho)\right)^{r+m_1+k} \\ &\quad \times t^{-(d+m_2+k)}\Gamma(d+m_2+k)\left(\frac{1}{\Omega_2(1-\rho)}\right)^{-(d+m_2+k)} {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \\ &= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times t^{m_1+m_2+2k} t^{-(r+m_1+k)} t^{-(d+m_2+k)} \left(\Omega_1(1-\rho)\right)^r \left(\Omega_2(1-\rho)\right)^d \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \\ &= t^{-(r+d)}(1-\rho)^{m_2}\left(\Omega_1(1-\rho)\right)^r \left(\Omega_2(1-\rho)\right)^d \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho). \end{aligned} \tag{4.29}$$

Substituting (4.29) into (4.25) leaves the final result. ■

Corollary 4.1 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.24) simplifies to:*

$$\begin{aligned} E_{normal}\left(U_1^r U_2^d\right) &= \Omega_1^r \Omega_2^d (1-\rho)^{r+d+m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \end{aligned} \tag{4.30}$$

where $r,d>0$, and $m_2,m_1,\Omega_1,\Omega_2>0$, $-1<\rho<1$, and $m_2\geq m_1$.

Corollary 4.2 *By choosing $\mathcal{W}(t)$ as (1.6) and using Result C.22, (4.24) simplifies to:*

$$\begin{aligned} E_t\left(U_1^r U_2^d\right) &= \Omega_1^r \Omega_2^d (1-\rho)^{r+d+m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \int_0^{\infty} t^{-(r+d)} \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-\frac{vt}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Omega_1^r \Omega_2^d (1-\rho)^{r+d+m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \int_0^{\infty} t^{\frac{v}{2}-(r+d)-1} \exp\left(-\frac{vt}{2}\right) dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \Omega_1^r \Omega_2^d (1-\rho)^{r+d+m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho)\, \Gamma\left(\frac{v}{2}-(r+d)\right)\left(\frac{v}{2}\right)^{-\left(\frac{v}{2}-(r+d)\right)} \\ &= \frac{\left(\frac{v}{2}\right)^{r+d}\Gamma\left(\frac{v}{2}-(r+d)\right)}{\Gamma\left(\frac{v}{2}\right)} \Omega_1^r \Omega_2^d (1-\rho)^{r+d+m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k}{k!}\rho^k \frac{\Gamma(r+m_1+k)\Gamma(d+m_2+k)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times {}_2F_1(m_2-m_1, d+m_2+k; m_2+k; \rho) \end{aligned} \tag{4.31}$$

and where $r,d>0$, $m_2,m_1,\Omega_1,\Omega_2,v>0$, $-1<\rho<1$, $m_2\geq m_1$, and $v>2(r+d)$ in order for the product moment to exist.

In the following figure the correlation coefficient of (U_1, U_2) with pdf (4.8) is illustrated using Result C.30 via (4.30) and (4.31) for arbitrary parameters. Note that ρ is defined as the correlation of (U_1, U_{2a}) (see (4.4)).

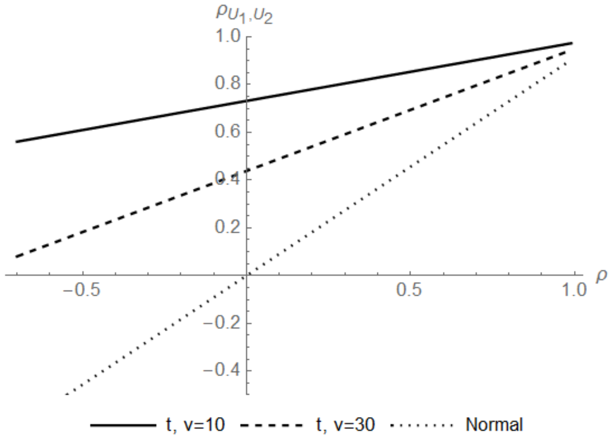


Figure 4.5 Correlation using (4.30) and (4.31) against ρ for $m_1 = 10$, $m_2 = 12$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, and $v = 10, 30$

Figure 4.5 illustrates a higher correlation coefficient for (4.31) when compared to (4.30). In particular it is observed that the correlation under the t distribution assumption approaches the correlation under the normal assumption for increased values of v. The same observation was made for the correlation of the bivariate gamma type I distribution (see Figure 3.6).

4.2.4 Cdf

In this section the cdf of the bivariate gamma type II distribution with pdf (4.8) is derived. This cdf is useful in evaluating the outage probability of a fading model subject to a bivariate gamma type II fading distribution (see (1.25)).

Theorem 4.5 *Suppose that (U_1, U_2) is distributed with pdf (4.8). Then the cdf of (U_1, U_2) is given by:*

$$
\begin{aligned}
F(u_1, u_2) &= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(m_1)_k (m_2 - m_1)_l}{(m_2 + k)_l \, k! l!} \frac{\rho^{k+l}}{\Gamma(m_1 + k)\Gamma(m_2 + k)} \\
&\quad \times \int_0^{\infty} \gamma\left(m_1 + k, \frac{t}{\Omega_1 (1-\rho)} u_1\right) \gamma\left(m_2 + k + l, \frac{t}{\Omega_2 (1-\rho)} u_2\right) \mathcal{W}(t)\, dt \qquad (4.32)
\end{aligned}
$$

where $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, $m_2 \geq m_1$ and where $\gamma(\cdot, \cdot)$ denotes the lower incomplete gamma function (see Result C.6).

Proof. *Consider from (4.8):*

$$\begin{aligned} F(u_1,u_2) &= \int_0^{u_1}\int_0^{u_2} f(c_1,c_2)\,dc_1dc_2 \\ &= \int_0^{u_1}\int_0^{u_2}\int_0^{\infty} f(c_1,c_2|t)\,\mathcal{W}(t)\,dtdc_1dc_2 \\ &= \int_0^{\infty}\int_0^{u_1}\int_0^{u_2} f(c_1,c_2|t)\,dc_1dc_2\mathcal{W}(t)\,dt \\ &= \int_0^{\infty} F(u_1,u_2|t)\,\mathcal{W}(t)\,dt. \end{aligned} \tag{4.33}$$

From (4.33) and (4.13):

$$\begin{aligned} F(u_1,u_2|t) &= \int_0^{u_1}\int_0^{u_2} f(c_1,c_2|t)\,dc_1dc_2 \\ &= \int_0^{u_1}\int_0^{u_2}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k} \\ &\quad\times\frac{c_1^{m_1+k-1}}{\Gamma(m_1+k)}\frac{c_2^{m_2+k-1}}{\Gamma(m_2+k)}\exp\left[-t\left(\frac{c_1}{\Omega_1(1-\rho)}\right)\right]\exp\left[-t\left(\frac{c_2}{\Omega_2(1-\rho)}\right)\right] \\ &\quad\times\,{}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}c_2\right)t^{m_1+m_2+2k}dc_1dc_2 \\ &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times\int_0^{u_1}c_1^{m_1+k-1}\exp\left[-t\left(\frac{c_1}{\Omega_1(1-\rho)}\right)\right]dc_1 \\ &\quad\times\int_0^{u_2}c_2^{m_2+k-1}\exp\left[-t\left(\frac{c_2}{\Omega_2(1-\rho)}\right)\right]\,{}_1F_1\left(m_2-m_1,m_2+k;\frac{t\rho}{\Omega_2(1-\rho)}c_2\right)dc_2 \\ &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times V_1\cdot V_2. \end{aligned} \tag{4.34}$$

Using Result C.23, consider from (4.34):

$$\begin{aligned} V_1 &= \int_0^{u_1}c_1^{m_1+k-1}\exp\left[-c_1\left(\frac{t}{\Omega_1(1-\rho)}\right)\right]dc_1 \\ &= \frac{\gamma\left(m_1+k,\frac{t}{\Omega_1(1-\rho)}u_1\right)}{\left(\frac{t}{\Omega_1(1-\rho)}\right)^{m_1+k}} \\ &= t^{-(m_1+k)}\left(\frac{1}{\Omega_1(1-\rho)}\right)^{-(m_1+k)}\gamma\left(m_1+k,\frac{t}{\Omega_1(1-\rho)}u_1\right) \end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
V_2 &= \int_0^{u_2} c_2^{m_2+k-1} \exp\left[-c_2\left(\frac{t}{\Omega_2(1-\rho)}\right)\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{t\rho}{\Omega_2(1-\rho)}c_2\right) dc_2 \\
&= \sum_{l=0}^{\infty} \frac{(m_2-m_1)_l}{(m_2+k)_l\, l!}\left(\frac{t\rho}{\Omega_2(1-\rho)}\right)^l \int_0^{u_2} c_2^{m_2+k+l-1} \exp\left[-c_2\left(\frac{t}{\Omega_2(1-\rho)}\right)\right] dc_2 \\
&= \sum_{l=0}^{\infty} \frac{(m_2-m_1)_l}{(m_2+k)_l\, l!}\left(\frac{t\rho}{\Omega_2(1-\rho)}\right)^l \frac{\gamma\left(m_2+k+l, \frac{t}{\Omega_2(1-\rho)}u_2\right)}{\left(\frac{t}{\Omega_2(1-\rho)}\right)^{m_2+k+l}} \\
&= t^{-(m_2+k)}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{-(m_2+k)} \sum_{l=0}^{\infty} \frac{(m_2-m_1)_l\, \rho^l}{(m_2+k)_l\, l!}\gamma\left(m_2+k+l, \frac{t}{\Omega_2(1-\rho)}u_2\right). \qquad (4.36)
\end{aligned}$$

Substituting (4.35) and (4.36) into (4.34) leaves:

$$\begin{aligned}
F(u_1,u_2|t) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\,\rho^k}{k!}\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad \times t^{-(m_1+k)}\left(\frac{1}{\Omega_1(1-\rho)}\right)^{-(m_1+k)}\gamma\left(m_1+k, \frac{t}{\Omega_1(1-\rho)}u_1\right) \\
&\quad \times t^{-(m_2+k)}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{-(m_2+k)}\sum_{l=0}^{\infty}\frac{(m_2-m_1)_l\,\rho^l}{(m_2+k)_l\, l!}\gamma\left(m_2+k+l, \frac{t}{\Omega_2(1-\rho)}u_2\right) \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad \times\gamma\left(m_1+k, \frac{t}{\Omega_1(1-\rho)}u_1\right)\gamma\left(m_2+k+l, \frac{t}{\Omega_2(1-\rho)}u_2\right). \qquad (4.37)
\end{aligned}$$

Substituting (4.37) into (4.33) leaves the final result. ■

Remark 4.8 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.32) simplifies to:*

$$\begin{aligned}
F_{normal}(u_1,u_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad \times\gamma\left(m_1+k, \frac{1}{\Omega_1(1-\rho)}u_1\right)\gamma\left(m_2+k+l, \frac{1}{\Omega_2(1-\rho)}u_2\right) \qquad (4.38)
\end{aligned}$$

where $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$.

Remark 4.9 *By choosing $\mathcal{W}(t)$ as (1.6), (4.32) simplifies to:*

$$\begin{aligned}
F_t(u_1,u_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad \times\int_0^{\infty}\gamma\left(m_1+k, \frac{t}{\Omega_1(1-\rho)}u_1\right)\gamma\left(m_2+k+l, \frac{t}{\Omega_2(1-\rho)}u_2\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad \times\int_0^{\infty}\gamma\left(m_1+k, \frac{t}{\Omega_1(1-\rho)}u_1\right)\gamma\left(m_2+k+l, \frac{t}{\Omega_2(1-\rho)}u_2\right)t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \qquad (4.39)
\end{aligned}$$

where $m_1, m_2, \Omega_1, \Omega_2, v > 0$, $-1 < \rho < 1$, *and* $m_2 \geq m_1$.

Remark 4.10 *Note that (4.39) can be rewritten as follows:*

$$\begin{aligned} F_t(u_1,u_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times\int_0^{\infty}\gamma\left(m_1+k,\frac{t}{\Omega_1(1-\rho)}u_1\right)\gamma\left(m_2+k+l,\frac{t}{\Omega_2(1-\rho)}u_2\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \\ &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times E_T\left(\gamma\left(m_1+k,\frac{t}{\Omega_1(1-\rho)}u_1\right)\gamma\left(m_2+k+l,\frac{t}{\Omega_2(1-\rho)}u_2\right)\right) \end{aligned}$$

where $T \sim Gamma\left(\frac{v}{2},\frac{v}{2}\right)$ *(see Result C.2).*

4.2.5 Pdf of noncentral counterpart

In this section, a bivariate *noncentral* gamma type II is proposed. This methodology emanates from Ferreira et. al. (2016), and is described below. In the univariate setting, Chen (2005) describes the noncentral gamma distribution as an infinite sum of a weighted central gamma distribution with Poisson weights:

$$\begin{aligned} f^{nc}(x) &= \sum_{k=0}^{\infty}\frac{x^{m+k-1}\exp\left(-\frac{1}{\Omega}x\right)}{\Omega^{m+k}\Gamma(m+k)}\frac{\exp\left(-\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)^k}{k!} \\ &= \sum_{k=0}^{\infty}f(x|k)\,g(k),\quad x>0 \end{aligned} \tag{4.40}$$

where $f(x|k)$ denotes a conditional central gamma pdf with shape and scale parameters $m, \Omega > 0$ respectively (see Result C.2), $g(k)$ denotes the Poisson probability mass function (pmf) with expected value $\frac{\theta}{2} > 0$ as weights (see Result C.3), and $f^{nc}(x)$ represents a noncentral gamma pdf with shape and scale parameters $m, \Omega > 0$ respectively and noncentrality parameter $\theta > 0$. Using Result C.11, the Laplace transform of the noncentral gamma distribution (see (4.40)) with noncentrality parameter $\theta > 0$ is given by:

$$\begin{aligned} \mathcal{L}^{nc}(t) &= \int_0^{\infty}\exp(-tx)\sum_{k=0}^{\infty}\frac{x^{m+k-1}\exp\left(-\frac{1}{\Omega}x\right)}{\Omega^{m+k}\Gamma(m+k)}\frac{\exp\left(-\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)^k}{k!}dx \\ &= \sum_{k=0}^{\infty}(1+\Omega t)^{-(m+k)}\frac{\exp\left(-\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right)^k}{k!} \\ &\quad\left(=\sum_{k=0}^{\infty}\mathcal{L}(t|k)\,g(k)\right) \\ &= (1+\Omega t)^{-m}\sum_{k=0}^{\infty}\frac{\exp\left(-\frac{\theta}{2}\right)\left(\frac{\theta}{2(1+\Omega t)}\right)^k}{k!} \\ &= (1+\Omega t)^{-m}\exp\left(-\frac{\theta}{2}\right)\exp\left(\frac{\theta}{2(1+\Omega t)}\right) \end{aligned}$$

where $\mathcal{L}(t|k) = (1+\Omega t)^{-(m+k)}$ denotes the Laplace transform of a conditional central gamma distribution with shape and scale parameters $m, \Omega > 0$ respectively (see Result C.2). This representation of weighted conditional Laplace transforms with Poisson weights is used to define a bivariate noncentral gamma type II distribution, stemming from the approach of Ferreira et. al. (2016).

Lemma 4.2.2 *A bivariate noncentral gamma type II pdf can be obtained from a conditional bivariate central gamma type II distribution with pdf $f(u_1,u_2|k_1,k_2)$ in the following manner:*

$$f^{nc}(u_1,u_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f(u_1,u_2|k_1,k_2)g(k_1)g(k_2) \tag{4.41}$$

where $g(k_i) = \frac{e^{-\frac{\theta_i}{2}}\left(\frac{\theta_i}{2}\right)^{k_i}}{k_i!}, i = 1,2$ are compounding factors - in this case, Poisson probabilities where θ_v denotes the noncentrality parameters, and $f(u_1,u_2|k_1,k_2)$ the pdf of a bivariate gamma type II distribution. In this regard, the Laplace transform of (U_1,U_2) in the noncentral case can be written as:

$$\mathcal{L}^{nc}(u_1,u_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathcal{L}(u_1,u_2|k_1,k_2)\,g(k_1)g(k_2). \tag{4.42}$$

Note that in this section, the methodology of Ferreira et. al. (2016) is employed, in conjunction with Lemma 4.2.2.

Consider (4.13). By conditioning on the shape parameters, m_1 and m_2:

$$\begin{aligned} f(u_1,u_2|t,k_1,k_2) &= (1-\rho)^{m_2+k_2}\sum_{k=0}^{\infty}\frac{(m_1+k_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k+k_1}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k+k_2} \\ &\quad\times\frac{u_1^{m_1+k+k_1-1}}{\Gamma(m_1+k+k_1)}\frac{u_2^{m_2+k+k_2-1}}{\Gamma(m_2+k+k_2)}\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}\right)\right] \\ &\quad\times {}_1F_1\left(m_2+k_2-m_1-k_1, m_2+k+k_2; \frac{t\rho}{\Omega_2(1-\rho)}u_2\right)t^{m_1+m_2+2k+k_1+k_2}. \end{aligned} \tag{4.43}$$

This means that the pdf of a bivariate noncentral gamma type II distribution can be constructed as:

$$f^{nc}(u_1,u_2) = \int_0^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f(u_1,u_2|t,k_1,k_2)g(k_1)g(k_2)\mathcal{W}(t)\,dt. \tag{4.44}$$

where $f(u_1,u_2|t,k_1,k_2)$ is given as in (4.43), and $g(k_i)$ denotes the Poisson pmf (see Result C.3) with expected value $\frac{\theta_i}{2} > 0$. Consider now the Laplace transform of (4.43).

By calculating the Laplace transform, the noncentral aspect of this bivariate gamma type II distribution (4.8) becomes clear:

$$\begin{aligned}
&\mathcal{L}\left(s_1, s_2|t, k_1, k_2\right) \\
= &\int_0^\infty\int_0^\infty \exp\left[-\left(s_1u_2+s_2u_2\right)\right]\left(1-\rho\right)^{m_2+k_2}\sum_{k=0}^{\infty}\frac{\left(m_1+k_1\right)_k}{k!}\rho^k\left(\frac{1}{\Omega_1\left(1-\rho\right)}\right)^{m_1+k+k_1}\left(\frac{1}{\Omega_2\left(1-\rho\right)}\right)^{m_2+k+k_2} \\
&\times\frac{u_1^{m_1+k+k_1-1}}{\Gamma\left(m_1+k+k_1\right)}\frac{u_2^{m_2+k+k_2-1}}{\Gamma\left(m_2+k+k_2\right)}\exp\left[-t\left(\frac{u_1}{\Omega_1\left(1-\rho\right)}+\frac{u_2}{\Omega_2\left(1-\rho\right)}\right)\right] \\
&\times {}_1F_1\left(m_2+k_2-m_1-k_1, m_2+k+k_2; \frac{t\rho}{\Omega_2\left(1-\rho\right)}u_2\right)t^{m_1+m_2+2k+k_1+k_2}du_1du_2 \\
= &\left(1-\rho\right)^{m_2+k_2}\sum_{k=0}^{\infty}\frac{\left(m_1+k_1\right)_k}{k!}\rho^k\left(\frac{t}{\Omega_1\left(1-\rho\right)}\right)^{m_1+k+k_1}\left(\frac{t}{\Omega_2\left(1-\rho\right)}\right)^{m_2+k+k_2}\frac{1}{\Gamma\left(m_1+k+k_1\right)\Gamma\left(m_2+k+k_2\right)} \\
&\times\int_0^\infty u_1^{m_1+k+k_1-1}\exp\left[-u_1\left(\frac{t}{\Omega_1\left(1-\rho\right)}+s_1\right)\right]du_1\int_0^\infty u_2^{m_2+k+k_2-1}\exp\left[-u_2\left(\frac{t}{\Omega_2\left(1-\rho\right)}+s_2\right)\right] \\
&\times {}_1F_1\left(m_2+k_2-m_1-k_1, m_2+k+k_2; \frac{t\rho}{\Omega_2\left(1-\rho\right)}u_2\right)du_2 \\
= &\left(1-\rho\right)^{m_2+k_2}\sum_{k=0}^{\infty}\frac{\left(m_1+k_1\right)_k}{k!}\rho^k\left(\frac{t}{\Omega_1\left(1-\rho\right)}\right)^{m_1+k+k_1}\left(\frac{t}{\Omega_2\left(1-\rho\right)}\right)^{m_2+k+k_2}\frac{1}{\Gamma\left(m_1+k+k_1\right)\Gamma\left(m_2+k+k_2\right)} \\
&\times V_1\cdot V_2. \qquad (4.45)
\end{aligned}$$

From (4.45) and using Result C.22, see that:

$$\begin{aligned}
V_1 &= \int_0^\infty u_1^{m_1+k+k_1-1}\exp\left[-u_1\left(\frac{t}{\Omega_1\left(1-\rho\right)}+s_1\right)\right]du_1 \qquad (4.46) \\
&= \frac{\Gamma\left(m_1+k+k_1\right)}{\left(\frac{t}{\Omega_1(1-\rho)}+s_1\right)^{m_1+k+k_1}}
\end{aligned}$$

and by using Result C.10 and Result C.22:

$$\begin{aligned}
V_2 &= \int_0^\infty u_2^{m_2+k+k_2-1}\exp\left[-u_2\left(\frac{t}{\Omega_2\left(1-\rho\right)}+s_2\right)\right]{}_1F_1\left(m_2+k_2-m_1-k_1, m_2+k+k_2; \frac{t\rho}{\Omega_2\left(1-\rho\right)}u_2\right)du_2 \\
&= \sum_{l=0}^{\infty}\frac{\left(m_2+k_2-m_1-k_1\right)_l}{\left(m_2+k+k_2\right)_l l!}\left(\frac{t\rho}{\Omega_2\left(1-\rho\right)}\right)^l\int_0^\infty u_2^{m_2+k+k_2+l-1}\exp\left[-u_2\left(\frac{t}{\Omega_2\left(1-\rho\right)}+s_2\right)\right]du_2 \\
&= \sum_{l=0}^{\infty}\frac{\left(m_2+k_2-m_1-k_1\right)_l}{\left(m_2+k+k_2\right)_l l!}\left(\frac{t\rho}{\Omega_2\left(1-\rho\right)}\right)^l\frac{\Gamma\left(m_2+k+k_2+l\right)}{\left(\frac{t}{\Omega_2(1-\rho)}+s_2\right)^{m_2+k+k_2+l}}. \qquad (4.47)
\end{aligned}$$

Substituting (4.46) and (4.47) into (4.45):

$$
\begin{aligned}
&\mathcal{L}(s_1, s_2|t, k_1, k_2) \\
= \; & (1-\rho)^{m_2+k_2} \sum_{k=0}^{\infty} \frac{(m_1+k_1)_k}{k!} \rho^k \left(\frac{t}{\Omega_1(1-\rho)} \right)^{m_1+k+k_1} \left(\frac{t}{\Omega_2(1-\rho)} \right)^{m_2+k+k_2} \frac{1}{\Gamma(m_1+k+k_1)\Gamma(m_2+k+k_2)} \\
& \times \frac{\Gamma(m_1+k+k_1)}{\left(\frac{t}{\Omega_1(1-\rho)} + s_1 \right)^{m_1+k+k_1}} \sum_{l=0}^{\infty} \frac{(m_2+k_2-m_1-k_1)_l}{(m_2+k+k_2)_l \, l!} \left(\frac{t\rho}{\Omega_2(1-\rho)} \right)^l \frac{\Gamma(m_2+k+k_2+l)}{\left(\frac{t}{\Omega_2(1-\rho)} + s_2 \right)^{m_2+k+k_2+l}} \\
= \; & (1-\rho)^{m_2+k_2} \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right)^{m_1+k_1} \left(\frac{\frac{t}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^{m_2+k_2} \sum_{k=0}^{\infty} \frac{(m_1+k_1)_k}{k!} \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right)^k \\
& \times \sum_{l=0}^{\infty} \frac{(m_2+k_2-m_1-k_1)_l}{(m_2+k+k_2)_l \, l!} \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^{k+l} \frac{\Gamma(m_1+k+k_1)\Gamma(m_2+k+k_2+l)}{\Gamma(m_1+k+k_1)\Gamma(m_2+k+k_2)} \\
= \; & (1-\rho)^{m_2+k_2} \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right)^{m_1+k_1} \left(\frac{\frac{t}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^{m_2+k_2} \sum_{k=0}^{\infty} \frac{(m_1+k_1)_k}{k!} \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right)^k \\
& \times \sum_{l=0}^{\infty} \frac{(m_2+k_2-m_1-k_1)_l}{\frac{\Gamma(m_2+k+k_2+l)}{\Gamma(m_2+k+k_2)} l!} \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^{k+l} \frac{\Gamma(m_2+k+k_2+l)}{\Gamma(m_2+k+k_2)} \\
= \; & (1-\rho)^{m_2+k_2} \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right)^{m_1+k_1} \left(\frac{\frac{t}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^{m_2+k_2} \sum_{k=0}^{\infty} \frac{(m_1+k_1)_k}{k!} \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right)^k \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^k \\
& \sum_{l=0}^{\infty} \frac{(m_2+k_2-m_1-k_1)_l}{l!} \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^l . \qquad (4.48)
\end{aligned}
$$

Using Result C.12, consider the first summation in (4.48):

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(m_1+k_1)_k}{k!} \left[\left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right) \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right) \right]^k \\
= \; & {}_1F_0 \left(m_1+k_1; \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right) \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right) \right) \\
= \; & \left(1 - \left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)} + s_1} \right) \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right) \right)^{-(m_1+k_1)} \qquad (4.49)
\end{aligned}
$$

and consider the second summation in (4.48):

$$
\begin{aligned}
& \sum_{l=0}^{\infty} \frac{(m_2+k_2-m_1-k_1)_l}{l!} \left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^l \\
= \; & {}_1F_0 \left(m_2+k_2-m_1-k_1; \frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right) \\
= \; & \left(1 - \frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)} + s_2} \right)^{-(m_2+k_2-m_1-k_1)} . \qquad (4.50)
\end{aligned}
$$

Substituting (4.49) and (4.50) into (4.48):

$$\begin{aligned}
&\mathcal{L}(s_1, s_2|t, k_1, k_2)\\
=\;& (1-\rho)^{m_2+k_2}\left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)}+s_1}\right)^{m_1+k_1}\left(\frac{\frac{t}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)}+s_2}\right)^{m_2+k_2}\\
&\times\left(1-\left(\frac{\frac{t}{\Omega_1(1-\rho)}}{\frac{t}{\Omega_1(1-\rho)}+s_1}\right)\left(\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)}+s_2}\right)\right)^{-(m_1+k_1)}\left(1-\frac{\frac{t\rho}{\Omega_2(1-\rho)}}{\frac{t}{\Omega_2(1-\rho)}+s_2}\right)^{-(m_2+k_2-m_1-k_1)}\\
=\;& (1-\rho)^{m_2+k_2}\left(\frac{\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-m_1}\left(\frac{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}{\frac{1}{\Omega_2t^{-1}(1-\rho)}}\right)^{-m_2}\left(\frac{\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-k_1}\left(\frac{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}{\frac{1}{\Omega_2t^{-1}(1-\rho)}}\right)^{-k_2}\\
&\times\left(1-\left(\frac{\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-1}\left(\frac{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}\right)^{-1}\right)^{-(m_1+k_1)}\left(1-\frac{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}\right)^{-(m_2+k_2-m_1-k_1)}\\
=\;& (1-\rho)^{m_2+k_2}\left(1+\frac{s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-m_1}\left(1+\frac{s_2}{\frac{1}{\Omega_2t^{-1}(1-\rho)}}\right)^{-m_2}\left(1+\frac{s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-k_1}\left(1+\frac{s_2}{\frac{1}{\Omega_2t^{-1}(1-\rho)}}\right)^{-k_2}\\
&\times\left(1-\left(1+\frac{s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-1}\left(\frac{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}\right)^{-1}\right)^{-(m_1+k_1)}\left(1-\frac{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}\right)^{-(m_2+k_2-m_1-k_1)}. \quad (4.51)
\end{aligned}$$

See that (4.51) can be seperated as follows:

$$\begin{aligned}
& (1-\rho)^{m_2}\left(1+\frac{s_1}{\frac{1}{\Omega_1t^{-1}(1-\rho)}}\right)^{-m_1}\left(1+\frac{s_2}{\frac{1}{\Omega_2t^{-1}(1-\rho)}}\right)^{-m_2}\\
&\times\left(1-\left(\frac{\frac{1}{\Omega_1t^{-1}(1-\rho)}}{\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1}\right)\left(\frac{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}\right)\right)^{-m_1}\left(\frac{\frac{1-\rho}{\Omega_2t^{-1}(1-\rho)}+s_2}{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}\right)^{-(m_2-m_1)}\\
=\;& (1-\rho)^{m_2}\left(1+s_1\Omega_1t^{-1}(1-\rho)\right)^{-m_1}\left(1+s_2\Omega_2t^{-1}(1-\rho)\right)^{-m_2}\\
&\times\left(1-\left(\frac{\frac{1}{\Omega_1t^{-1}(1-\rho)}}{\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1}\right)\left(\frac{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}\right)\right)^{-m_1}\left(\frac{1-\rho}{\Omega_2t^{-1}(1-\rho)}+s_2\right)^{-m_2+m_1}\left(\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2\right)^{m_2-m_1}\\
=\;& (1-\rho)^{m_2}\left(\Omega_1t^{-1}(1-\rho)\right)^{-m_1}\left(\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1\right)^{-m_1}\left(\Omega_2t^{-1}(1-\rho)\right)^{-m_2}\\
&\times\left(1-\left(\frac{\frac{1}{\Omega_1t^{-1}(1-\rho)}}{\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1}\right)\left(\frac{\frac{\rho}{\Omega_2t^{-1}(1-\rho)}}{\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2}\right)\right)^{-m_1}\left(\frac{(1-\rho)}{\Omega_2t^{-1}(1-\rho)}+s_2\right)^{-m_2+m_1}\left(\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2\right)^{-m_1}\\
=\;& (1-\rho)^{-m_1}\left(\Omega_1t^{-1}\right)^{-m_1}\left(\Omega_2t^{-1}\right)^{-m_2}\\
&\times\left(\left(\frac{1}{\Omega_1t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2t^{-1}(1-\rho)}+s_2\right)-\left(\frac{\rho}{\Omega_1t^{-1}\Omega_2t^{-1}(1-\rho)^2}\right)\right)^{-m_1}\left(\frac{1}{\Omega_2t^{-1}}+s_2\right)^{-(m_2-m_1)} \quad (4.52)
\end{aligned}$$

and

$$
\begin{aligned}
& (1-\rho)^{k_2}\left(1+\frac{s_1}{\frac{1}{\Omega_1 t^{-1}(1-\rho)}}\right)^{-k_1}\left(1+\frac{s_2}{\frac{1}{\Omega_2 t^{-1}(1-\rho)}}\right)^{-k_2} \\
& \times\left(1-\left(\frac{\frac{1}{\Omega_1 t^{-1}(1-\rho)}}{\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1}\right)\left(\frac{\frac{\rho}{\Omega_2 t^{-1}(1-\rho)}}{\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2}\right)\right)^{-k_1}\left(\frac{\frac{1-\rho}{\Omega_2 t^{-1}(1-\rho)}+s_2}{\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2}\right)^{-(k_2-k_1)} \\
= & (1-\rho)^{k_2}\left(1+s_1\Omega_1 t^{-1}(1-\rho)\right)^{-k_1}\left(1+s_2\Omega_2 t^{-1}(1-\rho)\right)^{-k_2} \\
& \times\left(1-\left(\frac{\frac{1}{\Omega_1 t^{-1}(1-\rho)}}{\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1}\right)\left(\frac{\frac{\rho}{\Omega_2 t^{-1}(1-\rho)}}{\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2}\right)\right)^{-k_1}\left(\frac{1-\rho}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)^{-k_2+k_1}\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)^{k_2-k_1} \\
= & (1-\rho)^{k_2}\left(\Omega_1 t^{-1}(1-\rho)\right)^{-k_1}\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)^{-k_1}\left(\Omega_2 t^{-1}(1-\rho)\right)^{-k_2} \\
& \times\left(1-\left(\frac{\frac{1}{\Omega_1 t^{-1}(1-\rho)}}{\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1}\right)\left(\frac{\frac{\rho}{\Omega_2 t^{-1}(1-\rho)}}{\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2}\right)\right)^{-k_1}\left(\frac{(1-\rho)}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)^{-k_2+k_1}\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)^{-k_1} \\
= & (1-\rho)^{-k_1}\left(\Omega_1 t^{-1}\right)^{-k_1}\left(\Omega_2 t^{-1}\right)^{-k_2} \\
& \times\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\left(\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)\right)^{-k_1}\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)^{-(k_2-k_1)} . \qquad (4.53)
\end{aligned}
$$

By using Result C.11, and by substituting (4.52) into (4.42), it can be observed that:

$$
\begin{aligned}
& \sum_{k_1=0}^{\infty}(1-\rho)^{-k_1}\left(\Omega_1 t^{-1}\right)^{-k_1}\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\left(\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)\right)^{-k_1} \\
& \times\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)^{k_1}\frac{\exp\left(-\frac{\theta_1}{2}\right)\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \\
= & \exp\left(-\frac{\theta_1}{2}\right)\sum_{k_1=0}^{\infty}(1-\rho)^{-k_1}\left(\Omega_1 t^{-1}\right)^{-k_1} \\
& \times\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\left(\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)\right)^{-k_1}\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)^{k_1}\frac{\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \\
= & \exp\left(-\frac{\theta_1}{2}\right)\sum_{k_1=0}^{\infty}\left(\frac{\frac{1}{\Omega_2 t^{-1}}+s_2}{(1-\rho)\left(\Omega_1 t^{-1}\right)\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)}\right)^{k_1}\frac{\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \\
= & \exp\left(-\frac{\theta_1}{2}\right)\sum_{k_1=0}^{\infty}\frac{1}{k_1!}\left(\frac{\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\frac{\theta_1}{2}}{(1-\rho)\left(\Omega_1 t^{-1}\right)\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)}\right)^{k_1} \\
= & \exp\left(-\frac{\theta_1}{2}\right)\exp\left(\frac{\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\frac{\theta_1}{2}}{(1-\rho)\left(\Omega_1 t^{-1}\right)\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)}\right) \qquad (4.54)
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{k_2=0}^{\infty}\left(\Omega_2 t^{-1}\right)^{-k_2}\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)^{-k_2}\frac{\exp\left(-\frac{\theta_2}{2}\right)\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\\
=&\ \exp\left(-\frac{\theta_2}{2}\right)\sum_{k_2=0}^{\infty}\frac{1}{\left(\left(\Omega_2 t^{-1}\right)\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\right)^{k_2}}\frac{\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\\
=&\ \exp\left(-\frac{\theta_2}{2}\right)\sum_{k_2=0}^{\infty}\frac{\left(\frac{\theta_2}{2\left(\left(\Omega_2 t^{-1}\right)\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\right)}\right)^{k_2}}{k_2!}\\
=&\ \exp\left(-\frac{\theta_2}{2}\right)\exp\left(\frac{\theta_2}{2\left(\left(\Omega_2 t^{-1}\right)\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\right)}\right).
\end{aligned}\tag{4.55}$$

Substituting (4.52), (4.54), and (4.55) into (4.51), leaves:

$$\begin{aligned}
\mathcal{L}\left(s_1,s_2|t\right) =&\ (1-\rho)^{-m_1}\left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}\\
&\times\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\left(\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)\right)^{-m_1}\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)^{-(m_2-m_1)}\\
&\times\exp\left(-\frac{\theta_1}{2}\right)\exp\left(\frac{\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\frac{\theta_1}{2}}{(1-\rho)\left(\Omega_1 t^{-1}\right)\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)}\right)\\
&\times\exp\left(-\frac{\theta_2}{2}\right)\exp\left(\frac{\theta_2}{2\left(\left(\Omega_2 t^{-1}\right)\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\right)}\right).
\end{aligned}\tag{4.56}$$

Finally, substituting (4.56) into (4.1) leaves:

$$\begin{aligned}
\mathcal{L}^{nc}\left(s_1,s_2\right) =&\ \int_0^{\infty}\mathcal{L}\left(s_1,s_2|t\right)\mathcal{W}(t)\,dt\\
=&\ \int_0^{\infty}(1-\rho)^{-m_1}\left(\Omega_1 t^{-1}\right)^{-m_1}\left(\Omega_2 t^{-1}\right)^{-m_2}\\
&\times\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\left(\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)\right)^{-m_1}\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)^{-(m_2-m_1)}\\
&\times\exp\left(-\frac{\theta_1}{2}\right)\exp\left(\frac{\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\theta_1}{2(1-\rho)\left(\Omega_1 t^{-1}\right)\left(\left(\frac{1}{\Omega_1 t^{-1}(1-\rho)}+s_1\right)\left(\frac{1}{\Omega_2 t^{-1}(1-\rho)}+s_2\right)-\frac{\rho}{\Omega_1 t^{-1}\Omega_2 t^{-1}(1-\rho)^2}\right)}\right)\\
&\times\exp\left(-\frac{\theta_2}{2}\right)\exp\left(\frac{\theta_2}{2\left(\left(\Omega_2 t^{-1}\right)\left(\frac{1}{\Omega_2 t^{-1}}+s_2\right)\right)}\right)\mathcal{W}(t)\,dt
\end{aligned}\tag{4.57}$$

which acts as the Laplace transform of a bivariate noncentral gamma type II distribution with pdf (4.44).

Remark 4.11 *When $\theta_1=\theta_2=0$, see that (4.57) simplifies to the Laplace transform of the bivariate gamma type II distribution (see (4.6)).*

Remark 4.12 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.44) simplifies to:*

$$\begin{aligned} f_{normal}^{nc}(u_1,u_2) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(1-\rho)^{m_2+k_2}\sum_{k=0}^{\infty}\frac{(m_1+k_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k+k_1}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k+k_2} \\ &\times\frac{u_1^{m_1+k+k_1-1}}{\Gamma(m_1+k+k_1)}\frac{u_2^{m_2+k+k_2-1}}{\Gamma(m_2+k+k_2)}\exp\left[-\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}\right)\right] \\ &\times{}_1F_1\left(m_2+k_2-m_1-k_1,m_2+k+k_2;\frac{\rho}{\Omega_2(1-\rho)}u_2\right)\frac{\exp\left(-\frac{\theta_1}{2}\right)\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{\exp\left(-\frac{\theta_2}{2}\right)\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \end{aligned} \tag{4.58}$$

Remark 4.13 *By choosing $\mathcal{W}(t)$ as (1.6) and using Result C.32, (4.44) simplifies to:*

$$\begin{aligned} f_t^{nc}(u_1,u_2) &= \int_0^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(1-\rho)^{m_2+k_2}\sum_{k=0}^{\infty}\frac{(m_1+k_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k+k_1}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k+k_2} \\ &\times\frac{u_1^{m_1+k+k_1-1}}{\Gamma(m_1+k+k_1)}\frac{u_2^{m_2+k+k_2-1}}{\Gamma(m_2+k+k_2)}\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}\right)\right] \\ &\times{}_1F_1\left(m_2+k_2-m_1-k_1,m_2+k+k_2;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)t^{m_1+m_2+2k+k_1+k_2} \\ &\times\frac{\exp\left(-\frac{\theta_1}{2}\right)\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{\exp\left(-\frac{\theta_2}{2}\right)\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\sum_{k=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(1-\rho)^{m_2+k_2}\frac{(m_1+k_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k+k_1}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k+k_2} \\ &\times\frac{u_1^{m_1+k+k_1-1}}{\Gamma(m_1+k+k_1)}\frac{u_2^{m_2+k+k_2-1}}{\Gamma(m_2+k+k_2)}\frac{\exp\left(-\frac{\theta_1}{2}\right)\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{\exp\left(-\frac{\theta_2}{2}\right)\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\ &\times\int_0^{\infty}t^{m_1+m_2+2k+k_1+k_2+\frac{v}{2}-1}\exp\left[-t\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right] \\ &\times{}_1F_1\left(m_2+k_2-m_1-k_1,m_2+k+k_2;\frac{t\rho}{\Omega_2(1-\rho)}u_2\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\sum_{k=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(1-\rho)^{m_2+k_2}\frac{(m_1+k_1)_k}{k!}\rho^k\left(\frac{1}{\Omega_1(1-\rho)}\right)^{m_1+k+k_1}\left(\frac{1}{\Omega_2(1-\rho)}\right)^{m_2+k+k_2} \\ &\times\frac{u_1^{m_1+k+k_1-1}}{\Gamma(m_1+k+k_1)}\frac{u_2^{m_2+k+k_2-1}}{\Gamma(m_2+k+k_2)}\frac{\exp\left(-\frac{\theta_1}{2}\right)\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{\exp\left(-\frac{\theta_2}{2}\right)\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\ &\times\Gamma\left(m_1+m_2+2k+k_1+k_2+\frac{v}{2}\right)\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{-\left(m_1+m_2+2k+k_1+k_2+\frac{v}{2}\right)} \\ &\times{}_2F_1\left(m_2+k_2-m_1-k_1,m_1+m_2+2k+k_1+k_2+\frac{v}{2};\right. \\ &\left. m_2+k+k_2;\frac{\rho}{\Omega_2(1-\rho)\left(\frac{u_1}{\Omega_1(1-\rho)}+\frac{u_2}{\Omega_2(1-\rho)}+\frac{v}{2}\right)}u_2\right). \end{aligned} \tag{4.59}$$

Remark 4.14 *This method of compounding a bivariate gamma distribution on the shape parameter (in this case, m_1 and m_2) with Poisson weights has been proposed and investigated by Ferreira et. al. (2016). In that paper, the authors paid close attention to a special case of a bivariate noncentral gamma distribution in the form of a bivariate noncentral chi-square distribution. In this thesis, the method is further generalised to include the derived bivariate gamma type II distribution (with pdf (4.8)). The methodology Ferreira et. al. (2016) provided* *acts thus as genesis to the body of work contained in this chapter.*

In the following figures the natures of pdfs (4.58) and (4.59) are illustrated for arbitrary choices of parameters.

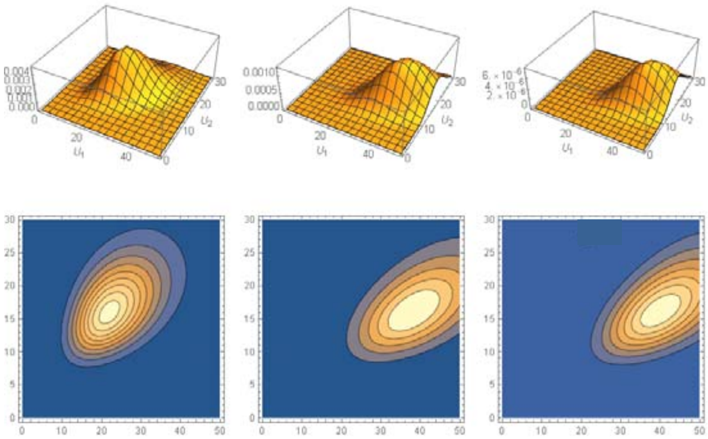


Figure 4.6 Pdf (4.58) for $m_1 = 10$, $m_2 = 12$, $\rho = 0.5$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, $\theta_2 = 0$, and varying $\theta_1 = 0, 20, 40$ (fltr)

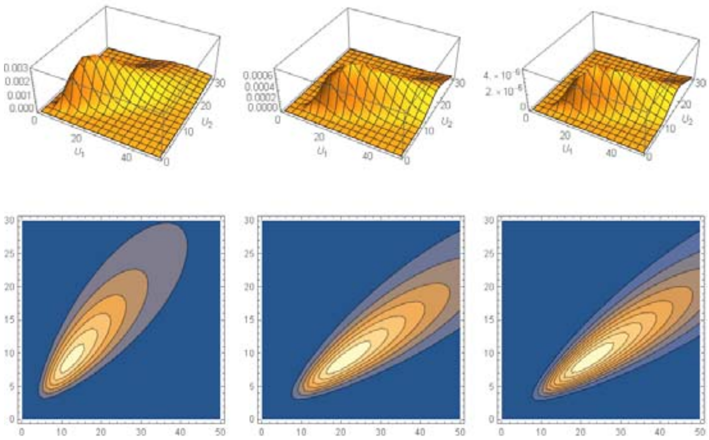


Figure 4.7 Pdf (4.59) for $v = 5$, $m_1 = 10$, $m_2 = 12$, $\rho = 0.5$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, $\theta_2 = 0$, and varying $\theta_1 = 0, 20, 40$ (fltr)

From the above figures, the following can be observed:

- In Figures 4.6 and 4.7 the effect of increasing θ_1 is observable, indicating a shift in the direction of variable U_1.

4.3 Bivariate Weibullised gamma type II distribution

In this section a bivariate Weibullised gamma type II distribution which emanates from (4.8) is proposed and some statistical properties studied. A special case of this bivariate Weibullised gamma type II distribution is of particular interest: when $\beta_1 = \beta_2 = 2$, the distribution is called a bivariate Nakagami *type* II distribution, which may act as a fading distribution within a communications system environment. Thus, the bivariate Weibullised gamma type II distribution is a generalisation of the bivariate gamma type II distribution which contains the bivariate Nakagami type distribution as a special case.

4.3.1 Pdf

In this section a bivariate Weibullised gamma type II distribution is introduced.

Theorem 4.6 *Suppose that* (U_1, U_2) *is bivariate gamma type II distributed with pdf (4.8). The pdf of* (W_1, W_2)*, where* $W_i = \left(\frac{U_i}{m_i}\right)^{\frac{1}{\beta_i}}$ *is given by:*

$$\begin{aligned} f(w_1, w_2) &= \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\ &\times\frac{w_1^{\beta_1 m_1+\beta_1 k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2 m_2+\beta_2 k-1}}{\Gamma(m_2+k)}\int_0^{\infty}t^{m_1+m_2+2k}\exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}\right)\right] \\ &\times\exp\left[-t\left(\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right]\ {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)\mathcal{W}(t)\,dt \end{aligned} \tag{4.60}$$

for $w_1, w_2 > 0$ *and where* $m_1, m_2, \Omega_1, \Omega_2, \beta_1, \beta_2 > 0$, $-1 < \rho < 1$, *and* $m_2 \geq m_1$*. This joint distribution is called a bivariate Weibullised gamma type II distribution.*

Proof. *Consider the transformations* $W_1 = \left(\frac{U_2}{m_2}\right)^{\frac{1}{\beta_1}}$ *and* $W_2 = \left(\frac{U_2}{m_2}\right)^{\frac{1}{\beta_2}}$ *with Jacobian (see Result C.1):*

$$\begin{aligned} J &= J(u_1, u_2 \to w_1, w_2) \\ &= \det\begin{pmatrix} \frac{du_1}{dw_1} & \frac{du_1}{dw_2} \\ \frac{du_2}{dw_1} & \frac{du_2}{dw_2} \end{pmatrix} \\ &= \det\begin{pmatrix} \beta_1 m_1 w_1^{\beta_1-1} & 0 \\ 0 & \beta_2 m_2 w_2^{\beta_2-1} \end{pmatrix} \\ &= \beta_1\beta_2 m_1 m_2 w_1^{\beta_1-1}w_2^{\beta_2-1}. \end{aligned}$$

The pdf of (W_1, W_2) *is obtained from (4.8) by:*

$$\begin{aligned} f(w_1, w_2) &= f\left(m_1 w_1^{\beta_1}, m_2 w_2^{\beta_2}\right)\ |J| \\ &= \int_0^{\infty} f\left(m_1 w_1^{\beta_1}, m_2 w_2^{\beta_2}|t\right)\mathcal{W}(t)\,dt\ \ |J| \\ &= \int_0^{\infty} f\left(m_1 w_1^{\beta_1}, m_2 w_2^{\beta_2}|t\right)\ \ |J|\,\mathcal{W}(t)\,dt \\ &= \int_0^{\infty} f(w_1, w_2|t)\,\mathcal{W}(t)\,dt. \end{aligned} \tag{4.61}$$

The pdf of $(W_1, W_2|t)$ *in (4.61) is obtained from (4.13):*

$$\begin{aligned}
f(w_1, w_2|t) &= f\left(m_1 w_1^{\beta_1}, m_2 w_2^{\beta_2}|t\right) \ |J| \\
&= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{1}{(\Omega_1 t^{-1})(1-\rho)}\right)^{m_1+k} \left(\frac{1}{(\Omega_2 t^{-1})(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{\left(m_1 w_1^{\beta_1}\right)^{m_1+k-1}}{\Gamma(m_1+k)} \frac{\left(m_2 w_2^{\beta_2}\right)^{m_2+k-1}}{\Gamma(m_2+k)} \exp\left[-\frac{m_1 w_1^{\beta_1}}{(\Omega_1 t^{-1})(1-\rho)}\right] \exp\left[-\frac{m_2 w_2^{\beta_2}}{(\Omega_2 t^{-1})(1-\rho)}\right] \\
&\quad \times {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{\rho}{(\Omega_2 t^{-1})(1-\rho)} m_2 w_2^{\beta_2}\right) \beta_1 \beta_2 m_1 m_2 w_1^{\beta_1 - 1} w_2^{\beta_2 - 1} \\
&= \beta_1 \beta_2 (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{\left(w_1^{\beta_1}\right)^{m_1+k-1} w_1^{\beta_1-1}}{\Gamma(m_1+k)} \frac{\left(w_2^{\beta_2}\right)^{m_2+k-1} w_2^{\beta_2-1}}{\Gamma(m_2+k)} \exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}\right)\right] \exp\left[-t\left(\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right] \\
&\quad \times {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{m_2 \rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) t^{m_1+m_2+2k} \\
&= \beta_1 \beta_2 (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{w_1^{\beta_1 m_1 + \beta_1 k - 1}}{\Gamma(m_1+k)} \frac{w_2^{\beta_2 m_2 + \beta_2 k - 1}}{\Gamma(m_2+k)} \exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}\right)\right] \exp\left[-t\left(\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right] \\
&\quad \times {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{m_2 \rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) t^{m_1+m_2+2k}. \qquad (4.62)
\end{aligned}$$

Substituting (4.62) into (4.61) leaves the final result. ■

Remark 4.15 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), (4.60) simplifies to:*

$$\begin{aligned}
f_{normal}(w_1, w_2) &= \beta_1 \beta_2 (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{w_1^{\beta_1 m_1 + \beta_1 k - 1}}{\Gamma(m_1+k)} \frac{w_2^{\beta_2 m_2 + \beta_2 k - 1}}{\Gamma(m_2+k)} \exp\left[-\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}\right] \exp\left[-\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}\right] \\
&\quad \times {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{m_2 \rho}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) \qquad (4.63)
\end{aligned}$$

where $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, *and* $m_2 \geq m_1$.

Remark 4.16 *By choosing $\mathcal{W}(t)$ as (1.6), (4.60) simplifies to:*

$$\begin{aligned}
f_t(w_1,w_2) &= \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{w_1^{\beta_1m_1+\beta_1k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2m_2+\beta_2k-1}}{\Gamma(m_2+k)}\int_0^{\infty}\exp\left[-t\left(\frac{m_1w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right] \\
&\quad\times{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho t}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)t^{m_1+m_2+2k}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}e^{-\frac{vt}{2}}dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{w_1^{\beta_1m_1+\beta_1k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2m_2+\beta_2k-1}}{\Gamma(m_2+k)}\int_0^{\infty}\exp\left[-t\left(\frac{m_1w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right] \\
&\quad\times{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho w_2^{\beta_2}}{\Omega_2(1-\rho)}t\right)t^{m_1+m_2+2k+\frac{v}{2}-1}dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{w_1^{\beta_1m_1+\beta_1k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2m_2+\beta_2k-1}}{\Gamma(m_2+k)}\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)\left(\frac{m_1w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{-\left(m_1+m_2+2k+\frac{v}{2}\right)} \\
&\quad\times{}_2F_1\left(m_2-m_1,m_1+m_2+2k+\frac{v}{2};m_2+k;\frac{\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}}{\left(\frac{m_1w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)}\rho\right)
\end{aligned} \tag{4.64}$$

where $m_1,m_2,\Omega_1,\Omega_2,v>0,\ -1<\rho<1,\ m_2\geq m_1,$ *and* $\left|\frac{\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}}{\left(\frac{m_1w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)}\right|<1.$

Remark 4.17 *Note that (4.64) can be rewritten as follows:*

$$\begin{aligned}
f_t(w_1,w_2) &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\,\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{w_1^{\beta_1 m_1+\beta_1 k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2 m_2+\beta_2 k-1}}{\Gamma(m_2+k)}\int_0^{\infty} t^{m_1+m_2+2k+\frac{v}{2}-1}\exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right] \\
&\quad\times\,{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho w_2^{\beta_2}}{\Omega_2(1-\rho)}t\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\,\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{w_1^{\beta_1 m_1+\beta_1 k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2 m_2+\beta_2 k-1}}{\Gamma(m_2+k)}\frac{\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)}{\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{m_1+m_2+2k+\frac{v}{2}}} \\
&\quad\int_0^{\infty}\frac{\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{m_1+m_2+2k+\frac{v}{2}}}{\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)}t^{m_1+m_2+2k+\frac{v}{2}-1} \\
&\quad\times\exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right]\,{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho w_2^{\beta_2}}{\Omega_2(1-\rho)}t\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\,\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{w_1^{\beta_1 m_1+\beta_1 k-1}}{\Gamma(m_1+k)}\frac{w_2^{\beta_2 m_2+\beta_2 k-1}}{\Gamma(m_2+k)}\frac{\Gamma\left(m_1+m_2+2k+\frac{v}{2}\right)}{\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)^{m_1+m_2+2k+\frac{v}{2}}} \\
&\quad\times E_T\left({}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho w_2^{\beta_2}}{\Omega_2(1-\rho)}t\right)\right)
\end{aligned}$$

where $T\sim Gamma\left(m_1+m_2+2k+\frac{v}{2},\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}+\frac{v}{2}\right)\right)$ *(see Result C.2).*

In the following figures the natures of pdfs (4.63) and (4.64) are illustrated for arbitrary parameters.

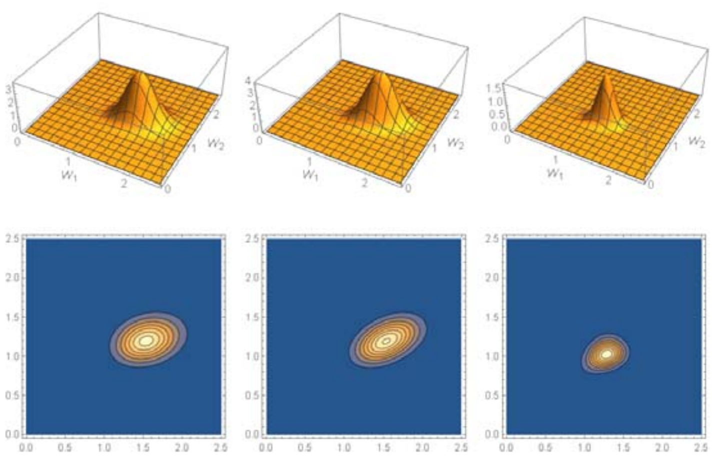


Figure 4.8 Pdf (4.63) for $m_1=10, m_2=12, \beta_1=\beta_2=2, \Omega_1=2.5, \Omega_2=1.5$, and varying $\rho=0.25, 0.5, 0.75$

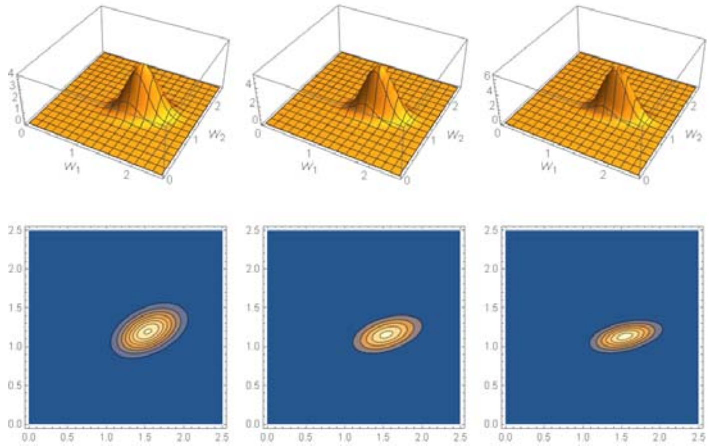


Figure 4.9 Pdf (4.63) for $m_1 = 10, m_2 = 12, \beta_1 = 2, \Omega_1 = 2.5, \Omega_2 = 1.5, \rho = 0.5$, and $\beta_2 = 2, 2.5, 3$

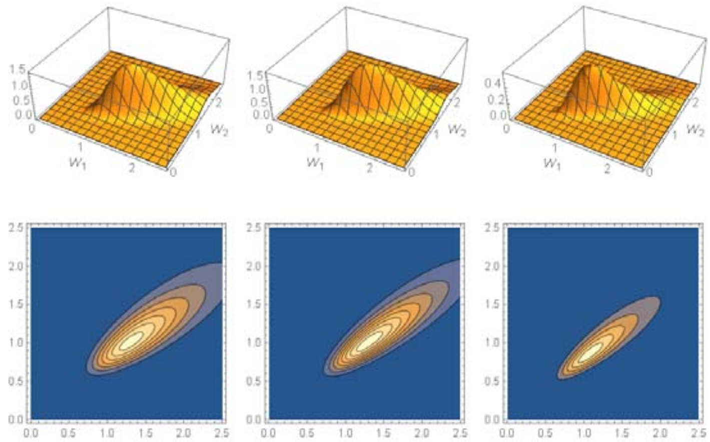


Figure 4.10 Pdf (4.64) for $v = 5$, $m_1 = 10, m_2 = 12, \beta_1 = \beta_2 = 2, \Omega_1 = 2.5, \Omega_2 = 1.5$, and varying $\rho = 0.25, 0.5, 0.75$

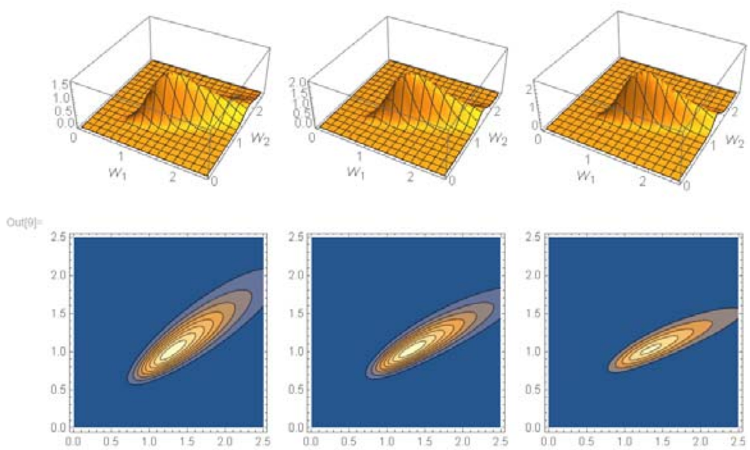


Figure 4.11 Pdf (4.64) for $v = 5, m_1 = 10, m_2 = 12, \beta_1 = 2, \Omega_1 = 2.5, \Omega_2 = 1.5, \rho = 0.5$, and $\beta_2 = 2, 2.5, 3$

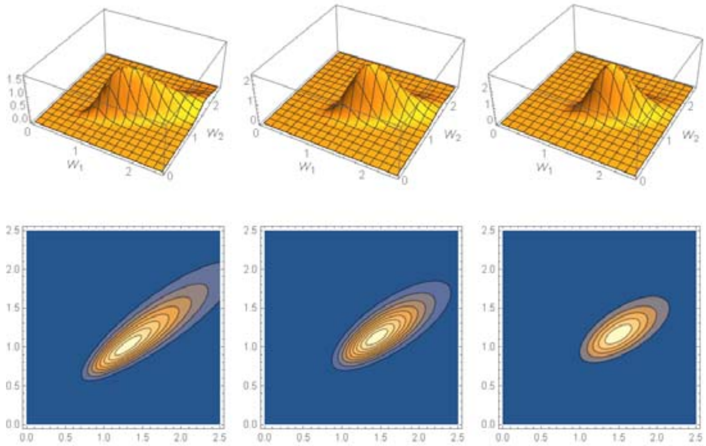


Figure 4.12 Pdf (4.64) for $m_1 = 10, m_2 = 12, \beta_1 = \beta_2 = 2, \Omega_1 = 2.5, \Omega_2 = 1.5, \rho = 0.5$, and varying $v = 5, 15, 30$

From the above figures, the following can be observed:

- In Figures 4.8 and 4.10, the effect of increasing ρ is observable, particularly indicating increased concentration (or correlation) between the variables.
- In Figures 4.9 and 4.11, the effect of increasing β_2 is observed. When β_2 increases, it renders the variable W_2 more "squashed" from the way W_2 has been defined; and this effect is observed in both the normal- and the t case.
- Figure 4.12 indicates the effect on the tails of (4.64) for increasing values of v. As v increases, (4.64) begins to illustrate the characteristics of (4.63) in Figure 4.8 (middle figure). For larger v, this should illustrate the shape of (4.63).

4.3.2 Laplace transform

In this section an expression for the Laplace transform of the bivariate Weibullised gamma type II distribution with pdf (4.60) is derived. The Laplace transform of the distribution under which a fading channel is operating is useful as it can be used to evaluate certain attributes of a communications system, particularly the average bit error rate (see Shankar (2012), Simon and Alouini (2005)).

Theorem 4.7 *Suppose that (W_1, W_2) is bivariate Weibullised gamma type II distributed with pdf (4.60). The Laplace transform of (W_1, W_2) is given by:*

$$
\begin{aligned}
&\mathcal{L}(s_1, s_2) \\
= \;& (2\pi)^{\frac{1}{2}(1-\beta_1)} (1-\rho)^{m_2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(m_1)_k \rho^k}{k! l!} \left(\frac{m_1}{\Omega_1 (1-\rho)} \right)^{m_1+k} \left(\frac{m_2}{\Omega_2 (1-\rho)} \right)^{-\left(\frac{l}{\beta_2}\right)} \frac{\Gamma\left(m_2 + k + \frac{l}{\beta_2}\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
& \times \beta_1^{\beta_1 m_1 + \beta_1 k - \frac{1}{2}} s_1^{-\beta_1 m_1 - \beta_1 k} (-s_2)^l \; {}_2F_1\left(m_2 - m_1, m_2 + k + \frac{l}{\beta_2}, m_2 + k; \rho\right) \\
& \times \int_0^{\infty} t^{m_1 + k - \frac{l}{\beta_2}} G_{\beta_1, 1}^{1, \beta_1} \left(\frac{\beta_1^{\beta_1} t m_1}{s_1^{\beta_1} \Omega_1 (1-\rho)} \;\middle|\; \begin{matrix} 0 \\ \frac{1-\beta_1 m_1 - \beta_1 k}{\beta_1}, \frac{1-\beta_1 m_1 - \beta_1 k + 1}{\beta_1}, \ldots, \frac{1-\beta_1 m_1 - \beta_1 k + \beta_1 - 1}{\beta_1} \end{matrix} \right) \mathcal{W}(t)\, dt
\end{aligned}
\tag{4.65}
$$

where $m_1, m_2, \Omega_1, \Omega_2, \beta_1, \beta_2 > 0$, $-1 < \rho < 1$, $m_2 \geq m_1$, *and* $G_{p,q}^{m,n}\left(x\middle|{}^{a_1,\ldots,a_p}_{b_1,\ldots,b_p}\right)$ *denotes Meijer's G function (see Result C.20).*

Proof. *From (4.60) and (4.61), the Laplace transform of* (W_1, W_2) *is given by:*

$$\begin{aligned}
\mathcal{L}(s_1, s_2) &= \int_0^\infty \int_0^\infty \exp\left[-s_1 w_2 - s_2 w_2\right] f(w_1, w_2)\, dw_1 dw_2 \\
&= \int_0^\infty \int_0^\infty \exp\left[-s_1 w_2 - s_2 w_2\right] \int_0^\infty f(w_1, w_2|t)\, \mathcal{W}(t)\, dt dw_1 dw_2 \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \exp\left[-s_1 w_2 - s_2 w_2\right] f(w_1, w_2|t)\, dw_1 dw_2 \mathcal{W}(t)\, dt \\
&= \int_0^\infty \mathcal{L}(s_1, s_2|t)\, \mathcal{W}(t)\, dt. \qquad (4.66)
\end{aligned}$$

From (4.66) and using (4.62):

$$\begin{aligned}
&\mathcal{L}(s_1, s_2|t) \\
&= \int_0^\infty \int_0^\infty \exp\left[-(s_1 w_2 + s_2 w_2)\right] f(w_1, w_2|t)\, dw_1 dw_2 \\
&= \int_0^\infty \int_0^\infty \exp\left[-(s_1 w_2 + s_2 w_2)\right] \beta_1 \beta_2 (1-\rho)^{m_2} \sum_{k=0}^\infty \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{w_1^{\beta_1 m_1 + \beta_1 k - 1}}{\Gamma(m_1+k)} \frac{w_2^{\beta_2 m_2 + \beta_2 k - 1}}{\Gamma(m_2+k)} \exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}\right)\right] \exp\left[-t\left(\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right] \\
&\quad \times {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{m_2 \rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) t^{m_1+m_2+2k} dw_1 dw_2 \\
&= \beta_1 \beta_2 (1-\rho)^{m_2} \sum_{k=0}^\infty \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad \times \frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \int_0^\infty w_1^{\beta_1 m_1 + \beta_1 k - 1} \exp\left[-\frac{t m_1}{\Omega_1(1-\rho)} w_1^{\beta_1}\right] \exp\left[-s_1 w_1\right] dw_1 \\
&\quad \times \int_0^\infty w_2^{\beta_2 m_2 + \beta_2 k - 1} \exp\left[-\frac{t m_2}{\Omega_2(1-\rho)} w_2^{\beta_2}\right] {}_1F_1\left(m_2 - m_1, m_2 + k; \frac{m_2 \rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) \exp\left[-s_2 w_2\right] dw_2 \\
&= \beta_1 \beta_2 (1-\rho)^{m_2} \sum_{k=0}^\infty \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} V_1 \cdot V_2 \qquad (4.67)
\end{aligned}$$

Using Result C.31 and Result C.21, consider V_1*:*

$$\begin{aligned}
V_1 &= \int_0^\infty w_1^{\beta_1 m_1+\beta_1 k-1} \exp\left[-\frac{tm_1}{\Omega_1(1-\rho)} w_1^{\beta_1}\right] \exp\left[-s_1 w_1\right] dw_1 \\
&= \int_0^\infty w_1^{\beta_1 m_1+\beta_1 k-1} G_{0,1}^{1,0}\left(\frac{tm_1}{\Omega_1(1-\rho)} w_1^{\beta_1} \mid \begin{array}{c} - \\ 0 \end{array}\right) G_{0,1}^{1,0}\left(s_1 w_1 \mid \begin{array}{c} - \\ 0 \end{array}\right) dw_1 \\
&= s_1^{-(\beta_1 m_1+\beta_1 k)} (2\pi)^{\frac{1}{2}(1-\beta_1)} \beta_1^{\frac{1}{2}+(\beta_1 m_1+\beta_1 k)-1} G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1}\frac{tm_1}{\Omega_1(1-\rho)}}{s_1^{\beta_1}} \mid \begin{array}{l} \Delta(1,0) \\ \Delta(\beta_1, 1-\beta_1 m_1-\beta_1 k) \end{array}\right) \\
&= s_1^{-\beta_1 m_1-\beta_1 k} (2\pi)^{\frac{1}{2}(1-\beta_1)} \beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}} \\
&\quad \times G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1} tm_1}{s_1^{\beta_1}\Omega_1(1-\rho)} \mid \begin{array}{l} 0 \\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1}, \frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1}, \ldots, \frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1} \end{array}\right). \qquad (4.68)
\end{aligned}$$

To obtain an expression for V_2*, consider the transformation* $z = w_2^{\beta_2}$*. Then* $w_2 = z^{\frac{1}{\beta_2}}$*, and* $\frac{dw_2}{dz} = \frac{1}{\beta_2} z^{\frac{1}{\beta_2}-1}$*, along with Result C.11 and Result C.32:*

$$\begin{aligned}
V_2 &= \int_0^\infty w_2^{\beta_2 m_2+\beta_2 k-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} w_2^{\beta_2}\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) \exp\left[-s_2 w_2\right] dw_2 \\
&= \int_0^\infty \left(z^{\frac{1}{\beta_2}}\right)^{\beta_2 m_2+\beta_2 k-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} z\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} z\right) \exp\left[-s_2 z^{\frac{1}{\beta_2}}\right] \frac{1}{\beta_2} z^{\frac{1}{\beta_2}-1} dz \\
&= \frac{1}{\beta_2}\int_0^\infty z^{m_2+k-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} z\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} z\right) \sum_{l=0}^\infty \frac{\left(-s_2 z^{\frac{1}{\beta_2}}\right)^l}{l!} dz \\
&= \frac{1}{\beta_2}\sum_{l=0}^\infty \frac{(-s_2)^l}{l!} \int_0^\infty z^{m_2+k+\frac{l}{\beta_2}-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} z\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} z\right) dz \\
&= \frac{1}{\beta_2}\sum_{l=0}^\infty \frac{(-s_2)^l}{l!} \Gamma\left(m_2+k+\frac{l}{\beta_2}\right) \left(\frac{tm_2}{\Omega_2(1-\rho)}\right)^{-\left(m_2+k+\frac{l}{\beta_2}\right)} {}_2F_1\left(m_2-m_1, m_2+k+\frac{l}{\beta_2}, m_2+k; \frac{\frac{m_2\rho t}{\Omega_2(1-\rho)}}{\frac{tm_2}{\Omega_2(1-\rho)}}\right) \\
&= \frac{1}{\beta_2}\sum_{l=0}^\infty \frac{(-s_2)^l}{l!} \Gamma\left(m_2+k+\frac{l}{\beta_2}\right) \left(\frac{tm_2}{\Omega_2(1-\rho)}\right)^{-\left(m_2+k+\frac{l}{\beta_2}\right)} {}_2F_1\left(m_2-m_1, m_2+k+\frac{l}{\beta_2}, m_2+k; \rho\right). \qquad (4.69)
\end{aligned}$$

Substituting (4.68) and (4.69) into (4.67) leaves:

$$\begin{aligned}
&\mathcal{L}\left(s_1, s_2|t\right) \\
=\;& \beta_1\beta_2\left(1-\rho\right)^{m_2}\sum_{k=0}^{\infty}\frac{\left(m_1\right)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1\left(1-\rho\right)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2\left(1-\rho\right)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma\left(m_1+k\right)\Gamma\left(m_2+k\right)}s_1^{-\beta_1 m_1-\beta_1 k} \\
&\times\left(2\pi\right)^{\frac{1}{2}(1-\beta_1)}\beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}}G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1}tm_1}{s_1^{\beta_1}\Omega_1\left(1-\rho\right)}\Big|\begin{matrix}0\\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1},\frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1},\ldots,\frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1}\end{matrix}\right) \\
&\times\frac{1}{\beta_2}\sum_{l=0}^{\infty}\frac{\left(-s_2\right)^l}{l!}\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)\left(\frac{tm_2}{\Omega_2\left(1-\rho\right)}\right)^{-\left(m_2+k+\frac{l}{\beta_2}\right)}{}_2F_1\left(m_2-m_1,m_2+k+\frac{l}{\beta_2},m_2+k;\rho\right) \\
=\;& \left(1-\rho\right)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left(m_1\right)_k\rho^k}{k!l!}\left(\frac{m_1}{\Omega_1\left(1-\rho\right)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2\left(1-\rho\right)}\right)^{-\left(\frac{l}{\beta_2}\right)}\frac{t^{m_1+k-\frac{l}{\beta_2}}\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma\left(m_1+k\right)\Gamma\left(m_2+k\right)}\left(2\pi\right)^{\frac{1}{2}(1-\beta_1)} \\
&\times\beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}}s_1^{-\beta_1 m_1-\beta_1 k}\left(-s_2\right)^l\;{}_2F_1\left(m_2-m_1,m_2+k+\frac{l}{\beta_2},m_2+k;\rho\right) \\
&\times G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1}tm_1}{s_1^{\beta_1}\Omega_1\left(1-\rho\right)}\Big|\begin{matrix}0\\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1},\frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1},\ldots,\frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1}\end{matrix}\right).
\end{aligned} \tag{4.70}$$

Substituting (4.70) into (4.66) leaves the final result. ■

Remark 4.18 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.65) simplifies to:*

$$\begin{aligned}
&\mathcal{L}_{normal}\left(s_1, s_2\right) \\
=\;& \left(1-\rho\right)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left(m_1\right)_k\rho^k}{k!l!}\left(\frac{m_1}{\Omega_1\left(1-\rho\right)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2\left(1-\rho\right)}\right)^{-\left(\frac{l}{\beta_2}\right)}\frac{\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma\left(m_1+k\right)\Gamma\left(m_2+k\right)}\left(2\pi\right)^{\frac{1}{2}(1-\beta_1)} \\
&\times\beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}}s_1^{-\beta_1 m_1-\beta_1 k}\left(-s_2\right)^l\;{}_2F_1\left(m_2-m_1,m_2+k+\frac{l}{\beta_2},m_2+k;\rho\right) \\
&\times G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1}m_1}{s_1^{\beta_1}\Omega_1\left(1-\rho\right)}\Big|\begin{matrix}0\\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1},\frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1},\ldots,\frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1}\end{matrix}\right)
\end{aligned} \tag{4.71}$$

where $m_1, m_2, \Omega_1, \Omega_2 > 0$ $-1 < \rho < 1$, and $m_2 \geq m_1$.

Remark 4.19 *By choosing $\mathcal{W}(t)$ as (1.6), (4.65) simplifies to:*

$$
\begin{aligned}
&\mathcal{L}_t(s_1, s_2) \\
= \;& (1-\rho)^{m_2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(m_1)_k \rho^k}{k!l!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{l}{\beta_2}\right)} \frac{\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} (2\pi)^{\frac{1}{2}(1-\beta_1)} \\
& \times \beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}} s_1^{-\beta_1 m_1-\beta_1 k} (-s_2)^l \; {}_2F_1\left(m_2-m_1, m_2+k+\frac{l}{\beta_2}, m_2+k; \rho\right) \int_0^{\infty} t^{m_1+k-\frac{l}{\beta_2}} \\
& \times G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1} t m_1}{s_1^{\beta_1}\Omega_1(1-\rho)} \,\middle|\, \begin{array}{c} 0 \\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1}, \frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1}, \ldots, \frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1} \end{array}\right) \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} t^{\frac{v}{2}-1} \exp\left(-\frac{vt}{2}\right) dt \\
= \;& \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} (1-\rho)^{m_2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(m_1)_k \rho^k}{k!l!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{l}{\beta_2}\right)} \frac{\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} (2\pi)^{\frac{1}{2}(1-\beta_1)} \\
& \times \beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}} s_1^{-\beta_1 m_1-\beta_1 k} (-s_2)^l \; {}_2F_1\left(m_2-m_1, m_2+k+\frac{l}{\beta_2}, m_2+k; \rho\right) \int_0^{\infty} t^{m_1+k+\frac{v}{2}-\frac{l}{\beta_2}-1} \\
& \times \exp\left(-\frac{vt}{2}\right) G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1} t m_1}{s_1^{\beta_1}\Omega_1(1-\rho)} \,\middle|\, \begin{array}{c} 0 \\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1}, \frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1}, \ldots, \frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1} \end{array}\right) dt \qquad (4.72)
\end{aligned}
$$

where $m_1, m_2, \Omega_1, \Omega_2, v > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$. By using Result C.31 and Result C.21, (4.72) can be simplified further as:

$$
\begin{aligned}
&\mathcal{L}_t(s_1, s_2) \\
= \;& \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} (1-\rho)^{m_2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(m_1)_k \rho^k}{k!l!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{l}{\beta_2}\right)} \frac{\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
& \times (2\pi)^{\frac{1}{2}(1-\beta_1)} \beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}} s_1^{-\beta_1 m_1-\beta_1 k} (-s_2)^l \; {}_2F_1\left(m_2-m_1, m_2+k+\frac{l}{\beta_2}, m_2+k; \rho\right) \int_0^{\infty} t^{m_1+k+\frac{v}{2}-\frac{l}{\beta_2}-1} \\
& \times G_{0,1}^{1,0}\left(\frac{v}{2}t \,\middle|\, \begin{array}{c} - \\ 0 \end{array}\right) G_{\beta_1,1}^{1,\beta_1}\left(\frac{\beta_1^{\beta_1} m_1}{s_1^{\beta_1}\Omega_1(1-\rho)} t \,\middle|\, \begin{array}{c} 0 \\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1}, \frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1}, \ldots, \frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1} \end{array}\right) dt \\
= \;& \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} (1-\rho)^{m_2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(m_1)_k \rho^k}{k!l!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{l}{\beta_2}\right)} \frac{\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
& \times (2\pi)^{\frac{1}{2}(1-\beta_1)} \beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}} s_1^{-\beta_1 m_1-\beta_1 k} (-s_2)^l \; {}_2F_1\left(m_2-m_1, m_2+k+\frac{l}{\beta_2}, m_2+k; \rho\right) \\
& \times \left(\frac{v}{2}\right)^{-\left(m_1+k+\frac{v}{2}-\frac{l}{\beta_2}\right)} G_{\beta_1+1,1}^{1,\beta_1+1}\left(\frac{\beta_1^{\beta_1} m_1}{s_1^{\beta_1}\Omega_1(1-\rho)\frac{v}{2}} \,\middle|\, \begin{array}{c} \Delta(1,0) \\ \Delta(1,d_{\beta_1}), \Delta\left(1, 1-\left(m_1+k+\frac{v}{2}-\frac{l}{\beta_2}\right)\right) \end{array}\right). \qquad (4.73)
\end{aligned}
$$

Therefore, (4.73) leaves:

$$\begin{aligned}\mathcal{L}(s_1,s_2) &= \frac{(1-\rho)^{m_2}}{\Gamma\left(\frac{v}{2}\right)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k\,\rho^k}{k!l!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{l}{\beta_2}\right)}\frac{\Gamma\left(m_2+k+\frac{l}{\beta_2}\right)}{\Gamma(m_1+k)\,\Gamma(m_2+k)}\\ &\quad\times(2\pi)^{\frac{1}{2}(1-\beta_1)}\,\beta_1^{\beta_1 m_1+\beta_1 k-\frac{1}{2}}\,s_1^{-\beta_1 m_1-\beta_1 k}\,(-s_2)^l\;{}_2F_1\left(m_2-m_1,m_2+k+\frac{l}{\beta_2},m_2+k;\rho\right)\left(\frac{v}{2}\right)^{-\left(m_1+k-\frac{l}{\beta_2}\right)}\\ &\quad\times G^{1,\beta_1+1}_{\beta_1+1,1}\left(\frac{\beta_1^{\beta_1}m_1}{s_1^{\beta_1}\Omega_1(1-\rho)\frac{v}{2}}\;\middle|\;\begin{matrix}0\\ \frac{1-\beta_1 m_1-\beta_1 k}{\beta_1},\frac{1-\beta_1 m_1-\beta_1 k+1}{\beta_1},\ldots,\frac{1-\beta_1 m_1-\beta_1 k+\beta_1-1}{\beta_1},1-\left(m_1+k+\frac{v}{2}-\frac{l}{\beta_2}\right)\end{matrix}\right).\end{aligned} \tag{4.74}$$

Specific values are evaluated for (4.71) and (4.74), and the machine runtime is listed as well. Furthermore, the integral representation of the Laplace transform (4.67) was also calculated numerically for the normal- and t case, together with its machine runtime. Similar as in Saboor et. al. (2012), the calculated values of (4.71), (4.74), and (4.67) for both normal- and t cases for select values of s_1 and s_2 are provided. The runtime is computed using Mathematica 11 on an Intel i7 processor. Whilst the runtime is affected by various factors of an individual processor, it gives some indication about the speed with which these expressions may be evaluated. The parameters for which these values are calculated are $m_1 = 10$, $m_2 = 12$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, $\rho = 0.5$, $v = 5$, and $\beta_1 = \beta_2 = 2$.

		Normal		**t**	
		Value	**Runtime (seconds)**	**Value**	**Runtime (seconds)**
$\mathcal{L}(0.5, 0.5)$	(4.71), (4.74)	0.125162	153.0130	0.113083	12.6415
	Integral (4.67)	0.126973	77.5169	0.121326	576.86
$\mathcal{L}(0.5, 1)$	(4.71), (4.74)	0.0695437	153.153	0.0665175	12.2351
	Integral (4.67)	0.0704006	77.8445	0.0612938	475.085
$\mathcal{L}(1, 1)$	(4.71), (4.74)	0.0329823	156.377	0.0320912	12.7365
	Integral (4.67)	0.0332989	75.3485	0.0269502	380.268

Table 4.1 Comparison between Laplace integral- and derived expressions for normal and t models

The following observations can be made from Table 4.1:

- The calculated values using expressions (4.71) and (4.74) illustrate the accuracy of computation when compared to (4.67);
- Upon comparing the calculated values of (4.71) and (4.74), it is observed that the values of (4.74) are consistently lower than the counterparts under ((4.71));
- Using (4.74) versus numerically integrating (4.67) indicates significant shortening of required runtime.

4.3.3 Product moment

In this section an expression for the product moment of the bivariate Weibullised gamma type II distribution with pdf (4.60) is derived.

Theorem 4.8 *The product moment of* (W_1, W_2) *with pdf (4.60) is given by:*

$$E\left(W_1^r W_2^d\right) = (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)} \frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \times {}_2F_1\left(m_2-m_1, \frac{d}{\beta_2}+m_2+k, m_2+k; \rho\right) \int_0^{\infty} t^{-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)} \mathcal{W}(t)\, dt \quad (4.75)$$

where $r, d > 0$, *and* $m_2, m_1, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, *and* $m_2 \geq m_1$.

Proof. *Consider from (4.60):*

$$\begin{aligned} E\left(W_1^r W_2^d\right) &= \int_0^{\infty}\int_0^{\infty} w_1^r w_2^d f_{W_1,W_2}(w_1,w_2)\, dw_1 dw_2 \\ &= \int_0^{\infty}\int_0^{\infty} w_1^r w_2^d \int_0^{\infty} f(w_1,w_2|t)\, \mathcal{W}(t)\, dt dw_1 dw_2 \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} w_1^r w_2^d f(w_1,w_2|t)\, dw_1 dw_2 \mathcal{W}(t)\, dt \\ &= \int_0^{\infty} E\left(W_1^r W_2^d | t\right) \mathcal{W}(t)\, dt. \end{aligned} \quad (4.76)$$

From (4.76) and (4.62):

$$\begin{aligned} E\left(W_1^r W_2^d | t\right) &= \int_0^{\infty}\int_0^{\infty} w_1^r w_2^d f(w_1,w_2|t)\, dw_1 dw_2 \\ &= \int_0^{\infty}\int_0^{\infty} w_1^r w_2^d \beta_1\beta_2 (1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\ &\quad \times \frac{w_1^{\beta_1 m_1+\beta_1 k-1}}{\Gamma(m_1+k)} \frac{w_2^{\beta_2 m_2+\beta_2 k-1}}{\Gamma(m_2+k)} \exp\left[-t\left(\frac{m_1 w_1^{\beta_1}}{\Omega_1(1-\rho)}\right)\right] \exp\left[-t\left(\frac{m_2 w_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right] \\ &\quad \times {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) t^{m_1+m_2+2k} dw_1 dw_2 \\ &= \beta_1\beta_2(1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times \int_0^{\infty} w_1^{r+\beta_1 m_1+\beta_1 k-1} \exp\left[-\frac{t m_1}{\Omega_1(1-\rho)} w_1^{\beta_1}\right] dw_1 \\ &\quad \times \int_0^{\infty} w_2^{d+\beta_2 m_2+\beta_2 k-1} \exp\left[-\frac{t m_2}{\Omega_2(1-\rho)} w_2^{\beta_2}\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) dw_2 \\ &= \beta_1\beta_2(1-\rho)^{m_2} \sum_{k=0}^{\infty} \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times V_1 \cdot V_2. \end{aligned} \quad (4.77)$$

Using Result C.24, consider from (4.77):

$$\begin{aligned} V_1 &= \int_0^\infty w_1^{r+\beta_1 m_1+\beta_1 k-1} \exp\left[-\frac{tm_1}{\Omega_1(1-\rho)} w_1^{\beta_1}\right] dw_1 \\ &= \frac{\Gamma\left(\frac{r+\beta_1 m_1+\beta_1 k}{\beta_1}\right)}{\beta_1\left(\frac{tm_1}{\Omega_1(1-\rho)}\right)^{\frac{r+\beta_1 m_1+\beta_1 k}{\beta_1}}} \\ &= \frac{1}{\beta_1}\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\left(\frac{tm_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}+m_1+k\right)}. \end{aligned} \tag{4.78}$$

To obtain an expression for V_2*, consider the transformation* $z = w_2^{\beta_2}$*. Then* $w_2 = z^{\frac{1}{\beta_2}}$*, and* $\frac{dw_2}{dz} = \frac{1}{\beta_2} z^{\frac{1}{\beta_2}-1}$*, and using Result C.32:*

$$\begin{aligned} V_2 &= \int_0^\infty w_2^{d+\beta_2 m_2+\beta_2 k-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} w_2^{\beta_2}\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} w_2^{\beta_2}\right) dw_2 \\ &= \int_0^\infty \left(z^{\frac{1}{\beta_2}}\right)^{d+\beta_2 m_2+\beta_2 k-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} z\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} z\right) \frac{1}{\beta_2} z^{\frac{1}{\beta_2}-1} dz \\ &= \frac{1}{\beta_2}\int_0^\infty z^{\frac{d}{\beta_2}+m_2+k-1} \exp\left[-\frac{tm_2}{\Omega_2(1-\rho)} z\right] {}_1F_1\left(m_2-m_1, m_2+k; \frac{m_2\rho t}{\Omega_2(1-\rho)} z\right) dz \\ &= \frac{1}{\beta_2}\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)\left(\frac{tm_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}+m_2+k\right)} {}_2F_1\left(m_2-m_1, \frac{d}{\beta_2}+m_2+k, m_2+k; \frac{\frac{m_2\rho t}{\Omega_2(1-\rho)}}{\frac{tm_2}{\Omega_2(1-\rho)}}\right) \\ &= \frac{1}{\beta_2}\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)\left(\frac{tm_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}+m_2+k\right)} {}_2F_1\left(m_2-m_1, \frac{d}{\beta_2}+m_2+k, m_2+k; \rho\right). \end{aligned} \tag{4.79}$$

Substituting (4.78) and (4.79) into (4.77) leaves:

$$\begin{aligned} E\left(W_1^r W_2^d | t\right) &= \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^\infty \frac{(m_1)_k \rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times\frac{1}{\beta_1}\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\left(\frac{tm_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}+m_1+k\right)} \\ &\quad \times\frac{1}{\beta_2}\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)\left(\frac{tm_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}+m_2+k\right)} {}_2F_1\left(m_2-m_1, \frac{d}{\beta_2}+m_2+k, m_2+k; \rho\right) \\ &= (1-\rho)^{m_2}\sum_{k=0}^\infty \frac{(m_1)_k \rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times t^{m_1+m_2+2k} t^{-\left(\frac{r}{\beta_1}+m_1+k\right)} t^{-\left(\frac{d}{\beta_2}+m_2+k\right)}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}+m_1+k\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}+m_2+k\right)} \\ &\quad \times {}_2F_1\left(m_2-m_1, \frac{d}{\beta_2}+m_2+k, m_2+k; \rho\right) \\ &= (1-\rho)^{m_2}\sum_{k=0}^\infty \frac{(m_1)_k \rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)}\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad \times t^{-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)} {}_2F_1\left(m_2-m_1, \frac{d}{\beta_2}+m_2+k, m_2+k; \rho\right) \end{aligned} \tag{4.80}$$

Finally, substituting (4.80) into (4.76) leaves the final result. ∎

Remark 4.20 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.76) simplifies to:*

$$E_{normal}\left(W_1^r W_2^d\right) = (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)} \times\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)}\,{}_2F_1\left(m_2-m_1,\frac{d}{\beta_2}+m_2+k,m_2+k;\rho\right) \quad (4.81)$$

for $r,d,m_1,m_2,\Omega_1,\Omega_2>0$, $-1<\rho<1$, and $m_2\geq m_1$.

Remark 4.21 *By choosing $\mathcal{W}(t)$ as (1.6) and by using Result C.22, (4.76) simplifies to:*

$$\begin{aligned}
E_t\left(W_1^r W_2^d\right) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)}\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times {}_2F_1\left(m_2-m_1,\frac{d}{\beta_2}+m_2+k,m_2+k;\rho\right)\int_0^\infty t^{-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)}\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times {}_2F_1\left(m_2-m_1,\frac{d}{\beta_2}+m_2+k,m_2+k;\rho\right)\int_0^\infty t^{\frac{v}{2}-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)-1}\exp\left(-\frac{vt}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)}\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times {}_2F_1\left(m_2-m_1,\frac{d}{\beta_2}+m_2+k,m_2+k;\rho\right)\Gamma\left(\frac{v}{2}-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)\right)\left(\frac{v}{2}\right)^{-\left(\frac{v}{2}-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)\right)} \\
&= \frac{\left(\frac{v}{2}\right)^{\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)}}{\Gamma\left(\frac{v}{2}\right)}\Gamma\left(\frac{v}{2}-\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)\right)(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-\left(\frac{r}{\beta_1}\right)}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-\left(\frac{d}{\beta_2}\right)} \\
&\quad\times\frac{\Gamma\left(\frac{r}{\beta_1}+m_1+k\right)\Gamma\left(\frac{d}{\beta_2}+m_2+k\right)}{\Gamma(m_1+k)\Gamma(m_2+k)}\,{}_2F_1\left(m_2-m_1,\frac{d}{\beta_2}+m_2+k,m_2+k;\rho\right)
\end{aligned} \quad (4.82)$$

for $r,d,m_1,m_2,\Omega_1,\Omega_2,v>0$, $-1<\rho<1$, $m_2\geq m_1$, and $v>2\left(\frac{r}{\beta_1}+\frac{d}{\beta_2}\right)$ in order for the product moment to exist.

In the following figure the correlation coefficient of (W_1,W_2) with pdf (4.60) is illustrated using Result C.30 via (4.81) and (4.82) for arbitrary parameters. Note that ρ is defined as the correlation of (U_1,U_{2a}) (see (4.4)), now where $W_1=\left(\frac{U_1}{m_1}\right)^{\frac{1}{\beta_1}}$ and $W_2=\left(\frac{U_2}{m_2}\right)^{\frac{1}{\beta_2}}$.

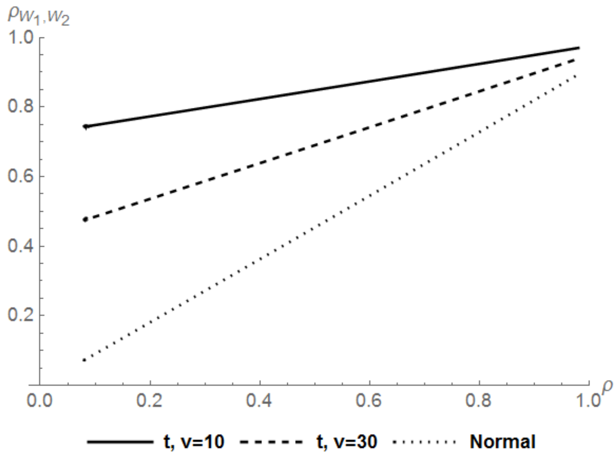


Figure 4.13 Correlation using (4.81) and (4.82) against ρ for $m_1 = 10$, $m_2 = 12$, $\Omega_1 = 2.5$, $\Omega_2 = 1.5$, $\beta_1 = \beta_2 = 2$, and $v = 10, 30$

Figure 4.13 illustrates a higher correlation coefficient for (4.81) when compared to (4.82). In particular it is observed that the correlation under the t distribution assumption approaches the correlation under the normal assumption for increased values of v. The same observation was made for the correlation of the bivariate gamma type II distribution (see Figure 4.5).

4.3.4 Cdf

In this section the cdf of the bivariate Weibullised gamma type II distribution with pdf (4.60) is derived.

Theorem 4.9 *Suppose that* (W_1, W_2) *is distributed as bivariate Weibullised gamma type II with pdf (4.60). The cdf of* (W_1, W_2) *is given by:*

$$\begin{aligned} F(w_1, w_2) &= (1-\rho)^{m_2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(m_1)_k (m_2 - m_1)_l}{(m_2 + k)_l \, k! l!} \frac{\rho^{k+l}}{\Gamma(m_1 + k) \Gamma(m_2 + k)} \\ &\times \int_0^{\infty} \gamma\left(m_1 + k, \frac{t m_1}{\Omega_1 (1-\rho)} w_1^{\beta_1}\right) \gamma\left(m_2 + k + l, \frac{t m_2}{\Omega_2 (1-\rho)} w_2^{\beta_2}\right) \mathcal{W}(t) \, dt \end{aligned} \tag{4.83}$$

where $m_2, m_1, \Omega_1, \Omega_2, \beta_1, \beta_2 > 0$, $-1 < \rho < 1$, $m_2 \geq m_1$, *and where* $\gamma(\cdot, \cdot)$ *denotes the lower incomplete gamma function (see Result C.6).*

Proof. *Consider from (4.60):*

$$\begin{aligned} F(w_1, w_2) &= \int_0^{w_1} \int_0^{w_2} f(c_1, c_2) \, dc_1 dc_2 \\ &= \int_0^{w_1} \int_0^{w_2} \int_0^{\infty} f(c_1, c_2 | t) \, \mathcal{W}(t) \, dt dc_1 dc_2 \\ &= \int_0^{\infty} \int_0^{w_1} \int_0^{w_2} f(c_1, c_2 | t) \, dc_1 dc_2 \mathcal{W}(t) \, dt \\ &= \int_0^{\infty} F(w_1, w_2 | t) \, \mathcal{W}(t) \, dt. \end{aligned} \tag{4.84}$$

From (4.84) and (4.62):

$$\begin{aligned}
F(w_1,w_2|t) &= \int_0^{w_1}\int_0^{w_2} f(c_1,c_2|t)\,dc_1dc_2 \\
&= \int_0^{w_1}\int_0^{w_2} \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k} \\
&\quad\times\frac{c_1^{\beta_1m_1+\beta_1k-1}}{\Gamma(m_1+k)}\frac{c_2^{\beta_2m_2+\beta_2k-1}}{\Gamma(m_2+k)}\exp\left[-t\left(\frac{m_1c_1^{\beta_1}}{\Omega_1(1-\rho)}+\frac{m_2c_2^{\beta_2}}{\Omega_2(1-\rho)}\right)\right] \\
&\quad\times{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho t}{\Omega_2(1-\rho)}c_2^{\beta_2}\right)t^{m_1+m_2+2k}dc_1dc_2 \\
&= \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times\int_0^{w_1}c_1^{\beta_1m_1+\beta_1k-1}\exp\left[-\frac{tm_1}{\Omega_1(1-\rho)}c_1^{\beta_1}\right]dc_1 \\
&\quad\times\int_0^{w_2}c_2^{\beta_2m_2+\beta_2k-1}\exp\left[-\frac{tm_2}{\Omega_2(1-\rho)}c_2^{\beta_2}\right]\,{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho t}{\Omega_2(1-\rho)}c_2^{\beta_2}\right)dc_2 \\
&= \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times V_1\cdot V_2 \qquad (4.85)
\end{aligned}$$

Using Result C.23, consider from (4.85):

$$\begin{aligned}
V_1 &= \int_0^{w_1}c_1^{\beta_1m_1+\beta_1k-1}\exp\left[-\frac{tm_1}{\Omega_1(1-\rho)}c_1^{\beta_1}\right]dc_1 \\
&= \frac{\gamma\left(\frac{\beta_1m_1+\beta_1k}{\beta_1},\frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)}{\beta_1\left(\frac{tm_1}{\Omega_1(1-\rho)}\right)^{\frac{\beta_1m_1+\beta_1k}{\beta_1}}} \\
&= \frac{t^{-(m_1+k)}}{\beta_1}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-(m_1+k)}\gamma\left(m_1+k,\frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right) \qquad (4.86)
\end{aligned}$$

and

$$\begin{aligned}
V_2 &= \int_0^{w_2}c_2^{\beta_2m_2+\beta_2k-1}\exp\left[-\frac{tm_2}{\Omega_2(1-\rho)}c_2^{\beta_2}\right]\,{}_1F_1\left(m_2-m_1,m_2+k;\frac{m_2\rho t}{\Omega_2(1-\rho)}c_2^{\beta_2}\right)dc_2 \\
&= \sum_{l=0}^{\infty}\frac{(m_2-m_1)_l}{(m_2+k)_l\,l!}\left(\frac{m_2\rho t}{\Omega_2(1-\rho)}\right)^l\int_0^{w_2}c_2^{\beta_2m_2+\beta_2k+\beta_2l-1}\exp\left[-\frac{tm_2}{\Omega_2(1-\rho)}c_2^{\beta_2}\right]dc_2 \\
&= \sum_{l=0}^{\infty}\frac{(m_2-m_1)_l}{(m_2+k)_l\,l!}\left(\frac{m_2\rho t}{\Omega_2(1-\rho)}\right)^l\frac{\gamma\left(\frac{\beta_2m_2+\beta_2k+\beta_2l}{\beta_2},\frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)}{\beta_2\left(\frac{tm_2}{\Omega_2(1-\rho)}\right)^{\frac{\beta_2m_2+\beta_2k+\beta_2l}{\beta_2}}} \\
&= \frac{1}{\beta_2}\sum_{l=0}^{\infty}t^{-(m_2+k)}\frac{(m_2-m_1)_l\rho^l}{(m_2+k)_l\,l!}\gamma\left(m_2+k+l,\frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-(m_2+k)}. \qquad (4.87)
\end{aligned}$$

Substituting (4.86) and (4.87) into (4.85) leaves:

$$\begin{aligned}
F(w_1, w_2|t) &= \beta_1\beta_2(1-\rho)^{m_2}\sum_{k=0}^{\infty}\frac{(m_1)_k\rho^k}{k!}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{m_1+k}\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{m_2+k}\frac{t^{m_1+m_2+2k}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times\frac{t^{-(m_1+k)}}{\beta_1}\left(\frac{m_1}{\Omega_1(1-\rho)}\right)^{-(m_1+k)}\gamma\left(m_1+k, \frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right) \\
&\quad\times\frac{1}{\beta_2}\sum_{l=0}^{\infty}t^{-(m_2+k)}\frac{(m_2-m_1)_l\rho^l}{(m_2+k)_l\, l!}\gamma\left(m_2+k+l, \frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)\left(\frac{m_2}{\Omega_2(1-\rho)}\right)^{-(m_2+k)} \\
&= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times\gamma\left(m_1+k, \frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)\gamma\left(m_2+k+l, \frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)
\end{aligned}$$

from where the result follows. ■

Remark 4.22 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.83) simplifies to:*

$$\begin{aligned}
F_{normal}(w_1, w_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times\gamma\left(m_1+k, \frac{m_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)\gamma\left(m_2+k+l, \frac{m_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right) \qquad (4.88)
\end{aligned}$$

for $m_1, m_2, \Omega_1, \Omega_2 > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$.

Remark 4.23 *By choosing $\mathcal{W}(t)$ as (1.6), (4.83) simplifies to:*

$$\begin{aligned}
F_t(w_1, w_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times\int_0^{\infty}\gamma\left(m_1+k, \frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)\gamma\left(m_2+k+l, \frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\, k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\
&\quad\times\int_0^{\infty}\gamma\left(m_1+k, \frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)\gamma\left(m_2+k+l, \frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \qquad (4.89)
\end{aligned}$$

for $m_1, m_2, \Omega_1, \Omega_2, v > 0$, $-1 < \rho < 1$, and $m_2 \geq m_1$.

Remark 4.24 *Note that (4.83), (4.88), and (4.89) are similar to (4.32), (4.38), and (4.39). This is a logical consequence due to the fact that for some variable X with cdf $F(x)$, a transformation $g(X)$ will have cdf $F(g(X))$.*

Remark 4.25 *Note that (4.89) can be rewritten as follows:*

$$\begin{aligned} F_t(w_1,w_2) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times\int_0^{\infty}\gamma\left(m_1+k,\frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)\gamma\left(m_2+k+l,\frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \\ &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times E_T\left(\gamma\left(m_1+k,\frac{tm_1}{\Omega_1(1-\rho)}w_1^{\beta_1}\right)\gamma\left(m_2+k+l,\frac{tm_2}{\Omega_2(1-\rho)}w_2^{\beta_2}\right)\right) \end{aligned}$$

where $T\sim Gamma\left(\frac{v}{2},\frac{v}{2}\right)$.

4.4 Illustrative application

4.4.1 Outage probability

To investigate the outage probability of a fading channel subject to the bivariate gamma type II distribution (4.8), the cdf of the maximum of (U_1,U_2) is of interest (see (1.25)). Using (4.32), the cdf of $\max(U_1,U_2)$ is:

$$\begin{aligned} F(u) &= P(U_1<u,U_2<u) \\ &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times\int_0^{\infty}\gamma\left(m_1+k,\frac{t}{\Omega_1(1-\rho)}u\right)\gamma\left(m_2+k+l,\frac{t}{\Omega_2(1-\rho)}u\right)\mathcal{W}(t)\,dt. \end{aligned} \tag{4.90}$$

Remark 4.26 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (4.90) simplifies to:*

$$\begin{aligned} F_{normal}(u) &= (1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times\gamma\left(m_1+k,\frac{1}{\Omega_1(1-\rho)}u\right)\gamma\left(m_2+k+l,\frac{1}{\Omega_2(1-\rho)}u\right). \end{aligned} \tag{4.91}$$

Remark 4.27 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), (4.90) simplifies to:*

$$\begin{aligned} F_t(u) &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}(1-\rho)^{m_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(m_1)_k(m_2-m_1)_l}{(m_2+k)_l\,k!l!}\frac{\rho^{k+l}}{\Gamma(m_1+k)\Gamma(m_2+k)} \\ &\quad\times\int_0^{\infty}\gamma\left(m_1+k,\frac{t}{\Omega_1(1-\rho)}u\right)\gamma\left(m_2+k+l,\frac{t}{\Omega_2(1-\rho)}u\right)t^{\frac{v}{2}-1}\exp\left(-\frac{vt}{2}\right)dt \end{aligned} \tag{4.92}$$

In the following figures the nature of cdf (4.92) and (4.91) is illustrated for arbitrary parameters $m_1=10, m_2=12, \rho=0.5, \Omega_1=2.5, \Omega_2=1.5$, and $v=5,15$ to investigate the effect of the assumed t distribution in a fading channel environment subject to a bivariate gamma type II distribution with pdf (4.8).

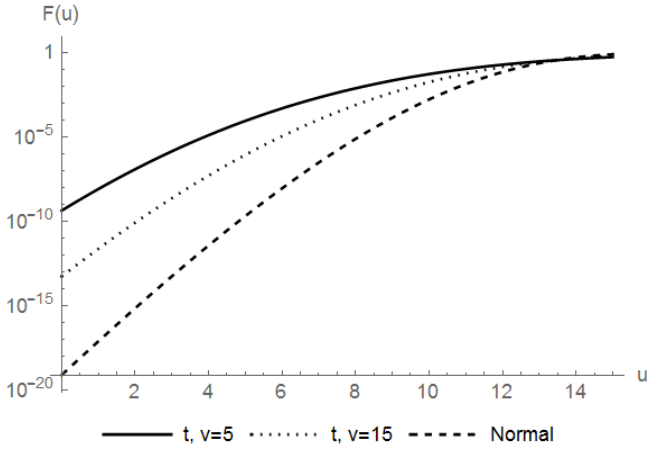


Figure 4.14 Analytical outage probabilities (4.91) and (4.92)

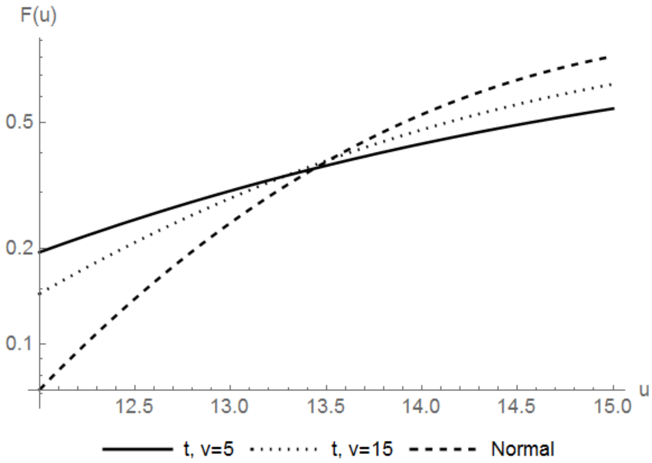


Figure 4.15 Analytical outage probabilities (4.91) and (4.92), for a subset of u

Figures 4.14 and 4.15 illustrates that (4.91) exhibits a lower outage probability for small outage thresholds than (4.92). However, for the scenario where the outage threshold is large, (4.92) exhibits a lower outage probability. This observation provides significant insight to the theoretical contribution of the candidacy of the underlying t distribution in comparison to the usual underlying normal case. The same behaviour of the outage probability is observed here that is observed in Figure 3.12 and Figure 3.13.

4.4.2 Percentiles of $F(u)$

Certain percentiles of the distribution of $F(u)$ in (4.90) are obtained numerically by solving the equation $F(u) = \alpha$. In particular, lower percentiles are computed for arbitrary parameters for (4.91) and (4.92). These values highlight the computational use of (4.90) as they act as possible critical values for testing hypothesis when a test statistic should follow a distribution with cdf (4.90).

ρ	$\alpha = 0.01$	0.025	0.05	0.1
0.25	11.0903	11.5503	11.9421	12.3904
0.75	10.5411	11.089	11.5543	12.0917

Table 4.2 Percentiles for (4.91), for $m_1 = 10, m_2 = 12, \Omega_1 = 2.5, \Omega_2 = 1.5$

m_2	$\alpha = 0.01$	0.025	0.05	0.1
12	10.8320	11.3295	11.7530	12.2367
14	11.2527	11.7092	12.0965	12.5376
16	11.6927	12.1188	12.4794	12.8889

Table 4.3 Percentiles for (4.91), for $m_1 = 10, \rho = 0.5, \Omega_1 = 2.5, \Omega_2 = 1.5$

ρ	$\alpha = 0.01$	0.025	0.05	0.1
0.25	6.9280	8.4101	10.1090	12.4440
0.75	6.6550	8.3351	10.5085	14.0959

Table 4.4 Percentiles for (4.92), for $m_1 = 10, m_2 = 12, \Omega_1 = 2.5, \Omega_2 = 1.5$ and $v = 5$

m_2	$\alpha = 0.01$	0.025	0.05	0.1
12	6.6450	8.1501	9.8099	12.1590
14	7.0955	8.6605	10.3805	12.801
16	7.6765	9.3050	11.0842	13.6288

Table 4.5 Percentiles for (4.92), for $m_1 = 10, \rho = 0.5, \Omega_1 = 2.5, \Omega_2 = 1.5$ and $v = 5$

v	$\alpha = 0.01$	0.025	0.05	0.1
5	6.9450	8.4001	10.1119	12.4590
15	9.5450	11.0721	12.6399	14.7690
30	10.8095	12.2616	13.8418	15.8477

Table 4.6 Percentiles for (4.92), for $m_1 = 10, m_2 = 12, \rho = 0.25, \Omega_1 = 2.5, \Omega_2 = 1.5$

The above tables of percentiles provides additional insight into (4.91) and (4.92). Particularly, it is interesting to note the percentiles under the t assumption in general are much lower than that of the normal assumption - this is in line with the commonly used z- and t scores in hypothesis testing (see Chapter 1).

4.5 Summary of results and conclusion

A summary of theoretical results in this chapter is provided for the convenience of the reader. (U_1, U_2) denotes the bivariate gamma type II distribution which originates from the elliptical class (see (4.8)), and (W_1, W_2) denotes the corresponding bivariate Weibullised gamma type II distribution emanating from (U_1, U_2).

	Pdf			**Moments**			**Cdf**		
	Elliptical	Normal	t	Elliptical	Normal	t	Elliptical	Normal	t
(U_1, U_2)	(4.8)	(4.14)	(4.15)	(4.24)	(4.30)	(4.31)	(4.32)	(4.38)	(4.39)
(W_1, W_2)	(4.60)	(4.63)	(4.64)	(4.75)	(4.81)	(4.82)	(4.83)	(4.88)	(4.89)

Table 4.7 Summary of derived results relating to this chapter

In addition, the pdf of a bivariate noncentral gamma type II distribution with pdf (4.44) was also derived, and the Laplace transform of (W_1, W_2) with pdf (4.60) is also derived, see (4.65).

This chapter proposes a bivariate gamma type II distribution with its origins in the elliptical class. In particular, a new bivariate gamma type II distribution has been derived, along with its pdf, cdf, and product moment. A bivariate noncentral gamma type II distribution stemming from this bivariate gamma type II distribution, is also proposed and derived. In addition, a bivariate Weibullised gamma type II distribution has also been derived and studied. This bivariate Weibullised gamma type II distribution contains the bivariate Nakagami type as a special case; of which the bivariate Nakagami distribution of Reig et. al. (2002) and Pibongungon (2005) is also a special case. Therefore, this bivariate Weibullised gamma type II distribution enriches the statistical distribution theory literature with its representation and origin from the elliptical class, and also provides a platform within the communications systems domain to assume underlying models othen than the normal for the bivariate Nakagami type distribution. Application has been discussed in terms of utilising the newly derived bivariate gamma type II distribution as a versatile bivariate gamma type distribution in the communications systems domain. In particular, the outage probability of a fading channel subject to such a bivariate gamma type II distribution under models other than the normal (in this case, t), provides significant insight into the behaviour of the bivariate gamma type distribution in such scenarios.

Chapter 5

Complex noncentral Wishart type distributions

Contents

5.1 Introduction

This chapter investigates the distribution of $\mathbf{S} = \mathbf{X}^H\mathbf{X} \in\mathbb{C}_2^{p \times p}$ assuming $\mathbf{X} \in\mathbb{C}_1^{n \times p}$ to follow the complex matrix variate elliptical distribution with $E(\mathbf{X}) = \mathbf{M}$. In certain practical scenarios, $E(\mathbf{X}) = \mathbf{0}$ (as considered in Chapter 2), which reflects the Rayleigh fading assumption (see (1.12)). However, in practice MIMO channels don't always exhibit this, stemming from a LOS connection between transmitters and receivers (Kang and Alouini (2006)b, Zhou et. al. (2015)). Jayaweera and Poor (2003) motivates the channel matrix $\mathbf{X}$ to be modelled having non-zero mean, to account for environments with strong LOS paths between transmitters and receivers. In order to encompass all channel characteristics, Taricco and Riegler (2011) suggests employing correlated Rician fading models - which directly pertains to modeling $\mathbf{X}$ with a non-zero mean. It is with these thoughts in mind that this chapter focus on assuming $E(\mathbf{X}) = \mathbf{M} \neq \mathbf{0}$ (see (1.11)). The resulting distribution of $\mathbf{S}$ is called a complex *noncentral* Wishart *type* distribution.

Particular interest lies with the distribution of the minimum eigenvalue of this complex noncentral Wishart type distribution. However, assuming $E(\mathbf{X}) = \mathbf{M}$ leaves a number of challenges analytically as well as computationally. In this chapter, the noncentral matrix parameter is assumed to have rank one. This low rank assumption is reportedly well modelled in practice (see Hansen and Bolcskei (2004)) and has been studied previously by Dharmawansa and McKay (2011).

5.2 Distributions of complex noncentral Wishart type and joint eigenvalues

In this section the distribution of $\mathbf{S} = \mathbf{X}^H\mathbf{X}$ is of interest, where $\mathbf{X}$ is distributed according to the complex matrix variate elliptical distribution (see (1.3)). The case where $\mathbf{\Phi} = \mathbf{I}_n$ is of interest (in Lemma 1.3.1), as considered by McKay and Collings (2005) and Dharmawansa and McKay (2011).

5.2.1 Pdf of the complex noncentral Wishart type

In this section the pdf of the complex noncentral Wishart type distribution is derived.

Theorem 5.1 *Suppose that* $n \geq p$ *and that* $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ *is distributed as* $\mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{I}_n \otimes \mathbf{\Sigma}, h)$. *Then* $\mathbf{S} = \mathbf{X}^H\mathbf{X} \in \mathbb{C}_2^{p\times p}$ *has a complex noncentral Wishart type distribution with pdf:*

$$f^{nc}(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\left(\mathbf{\Sigma}^{-1}\mathbf{S} + \mathbf{\Delta}\right)\right)\ {}_0\mathcal{C}F_1\left(n; t^2\mathbf{\Delta}\mathbf{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}(t)\,dt \tag{5.1}$$

where $\mathbf{\Delta} = \mathbf{\Sigma}^{-1}\mathbf{M}^H\mathbf{M}$ *denotes the noncentral matrix parameter and* ${}_0\mathcal{C}F_1(\cdot)$ *denotes the complex hypergeometric function of Hermitian matrix argument (see Result D.51). This distribution is denoted by* $\mathbf{S} \sim ISCW_p(n, \mathbf{M}, \mathbf{I}_n \otimes \mathbf{\Sigma})$.

Proof. *The pdf of* $\mathbf{X}|t$ *is given by (see James (1964)):*

$$\begin{aligned}
f(\mathbf{X}|t) &= \pi^{-np}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-(\mathbf{X}-\mathbf{M})\left(t^{-1}\mathbf{\Sigma}\right)^{-1}(\mathbf{X}-\mathbf{M})^H\right)\\
&= \pi^{-np}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)(\mathbf{X}-\mathbf{M})^H(\mathbf{X}-\mathbf{M})\right)\\
&= \pi^{-np}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{X}^H\mathbf{X}\right)\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{M}\right)\operatorname{etr}\left(2\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{X}\right).
\end{aligned}$$

Let $\mathbf{X} = \mathbf{E}\mathbf{T}$, *where* $\mathbf{E} : n \times p \in \mathcal{C}V_{p,n}$, *the Stiefel manifold (see Result D.37) such that* $\mathbf{E}^H\mathbf{E} = \mathbf{I}_p$ *and* $\mathbf{T}$ *is an upper triangular matrix with real and positive diagonal elements. Consider the Cholesky decomposition of* $\mathbf{S}$:

$$\mathbf{S} = \mathbf{X}^H\mathbf{X} = (\mathbf{E}\mathbf{T})^H\mathbf{E}\mathbf{T} = \mathbf{T}^H\mathbf{E}^H\mathbf{E}\mathbf{T} = \mathbf{T}^H\mathbf{T}$$

Using Result D.41) leaves:

$$f^{nc}(\mathbf{S}, \mathbf{E}|t) = 2^{-p}\pi^{-np}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{S}\right)\det(\mathbf{S})^{n-p}\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{M}\right)\operatorname{etr}\left(2\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{E}\mathbf{T}\right). \tag{5.2}$$

Subsequently the marginal pdf of $\mathbf{S}|t$ *can be obtained from (5.2) as:*

$$\begin{aligned}
f^{nc}(\mathbf{S}|t) &= \int_{\mathcal{C}V_{p,n}} f(\mathbf{S}, \mathbf{E}|t)\left(\mathbf{E}^H d\mathbf{E}\right)\\
&= \int_{\mathcal{C}V_{p,n}} 2^{-p}\pi^{-np}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{S}\right)\det(\mathbf{S})^{n-p}\\
&\quad\times\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{M}\right)\operatorname{etr}\left(2\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{E}\mathbf{T}\right)\left(\mathbf{E}^H d\mathbf{E}\right)\\
&= 2^{-p}\pi^{-np}\det\left(t^{-1}\mathbf{\Sigma}\right)^{-n}\operatorname{etr}\left(-\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{S}\right)\det(\mathbf{S})^{n-p}\operatorname{etr}(-t\mathbf{\Delta})\\
&\quad\times\int_{\mathcal{C}V_{p,n}}\operatorname{etr}\left(2\left(t\mathbf{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{E}\mathbf{T}\right)\left(\mathbf{E}^H d\mathbf{E}\right).
\end{aligned} \tag{5.3}$$

Using Result D.40, see that the integral in (5.3) can be written as:

$$\begin{aligned}\int_{\mathcal{C}V_{p,n}} \operatorname{etr}\left(2\left(t\boldsymbol{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{E}\mathbf{T}\right)\left(\mathbf{E}^H d\mathbf{E}\right) &= \int_{\mathcal{C}V_{p,n}} \operatorname{etr}\left(2\mathbf{T}\left(t\boldsymbol{\Sigma}^{-1}\right)\mathbf{M}^H\mathbf{E}\right)\left(\mathbf{E}^H d\mathbf{E}\right)\\
&= \frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}\; {}_0\mathcal{C}F_1\left(n; \frac{1}{4}\left(\mathbf{T}\left(2t\boldsymbol{\Sigma}^{-1}\right)\mathbf{M}^H\right)\left(\mathbf{T}\left(2t\boldsymbol{\Sigma}^{-1}\right)\mathbf{M}^H\right)^H\right)\\
&= \frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}\; {}_0\mathcal{C}F_1\left(n; t^2\mathbf{T}\boldsymbol{\Sigma}^{-1}\mathbf{M}^H\mathbf{M}\left(\boldsymbol{\Sigma}^{-1}\right)^H\mathbf{T}^H\right)\\
&= \frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Sigma}^{-1}\mathbf{M}^H\mathbf{M}\boldsymbol{\Sigma}^{-1}\mathbf{T}^H\mathbf{T}\right)\\
&= \frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right). \qquad (5.4)\end{aligned}$$

Substituting (5.4) into (5.3) leaves:

$$\begin{aligned}f^{nc}\left(\mathbf{S}|t\right) &= 2^{-p}\pi^{-np}\det\left(t^{-1}\boldsymbol{\Sigma}\right)^{-n}\operatorname{etr}\left(-\left(t\boldsymbol{\Sigma}^{-1}\right)\mathbf{S}\right)\det\left(\mathbf{S}\right)^{n-p}\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\\
&= \frac{\det\left(\mathbf{S}\right)^{n-p}}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}t^{np}\operatorname{etr}\left(-t\left(\boldsymbol{\Sigma}^{-1}\mathbf{S}+\boldsymbol{\Delta}\right)\right)\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right). \qquad (5.5)\end{aligned}$$

Finally, substituting (5.5) into (1.4) leaves the final result:

$$\begin{aligned}f^{nc}\left(\mathbf{S}\right) &= \int_{\mathbb{R}^+} f^{nc}\left(\mathbf{S}|t\right)\mathcal{W}\left(t\right)dt\\
&= \frac{\det\left(\mathbf{S}\right)^{n-p}}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\left(\boldsymbol{\Sigma}^{-1}\mathbf{S}+\boldsymbol{\Delta}\right)\right)\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}\left(t\right)dt\end{aligned}$$

which leaves the final result. ∎

Remark 5.1 *Suppose that* $\mathbf{M} = \mathbf{0}$*. Then* $\boldsymbol{\Delta} = \boldsymbol{\Sigma}^{-1}\mathbf{M}^H\mathbf{M} = \mathbf{0}$*, and the pdf in (5.1) simplifies to:*

$$f\left(\mathbf{S}\right) = \int_{\mathbb{R}^+} \frac{\det\left(\mathbf{S}\right)^{n-p}\operatorname{etr}\left(-t\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)}{\mathcal{C}\Gamma_p(n)\det\left(t^{-1}\boldsymbol{\Sigma}\right)^n}\mathcal{W}\left(t\right)dt \qquad (5.6)$$

where $\mathbf{S} \in \mathbb{C}_2^{p\times p}$*, which is the distribution in (2.3) and (2.4).*

Remark 5.2 *The complex noncentral Wishart type distribution (see (5.1)) can be written in terms of the complex central Wishart type distribution:*

$$\begin{aligned}f^{nc}\left(\mathbf{S}\right) &= \frac{\det\left(\mathbf{S}\right)^{n-p}}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\left(\boldsymbol{\Sigma}^{-1}\mathbf{S}+\boldsymbol{\Delta}\right)\right)\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}\left(t\right)dt\\
&= \int_{\mathbb{R}^+} \frac{\det\left(\mathbf{S}\right)^{n-p}}{\mathcal{C}\Gamma_p(n)\det\left(t^{-1}\boldsymbol{\Sigma}\right)^n}\operatorname{etr}\left(-t\left(\boldsymbol{\Sigma}^{-1}\mathbf{S}+\boldsymbol{\Delta}\right)\right)\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}\left(t\right)dt\\
&= \int_{\mathbb{R}^+} \frac{\det\left(\mathbf{S}\right)^{n-p}\operatorname{etr}\left(-t\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)}{\mathcal{C}\Gamma_p(n)\det\left(t^{-1}\boldsymbol{\Sigma}\right)^n}\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}\left(t\right)dt\\
&= \int_{\mathbb{R}^+} f\left(\mathbf{S}\right)\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\; {}_0\mathcal{C}F_1\left(n; t^2\boldsymbol{\Delta}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right)\mathcal{W}\left(t\right)dt.\end{aligned}$$

where $\mathbf{S} \in \mathbb{C}_2^{p\times p}$*, and where* $f(\mathbf{S})$ *denotes the pdf of the complex central Wishart type distribution (see (5.6)).*

Special cases of the distribution in (5.1) are highlighted next.

Corollary 5.1 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), (5.1) simplifies to:*

$$f_{normal}^{nc}(\mathbf{S}) = \frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\operatorname{etr}\left(-\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Delta}\right)\right)\ {}_0\mathcal{C}F_1\left(n;\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right) \tag{5.7}$$

where $\mathbf{S} \in \mathbb{C}_2^{p\times p}$*, which is the distribution as reported by James (1964).*

Corollary 5.2 *By choosing* $\mathcal{W}(t)$ *as the t distribution weight (1.6), (5.1) simplifies to:*

$$\begin{aligned} f_t^{nc}(\mathbf{S}) &= \frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Delta}\right)\right)\ {}_0\mathcal{C}F_1\left(n;t^2\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\exp\left[-t\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Delta}+\frac{v}{2}\right)\right]\ {}_0\mathcal{C}F_1\left(n;t^2\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right)dt. \end{aligned}$$

Expanding the hypergeometric function ${}_0\mathcal{C}F_1(\cdot)$ *using Result D.51, see that:*

$$\begin{aligned} {}_0\mathcal{C}F_1\left(n;t^2\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right) &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!\,[n]_\kappa}C_\kappa\left(t^2\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right) \\ &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(t^2\right)^k}{k!\,[n]_\kappa}C_\kappa\left(\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right) \end{aligned}$$

and together with Result C.22 leaves:

$$\begin{aligned} f_t^{nc}(\mathbf{S}) &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\exp\left[-t\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Delta}+\frac{v}{2}\right)\right]\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(t^2\right)^k}{k!\,[n]_\kappa}C_\kappa\left(\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right)}{k!\,[n]_\kappa}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}+2k-1}\exp\left[-t\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Delta}+\frac{v}{2}\right)\right]dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\frac{\det(\mathbf{S})^{n-p}}{\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(\mathbf{\Delta\Sigma}^{-1}\mathbf{S}\right)}{k!\,[n]_\kappa}\frac{\Gamma\left(np+\frac{v}{2}+2k\right)}{\left(\operatorname{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{S}+\mathbf{\Delta}+\frac{v}{2}\right)\right)^{np+\frac{v}{2}+2k}} \end{aligned} \tag{5.8}$$

where $\mathbf{S} \in \mathbb{C}_2^{p\times p}$.

5.2.2 Pdf of the joint eigenvalues

In this section, expressions for the joint pdf of the eigenvalues of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ (see (5.1)) and some special cases are derived. The eigenvalues of the noncentral matrix parameter $\mathbf{\Delta}$ is denoted by $\mu_1 > \mu_2 > ... > \mu_p > 0$.

Theorem 5.2 *Suppose that* $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ *is distributed with pdf (5.1), and let* $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ *represent* *the ordered eigenvalues of* $\mathbf{S}$*. Then the eigenvalues of* $\mathbf{S}$*,* $\mathbf{\Lambda} = diag\left(\lambda_1, \lambda_2, ..., \lambda_p\right)$*, has joint pdf:*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)\det\left(\mathbf{\Lambda}\right)^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\det\left(\mathbf{\Sigma}\right)^n}\int\limits_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\mathbf{\Delta}\right) \\ &\times \int\limits_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)\ {}_0\mathcal{C}F_1\left(n; t^2\mathbf{\Delta}\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E}\mathcal{W}\left(t\right)dt \end{aligned} \tag{5.9}$$

where $\mathbf{\Delta}$ *denotes the noncentral matrix parameter and* $U\left(p\right)$ *denotes the unitary space (see Result D.38).*

Proof. *Using (5.1) and Result D.59, the joint pdf of the eigenvalues* $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ *of* $\mathbf{S}$ *is given by:*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)}{\mathcal{C}\Gamma_p(p)}\int\limits_{\mathbf{E}\in U(p)} f\left(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E} \\ &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)}{\mathcal{C}\Gamma_p(p)} \\ &\times \int\limits_{\mathbf{E}\in U(p)} \frac{\det\left(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)^{n-p}}{\mathcal{C}\Gamma_p(n)\det\left(\mathbf{\Sigma}\right)^n}\int\limits_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\left(\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H+\mathbf{\Delta}\right)\right)\ {}_0\mathcal{C}F_1\left(n; t^2\mathbf{\Delta}\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)\mathcal{W}\left(t\right)dtd\mathbf{E} \\ &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)\det\left(\mathbf{\Lambda}\right)^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\det\left(\mathbf{\Sigma}\right)^n}\int\limits_{\mathbb{R}^+} t^{np}\operatorname{etr}\left(-t\mathbf{\Delta}\right) \\ &\times \int\limits_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)\ {}_0\mathcal{C}F_1\left(n; t^2\mathbf{\Delta}\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right) d\mathbf{E}\mathcal{W}\left(t\right)dt \end{aligned}$$

which completes the proof. ■

Remark 5.3 *By choosing* $\mathcal{W}\left(t\right)$ *as the dirac delta function (1.5), (5.9) simplifies to:*

$$f_{normal}(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)\det\left(\mathbf{\Lambda}\right)^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\det\left(\mathbf{\Sigma}\right)^n}\operatorname{etr}\left(-\mathbf{\Delta}\right)\int\limits_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)\ {}_0\mathcal{C}F_1\left(n; \mathbf{\Delta}\mathbf{\Sigma}^{-1}\mathbf{E}\mathbf{\Lambda}\mathbf{E}^H\right)d\mathbf{E}. \tag{5.10}$$

Remark 5.4 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), (5.9) simplifies to:*

$$\begin{aligned}
f_t(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}(-t\mathbf{\Delta}) \\
&\times \int_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{E\Lambda E}^H\right)\ {}_0\mathcal{C}F_1\left(n;t^2\mathbf{\Delta\Sigma}^{-1}\mathbf{E\Lambda E}^H\right)d\mathbf{E}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\
&= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\det(\mathbf{\Sigma})^n}\int_{\mathbb{R}^+} t^{np+\frac{v}{2}-1}\operatorname{etr}(-t\mathbf{\Delta}) \\
&\times \int_{\mathbf{E}\in U(p)} \operatorname{etr}\left(-t\left(\mathbf{\Sigma}^{-1}\mathbf{E\Lambda E}^H+\frac{v}{2}\right)\right)\ {}_0\mathcal{C}F_1\left(n;t^2\mathbf{\Delta\Sigma}^{-1}\mathbf{E\Lambda E}^H\right)d\mathbf{E}dt. \qquad (5.11)
\end{aligned}$$

In the following corollary, the special case of $\mathbf{\Sigma}=\sigma^2\mathbf{I}_p$ in (5.9) is presented.

Corollary 5.3 *Suppose that $\mathbf{S}\in\mathbb{C}_2^{p\times p}$ is distributed with pdf (5.1), and let $\lambda_1>\lambda_2>...>\lambda_p>0$ represent the ordered eigenvalues of $\mathbf{S}$. Furthermore suppose that $\mathbf{\Sigma}=\sigma^2\mathbf{I}_p$. Then the eigenvalues of $\mathbf{S}$, $\mathbf{\Lambda}=diag(\lambda_1,\lambda_2,...,\lambda_p)$, has joint pdf:*

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}}{((n-p)!)^p}\frac{\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{\left(\prod_{k<l}^{p}(\mu_k-\mu_l)\right)\sigma^{2np-p^2+1}} \\
&\times\int_{\mathbb{R}^+} t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)\mathcal{W}(t)\,dt \qquad (5.12)
\end{aligned}$$

where $\mathbf{\Delta}$ denotes the noncentral matrix parameter and ${}_0F_1$ denotes the hypergeometric function of scalar argument (see Result C.10).

Proof. *Substituting $\mathbf{\Sigma}=\sigma^2\mathbf{I}_p$ into (5.9) and using Result D.62, observe that:*

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\det(\sigma^2\mathbf{I})^n}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}(-t\mathbf{\Delta}) \\
&\times\int_{\mathbf{E}\in U(p)}\operatorname{etr}\left(-t\left(\sigma^2\mathbf{I}\right)^{-1}\mathbf{E\Lambda E}^H\right)\ {}_0\mathcal{C}F_1\left(n;t^2\mathbf{\Delta}\left(\sigma^2\mathbf{I}\right)^{-1}\mathbf{E\Lambda E}^H\right)d\mathbf{E}\mathcal{W}(t)\,dt \\
&= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\sigma^{2np}}\int_{\mathbb{R}^+} t^{np}\operatorname{etr}(-t\mathbf{\Delta})\operatorname{etr}\left(-t\sigma^{-2}\mathbf{\Lambda}\right) \\
&\times\int_{\mathbf{E}\in U(p)}{}_0\mathcal{C}F_1\left(n;t^2\sigma^{-2}\mathbf{\Delta E\Lambda E}^H\right)d\mathbf{E}\mathcal{W}(t)\,dt \\
&= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\sigma^{2np}} \\
&\times\int_{\mathbb{R}^+} t^{np}\operatorname{etr}(-t\mathbf{\Delta})\operatorname{etr}\left(-t\sigma^{-2}\mathbf{\Lambda}\right)\ {}_0\mathcal{C}F_1^{(p)}\left(n;t^2\sigma^{-2}\mathbf{\Delta},\mathbf{\Lambda}\right)\mathcal{W}(t)\,dt \qquad (5.13)
\end{aligned}$$

where ${}_0\mathcal{C}F_1^{(p)}(\cdot,\cdot)$ *denotes the complex hypergeometric function of double Hermitian matrix argument (see Result D.53). Using Result D.66 and D.57, see that:*

$$\begin{aligned}
{}_0\mathcal{C}F_1^{(p)}\left(n;\mathbf{\Delta},t^2\sigma^{-2}\mathbf{\Lambda}\right) &= \frac{\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{\prod\limits_{k<l}^{p}\left(t^2\sigma^{-2}\lambda_k-t^2\sigma^{-2}\lambda_l\right)\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)}\frac{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)}{\left((n-p)!\right)^p} \\
&= \frac{\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{\prod\limits_{k<l}^{p}\left(t^2\sigma^{-2}\left(\lambda_k-\lambda_l\right)\right)\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)}\frac{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)}{\left((n-p)!\right)^p} \\
&= \frac{\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{t^{2\frac{p(p-1)}{2}}\sigma^{-2\frac{p(p-1)}{2}}\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)}\frac{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)}{\left((n-p)!\right)^p} \\
&= \frac{\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{t^{p(p-1)}\sigma^{-p(p-1)}\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)}\frac{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)}{\left((n-p)!\right)^p} \qquad (5.14)
\end{aligned}$$

where ${}_0F_1(\cdot)$ *represents the hypergeometric function of scalar argument (see Result C.10). Substituting (5.14) into (5.13) results in:*

$$\begin{aligned}
f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)^2\right)\det(\mathbf{\Lambda})^{n-p}}{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)\sigma^{2np}\sigma^{-p(p-1)}}\int\limits_{\mathbb{R}^+} t^{np}\,\mathrm{etr}\left(-t\mathbf{\Delta}\right)\mathrm{etr}\left(-t\sigma^{-2}\mathbf{\Lambda}\right) \\
&\quad\times\frac{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)}{\left((n-p)!\right)^p}\frac{\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{t^{p(p-1)}\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)}\mathcal{W}(t)\,dt \\
&= \frac{\pi^{p(p-1)}}{\left((n-p)!\right)^p}\frac{\left(\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)\right)\det(\mathbf{\Lambda})^{n-p}}{\left(\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)\right)\sigma^{2np-p^2+1}} \\
&\quad\times\int\limits_{\mathbb{R}^+} t^{np-p^2+1}\,\mathrm{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)\mathcal{W}(t)\,dt
\end{aligned}$$

leaving the final result. ■

Corollary 5.4 *By choosing* $\mathcal{W}(t)$ *as the dirac delta function (1.5), (5.12) simplifies to:*

$$f_{normal}(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)}}{\left((n-p)!\right)^p}\frac{\left(\prod\limits_{k<l}^{p}\left(\lambda_k-\lambda_l\right)\right)\det(\mathbf{\Lambda})^{n-p}}{\left(\prod\limits_{k<l}^{p}\left(\mu_k-\mu_l\right)\right)\sigma^{2np-p^2+1}}\,\mathrm{etr}\left(-\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det\left({}_0F_1\left(n-p+1;\sigma^{-2}\mu_j\lambda_i\right)\right). \qquad (5.15)$$

When $\sigma^2=1$*, this result simplifies to p. 41, eq. 2.52 of McKay (2006).*

Corollary 5.5 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), (5.12) simplifies to:*

$$\begin{aligned} f_t(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}}{((n-p)!)^p} \frac{\left(\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{\left(\prod\limits_{k<l}^{p}(\mu_k-\mu_l)\right)\sigma^{2np-p^2+1}} \\ &\quad\times\int\limits_{\mathbb{R}^+} t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}\pi^{p(p-1)}}{\Gamma(\frac{v}{2})((n-p)!)^p} \frac{\left(\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{\left(\prod\limits_{k<l}^{p}(\mu_k-\mu_l)\right)\sigma^{2np-p^2+1}} \\ &\quad\times\int\limits_{\mathbb{R}^+} t^{np-p^2+\frac{v}{2}1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}+\frac{v}{2}\right)\right)\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)dt. \end{aligned} \tag{5.16}$$

Suppose now the noncentral matrix $\mathbf{\Delta}$ has $L\le p$ non-zero eigenvalues, thus, $rank(\mathbf{\Delta})=L\le p$. For the case $\mathbf{\Sigma}=\sigma^2\mathbf{I}_p$, the joint pdf of eigenvalues of $\mathbf{S}$, $\mathbf{\Lambda}=diag(\lambda_1,\lambda_2,...,\lambda_p)$, is presented in the following theorem.

Theorem 5.3 *Suppose that $\mathbf{S}\in\mathbb{C}_2^{p\times p}$ is distributed with pdf (5.1), and let $\lambda_1>\lambda_2>...>\lambda_p>0$ represent the ordered eigenvalues of $\mathbf{S}\in\mathbb{C}_2^{p\times p}$. Furthermore suppose that $\mathbf{\Sigma}=\sigma^2\mathbf{I}_p$, and that $\mathbf{\Delta}$ has arbitrary rank $L<p$ with eigenvalues $\mu_1>\mu_2>...>\mu_L>0$. Then the eigenvalues of $\mathbf{S}$, $\mathbf{\Lambda}=diag(\lambda_1,\lambda_2,...,\lambda_p)$, has joint pdf:*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\left(\prod\limits_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod\limits_{i=1}^{L}\mu_i^{p-L}\right)\mathcal{C}\Gamma_{p-L}(p-L)\sigma^{2np-p^2+1}} \\ &\quad\times\int\limits_{\mathbb{R}^+} t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det(\mathbf{T})\,\mathcal{W}(t)\,dt \end{aligned} \tag{5.17}$$

where $\mathbf{\Delta}$ denotes the noncentral matrix parameter, and where $\mathbf{T}$ is a $p\times p$ matrix with $(i,j)^{th}$ entry

$$\{\mathbf{T}\}_{i,j}=\begin{cases} {}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_i\lambda_j\right) & i=1,...,p \quad j=1,...,L \\ \frac{\left(t^2\lambda_i\right)^k(n-p)!}{(n-p+k)!} & i=1,...,p \quad j=L+1,...,p \end{cases}.$$

Proof. *Consider from (5.12):*

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}}{((n-p)!)^p} \frac{\left(\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{\left(\prod\limits_{k<l}^{p}(\mu_k-\mu_l)\right)\sigma^{2np-p^2+1}} \\ &\quad\times\int\limits_{\mathbb{R}^+} t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)\mathcal{W}(t)\,dt \\ &= \int\limits_{\mathbb{R}^+}\frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\sigma^{2np-p^2+1}}\left(\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\right)t^{np-p^2+1} \\ &\quad\times\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\frac{\det\left({}_0F_1\left(n-p+1;t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{\left(\prod\limits_{k<l}^{p}(\mu_k-\mu_l)\right)}\mathcal{W}(t)\,dt. \end{aligned} \tag{5.18}$$

From (5.18), consider:

$$\begin{aligned}\mathcal{J} &= \lim_{\mu_{L+1},\dots,\mu_p \to 0} \frac{\det\left({}_0F_1\left(n-p+1; t^2\sigma^{-2}\mu_j\lambda_i\right)\right)}{\prod\limits_{k<l}^{p}(\mu_k-\mu_l)} \\ &= \lim_{\mu_{L+1},\dots,\mu_p \to 0} \frac{\det\left(f_i\left(\mu_j\right)_{i,j=1,\dots,p}\right)}{\prod\limits_{k<l}^{p}(\mu_k-\mu_l)} \end{aligned} \tag{5.19}$$

where $f_i\left(\mu_j\right) = {}_0F_1\left(n-p+1; t^2\sigma^{-2}\mu_i\lambda_j\right)$. *Applying Lemma 5, p. 340 of Chiani et. al. (2010) (see also McKay (2006)) to (5.19) leaves:*

$$\mathcal{J} = \frac{\det\begin{bmatrix} f_1(\mu_1) & \cdots & f_1(\mu_L) & f_1^{(p-L-1)}(0) & \cdots & f_1^{(0)}(0) \\ \vdots & & & & & \vdots \\ f_p(\mu_1) & \cdots & f_p(\mu_L) & f_p^{(p-L-1)}(0) & \cdots & f_p^{(0)}(0) \end{bmatrix}}{\mathcal{C}\Gamma_{p-L}(p-L)\left(\prod\limits_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod\limits_{i=1}^{L}\mu_i^{p-L}\right)} \tag{5.20}$$

where

$$f_i^{(k)}(0) = \frac{\left(t^2\sigma^{-2}\lambda_i\right)^k (n-p)!}{(n-p+k)!}.$$

Substituting (5.20) into (5.18) leaves:

$$\begin{aligned} &= \int_{\mathbb{R}^+} \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\,\sigma^{2np-p^2+1}} \left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right) t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\mathcal{J}\mathcal{W}(t)\,dt \\ &= \int_{\mathbb{R}^+} \frac{\pi^{p(p-1)}\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\,\sigma^{2np-p^2+1}} \left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right) t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right) \\ &\quad\times \frac{\det\begin{bmatrix} f_1(\mu_1) & \cdots & f_1(\mu_L) & f_1^{(p-L-1)}(0) & \cdots & f_1^{(0)}(0) \\ \vdots & & & & & \vdots \\ f_p(\mu_1) & \cdots & f_p(\mu_L) & f_p^{(p-L-1)}(0) & \cdots & f_p^{(0)}(0) \end{bmatrix}}{\mathcal{C}\Gamma_{p-L}(p-L)\left(\prod\limits_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod\limits_{i=1}^{L}\mu_i^{p-L}\right)}\mathcal{W}(t)\,dt \\ &= \frac{\pi^{p(p-1)}\left(\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\left(\prod\limits_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod\limits_{i=1}^{L}\mu_i^{p-L}\right)\mathcal{C}\Gamma_{p-L}(p-L)\sigma^{2np-p^2+1}} \int_{\mathbb{R}^+} t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det(\mathbf{T})\,\mathcal{W}(t)\,dt \end{aligned}$$

where $\mathbf{T}$ *is a* $p \times p$ *matrix with* $(i,j)^{th}$ *entry*

$$\{\mathbf{T}\}_{i,j} = \begin{cases} {}_0F_1\left(n-p+1; t^2\sigma^{-2}\mu_i\lambda_j\right) & i=1,\dots,p \quad j=1,\dots,L \\ \frac{\left(t^2\sigma^{-2}\lambda_i\right)^k (n-p)!}{(n-p+k)!} & i=1,\dots,p \quad j=L+1,\dots,p \end{cases}$$

which leaves the final result. ■

Remark 5.5 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (5.17) simplifies to:*

$$f_{normal}(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\left(\prod_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod_{i=1}^{L}\mu_i^{p-L}\right)\mathcal{C}\Gamma_{p-L}(p-L)\sigma^{2np-p^2+1}} \times \operatorname{etr}\left(-\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det(\mathbf{T}). \tag{5.21}$$

Remark 5.6 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), (5.17) simplifies to:*

$$\begin{aligned} f_t(\mathbf{\Lambda}) &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}}{((n-p)!)^p\left(\prod_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod_{i=1}^{L}\mu_i^{p-L}\right)\mathcal{C}\Gamma_{p-L}(p-L)\sigma^{2np-p^2+1}} \\ &\quad \times \int_{\mathbb{R}^+} t^{np-p^2+1}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}\right)\right)\det(\mathbf{T})\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)\right)\det(\mathbf{\Lambda})^{n-p}\left(\frac{v}{2}\right)^{\frac{v}{2}}}{((n-p)!)^p\left(\prod_{k<l}^{L}(\mu_k-\mu_l)\right)\left(\prod_{i=1}^{L}\mu_i^{p-L}\right)\mathcal{C}\Gamma_{p-L}(p-L)\Gamma(\frac{v}{2})\sigma^{2np-p^2+1}} \\ &\quad \times \int_{\mathbb{R}^+} t^{np-p^2+\frac{v}{2}}\operatorname{etr}\left(-t\left(\mathbf{\Delta}+\sigma^{-2}\mathbf{\Lambda}+\frac{v}{2}\right)\right)\det(\mathbf{T})\,dt. \end{aligned} \tag{5.22}$$

5.3 Minimum eigenvalue cdf under rank one assumption

In this section, the cdf of the minimum eigenvalue of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ with pdf (5.1) is derived. Under the assumption of rank one matrices, $\mathbf{\Delta\Sigma}^{-1} \in \mathbb{C}_1^{p\times p}$ has rank one and is represented via its eigendecomposition as

$$\mathbf{\Delta\Sigma}^{-1} = \mu\boldsymbol{\gamma}\boldsymbol{\gamma}^H \tag{5.23}$$

where $\boldsymbol{\gamma} \in \mathbb{C}_1^{p\times 1}$ and $\boldsymbol{\gamma}^H\boldsymbol{\gamma} = 1$. In (5.23), μ denotes the single eigenvalue of $\mathbf{\Delta\Sigma}^{-1}$. To derive the cdf of the minimum eigenvalue of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ under this assumption, the following approach is employed:

$$\begin{aligned} F_{\min}(y) &= P(\lambda_{\min}(\mathbf{S}) \leq y) \\ &= 1 - P(\lambda_{\min}(\mathbf{S}) > y). \end{aligned}$$

Knowing that

$$P(\lambda_{\min}(\mathbf{S}) > y) = P(\mathbf{S} > y\mathbf{I}_p)$$

the cdf of the minimum eigenvalue can be found using (5.1) directly, therefore avoiding cumbersome derivations and computations of deriving the marginal distributions of random variables with pdfs like (5.9).

5.3.1 Cdf of minimum eigenvalue of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ where $\mathbf{X} \in \mathbb{C}_1^{n\times p}$

For the complex noncentral Wishart type distribution with pdf (5.1), the cdf for the minimum eigenvalue of $\mathbf{S}$ is derived next.

Theorem 5.4 *Suppose that* $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ *is distributed as* $\mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{I}_n \otimes \mathbf{\Sigma}, h)$, *where* $\mathbf{M} \in \mathbb{C}_1^{n\times p}$ *has rank one, and* $\mathbf{S} = \mathbf{X}^H\mathbf{X} \sim ISCW_p(n, \mathbf{M}, \mathbf{I}_n \otimes \mathbf{\Sigma})$ *with pdf (5.1). The cdf of* $\lambda_{\min}(\mathbf{S})$ *is given by:*

$$F_{\min}(y) = 1 - \int_{\mathbb{R}^+} y^{np} \frac{\operatorname{etr}(-t\mathbf{\Delta}) \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n) \det(\mathbf{\Sigma})^n} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{t^{np+2k} (y\mu)^k}{k! (n)_k} \binom{k}{r} \mathcal{Q}_{n,p,t}^r(y) \mathcal{W}(t) \, dt \tag{5.24}$$

where $y > 0$, $\mathbf{\Delta}$ *denotes the noncentral matrix parameter, and*

$$\mathcal{Q}_{n,p,t}^r(y) = \int_{\mathbf{Y}} \det(\mathbf{I}_p + \mathbf{Y})^{n-p} \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^r\left(\gamma\gamma^H \mathbf{Y}\right) d\mathbf{Y} \tag{5.25}$$

where $\mathbf{Y} \in \mathbb{C}_2^{p\times p}$.

Proof. *See that* $\mathbf{S} - y\mathbf{I}_p \in \mathbb{C}_2^{p\times p}$, *and consider from (5.1):*

$$\begin{aligned} P(\lambda_{\min}(\mathbf{S}) > y) &= \int_{\mathbf{S}-y\mathbf{I}_p} f(\mathbf{S}) \, d\mathbf{S} \\ &= \int_{\mathbb{R}^+} t^{np} \frac{\operatorname{etr}(-t\mathbf{\Delta})}{\mathcal{C}\Gamma_p(n) \det(\mathbf{\Sigma})^n} \\ &\quad \times \int_{\mathbf{S}-y\mathbf{I}_p} \det(\mathbf{S})^{n-p} \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\mathbf{S}\right) {}_0\mathcal{C}F_1\left(n; t^2 \mathbf{\Delta}\mathbf{\Sigma}^{-1}\mathbf{S}\right) d\mathbf{S} \mathcal{W}(t) \, dt \end{aligned} \tag{5.26}$$

where $\mathbf{S} - y\mathbf{I}_p \in \mathbb{C}_2^{p\times p}$. *Consider the transformation* $\mathbf{S} = y(\mathbf{I}_p + \mathbf{Y})$, *thus* $\mathbf{Y} \in \mathbb{C}_2^{p\times p}$. *Using Result D.45, from (5.26) it follows that:*

$$\begin{aligned} P(\lambda_{\min}(\mathbf{S}) > y) &= \int_{\mathbb{R}^+} t^{np} \frac{\operatorname{etr}(-t\mathbf{\Delta})}{\mathcal{C}\Gamma_p(n) \det(\mathbf{\Sigma})^n} \int_{\mathbf{Y}} y^{(n-p)p} \det(\mathbf{I}_p + \mathbf{Y})^{n-p} \\ &\quad \times \operatorname{etr}\left(-t\mathbf{\Sigma}^{-1}\left(y(\mathbf{I}_p + \mathbf{Y})\right)\right) {}_0\mathcal{C}F_1\left(n; t^2 \mathbf{\Delta}\mathbf{\Sigma}^{-1} y (\mathbf{I}_p + \mathbf{Y})\right) y^{p^2} d\mathbf{Y} \mathcal{W}(t) \, dt \\ &= \int_{\mathbb{R}^+} t^{np} y^{np} \frac{\operatorname{etr}(-t\mathbf{\Delta}) \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n) \det(\mathbf{\Sigma})^n} \\ &\quad \times \int_{\mathbf{Y}} \det(\mathbf{I}_p + \mathbf{Y})^{n-p} \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\mathbf{Y}\right) {}_0\mathcal{C}F_1\left(n; yt^2 \mathbf{\Delta}\mathbf{\Sigma}^{-1} (\mathbf{I}_p + \mathbf{Y})\right) d\mathbf{Y} \mathcal{W}(t) \, dt. \end{aligned} \tag{5.27}$$

By using Result D.51 and applying the assumption of rank one for the noncentral matrix parameter (see Result 5.23) to (5.27), the following is obtained:

$$\begin{aligned} P(\lambda_{\min}(\mathbf{S}) > y) &= \int_{\mathbb{R}^+} t^{np} y^{np} \frac{\operatorname{etr}(-t\mathbf{\Delta}) \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n) \det(\mathbf{\Sigma})^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k! [n]_\kappa} \\ &\quad \times \int_{\mathbf{Y}} \det(\mathbf{I}_p + \mathbf{Y})^{n-p} \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\mathbf{Y}\right) C_\kappa\left(yt^2 \mathbf{\Delta}\mathbf{\Sigma}^{-1} (\mathbf{I}_p + \mathbf{Y})\right) d\mathbf{Y} \mathcal{W}(t) \, dt \\ &= \int_{\mathbb{R}^+} t^{np} y^{np} \frac{\operatorname{etr}(-t\mathbf{\Delta}) \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n) \det(\mathbf{\Sigma})^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k! [n]_\kappa} \\ &\quad \times \int_{\mathbf{Y}} \det(\mathbf{I}_p + \mathbf{Y})^{n-p} \operatorname{etr}\left(-ty\mathbf{\Sigma}^{-1}\mathbf{Y}\right) C_\kappa\left(yt^2 \mu \gamma^H (\mathbf{I}_p + \mathbf{Y}) \gamma\right) d\mathbf{Y} \mathcal{W}(t) \, dt. \end{aligned} \tag{5.28}$$

Since having only one eigenvalue results in the partition κ *to reduce to a single partition, per definition of zonal polynomials it follows that* $[n]_\kappa = (n)_k$ *and* $C_\kappa(\mathbf{A}) = \operatorname{tr}(\mathbf{A})^k$ *(see Result D.51). Thus, by using Result C.29,* *the fact that the trace of a scalar value is the scalar value itself, and Result D.56, from (5.28) it follows that:*

$$\begin{aligned} C_\kappa\left(yt^2\mu\boldsymbol{\gamma}^H\left(\mathbf{I}_p+\mathbf{Y}\right)\boldsymbol{\gamma}\right) &= \left(yt^2\mu\right)^k C_\kappa\left(\boldsymbol{\gamma}^H\left(\mathbf{I}_p+\mathbf{Y}\right)\boldsymbol{\gamma}\right) \\ &= \left(yt^2\mu\right)^k C_\kappa\left(\boldsymbol{\gamma}^H\boldsymbol{\gamma}+\boldsymbol{\gamma}^H\mathbf{Y}\boldsymbol{\gamma}\right) \\ &= \left(yt^2\mu\right)^k C_\kappa\left(1+\boldsymbol{\gamma}^H\mathbf{Y}\boldsymbol{\gamma}\right) \\ &= \left(yt^2\mu\right)^k \operatorname{tr}\left(1+\boldsymbol{\gamma}^H\mathbf{Y}\boldsymbol{\gamma}\right)^k \\ &= \left(yt^2\mu\right)^k \left(1+\boldsymbol{\gamma}^H\mathbf{Y}\boldsymbol{\gamma}\right)^k \\ &= \left(yt^2\mu\right)^k \sum_{r=0}^{k}\binom{k}{r}\left(\boldsymbol{\gamma}^H\mathbf{Y}\boldsymbol{\gamma}\right)^r \\ &= \left(yt^2\mu\right)^k \sum_{r=0}^{k}\binom{k}{r}\operatorname{tr}^r\left(\boldsymbol{\gamma}^H\mathbf{Y}\boldsymbol{\gamma}\right) \\ &= \left(yt^2\mu\right)^k \sum_{r=0}^{k}\binom{k}{r}\operatorname{tr}^r\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^H\mathbf{Y}\right). \end{aligned} \tag{5.29}$$

Substituting (5.29) into (5.28) leaves:

$$\begin{aligned} P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right) &= \int_{\mathbb{R}^+} t^{np}y^{np}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{1}{k!\,(n)_k} \\ &\quad\times\int_{\mathbf{Y}}\det\left(\mathbf{I}_p+\mathbf{Y}\right)^{n-p}\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right)\left(yt^2\mu\right)^k\binom{k}{r}\operatorname{tr}^r\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^H\mathbf{Y}\right)d\mathbf{Y}\mathcal{W}\left(t\right)dt \\ &= \int_{\mathbb{R}^+} y^{np}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{np+2k}\left(y\mu\right)^k}{k!\,(n)_k}\binom{k}{r} \\ &\quad\times\int_{\mathbf{Y}}\det\left(\mathbf{I}_p+\mathbf{Y}\right)^{n-p}\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right)\operatorname{tr}^r\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^H\mathbf{Y}\right)d\mathbf{Y}\mathcal{W}\left(t\right)dt \\ &= \int_{\mathbb{R}^+} y^{np}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{np+2k}\left(y\mu\right)^k}{k!\,(n)_k}\binom{k}{r}\mathcal{Q}^r_{n,p,t}\left(y\right)\mathcal{W}\left(t\right)dt \end{aligned}$$

which leaves the final result. ■

Corollary 5.6 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (5.24) simplifies to:*

$$P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right)=\int_{\mathbb{R}^+} y^{np}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{np+2k}\left(y\mu\right)^k}{k!\,(n)_k}\binom{k}{r}\mathcal{Q}^r_{n,p,t}\left(y\right)\delta(t-1)dt.$$

Let $x=t-1$, then $t=x+1$ and $dx=dt$:

$$P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right)=\int_{\mathbb{R}^+} y^{np}\frac{\operatorname{etr}\left(-\left(x+1\right)\boldsymbol{\Delta}\right)\operatorname{etr}\left(-\left(x+1\right)y\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_p(n)\det\left(\boldsymbol{\Sigma}\right)^n}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(x+1\right)^{np+2k}\left(y\mu\right)^k}{k!\,(n)_k}\binom{k}{r}\mathcal{Q}^r_{n,p,x+1}\left(y\right)\delta(x)dx$$

with

$$\mathcal{Q}^r_{n,p,x+1}\left(y\right)=\int_{\mathbf{Y}}\det\left(\mathbf{I}_p+\mathbf{Y}\right)^{n-p}\operatorname{etr}\left(-\left(x+1\right)y\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right)\operatorname{tr}^r\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^H\mathbf{Y}\right)d\mathbf{Y}.$$

This leaves

$$P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right)=y^{np}\frac{\operatorname{etr}\left(-\boldsymbol{\Delta}\right)\operatorname{etr}\left(-y\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_{p}(n)\det\left(\boldsymbol{\Sigma}\right)^{n}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(y\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\mathcal{Q}_{n,p}^{r}\left(y\right)$$

with

$$\mathcal{Q}_{n,p}^{r}\left(y\right)=\int_{\mathbf{Y}}\det\left(\mathbf{I}_{p}+\mathbf{Y}\right)^{n-p}\operatorname{etr}\left(-y\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right)\operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right)d\mathbf{Y},$$

and finally

$$F_{\min}\left(y\right)=1-\frac{y^{np}\operatorname{etr}\left(-\boldsymbol{\Delta}\right)\operatorname{etr}\left(-y\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_{p}(n)\det\left(\boldsymbol{\Sigma}\right)^{n}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(y\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\mathcal{Q}_{n,p}^{r}\left(y\right)\tag{5.30}$$

which reflects the result by Dharmawansa and McKay (2011).

Corollary 5.7 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6), (5.24) simplifies to:*

$$\begin{aligned}P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right) &= \int_{\mathbb{R}^{+}}y^{np}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_{p}(n)\det\left(\boldsymbol{\Sigma}\right)^{n}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{np+2k}\left(y\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\mathcal{Q}_{n,p,t}^{r}\left(y\right)\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt\\ &= \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\int_{\mathbb{R}^{+}}y^{np}\frac{\operatorname{etr}\left(-t\left(\boldsymbol{\Delta}+\frac{v}{2}\right)\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_{p}(n)\det\left(\boldsymbol{\Sigma}\right)^{n}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{np+2k+\frac{v}{2}}\left(y\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\mathcal{Q}_{n,p,t}^{r}\left(y\right)dt\end{aligned}$$

where $\mathcal{Q}_{n,p,t}^{r}\left(y\right)$ is given by (5.25). Thus:

$$\begin{aligned}F_{\min}\left(y\right) &= 1-P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right)\\ &= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\int_{\mathbb{R}^{+}}y^{np}\frac{\operatorname{etr}\left(-t\left(\boldsymbol{\Delta}+\frac{v}{2}\right)\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_{p}(n)\det\left(\boldsymbol{\Sigma}\right)^{n}}\\ &\quad\times\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{np+2k+\frac{v}{2}}\left(y\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\mathcal{Q}_{n,p,t}^{r}\left(y\right)dt.\end{aligned}\tag{5.31}$$

5.3.2 Cdf of minimum eigenvalue of $\mathbf{S}\in\mathbb{C}_{2}^{n\times n}$ where $\mathbf{X}\in\mathbb{C}_{1}^{n\times n}$

For the complex noncentral Wishart type distribution with pdf (5.1), particularly when $n=p$, the cdf for the minimum eigenvalue of $\mathbf{S}$ is derived next.

Theorem 5.5 *Suppose that $\mathbf{X}\in\mathbb{C}_{1}^{n\times n}$ is distributed as $\mathcal{C}E_{n\times n}(\mathbf{M},\mathbf{I}_{n}\otimes\boldsymbol{\Sigma},h)$, where $\mathbf{M}\in\mathbb{C}_{1}^{n\times n}$ has rank one, and $\mathbf{S}=\mathbf{X}^{H}\mathbf{X}\sim ISCW_{n}\left(n,\mathbf{M},\mathbf{I}_{n}\otimes\boldsymbol{\Sigma}\right)$ with pdf (5.1). The cdf of $\lambda_{\min}\left(\mathbf{S}\right)$ is given by:*

$$F_{\min}\left(y\right)=1-\int_{\mathbb{R}^{+}}\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)\sum_{j=0}^{\infty}\frac{\left(yt^{2}\mu\right)^{j}}{j!\left(n\right)_{j}}\,{}_{1}F_{1}\left(n;n+j,t\operatorname{tr}\boldsymbol{\Delta}\right)\mathcal{W}\left(t\right)dt\tag{5.32}$$

where ${}_{1}F_{1}\left(\cdot\right)$ denotes the confluent hypergeometric function (see Result C.10).

Proof. *Let $n=p$. See from (5.24) and (5.25) by using Result D.63 and Result D.49 that:*

$$P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right)=\int_{\mathbb{R}^{+}}y^{n^{2}}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_{n}(n)\det\left(\boldsymbol{\Sigma}\right)^{n}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{n^{2}+2k}\left(y\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\mathcal{Q}_{n,n,t}^{r}\left(y\right)\mathcal{W}\left(t\right)dt\tag{5.33}$$

where

$$\begin{aligned}
\mathcal{Q}^{r}_{n,n,t}(y) &= \int_{\mathbf{Y}} \det\left(\mathbf{I}_p+\mathbf{Y}\right)^{n-n} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y} \\
&= \int_{\mathbf{Y}} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) C_r\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y} \\
&= \mathcal{C}\Gamma_n\left(n,r\right)\left(\det\left(ty\boldsymbol{\Sigma}^{-1}\right)\right)^{-n} C_r\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\left(ty\boldsymbol{\Sigma}^{-1}\right)^{-1}\right) \\
&= \mathcal{C}\Gamma_n\left(n,r\right)\left(t^n y^n \det\left(\boldsymbol{\Sigma}^{-1}\right)\right)^{-n} C_r\left(\frac{\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\boldsymbol{\Sigma}}{ty}\right) \\
&= \mathcal{C}\Gamma_n\left(n,r\right) t^{-n^2} y^{-n^2}\left(\det\boldsymbol{\Sigma}\right)^{n} C_r\left(\frac{\boldsymbol{\Delta}}{\mu t y}\right) \\
&= \frac{\mathcal{C}\Gamma_n\left(n,r\right)\left(\det\boldsymbol{\Sigma}\right)^{n}}{t^{n^2}y^{n^2}}\left(\frac{1}{\mu t y}\right)^{r} C_r\left(\boldsymbol{\Delta}\right) \\
&= \frac{\mathcal{C}\Gamma_n\left(n\right)\left(n\right)_r\left(\det\boldsymbol{\Sigma}\right)^{n}}{t^{n^2}y^{n^2}}\left(\frac{1}{\mu t y}\right)^{r}\left(\operatorname{tr}^{r}\boldsymbol{\Delta}\right) \qquad (5.34)
\end{aligned}$$

using the fact that with a single eigenvalue the partition of κ falls away. Substituting (5.34) into (5.33) leaves:

$$\begin{aligned}
P\left(\lambda_{\min}\left(\mathbf{S}\right)>y\right) &= \int_{\mathbb{R}^+} y^{n^2}\frac{\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_n(n)\det\left(\boldsymbol{\Sigma}\right)^{n}} \\
&\quad\times\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{n^2+2k}\left(y\mu\right)^{k}}{k!\left(n\right)_k}\binom{k}{r}\frac{\mathcal{C}\Gamma_n\left(n\right)\left(n\right)_r\left(\det\boldsymbol{\Sigma}\right)^{n}}{t^{n^2}y^{n^2}}\left(\frac{1}{\mu t y}\right)^{r}\left(\operatorname{tr}^{r}\boldsymbol{\Delta}\right)\mathcal{W}\left(t\right)dt \\
&= \int_{\mathbb{R}^+}\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(yt^2\mu\right)^{k}}{k!\left(n\right)_k}\binom{k}{r}\left(n\right)_r\left(\frac{1}{\mu t y}\right)^{r}\left(\operatorname{tr}^{r}\boldsymbol{\Delta}\right)\mathcal{W}\left(t\right)dt. \qquad (5.35)
\end{aligned}$$

Consider the double summation component in (5.35). Using Result C.10, this component can be rewritten as *follows:*

$$\begin{aligned}
& \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(yt^{2}\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\left(n\right)_{r}\left(\frac{1}{\mu ty}\right)^{r}\left(\operatorname{tr}^{r}\boldsymbol{\Delta}\right) && (5.36)\\
&= \sum_{k=0}^{\infty}\frac{\left(yt^{2}\mu\right)^{k}}{k!\left(n\right)_{k}}\sum_{r=0}^{k}\binom{k}{r}\left(n\right)_{r}\left(\frac{\operatorname{tr}\boldsymbol{\Delta}}{\mu ty}\right)^{r}\\
&= \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\frac{\left(yt^{2}\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\left(n\right)_{r}\left(\frac{\operatorname{tr}\boldsymbol{\Delta}}{\mu ty}\right)^{r}\\
&= \sum_{r=0}^{\infty}\sum_{k-r=0}^{\infty}\frac{\left(yt^{2}\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\left(n\right)_{r}\left(\frac{\operatorname{tr}\boldsymbol{\Delta}}{\mu ty}\right)^{r}\\
&= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(yt^{2}\mu\right)^{j+r}}{\left(j+r\right)!\left(n\right)_{j+r}}\binom{j+r}{r}\left(n\right)_{r}\left(\frac{\operatorname{tr}\boldsymbol{\Delta}}{\mu ty}\right)^{r}\\
&= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^{j+r}\left(yt\mu\right)^{j+r}}{\left(j+r\right)!\left(n\right)_{j+r}}\frac{\left(j+r\right)!}{\left(j+r-r\right)!r!}\left(n\right)_{r}\left(\frac{1}{\mu ty}\right)^{r}\operatorname{tr}^{r}\boldsymbol{\Delta}\\
&= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^{j+r}\left(yt\mu\right)^{j}}{j!r!\frac{\Gamma(n+r+j)}{\Gamma(n)}}\frac{\Gamma\left(n+r\right)}{\Gamma\left(n\right)}\operatorname{tr}^{r}\boldsymbol{\Delta}\\
&= \sum_{j=0}^{\infty}\frac{t^{j}\left(yt\mu\right)^{j}}{j!}\frac{\Gamma\left(n\right)}{\Gamma\left(n+j\right)}\sum_{r=0}^{\infty}\frac{\left(n\right)_{r}}{\left(n+j\right)_{r}}\frac{\left(t\operatorname{tr}\boldsymbol{\Delta}\right)^{r}}{r!}\\
&= \sum_{j=0}^{\infty}\frac{\left(yt^{2}\mu\right)^{j}}{j!\left(n\right)_{j}}\,{}_{1}F_{1}\left(n;n+j,t\operatorname{tr}\boldsymbol{\Delta}\right). && (5.37)
\end{aligned}$$

Substituting (5.37) into (5.35) leaves the final result. ∎

Remark 5.7 *See that (5.36) can also be expressed as:*

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(yt^{2}\mu\right)^{k}}{k!\left(n\right)_{k}}\binom{k}{r}\left(n\right)_{r}\left(\frac{1}{\mu ty}\right)^{r}\left(\operatorname{tr}^{r}\boldsymbol{\Delta}\right) &= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^{j+r}\left(yt\mu\right)^{j}}{j!r!\frac{\Gamma(n+r+j)}{\Gamma(n)}}\frac{\Gamma\left(n+r\right)}{\Gamma\left(n\right)}\operatorname{tr}^{r}\boldsymbol{\Delta}\\
&= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{\frac{\Gamma(n+r)}{\Gamma(n)}}{\frac{\Gamma(n+r+j)}{\Gamma(n)}}\frac{\left(yt^{2}\mu\right)^{j}\left(t\operatorname{tr}\boldsymbol{\Delta}\right)^{r}}{j!r!}\\
&= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(n\right)_{r}}{\left(n\right)_{r+j}}\frac{\left(yt^{2}\mu\right)^{j}\left(t\operatorname{tr}\boldsymbol{\Delta}\right)^{r}}{j!r!}\\
&= \Phi_{3}\left(n,n,t\operatorname{tr}\boldsymbol{\Delta},yt^{2}\mu\right)
\end{aligned}$$

where $\Phi_3(\cdot)$ denotes the Humbert confluent hypergeometric function of two variables (see Result C.35). Thus (5.32) can be written as:

$$F_{\min}\left(y\right)=1-\int_{\mathbb{R}^{+}}\operatorname{etr}\left(-t\boldsymbol{\Delta}\right)\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)\Phi_{3}\left(n,n,t\operatorname{tr}\boldsymbol{\Delta},yt^{2}\mu\right)\mathcal{W}\left(t\right)dt.$$

Remark 5.8 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (5.32) simplifies to:*

$$F_{\min}\left(y\right)=1-\operatorname{etr}\left(-\boldsymbol{\Delta}\right)\operatorname{etr}\left(-y\boldsymbol{\Sigma}^{-1}\right)\sum_{j=0}^{\infty}\frac{\left(y\mu\right)^{j}}{j!\left(n\right)_{j}}\,{}_{1}F_{1}\left(n;n+j,\operatorname{tr}\boldsymbol{\Delta}\right) \tag{5.38}$$

which reflects the result by Dharmawansa and McKay (2011).

Remark 5.9 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6) and using Result C.32, (5.32) simplifies to:*

$$\begin{aligned}
F_{\min}(y) &= 1-\int_{\mathbb{R}^+} \operatorname{etr}(-t\boldsymbol{\Delta})\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)\sum_{j=0}^{\infty}\frac{(yt^2\mu)^j}{j!\,(n)_j}\,{}_1F_1(n;n+j,t\operatorname{tr}\boldsymbol{\Delta})\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\
&= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\int_{\mathbb{R}^+}\operatorname{etr}\left(-t\left(y\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Delta}+\frac{v}{2}\right)\right)\sum_{j=0}^{\infty}\frac{t^{\frac{v}{2}+2j-1}(y\mu)^j}{j!\,(n)_j}\,{}_1F_1(n;n+j,t\operatorname{tr}\boldsymbol{\Delta})\,dt \\
&= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\sum_{j=0}^{\infty}\frac{(y\mu)^j}{j!\,(n)_j}\int_{\mathbb{R}^+}t^{\frac{v}{2}+2j-1}\operatorname{etr}\left(-t\left(y\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Delta}+\frac{v}{2}\right)\right)\,{}_1F_1(n;n+j,t\operatorname{tr}\boldsymbol{\Delta})\,dt \\
&= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\sum_{j=0}^{\infty}\frac{(y\mu)^j}{j!\,(n)_j}\int_{\mathbb{R}^+}t^{\frac{v}{2}+2j-1}\exp\left(-t\operatorname{tr}\left(y\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Delta}+\frac{v}{2}\right)\right)\,{}_1F_1(n;n+j,t\operatorname{tr}\boldsymbol{\Delta})\,dt \\
&= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\sum_{j=0}^{\infty}\frac{(y\mu)^j}{j!\,(n)_j}\frac{\Gamma(v+2j)}{\left(\operatorname{tr}\left(y\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Delta}+\frac{v}{2}\right)\right)^{\frac{v}{2}+2j}}\,{}_2F_1\left(n,v+2j;n+j,\frac{\operatorname{tr}\boldsymbol{\Delta}}{\operatorname{tr}\left(y\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Delta}+\frac{v}{2}\right)}\right) \qquad (5.39)
\end{aligned}$$

where $\left|\frac{\operatorname{tr}\boldsymbol{\Delta}}{\operatorname{tr}\left(y\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Delta}+\frac{v}{2}\right)}\right|<1$, *and noting that the diagonal entries of a complex matrix is real, and the trace of a matrix is a scalar.*

5.3.3 Cdf of minimum eigenvalue of $\mathbf{S}\in\mathbb{C}_2^{2\times 2}$ where $\mathbf{X}\in\mathbb{C}_1^{n\times 2}$

For the complex noncentral Wishart type distribution with pdf (5.1), particularly when $p=2$, the cdf for the minimum eigenvalue of $\mathbf{S}$ is derived next. Thus, this result gives the cdf of the minimum eigenvalue of a 2×2 complex noncentral Wishart type matrix with n arbitrary degrees of freedom.

Theorem 5.6 *Suppose that $\mathbf{X}\in\mathbb{C}_1^{n\times 2}$ is distributed as $\mathcal{C}E_{n\times 2}(\mathbf{M},\mathbf{I}_n\otimes\boldsymbol{\Sigma},h)$, where $\mathbf{M}\in\mathbb{C}_1^{n\times 2}$ has rank one, and $\mathbf{S}=\mathbf{X}^H\mathbf{X}\sim ISCW_2(n,\mathbf{M},\mathbf{I}_n\otimes\boldsymbol{\Sigma})$ with pdf (5.1). Thus, $\mathbf{S}$ is a 2×2 complex noncentral Wishart type matrix with arbitrary degrees of freedom n. The cdf of $\lambda_{\min}(\mathbf{S})$ is given by:*

$$F_{\min}(y)=1-\int_{\mathbb{R}^+}\frac{\operatorname{etr}(-t\boldsymbol{\Delta})\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(yt^2\mu)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{yt\mu}\right)^r\rho(r,y,t)\,\mathcal{W}(t)\,dt \qquad (5.40)$$

with

$$\begin{aligned}
\rho(r,y,t) &= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2) \\
&\quad\times\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h(\det\boldsymbol{\Sigma})^{i_1+\frac{h}{2}-\frac{i_2}{2}}\,\mathfrak{C}_{i_2-h}^{i_1-i_2+2+r}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)(ty)^{2n+i_2-2i_1-4} \qquad (5.41)
\end{aligned}$$

where $\mathfrak{C}_n^v(\cdot)$ denotes the Gegenbauer polynomial (see Result C.36).

Proof. *Substituting $p=2$ into (5.24) and (5.25), see that:*

$$P(\lambda_{\min}(\mathbf{S})>y)=\int_{\mathbb{R}^+}y^{2n}\frac{\operatorname{etr}(-t\boldsymbol{\Delta})\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{2n+2k}(y\mu)^k}{k!\,(n)_k}\binom{k}{r}\mathcal{Q}_{n,2,t}^{r}(y)\,\mathcal{W}(t)\,dt \qquad (5.42)$$

where

$$\mathcal{Q}^{r}_{n,2,t}\left(y\right) = \int\limits_{\mathbf{Y}} \det\left(\mathbf{I}_2+\mathbf{Y}\right)^{n-2} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y} \tag{5.43}$$

noting the dimensions of $\mathbf{Y}$ *is* 2×2. *Consider the determinant in (5.43) using Result D.65:*

$$\det\left(\mathbf{I}_2+\mathbf{Y}\right)^{n-2} = \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\binom{n-2}{i_1}\binom{i_1}{i_2}\operatorname{tr}^{i_2}\left(\mathbf{Y}\right)\det\left(\mathbf{Y}\right)^{i_1-i_2}. \tag{5.44}$$

Substituting (5.44) into (5.43):

$$\begin{aligned}
\mathcal{Q}^{r}_{n,2,t}\left(y\right) &= \int\limits_{\mathbf{Y}} \det\left(\mathbf{I}_2+\mathbf{Y}\right)^{n-2} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y} \\
&= \int\limits_{\mathbf{Y}} \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\binom{n-2}{i_1}\binom{i_1}{i_2}\operatorname{tr}^{i_2}\left(\mathbf{Y}\right)\det\left(\mathbf{Y}\right)^{i_1-i_2} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y} \\
&= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\binom{n-2}{i_1}\binom{i_1}{i_2}\int\limits_{\mathbf{Y}} \operatorname{tr}^{i_2}\left(\mathbf{Y}\right)\det\left(\mathbf{Y}\right)^{i_1-i_2} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y}.
\end{aligned} \tag{5.45}$$

By using Result D.64 and setting $p = i_2, a = i_1 - i_2 + 2, t = r, \mathbf{A} = ty\boldsymbol{\Sigma}^{-1}$ *and* $\mathbf{R} = \boldsymbol{\gamma}\boldsymbol{\gamma}^{H}$, *see that (5.45) simplifies as:*

$$\begin{aligned}
&\int\limits_{\mathbf{Y}} \operatorname{tr}^{i_2}\left(\mathbf{Y}\right)\det\left(\mathbf{Y}\right)^{i_1-i_2} \operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\mathbf{Y}\right) \operatorname{tr}^{r}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\mathbf{Y}\right) d\mathbf{Y} \\
&= \frac{i_2!\left(i_1-i_2+2\right)_r \mathcal{C}\Gamma_2\left(i_1-i_2+2\right)}{\left(\det\left(ty\boldsymbol{\Sigma}^{-1}\right)\right)^{i_1-i_2+2+\frac{i_2}{2}}} \sum_{h=0}^{\min(i_2,r)} \frac{(-1)^h\binom{r}{h}}{\left(\det\left(ty\boldsymbol{\Sigma}^{-1}\right)\right)^{\frac{h}{2}}} \\
&\quad \times \operatorname{tr}^{r-h}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}ty\boldsymbol{\Sigma}^{-1}\right)\operatorname{tr}^{h}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^{H}\right)\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{\operatorname{tr}\left(ty\boldsymbol{\Sigma}^{-1}\right)}{2\sqrt{\det\left(ty\boldsymbol{\Sigma}^{-1}\right)}}\right)
\end{aligned} \tag{5.46}$$

Noting that $\boldsymbol{\gamma}^H\boldsymbol{\gamma}=1$ *(see (5.23)) and substituting (5.46) into (5.45), it follows that:*

$$\begin{aligned}
\mathcal{Q}^r_{n,2,t}(y) &= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\binom{n-2}{i_1}\binom{i_1}{i_2}\frac{i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)}{\left(\det\left(ty\boldsymbol{\Sigma}^{-1}\right)\right)^{i_1-i_2+2+\frac{i_2}{2}}} \\
&\times\sum_{h=0}^{\min(i_2,r)}\frac{(-1)^h\binom{r}{h}}{\left(\det\left(ty\boldsymbol{\Sigma}^{-1}\right)\right)^{\frac{h}{2}}}\operatorname{tr}^{r-h}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^H\left(ty\boldsymbol{\Sigma}^{-1}\right)^{-1}\right)\operatorname{tr}^h\left(\boldsymbol{\gamma}^H\boldsymbol{\gamma}\right)\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{\operatorname{tr}\left(ty\boldsymbol{\Sigma}^{-1}\right)}{2\sqrt{\det\left(ty\boldsymbol{\Sigma}^{-1}\right)}}\right) \\
&= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}\frac{i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)}{\left(\det\left(ty\boldsymbol{\Sigma}^{-1}\right)\right)^{i_1-i_2+2+\frac{i_2}{2}+\frac{h}{2}}} \\
&\times\left(t^{-1}y^{-1}\right)^{r-h}\operatorname{tr}^{r-h}\left(\boldsymbol{\gamma}\boldsymbol{\gamma}^H\boldsymbol{\Sigma}\right)\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\frac{ty\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)}{\sqrt{(ty)^2\det\left(\boldsymbol{\Sigma}^{-1}\right)}}\right) \\
&= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}\frac{i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)}{\left((ty)^2\det\left(\boldsymbol{\Sigma}^{-1}\right)\right)^{i_1-i_2+2+\frac{i_2}{2}+\frac{h}{2}}} \\
&\times\left(t^{-1}y^{-1}\right)^{r-h}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{\mu}\right)^{r-h}\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\det(\boldsymbol{\Sigma})^{\frac{1}{2}}\right) \\
&= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}\frac{i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)}{\left(\det\left(\boldsymbol{\Sigma}^{-1}\right)\right)^{i_1-i_2+2+\frac{i_2}{2}+\frac{h}{2}}} \\
&\times\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{\mu}\right)^{r-h}\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\det(\boldsymbol{\Sigma})^{\frac{1}{2}}\right)t^{-2\left(i_1-i_2+2+\frac{i_2}{2}+\frac{h}{2}+r-h\right)}y^{-2\left(i_1-i_2+2+\frac{i_2}{2}+\frac{h}{2}+r-h\right)} \\
&= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)\left(\det\boldsymbol{\Sigma}\right)^{i_1-\frac{i_2}{2}+\frac{h}{2}+2} \\
&\times\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{\mu}\right)^{r-h}\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)t^{-2i_1+i_2-4-h-r+h}y^{-2i_1+i_2-4-h-r+h} \\
&= \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)\left(\det\boldsymbol{\Sigma}\right)^{i_1-\frac{i_2}{2}+\frac{h}{2}+2} \\
&\times\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{\mu}\right)^{r-h}\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)t^{i_2-2i_1-4-r}y^{i_2-2i_1-4-r}.
\end{aligned}\tag{5.47}$$

Substituting (5.47) into (5.42) leaves:

$$\begin{aligned}
P\left(\lambda_{\min}(\mathbf{S})>y\right) &= \int_{\mathbb{R}^+}y^{2n}\frac{\operatorname{etr}(-t\boldsymbol{\Delta})\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^n}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{t^{2n+2k}(y\mu)^k}{k!\,(n)_k}\binom{k}{r}\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h \\
&\times\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)\left(\det\boldsymbol{\Sigma}\right)^{i_1-\frac{i_2}{2}+\frac{h}{2}+2} \\
&\times\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{\mu}\right)^{r-h}\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)t^{i_2-2i_1-4-r}y^{i_2-2i_1-4-r}\mathcal{W}(t)\,dt \\
&= \int_{\mathbb{R}^+}\frac{\operatorname{etr}(-t\boldsymbol{\Delta})\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{\left(yt^2\mu\right)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{yt\mu}\right)^r \\
&\times\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2) \\
&\times\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h\left(\det\boldsymbol{\Sigma}\right)^{i_1+\frac{h}{2}-\frac{i_2}{2}}\mathfrak{C}^{i_1-i_2+2+r}_{i_2-h}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)(ty)^{2n+i_2-2i_1-4}\mathcal{W}(t)\,dt
\end{aligned}$$

which leaves the final result. ∎

Corollary 5.8 *By choosing $\mathcal{W}(t)$ as the dirac delta function (1.5), (5.40) and (5.41) simplifies to:*

$$\begin{aligned} F_{\min}(y) &= 1-\frac{\operatorname{etr}(-\boldsymbol{\Delta})\operatorname{etr}\left(-y\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(y\mu)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{y\mu}\right)^r \\ &\quad\times\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2) \\ &\quad\times\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h(\det\boldsymbol{\Sigma})^{i_1+\frac{h}{2}-\frac{i_2}{2}}\,\mathfrak{C}_{i_2-h}^{i_1-i_2+2+r}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)y^{2n+i_2-2i_1-4} \end{aligned} \tag{5.48}$$

which reflects the result by Dharmawansa and McKay (2011).

Corollary 5.9 *By choosing $\mathcal{W}(t)$ as the t distribution weight (1.6) and using Result C.22, (5.40) and (5.41) simplifies to:*

$$\begin{aligned} F_{\min}(y) &= 1-\int_{\mathbb{R}^+}\frac{\operatorname{etr}(-t\boldsymbol{\Delta})\operatorname{etr}\left(-ty\boldsymbol{\Sigma}^{-1}\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(yt^2\mu)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{yt\mu}\right)^r \\ &\quad\times\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h \\ &\quad\times(\det\boldsymbol{\Sigma})^{i_1+\frac{h}{2}-\frac{i_2}{2}}\,\mathfrak{C}_{i_2-h}^{i_1-i_2+2+r}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)(ty)^{2n+i_2-2i_1-4}\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}t^{\frac{v}{2}-1}\exp\left(-t\frac{v}{2}\right)dt \\ &= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\int_{\mathbb{R}^+}\frac{\operatorname{etr}\left(-t\left(\boldsymbol{\Delta}+y\boldsymbol{\Sigma}^{-1}+\frac{v}{2}\right)\right)}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(yt^2\mu)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{yt\mu}\right)^r \\ &\quad\times\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2) \\ &\quad\times\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h(\det\boldsymbol{\Sigma})^{i_1+\frac{h}{2}-\frac{i_2}{2}}\,\mathfrak{C}_{i_2-h}^{i_1-i_2+2+r}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)t^{2n+i_2-2i_1-4+\frac{v}{2}-1}y^{2n+i_2-2i_1-4}dt \\ &= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{1}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(y\mu)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{yt\mu}\right)^r \\ &\quad\times\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1}\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r \\ &\quad\times\mathcal{C}\Gamma_2(i_1-i_2+2)\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h(\det\boldsymbol{\Sigma})^{i_1+\frac{h}{2}-\frac{i_2}{2}}\,\mathfrak{C}_{i_2-h}^{i_1-i_2+2+r}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)y^{2n+i_2-2i_1-4} \\ &\quad\times\int_{\mathbb{R}^+}t^{2n+2k-r+i_2-2i_1-4+\frac{v}{2}-1}\exp\left(-t\operatorname{tr}\left(\boldsymbol{\Delta}+y\boldsymbol{\Sigma}^{-1}+\frac{v}{2}\right)\right)dt \\ &= 1-\frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{1}{\mathcal{C}\Gamma_2(n)\det(\boldsymbol{\Sigma})^{n-2}}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(y\mu)^k}{k!\,(n)_k}\binom{k}{r}\left(\frac{\operatorname{tr}(\boldsymbol{\Delta})}{yt\mu}\right)^r\sum_{i_1=0}^{n-2}\sum_{i_2=0}^{i_1}\sum_{h=0}^{\min(i_2,r)}(-1)^h\binom{n-2}{i_1} \\ &\quad\times\binom{i_1}{i_2}\binom{r}{h}i_2!\,(i_1-i_2+2)_r\,\mathcal{C}\Gamma_2(i_1-i_2+2)\left(\frac{\mu}{\operatorname{tr}(\boldsymbol{\Delta})}\right)^h(\det\boldsymbol{\Sigma})^{i_1+\frac{h}{2}-\frac{i_2}{2}} \\ &\quad\times\mathfrak{C}_{i_2-h}^{i_1-i_2+2+r}\left(\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\right)\sqrt{\det(\boldsymbol{\Sigma})}\right)y^{2n+i_2-2i_1-4}\frac{\Gamma\left(2n+2k-r+i_2-2i_1-4+\frac{v}{2}\right)}{\left(\operatorname{tr}\left(\boldsymbol{\Delta}+y\boldsymbol{\Sigma}^{-1}+\frac{v}{2}\right)\right)^{2n+2k-r+i_2-2i_1-4+\frac{v}{2}}} \end{aligned} \tag{5.49}$$

which concludes the proof.

5.4 Illustrative application

In this section, simulation results are presented to illustrate the accuracy of the derived results. For the cdfs (5.32) and (5.40), the covariance matrix $\mathbf{\Sigma}$ is assumed to be given by:

$$\{\mathbf{\Sigma}\}_{i,j} = \exp\left(-\frac{\pi^3}{32}(i-j)^2\right)$$

where $1 \leq i, j \leq p$. The mean matrix $\mathbf{M}$ is constructed as:

$$\mathbf{M} = \mathbf{a}^H \mathbf{b}$$

where $\mathbf{a} \in \mathbb{C}_1^{1\times n}$ and $\mathbf{b} \in \mathbb{C}_1^{1\times p}$ is given by:

$$\begin{aligned} \{a\}_i &= \exp\left(2\left(i-1\right) l\pi \cos\left(\theta\right)\right) \\ \{b\}_j &= \exp\left(2\left(j-1\right) l\pi \cos\left(\theta\right)\right) \end{aligned}$$

where $l = \sqrt{-1}$, $\theta = \frac{\pi}{4}$, and $i = 1, ..., n$ and $j = 1, ..., p$. These specific constructions of the covariance and mean matrices are meaningful when modeling practical MIMO channels with a nonzero mean (see McKay and Collings (2005) and Dharmawansa and McKay (2011)). The simulation is done in a similar fashion as that of Dharmawansa and McKay (2011) and the code can be found in the Appendix. Table 5.1 compares the analytical values of the cdf of $\lambda_{\min}(\mathbf{S})$ where $\mathbf{X} \in \mathbb{C}_1^{2\times 2}$ for the underlying t distribution (see (5.39)) and the underlying normal distribution (see (5.38)).

y	$t,\ v = 15$		Normal	
	Analytical	**Simulation**	**Analytical**	**Simulation**
0	0	0	0	0
0.5	0.542821	0.47118	0.559005	0.53030
1	0.782587	0.74190	0.808532	0.83326
1.5	0.892376	0.86648	0.918016	0.92842
2	0.94458	0.92544	0.965325	0.96836

Table 5.1 Analytical ((5.38) and (5.39)) and simulated values of cdf of $\lambda_{\min}(\mathbf{S})$

The following figures illustrate the cdfs (5.32) and (5.40) for various dimension combinations.

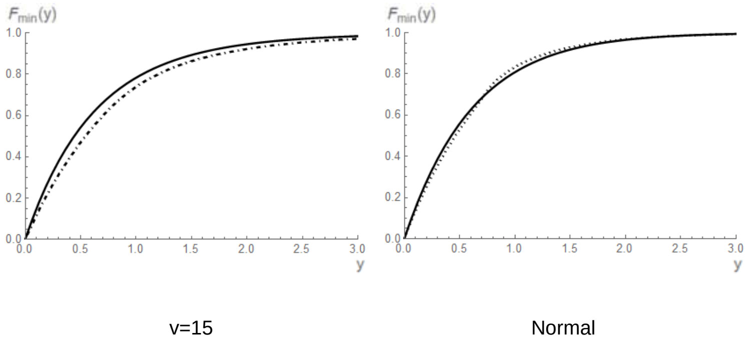


Figure 5.1 Cdf (5.38) and (5.39) for $v = 15$ and simulated values

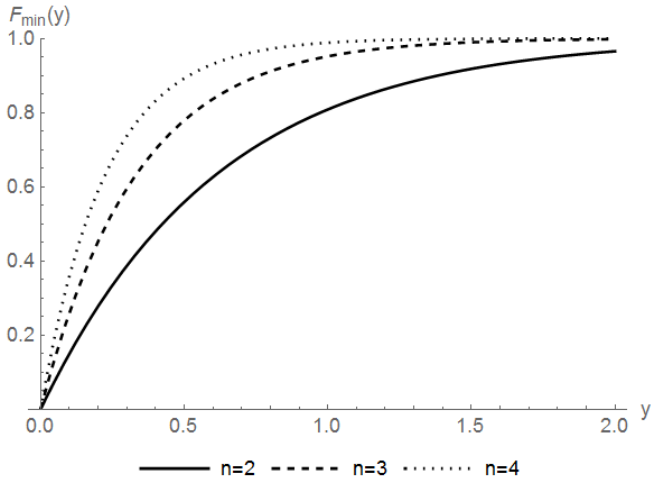


Figure 5.2 Cdf (5.38) for $n = 2, 3, 4$

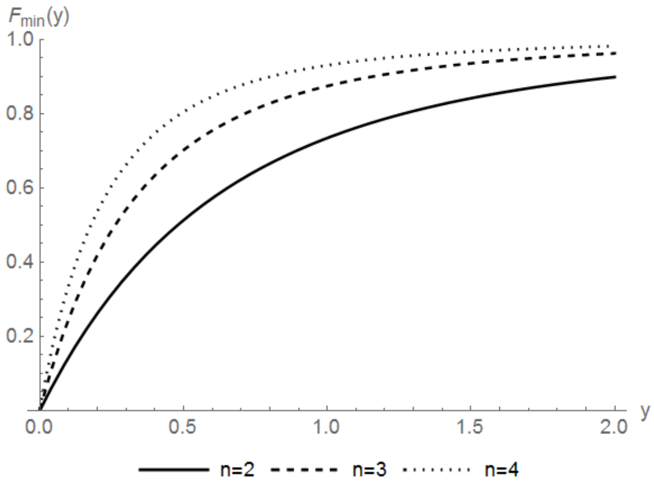


Figure 5.3 Cdf (5.39) for $n = 2, 3, 4$ when $v = 5$

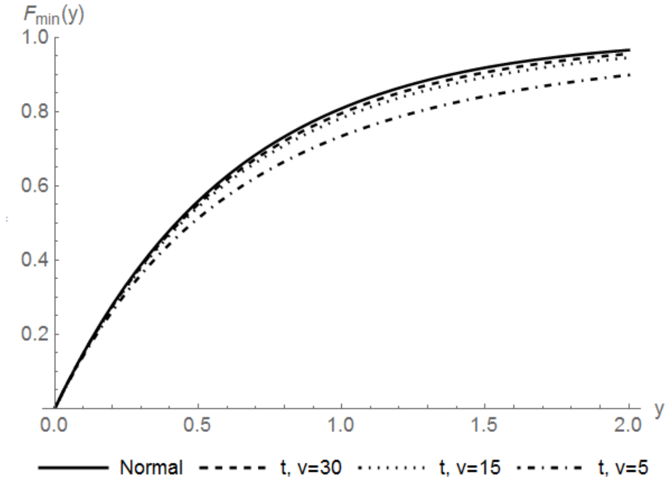


Figure 5.4 Cdf (5.38) and (5.39) for different values of $v = 5, 15, 30$ when $n = 2$

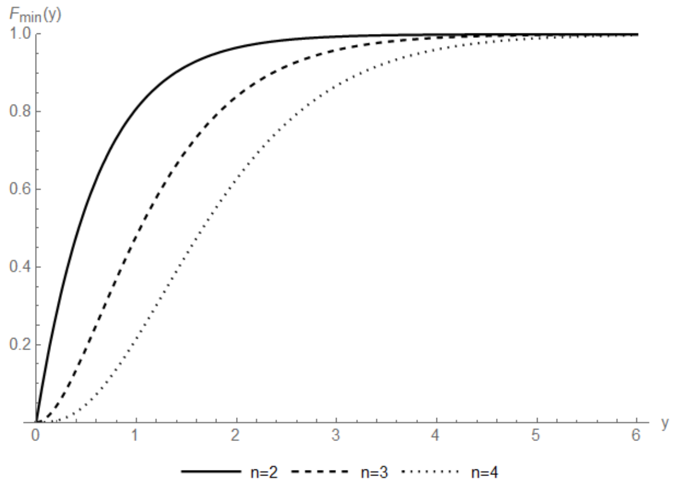


Figure 5.5 Cdf (5.48) for $n = 2, 3, 4$

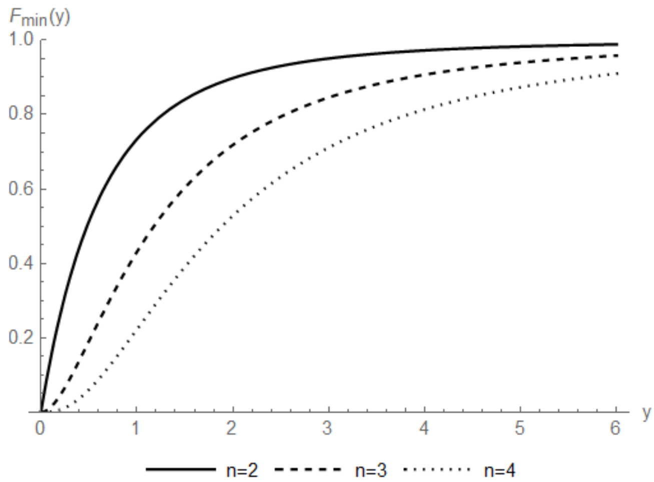


Figure 5.6 Cdf (5.49) for $n = 2, 3, 4$ when $v = 5$

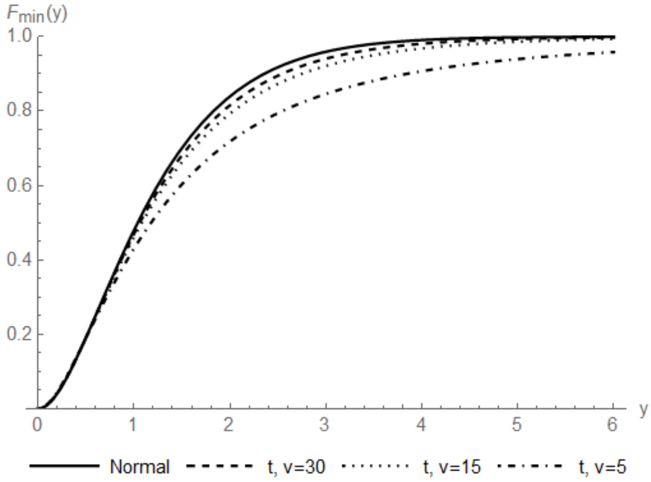


Figure 5.7 Cdf (5.48) and (5.49) for different values of $v = 5, 15, 30$ when $n = 3$

From these figures, the following observations are made:

- Figure 5.2 and 5.5 reflect the same results of (5.38) and (5.48) as reported by Dharmawansa and McKay (2011).
- Figure 5.3 and 5.6 reflects similar behaviour of (5.39) and (5.49) to that of (5.38) and (5.48) respectively.
- In Figure 5.4, it is observed that (5.39) tends to (5.38) as the value of v increases - the same behaviour is observed in Figure 5.7 for (5.48) and (5.49).

5.5 Summary of results and conclusion

A summary of theoretical results in this chapter is provided for the convenience of the reader.

Distribution of X	Pdf of $\mathbf{S}$	Pdf of $\mathbf{\Lambda}$ for $\mathbf{\Sigma}$	Pdf of $\mathbf{\Lambda}$ for $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_p$	Pdf of $\mathbf{\Lambda}$ for arbitrary rank of $\mathbf{\Delta}$
Elliptical	(5.1)	(5.9)	(5.12)	(5.17)
Normal	(5.7)	(5.10)	(5.15)	(5.21)
t	(5.8)	(5.11)	(5.16)	(5.22)

Table 5.2 Pdfs of complex noncentral matrix variate Wishart type distributions in this chapter

Distribution of X	Cdf of $\lambda_{\min}(\mathbf{S})$ where $\mathbf{X} \in \mathbb{C}_1^{n \times p}$	Cdf of $\lambda_{\min}(\mathbf{S})$ where $\mathbf{X} \in \mathbb{C}_1^{n \times n}$	Cdf of $\lambda_{\min}(\mathbf{S})$ where $\mathbf{X} \in \mathbb{C}_1^{n \times 2}$
Elliptical	(5.24)	(5.32)	(5.40)
Normal	(5.30)	(5.38)	(5.48)
t	(5.31)	(5.39)	(5.49)

Table 5.3 Cdfs of minimum eigenvalue of complex noncentral matrix variate Wishart type distributions in this chapter

In this chapter, results were derived and presented for characteristics pertaining to a complex noncentral Wishart type distribution. In particular, the pdf of a complex noncentral Wishart type matrix $\mathbf{S} = \mathbf{X}^H\mathbf{X}$, where $\mathbf{X} \in \mathbb{C}_1^{n\times p} \sim \mathcal{C}E_{n\times p}(\mathbf{M}, \mathbf{I}\otimes\mathbf{\Sigma}, h)$ and the joint pdf of its associated ordered eigenvalues have been derived. Some special cases were investigated, of which the pdf of the eigenvalues when $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ and the noncentral matrix has rank $L < p$, is a noteworthy contribution. Subsequently, the cdf of the minimum eigenvalue of $\mathbf{S}$ was derived for the case when $\mathbf{X} \in \mathbb{C}_1^{n\times p}$, $\mathbf{X} \in \mathbb{C}_1^{n\times n}$, and $\mathbf{X} \in \mathbb{C}_1^{n\times 2}$. These cdfs were derived under the assumption that the noncentral matrix has rank one, which is a practical assumption. This theoretical investigation has proposed impact in communications systems to allow the user a flexible choice of underlying model for $\mathbf{X}$, and thus $\mathbf{S}$; thereby alleviating the restricted assumption of normality available in literature.

Chapter 6

Conclusions

This thesis made significant contributions to the advancement of statistical distribution theory, being inspired from communications systems. Broadly speaking, the assumption of normality as an underlying distribution within communications systems is questioned. From a statistical viewpoint, the t distribution is known particularly for its meaningful connection to the normal distribution. As both these distributions are members of the class of elliptical distributions, this thesis investigates the candidacy of the t distribution as an underlying model within the communications systems framework. Particular results relating to the field of communications systems which has their origin stemming from the normal assumption, are used as a point of departure for their generalisation into the complex elliptical class of distributions. Therefore, by using the representation of the complex elliptical class in (1.4) as basis for this generalisation, this thesis paves the way to investigate the candidacy of the t distribution as viable alternative to that of the well-studied normal.

In Chapter 2 the distribution of $\mathbf{S} = \mathbf{X}^H \mathbf{A} \mathbf{X} \in \mathbb{C}_2^{p \times p}$ is investigated, where $\mathbf{X} \in \mathbb{C}_1^{n \times p}$ and $\mathbf{A} \in \mathbb{C}_2^{n \times n}$. This is commonly known as the quadratic form of $\mathbf{X}$, and in this case, $\mathbf{X}$ is assumed to follow the complex matrix variate elliptical distribution. The central case is of interest, meaning $E(\mathbf{X}) = \mathbf{0}$. In particular, both cases when $\mathbf{X}$ is nonsingular and singular is investigated. The joint distribution of the eigenvalues of $\mathbf{S}$ is also studied. These new results are used to evaluate the capacity of multiple input, multiple output (MIMO) communications systems subject to Rayleigh type fading of wireless signals (see (1.10)). In particular, the complex matrix variate t distribution is comparatively investigated as underlying distribution for $\mathbf{X}$ against the well-studied complex matrix variate normal distribution. It is observed that, under the complex matrix variate t distribution, the existence of correlation between signal transmitters degrades the system capacity. However, under this t assumption, the capacity of the system is higher than that of a system with an underlying complex matrix variate normal distribution.

In Chapter 3, the complex matrix variate inverse Wishart type distribution, emanating from Chapter 2, is proposed. This contribution is used as the platform to derive a bivariate gamma type I distribution stemming from the inverse of the diagonal elements of the complex matrix variate inverse Wishart type distribution. Key characteristics of the new bivariate gamma type I distribution are studied. A bivariate Weibullised gamma type I distribution emanating from the proposed bivariate gamma type I distribution is also introduced. This contribution contains a bivariate Nakagami distribution as a special case, which is a well documented fading model within communications systems. Therefore, this bivariate Weibullised gamma type I distribution is a contribution within the statistical distribution theory domain, and may also act as a generalisation of a bivariate Nakagami distribution to employ within communications systems. Application of the derived results are proposed with regards to the outage probability and the EGC diversity of an $n \times 2$ MIMO system; which is analyzed in a broad generality from an elliptical viewpoint, and comparatively investigated for the underlying complex matrix variate normal- and t cases. Simulation results confirm the validity of the derived analytic expressions. In particular, it is observed that the outage probability is lower for certain subsets of the outage threshold for a fading channel subject to the bivariate gamma type I distribution when the complex matrix variate t distribution is assumed; and the mean EGC diversity is higher under the complex matrix variate t assumption, compared to the usual underlying complex matrix variate normal distribution.

Chapter 4 continues on a bivariate gamma type distribution path, although the point of origin is different from that of Chapter 3. In this chapter, the systematic construction of a bivariate gamma type II distribution from an exponential type distribution is motivated and described. This exponential type distribution emanates from the Rayleigh type distribution (see (1.10)). Key characteristics of this bivariate gamma type II distribution are studied. In addition, a bivariate noncentral gamma type II distribution is also proposed and derived. The bivariate Weibullised gamma type II distribution which emanates from this bivariate gamma type II distribution is also proposed. Corresponding characteristics of this bivariate Weibullised gamma type II distribution are studied. The outage probability of a fading channel operating under this proposed bivariate gamma type II distribution is investigated, and comparatively investigated for the underlying complex matrix variate normal- and t cases. As before, it is observed that outage probability is lower for certain subsets of the outage threshold under the complex matrix variate t assumption. Certain percentiles for the outage probability under this assumption is also calculated to illustrate the statistical contribution of this distribution. Together with the observations in Chapter 3, significant insight is gained into the behaviour of communications systems when subject to a bivariate gamma type I or type II distribution with an underlying complex matrix variate t distribution.

Chapter 5 investigates the distribution of $\mathbf{S} = \mathbf{X}^H\mathbf{X} \in \mathbb{C}_2^{p\times p}$ assuming $\mathbf{X} \in \mathbb{C}_1^{n\times p}$ to follow the complex matrix variate elliptical distribution with $E(\mathbf{X}) = \mathbf{M}$. This assumption of noncentrality is useful when the fading channel is of Rician type, which stems from a direct LOS component between transmitters and receivers. The pdf of the joint eigenvalues of $\mathbf{S}$ are studied, and special cases highlighted. Valuable results pertaining to the assumed behaviour of the noncentrality matrix parameter are given and studied. In particular, the assumption of the noncentrality matrix parameter having rank one is of practical (and hence theoretical) interest. The distribution of the minimum eigenvalue of $\mathbf{S}$ is investigated for some special cases. The chapter concludes with simulation results to confirm the validity of the derived analytic expressions.

The contribution of this thesis to the realm of statistical distribution theory and its advances is significant and useful. Furthermore, the proposal of the complex elliptical class within the communications systems domain paves the way for future innovation and research to be able to consider alternative fading models- and their related engineering tools under an assumption other than that of normality. In future, different members of the complex elliptical class may be explored as possible candidates versus the well-studied-and assumed normal model in this MIMO field (Clavier (2017)).

Several avenues remain which may be worthwhile pursuing for future research which emanates from this thesis. Weight functions alternative to that of the t distribution may be considered such as the generalised slash, Pearson type VII, and mixture normal, which are all contained within the class of complex elliptical distributions. The candidacy of these distribution may provide further insight into the behaviour of communications systems and the statistical modeling thereof. Different correlation assumptions in the analysis of channel capacity may be considered to account for negative correlation between transmitters and receivers (see Chapter 2). The bivariate noncentral gamma type II distribution in Chapter 4 can provide further significant insight into the theory of fading channels, as the noncentrality component sets the platform for investigation of Rician type fading subject to such a bivariate noncentral gamma type II distribution. Within a broad spectrum, the bivariate gamma type distributions in Chapters 3 and 4 can be investigated more closely to consider further physical interpretations of the parameters; along with statistical inference regarding the parameters in question. Investigating the behaviour of condition numbers of complex Wishart type distributions when subject to an underlying complex matrix variate t distribution (see Ratnarajah and Vaillancourt (2003)) may be of interest. In addition, the work of Dharmawansa et. al. (2009) may be generalised to random variables with origins in the complex elliptical class.

APPENDICES

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C. Scalar special functions and theory

Result 1 *(Bain and Engelhardt (1992), p. 206).*

For a transformation of k variables $y = \mathbf{u}(\mathbf{x})$ with a unique solution $x = (x_1, x_2, ..., x_k)$, the Jacobian is the determinant of the $k \times k$ matrix of partial derivatives:

$$J = J(x_1, x_2, ..., x_k \rightarrow y_1, y_2, ..., y_k) = \operatorname{mod}\det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_k} \\ \frac{\partial x_2}{\partial y_1} & \ddots & & \vdots \\ \vdots & & & \\ \frac{\partial x_k}{\partial y_1} & \cdots & & \frac{\partial x_k}{\partial y_k} \end{pmatrix}. \tag{C.1}$$

Result 2 *(Bain and Engelhardt (1992), p. 111).*

A random variable X is said to follow a gamma distribution, denoted by $X \sim Gamma(\Omega, m)$, if X has pdf:

$$f(x) = \frac{1}{\Omega^m \Gamma(m)} x^{m-1} \exp\left(-\frac{x}{\Omega}\right) \tag{C.2}$$

where $x > 0$ and $\Omega, m > 0$. When $m = 1$, then X is said to an exponential distribution with parameter $\Omega > 0$, denoted by $X \sim Exp(\Omega)$. The Laplace transform of X is given by:

$$\mathcal{L}(z) = \frac{1}{(1 + \Omega z)^m}.$$

Result 3 *(Bain and Engelhardt (1992), p. 104).*

A random variable K is said to follow a Poisson distribution, denoted by $K \sim Poi(\theta)$, if K has pmf:

$$g(k) = \frac{\exp(-\theta)\,\theta^k}{k!}, \quad k = 0, 1, 2, 3... \tag{C.3}$$

where $\theta > 0$, and with $E(K) = \theta$.

Result 4 *(Arashi et. al. (2012)).*

The dirac delta function is the function $\delta(x)$ with the property such that:

$$\int_{\mathbb{R}} f(x)\delta(x)dx = f(0) \tag{C.4}$$

for every Borel-measurable function $f(x)$.

Result 5 *(Gradshteyn and Rhyzik (2007), p. 892, eq. 8.310.1).*

The gamma function, denoted $\Gamma(\alpha)$, is defined as:

$$\Gamma(\alpha) = \int_0^\infty \exp(-t)\, t^{\alpha-1} dt \tag{C.5}$$

where $\operatorname{Re}(\alpha) > 0$.

Result 6 *(Gradshteyn and Rhyzik (2007), p. 899, eq. 8.350.1).*

The lower incomplete gamma function, denoted $\gamma(x, \alpha)$, is defined as:

$$\gamma(x, \alpha) = \int_0^x \exp(-t)\, t^{\alpha-1} dt \tag{C.6}$$

where $\operatorname{Re}(\alpha) > 0$.

Result 7 *(Gradshteyn and Rhyzik (2007), p. 899, eq. 8.350.2).*

The upper incomplete gamma function, denoted $\Gamma(x,\alpha)$, is defined as:

$$\Gamma(x,\alpha) = \int_x^\infty \exp(-t)\, t^{\alpha-1} dt \tag{C.7}$$

where $\operatorname{Re}(\alpha) > 0$.

Result 8 *(Gradshteyn and Rhyzik (2007), p. 908; p. 909).*

The beta function, denoted $B(\alpha,\beta)$, is defined as:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{C.8}$$

where $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$, and $\Gamma(\alpha)$ denotes the gamma function (see C.5).

Result 9 *(Mathai (1993), p. 96).*

The Pochhammer coefficient, denoted $(\alpha)_j$, is defined as:

$$(\alpha)_j = \alpha(\alpha+1)\dots(\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \tag{C.9}$$

where $j = 1, 2, \dots$, $(\alpha)_0 = 1$, $\alpha \neq 0$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\alpha+j) > 0$ and $\Gamma(\alpha)$ denotes the gamma function (see C.5).

Result 10 *(Mathai (1993), p. 96).*

The hypergeometric series with p upper parameters and q lower parameters of scalar argument is defined as:

$${}_pF_q(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_q;x) = \sum_{j=0}^{\infty} \frac{(\alpha_1)_j \dots (\alpha_p)_j}{(\beta_1)_j \dots (\beta_q)_j} \frac{x^j}{j!} \tag{C.10}$$

where $(\alpha)_j$ is the Pochhammer coefficient (see C.9).

The following holds for the series:

(i) if any α_i, $i = 1, \dots p$, is a negative integer or zero the series terminates and ${}_pF_q$ becomes a polynomial in x provided none of β_k, $k = 1, \dots, q$, is zero or a negative integer;

(ii) if any β_k, $k = 1, \dots q$, is zero or a negative integer then the series is not defined unless there is an α_i, $i = 1, \dots p$, such that $(\alpha_i)_j$ becomes zero first. That is, suppose α_i and β_k are two negative integers such that $(\alpha_i)_\ell = 0$ for $\ell \geq j$ and $(\beta_k)_\ell = 0$ for $\ell \geq n$. Then in order for ${}_pF_q$ to be defined j must be less than n;

(iii) the series converges for all x if $p \leq q$ and for $|x| < 1$ if $p = q + 1$;

(iv) the series diverges for all x, $x \neq 0$ for $p > q + 1$;

(v) if $p = q + 1$ and $|x| = 1$, the series is absolutely convergent if $\operatorname{Re}(\gamma) < 0$ where $\gamma = \sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j$; divergent if $\operatorname{Re}(\gamma) \geq 1$; and if $p = q + 1$ and $|x| = 1$, $x \neq 1$, the series is conditionally convergent if $0 \leq \operatorname{Re}(\gamma) < 1$.

Some special cases of the hypergeometric series:

- The exponential function:

$${}_0F_0(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} = \exp(x). \tag{C.11}$$

- The binomial series:

$$ {}_1F_0(\alpha;x)=\sum_{j=0}^{\infty}(\alpha)_j\frac{x^j}{j!}=(1-x)^{-\alpha}\quad ,\ |x|<1. \tag{C.12}$$

-

$$ {}_0F_1(\beta;x)=\sum_{j=0}^{\infty}\frac{x^j}{(\beta)_j\,j!}. \tag{C.13}$$

- The confluent hypergeometric series or Kummer's hypergeometric series:

$$ {}_1F_1(\alpha;\beta;x)=\sum_{j=0}^{\infty}\frac{(\alpha)_j}{(\beta)_j}\frac{x^j}{j!}. \tag{C.14}$$

- The Gauss hypergeometric function:

$$ {}_2F_1(\alpha,\beta;\varsigma;x)=\sum_{j=0}^{\infty}\frac{(\alpha)_j(\beta)_j}{(\varsigma)_j}\frac{x^j}{j!}\quad ,\ |x|<1. \tag{C.15}$$

Result 11 *(Nakagami (1960), p. 30, eq. 123).*

Suppose that (U_1,U_2) follows a bivariate gamma distribution as proposed in Nakagami (1960). The Laplace transform of (U_1,U_2) is given by:

$$\begin{aligned}\mathcal{L}(s_1,s_2) &= (\Omega_1)^{-m_1}(\Omega_2)^{-m_1}(1-\rho)^{-m_1} \\ &\quad\times\left[\left(s_1+\frac{1}{(\Omega_1)(1-\rho)}\right)\left(s_2+\frac{1}{(\Omega_2)(1-\rho)}\right)-\frac{\rho}{(\Omega_1)(\Omega_2)(1-\rho^2)}\right]^{-m_1}\end{aligned} \tag{C.16}$$

for $m_1,m_2,\Omega_1,\Omega_2>0$ and $-1\le\rho\le1$.

Result 12 *(Gradshteyn and Rhyzik (2007), p. 919, eq. 8.445).*

The modified Bessel function of the first kind, denoted by $I_v(x)$, is defined as:

$$I_v(x)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(v+k+1)}\left(\frac{x}{2}\right)^{v+2k} \tag{C.17}$$

for $v\ge0$.

Result 13 *(Mathai (1993), p. 87).*

If $f(x)$ is a real function which is single valued almost everywhere for $x\ge0$ and if the integral:

$$\int_0^{\infty}\exp(tx)\,|f(x)|\,dx$$

converges for some value of t then the Laplace transform of $f(x)$ is defined as follows:

$$\mathcal{L}_f(s)\equiv\int_0^{\infty}\exp(-sx)\,f(x)\,dx \tag{C.18}$$

where $\mathcal{L}_f(s)$ is the Laplace transform of f with respect to the parameter s. The inverse Laplace transform is given by the inverse integral:

$$f(s)=\tfrac{1}{2\pi l}\int_{\omega-l\infty}^{\omega+l\infty}\mathcal{L}_f(s)\exp(sx)\,dh \tag{C.19}$$

where $l=\sqrt{-1}$ and ω is a real number in the strip of analyticity of $\mathcal{L}_f(h)$.

Result 14 *(Mathai (1993), p. 60).*

Meijer's G-function with the parameters $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ is defined as:

$$G_{r,s}^{m,n}\left(x|_{\beta_1,\ldots,\beta_s}^{\alpha_1,\ldots,\alpha_r}\right) = \frac{1}{2\pi l}\int_L g(h)\, x^{-h} dh \tag{C.20}$$

where $l = \sqrt{-1}$, L is a suitable contour, $x \neq 0$, and:

$$g(h) = \frac{\prod_{j=1}^{m}\Gamma(\beta_j+h)\prod_{j=1}^{n}\Gamma(1-\alpha_j-h)}{\prod_{j=m+1}^{s}\Gamma(1-\beta_j-h)\prod_{j=n+1}^{r}\Gamma(\alpha_j+h)}$$

where m, n, r and s are integers with $0 \leq n \leq r$ and $0 \leq m \leq s$. The parameters $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ are complex numbers such that no pole of $\Gamma\left(\beta_j + h\right)$, $j = 1, \ldots, m$ coincides with any pole of $\Gamma(1 - \alpha_k - h)$, $k = 1, \ldots, n$. The empty product is interpreted as 0. A special case of C.20 is:

$$\exp\left(-\theta x^k\right) = G_{0,1}^{1,0}\left(\theta x^k|_{0}^{-}\right). \tag{C.21}$$

Result 15 *(Gradshteyn and Rhyzik (2007), p. 337).*

$$\int_0^\infty x^{\alpha-1}\exp(-\beta x)\, dx = \beta^{-\alpha}\Gamma(\alpha) \tag{C.22}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ *and* $\Gamma(\alpha)$ *denotes the gamma function (see C.5).*

Result 16 *(Gradshteyn and Rhyzik (2007), p. 346, eq. 3.381.8).*

$$\int_0^u x^m \exp(-\beta x^n)\, dx = \frac{\gamma(v, \beta u^n)}{n\beta^v} \tag{C.23}$$

where $v = \frac{m+1}{n}$, $\gamma(\cdot,\cdot)$ *denotes the lower incomplete gamma function (see C.6), and* $u, v, n, \beta > 0$.

Result 17 *(Gradshteyn and Rhyzik (2007), p. 347, eq. 3.381.11).*

$$\int_0^\infty x^{2m}\exp\left(-\beta x^{2n}\right) dx = \frac{\Gamma(v)}{2n\beta^v} \tag{C.24}$$

where $v = \frac{2m+1}{2n}$, $\Gamma(\alpha)$ *denotes the gamma function (see C.5), and* $u, v, n, \beta > 0$.

Result 18 *(Gradshteyn and Rhyzik (2007), p. 346, eq. 3.381.1).*

$$\int_0^u x^{\alpha-1}\exp(-\beta x)\, dx = \mu^{-\alpha}\gamma(\alpha, \beta u) \tag{C.25}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ *and* $\gamma(\cdot,\cdot)$ *denotes the lower incomplete gamma function (see C.6).*

Result 19 *(Gradshteyn and Rhyzik (2007), p. 346, eq. 3.381.3).*

$$\int_u^\infty x^{\alpha-1}\exp(-\beta x)\, dx = \mu^{-\alpha}\Gamma(\alpha, \beta u) \tag{C.26}$$

where $\Gamma(\cdot,\cdot)$ *denotes the upper incomplete gamma function (see C.7), and* $u, v, \mu > 0$.

Result 20 *(Gradshteyn and Rhyzik (2007), p. 315, eq. 3.191.2).*

$$\int_{1}^{\infty} \frac{(y-1)^k}{y^{n+m+k}} dy = B(n+m-1, k+1) \tag{C.27}$$

where $B(\cdot,\cdot)$ *is the beta function (see C.8) and* $n, m, k > 0$.

Result 21 *(Gradshteyn and Rhyzik (2007), p. 657, eq. 6.455.1).*

$$\int_{0}^{\infty} x^{\alpha-1} \exp(-\beta x) \Gamma(v, \theta x) dx = \frac{\theta^v \Gamma(\alpha+v)}{\alpha(\theta+\beta)^{\alpha+v}} {}_2F_1\left(1, \alpha+v; \alpha+1; \frac{\beta}{\theta+\beta}\right) \tag{C.28}$$

where ${}_2F_1(\cdot)$ *is the Gauss hypergeometric function (see C.15), and* $\alpha, \beta, \mu, v > 0$.

Result 22 *(Gradshteyn and Rhyzik (2007), p. 25, eq. 1.111).*

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}. \tag{C.29}$$

Result 23 *(Bain and Engelhardt (1992), p. 178, eq. 5.3.1).*

The correlation coefficient between two variables X_1 and X_2 is denoted by ρ, and is given by:

$$\rho = \frac{E(X_1X_2) - E(X_1)E(X_2)}{\sqrt{E(X_1^2) - E(X_1)^2}\sqrt{E(X_2^2) - E(X_2)^2}}. \tag{C.30}$$

Result 24 *(Mathai (1993), p. 80, eq. 3.2.2).*

$$\begin{aligned} &\int_{0}^{\infty} x^{\sigma-1} G_{p,q}^{m,n}\left(\omega x | \begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix}\right) G_{\gamma,\delta}^{\alpha,\beta}\left(\eta x^{\frac{k}{\rho}} | \begin{matrix} c_1, c_2, ..., c_\gamma \\ d_1, d_2, ..., d_\delta \end{matrix}\right) dx \\ = \;& \omega^{-\sigma} (2\pi)^{c^*(1-k)+b^*(1-\rho)} k^{U+\sigma(q-p)-1} \rho^V \\ &\times G_{\rho\gamma+kq,\rho\delta+kp}^{\rho\alpha+kn,\rho\beta+km}\left(W | \begin{matrix} \Delta(\rho,c_1),...,\Delta(\rho,c_\beta),\Delta(k,1-b_q-\sigma),\Delta(\rho,c_{\beta+1}),...,\Delta(\rho,c_\gamma) \\ \Delta(\rho,d_1),...,\Delta(\rho,d_\alpha),\Delta(k,1-a_p-\sigma),\Delta(\rho,d_{\alpha+1}),...,\Delta(\rho,d_\delta) \end{matrix}\right) \end{aligned} \tag{C.31}$$

where k, ρ *are positive integers,* $c^* = m+n-\frac{p}{2}-\frac{q}{2}$, $b^* = \alpha+\beta-\frac{\gamma}{2}-\frac{\delta}{2}$, $U = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p}{2} - \frac{q}{2} + 1$, $V = \sum_{j=1}^{\delta} d_j - \sum_{j=1}^{\gamma} c_j + \frac{\gamma}{2} - \frac{\delta}{2} + 1$, *and* $W = \frac{\eta^\rho \rho^{\rho(\gamma-\delta)}}{\omega^k k^{k(p-q)}}$. *Particularly, note that* $\Delta(b,q)$ *denotes the set of* b *total values* $\left\{\frac{q}{b}, \frac{q+1}{b}, ..., \frac{q+b-1}{b}\right\}$.

Result 25 *(Gradshteyn and Rhyzik (2007), p. 815, eq. 7.522.9, eq. 7.525.1).*

$$\int_{0}^{\infty} x^{\sigma-1} \exp(-\mu x) \; {}_pF_q(a_1,...,a_p; b_1,...,b_q; \lambda x) dx = \Gamma(\sigma) \mu^{-\sigma} \; {}_{p+1}F_q\left(a_1,...,a_p,\sigma; b_1,...,b_q; \frac{\lambda}{\mu}\right) \tag{C.32}$$

where $p \leq q$, $\sigma, \mu > 0$, ${}_pF_q(\cdot)$ *denotes the hypergeometric function (see C.10) and* $\Gamma(\alpha)$ *denotes the gamma function (see C.5).*

Result 26 *(Erdelyi et. al. (1954), p. 137, eq. 4.3.1, p. 217, eq. 4.23.1).*

$$\frac{1}{2\pi l} \int_{c-l\infty}^{c+l\infty} \exp(ps) \Gamma(v+1) p^{-v-1} dp = s^v \tag{C.33}$$

where $v > 0$ and $\Gamma(\cdot)$ denotes the gamma function (see C.5). Furthermore:

$$\frac{1}{2\pi l}\int_{c-l\infty}^{c+l\infty} \exp(ps)\,\Gamma(\gamma)\,p^{\alpha-\gamma}(p-\lambda)^{-\alpha}\,dp = s^{\gamma-1}\ {}_1F_1(\alpha,\gamma;\lambda s) \tag{C.34}$$

where ${}_1F_1(\cdot)$ denotes the confluent hypergeometric function (see C.14), and where $\alpha, \gamma, \lambda > 0$.

Result 27 *(Gradshteyn and Rhyzik (2007), p. 1031, eq. 9.261.3).*

The Humbert hypergeometric series of two variables x and y is given by:

$$\Phi_3(a,b;x,y) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_r}{(b)_{r+j}\,j!r!}x^j y^r \tag{C.35}$$

where $(\alpha)_j$ is the Pochhammer coefficient (see C.9).

Result 28 *(Gradshteyn and Rhyzik (2007), p. 991, eq. 8.932.1).*

The Gegenbauer polynomial of degree n can be expressed in terms of the Gauss hypergeometric function ${}_2F_1(\cdot)$ (see C.15) as:

$$\mathfrak{C}_n^v(t) = \frac{\Gamma(2v+n)}{\Gamma(n+1)\,\Gamma(2v)}\ {}_2F_1\left(2v+n,-n,v+\frac{1}{2};\frac{1-t}{2}\right), \tag{C.36}$$

where $\Gamma(\alpha)$ denotes the gamma function (see C.5) and ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function (see C.15).

D. Matrix special functions and theory

Let $\mathbb{C}_1^{n\times p}$ denote the space of $n\times p$ complex matrices, and $\mathbb{C}_2^{p\times p}$ denote the space of $p\times p$ Hermitian positive definite matrices.

Jacobians of transformations

Result 29 *(Ratnarajah (2003), Ratnarajah et. al. (2004)).*

The set of all $n\times p$ $(n\geq p)$ matrices, $\mathbf{E}$, with orthonormal columns is called the Stiefel manifold, denoted by $\mathcal{C}V_{p,n}$. Thus $\mathcal{C}V_{p,n}=\left\{\mathbf{E}\left(n\times p\right);\mathbf{E}^H\mathbf{E}=\mathbf{I}_p\right\}$. The volume of this manifold is given by:

$$Vol\left(\mathcal{C}V_{p,n}\right)=\int\limits_{\mathcal{C}V_{p,n}}\left(\mathbf{E}^H d\mathbf{E}\right)=\frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}. \tag{D.37}$$

If $n=p$ then a special case of the Stiefel manifold is obtained:

$$\mathcal{C}V_{p,p}=\left\{\mathbf{E}\left(p\times p\right);\mathbf{E}^H\mathbf{E}=\mathbf{I}_p\right\}\equiv U\left(p\right) \tag{D.38}$$

where $U\left(p\right)$ denotes the space of unitary $p\times p$ matrices. The volume of $U\left(p\right)$ is given by:

$$Vol\left(U\left(p\right)\right)=\int\limits_{U(p)}\left(\mathbf{E}^H d\mathbf{E}\right)=\frac{2^p\pi^{p^2}}{\mathcal{C}\Gamma_p(p)}. \tag{D.39}$$

Furthermore:

$$\int\limits_{\mathcal{C}V_{p,n}}\text{etr}\left(\mathbf{X}\mathbf{E}\right)\left(\mathbf{E}^H d\mathbf{E}\right)=\frac{2^p\pi^{np}}{\mathcal{C}\Gamma_p(n)}\,{}_0\mathcal{C}F_1\left(n;\frac{1}{4}\mathbf{X}\mathbf{X}^H\right) \tag{D.40}$$

where $\mathbf{X}\in\mathbb{C}_1^{p\times n}$.

Result 30 *(Ratnarajah (2003) (p. 41), Ratnarajah et. al. (2004), Ratnarajah (2005)).*

Suppose that $\mathbf{X}\in\mathbb{C}_1^{n\times p}$ and $\mathbf{S}=\mathbf{X}^H\mathbf{X}\mathbb{C}_2^{p\times p}$. The Jacobian of the transformation from $\mathbf{X}$ to $\mathbf{S},\mathbf{E}$ with $\mathbf{E}\in\mathcal{C}V_{p,n}$ is given by:

$$J\left(\mathbf{X}\to\mathbf{S},\mathbf{E}\right)=2^{-p}\det\left(\mathbf{S}\right)^{n-p}\left(\mathbf{E}^H d\mathbf{E}\right) \tag{D.41}$$

In the singular case, the Jacobian is given by:

$$J\left(\mathbf{X}\to\mathbf{S},\mathbf{E}\right)=2^{-n}\det\left(\mathbf{\Lambda}\right)^{n-p}\left(\mathbf{E}^H d\mathbf{E}\right). \tag{D.42}$$

Result 31 *(Nagar and Gupta (2009)).*

Suppose that $\mathbf{Y},\mathbf{X}\in\mathbb{C}_1^{n\times p}$ and $\mathbf{A}\in\mathbb{C}_1^{n\times n}$. If $\mathbf{Y}=\mathbf{A}^{-\frac{1}{2}}\mathbf{X}$, then the Jacobian is given by:

$$J\left(\mathbf{Y}\to\mathbf{X}\right)=\det\left(\mathbf{A}\right)^{-p}. \tag{D.43}$$

Result 32 *(Maiwald and Kraus (1997)).*

Suppose that $\mathbf{S}=\mathbf{H}^H\mathbf{H}$, with $\mathbf{H}\in\mathbb{C}_1^{n\times 2}$ and thus $\mathbf{S}\in\mathbb{C}_2^{2\times 2}$. If $\mathbf{W}=\mathbf{S}^{-1}\in\mathbb{C}_2^{2\times 2}$, then the Jacobian is given by:

$$J(\mathbf{S}\to\mathbf{W}^{-1})=\det\left(\mathbf{W}\right)^{-2n}. \tag{D.44}$$

Result 33 *(Dharmawansa and McKay (2011)).*

Suppose that $\mathbf{S}\in\mathbb{C}_2^{p\times p}$. Let $y>0$ be some constant and $\mathbf{I}_p$ is the identity matrix of dimension p. If $\mathbf{S}=y\left(\mathbf{I}_p+\mathbf{Y}\right)$, then the Jacobian is given by:

$$d\mathbf{S}=y^{p^2}d\mathbf{Y}. \tag{D.45}$$

Other results

Result 34 *(Andersen et. al. (1995), p. 170).*

Let $\mathbf{A}$ denote a complex matrix. If $\mathbf{A} = \mathbf{A}^H$ then $\mathbf{A}$ is called a Hermitian matrix, and denoted to be within the space $\mathbb{C}_2^{p \times p}$.

Result 35 *(Gupta and Nagar (2000), p. 7).*

If $\mathbf{A} \in \mathbb{C}_2^{p \times p}$ then there exists a Hermitian positive definite matrix $\mathbf{B} \in \mathbb{C}_2^{p \times p}$ such that $\mathbf{A} = \mathbf{B}^2$. Furthermore, the square root of $\mathbf{A}$ is defined as:

$$\mathbf{A}^{\frac{1}{2}} = \mathbf{B}. \tag{D.46}$$

Result 36 *(Diaz-Garcia (2009), eq. 2.3).*

The complex multivariate gamma function, denoted $\mathcal{C}\Gamma_p(\alpha)$, is defined as:

$$\mathcal{C}\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{2}} \prod_{i=1}^{p} \Gamma[\alpha - (i-1)] \tag{D.47}$$

where $\mathrm{Re}(\alpha) > p - 1$ and $\Gamma(\alpha)$ denotes the gamma function (see C.5). For $p = 1$ it simplifies to the gamma function (see C.5).

Result 37 *(Mathai (1997), p. 365, eq. 6.1.10, Diaz-Garcia (2009), eq. 2.4).*

Let $\kappa = (k_1, \ldots, k_p)$ denote a partition of the nonnegative integer k such that $k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$. The generalised hypergeometric coefficient $[\alpha]_\kappa$ is defined as:

$$\begin{aligned} [\alpha]_\kappa &= \prod_{i=1}^{p} (\alpha - (i-1))_{k_i} \\ &= \frac{\mathcal{C}\Gamma_p(\alpha, \kappa)}{\mathcal{C}\Gamma_p(\alpha)} \end{aligned} \tag{D.48}$$

where $\mathrm{Re}(\alpha) > (p-1) - k_p$, $(\alpha)_m$ is the Pochhammer coefficient (see C.9), $\mathcal{C}\Gamma_p(\alpha)$ is the complex multivariate gamma function (see D.47) and $\mathcal{C}\Gamma_p(\alpha, \kappa)$ is the partitioned complex multivariate gamma function with weight κ, where:

$$\begin{aligned} \mathcal{C}\Gamma_p(\alpha, \kappa) &= \pi^{\frac{p(p-1)}{2}} \prod_{i=1}^{p} \Gamma[\alpha + k_i - (i-1)] \\ &= \mathcal{C}\Gamma_p(\alpha) \prod_{i=1}^{p} (\alpha + k_i - (i-1))_{k_i}. \end{aligned} \tag{D.49}$$

For $p = 1$ it simplifies to the Pochhammer coefficient (see C.9).

A brief description of complex zonal polynomials and results involving complex zonal polynomials are given next. For a more detailed discussion see James (1964) and Constantine (1963).

Result 38 *(Diaz-Garcia (2009), Ratnarajah et. al. (2004)).*

Let $\mathbf{S} \in \mathbb{C}_2^{p \times p}$ and let V_k be the vector space of homogeneous polynomials $\phi(\mathbf{S})$ of degree k in the elements of $\mathbf{S}$. The space V_k can be decomposed into a direct sum of irreducible invariant subspaces V_κ where $\kappa = (k_1, \ldots, k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$. The polynomial $(\mathrm{tr}\,\mathbf{S})^k \in V_k$ has a unique decomposition into polynomials $C_\kappa(\mathbf{S}) \in V_k$ as:

$$(\mathrm{tr}\,\mathbf{S})^k = \sum_\kappa C_\kappa(\mathbf{S}). \tag{D.50}$$

The zonal polynomial $C_\kappa(\mathbf{S})$ is defined as the component of $(\mathrm{tr}\,\mathbf{S})^k$ in the subspace V_κ. It is a symmetric homogeneous polynomial of degree k in the latent roots of $\mathbf{S}$ and holds for all p. If the partition κ has more than p parts, the corresponding zonal polynomial will be identically zero.

Result 39 *(Constantine (1963), , Ratnarajah et. al. (2004)). The hypergeometric function of Hermitian matrix argument is defined as:*

$$ {}_r\mathcal{C}F_s\left(a_1,...,a_r;b_1,...,b_s;\mathbf{X}\right)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa [a_2]_\kappa ...[a_r]_\kappa}{[b_1]_\kappa [b_2]_\kappa ...[b_s]_\kappa}\frac{C_\kappa(\mathbf{X})}{k!} \tag{D.51}$$

where $\mathbf{X}\in\mathbb{C}_2^{p\times p}$, $[a]_\kappa=\prod_{j=1}^{p}(a-j+1)_{k_j}$, $\kappa=(k_1,k_2,...,k_p)$ *is a partition of* k *such that* $k_1\geq k_2\geq ...\geq k_p\geq 0$, *and* $\sum_{i=1}^{p}k_i=k$, *and* $(a)_k$ *denotes the Pochhammer symbol.* $C_\kappa(\mathbf{X})$ *denotes the zonal polynomial of* $\mathbf{X}$ *(see D.50) and* $(a)_\kappa$ *denotes the complex generalized hypergeometric coefficient (see D.48). Conditions for convergence of the series in (D.51) are given below:*

(i) the series converges for all $\mathbf{X}$ if $r<s+1$, otherwise the series may only converge for $\mathbf{X}=\mathbf{0}$;

(ii) for $r=s+1$ the series converges for $\|\mathbf{X}\|<1$ (where $\|\mathbf{X}\|$ denotes the maximum of the absolute values of the characteristic roots of $\mathbf{X}$);

(iii) for $r\leq s$ the series converges for all $\mathbf{X}$;

(iv) for $r>s+1$ the series diverges for all $\mathbf{X}\neq\mathbf{0}$ unless the series terminates;

(v) none of the β_j is zero, an integer or half integer $\leq\frac{1}{2}(s-1)$ (otherwise some of the denominators in (D.51) will vanish);

(vi) if α_i is a negative integer, say $-w$, then for $t\geq sw+1$, all coefficients in (D.51) vanish and the function reduces to a finite polynomial of degree sw.

A special case of (D.51) is given by:

$$ {}_0\mathcal{C}F_0(\mathbf{X})=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{X})}{k!}=\operatorname{etr}(\mathbf{X}). \tag{D.52}$$

The hypergeometric function of double Hermitian matrix argument is defined as (James (1964), eq. 88, p. 488):

$$ {}_r\mathcal{C}F_s^{(p)}\left(a_1,...,a_r;b_1,...,b_s;\mathbf{X},\mathbf{Y}\right)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa [a_2]_\kappa ...[a_r]_\kappa}{[b_1]_\kappa [b_2]_\kappa ...[b_s]_\kappa}\frac{C_\kappa(\mathbf{X})C_\kappa(\mathbf{Y})}{C_\kappa(\mathbf{I}_p)\,k!} \tag{D.53}$$

where $\mathbf{X},\mathbf{Y}\in\mathbb{C}_2^{p\times p}$.

A special case of (D.53) is given by:

$$ {}_0\mathcal{C}F_0^{(p)}(\mathbf{X},\mathbf{Y})=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{X})C_\kappa(\mathbf{Y})}{k!C_\kappa(\mathbf{I}_p)}. \tag{D.54}$$

For $\mathbf{X}\in\mathbb{C}_2^{n\times n}$ and $\mathbf{Y}\in\mathbb{C}_2^{p\times p}$, and $n<p$, then (see Ratnarajah (2005)):

$$ {}_0\mathcal{C}F_0^{(n)}(\mathbf{X},\mathbf{Y})=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{X})C_\kappa(\mathbf{Y})}{k!C_\kappa(\mathbf{I}_p)}. \tag{D.55}$$

Furthermore:

$$ C_\kappa\left(-t\boldsymbol{\Sigma}^{-1}\right)=t^kC_\kappa\left(-\boldsymbol{\Sigma}^{-1}\right). \tag{D.56}$$

For the Vandermonde determinant $\prod_{k<l}^{p}(\lambda_k-\lambda_l)$, note that:

$$ \prod_{k<l}^{p}(c\lambda_k-c\lambda_l)=\prod_{k<l}^{p}\left(c(\lambda_k-\lambda_l)\right)=c^{\frac{p(p-1)}{2}}\prod_{k<l}^{p}(\lambda_k-\lambda_l) \tag{D.57}$$

for some constant c. Khatri (1969) gives a result where the hypergeometric function of two Hermitian matrix arguments (see D.54), ${}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Upsilon}^{-1}\right)$ can be written as follows (using D.57):

$$\begin{aligned} {}_0\mathcal{C}F_0^{(p)}\left(\mathbf{\Lambda}, -t\mathbf{\Upsilon}^{-1}\right) &= \frac{\mathcal{C}\Gamma_p(p)\det\left(\exp\left(-ta_i\lambda_j\right)\right)}{\pi^{\frac{p(p-1)}{2}}\prod_{k<l}^{p}(\lambda_k-\lambda_l)\prod_{k<l}^{p}(ta_l-ta_k)} \\ &= \frac{\mathcal{C}\Gamma_p(p)\det\left(\exp\left(-ta_i\lambda_j\right)\right)}{\pi^{\frac{p(p-1)}{2}}t^{\frac{p(p-1)}{2}}\prod_{k<l}^{p}(\lambda_k-\lambda_l)\prod_{k<l}^{p}(a_l-a_k)} \end{aligned} \quad \text{(D.58)}$$

where $\lambda_1 > ... > \lambda_p > 0$ denotes the eigenvalues of $\mathbf{\Lambda}$, and $a_1 > ... > a_p$ denotes the eigenvalues of $\mathbf{\Upsilon}$. Note that $\det\left(\exp\left(-ta_i\lambda_j\right)\right)$ denotes the determinant of a matrix with entries $\exp\left(-ta_i\lambda_j\right)$.

Result 40 *(James (1964), eq. 93, p. 488).*

The joint pdf of the eigenvalues $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ of $\mathbf{S} \in \mathbb{C}_2^{p\times p}$ is given by:

$$f(\mathbf{\Lambda}) = \frac{\pi^{p(p-1)}\left(\prod_{k<l}^{p}(\lambda_k-\lambda_l)^2\right)}{\mathcal{C}\Gamma_p(p)} \int_{\mathbf{E}\in U(p)} f\left(\mathbf{E\Lambda E}^H\right)d\mathbf{E} \quad \text{(D.59)}$$

where $U(p)$ denotes the unitary space (see D.38).

Result 41 *(Diaz-Garcia (2009), eq. 3.11).*

If $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{C}_2^{p\times p}$, then:

$$\int_{\mathbf{H}\in U(p)} C_\kappa(\mathbf{X}_1\mathbf{H}\mathbf{X}_2\mathbf{H}^H)d\mathbf{H} = \frac{C_\kappa(\mathbf{X}_1)C_\kappa(\mathbf{X}_2)}{C_\kappa(\mathbf{I}_p)}. \quad \text{(D.60)}$$

Result 42 *(Analogous to Bekker et. al. (2011)).*

Suppose $\mathbf{Q} \in \mathbb{C}_2^{n\times n}, \mathbf{R} \in \mathbb{C}_2^{p\times p}$, and $\mathbf{E}_1 \in U(n), \mathbf{E}_2 \in U(p)$. Consider the transformations $\mathbf{B} \to \mathbf{E}_1\mathbf{B}\mathbf{E}_1^H$ and $\mathbf{B} \to \mathbf{E}_2\mathbf{B}\mathbf{E}_2^H$. Define the following:

$$f(\mathbf{Q}, \mathbf{R}) = \int_{\mathbf{X}^H\mathbf{X}=\mathbf{S}} C_\kappa(-\mathbf{QXRX}^H)d\mathbf{X}$$

where $C_\kappa(\cdot)$ denotes the complex zonal polynomial (see Result D.50). This function remains invariant under a unitary transformation:

$$\begin{aligned} f\left(\mathbf{E}_1\mathbf{Q}\mathbf{E}_1^H, \mathbf{E}_2\mathbf{R}\mathbf{E}_2^H\right) &= \int_{\mathbf{X}^H\mathbf{X}=\mathbf{S}} C_\kappa(-\mathbf{E}_1\mathbf{Q}\mathbf{E}_1^H\mathbf{X}\mathbf{E}_2\mathbf{R}\mathbf{E}_2^H\mathbf{X}^H)d\mathbf{X} \\ &= \int_{\mathbf{X}^H\mathbf{X}=\mathbf{S}} C_\kappa(-\mathbf{Q}\mathbf{E}_1^H\mathbf{X}\mathbf{E}_1\mathbf{R}\mathbf{E}_2^H\mathbf{X}^H\mathbf{E}_2)d\mathbf{X} \\ &= \int_{\mathbf{U}^H\mathbf{U}=\mathbf{S}} C_\kappa(-\mathbf{QURU}^H)d\mathbf{U}. \end{aligned} \quad \text{(D.61)}$$

Result 43 *(Diaz-Garcia (2009), eq. 4.8).*

If $\mathbf{X}, \mathbf{Y} \in \mathbb{C}_2^{p\times p}$, then:

$$\int_{\mathbf{H}\in U(p)} {}_r\mathcal{C}F_s\left(a_1,...,a_r; b_1,...,b_s; \mathbf{XHYH}\right)d\mathbf{H} = {}_r\mathcal{C}F_s^{(p)}\left(a_1,...,a_r; b_1,...,b_s; \mathbf{X}, \mathbf{Y}\right) \quad \text{(D.62)}$$

where ${}_r\mathcal{C}F_s^{(p)}\left(a_1,...,a_r; b_1,...,b_s; \mathbf{X}, \mathbf{Y}\right)$ denotes the hypergeometric function of two Hermitian matrix arguments (see D.54).

Result 44 *(McKay and Collings (2005), eq. 115).*

For $\mathbf{T}=\mathbf{T}^H, \mathbf{Z}, \mathbf{A} \in \mathbb{C}_1^{n\times n}$, the following holds:

$$\begin{aligned} & \int \det(\mathbf{T})^{c-n} \operatorname{etr}(-\mathbf{ZT}) \ {}_p\mathcal{C}F_q(a_1,...,a_p;b_1,...,b_q,\mathbf{AT})\,d\mathbf{T} \\ = \ & \det(\mathbf{Z})^{-c}\,\mathcal{C}\Gamma_n(c)\ {}_{p+1}\mathcal{C}F_q\left(a_1,...,a_p,c;b_1,...,b_q,\mathbf{AZ}^{-1}\right), \end{aligned}$$

where $\mathcal{C}\Gamma_n(c)$ denotes the complex multivariate gamma function (see D.47).

Result 45 *(Mathai (1997), p. 365, eq. 6.1.20).*

Let $\mathbf{Z}=\mathbf{Z}^{\mathbf{H}}, \mathbf{S} \in \mathbb{C}_2^{p\times p}$. Then:

$$\int \operatorname{etr}(\mathbf{ZS}) \det(\mathbf{Z})^{\alpha-p} C_\kappa(\mathbf{ZT})\,d\mathbf{Z} = \mathcal{C}\Gamma_p(\alpha,\kappa) \det(\mathbf{S})^{-\alpha} C_\kappa\left(\mathbf{TS}^{-1}\right) \tag{D.63}$$

where $\mathcal{C}\Gamma_p(\alpha,\kappa)$ denotes the complex multivariate gamma function pertaining to partition κ (see D.49).

Result 46 *(Dharmawansa and McKay (2011), eq. 17).*

Suppose $\mathbf{A} \in \mathbb{C}_2^{2\times 2}$, let $\mathbf{R} \in \mathbb{C}_1^{2\times 2}$ with unit rank. For p, t integers and $a > 1$, then:

$$\begin{aligned} & \int\limits_{\mathbf{X}>\mathbf{0}} \operatorname{etr}(-\mathbf{AX}) \operatorname{tr}^p(\mathbf{X}) \det(\mathbf{X})^{a-2} \operatorname{tr}^t(\mathbf{RX})\,d\mathbf{X} \\ = \ & p!\frac{(a)_t\,\mathcal{C}\Gamma_2(a)}{\det(\mathbf{A})^{a+\frac{p}{2}}} \sum_{k=0}^{\min(p,t)} \frac{(-1)^k \binom{t}{k}}{\det(\mathbf{A})^{\frac{k}{2}}} \operatorname{tr}^{t-k}\left(\mathbf{RA}^{-1}\right) \operatorname{tr}^k(\mathbf{R})\, \mathfrak{C}_{p-k}^{a+t}\left(\frac{\operatorname{tr}(\mathbf{A})}{2\sqrt{\det(\mathbf{A})}}\right) \end{aligned} \tag{D.64}$$

where $\mathfrak{C}_n^v(\cdot)$ denotes the Gegenbauer polynomial (see C.36).

Result 47 *(Dharmawansa and McKay (2011), eq. 41).*

Suppose $\mathbf{Y} \in \mathbb{C}_2^{2\times 2}$. Then:

$$\det(\mathbf{I}_2+\mathbf{Y})^{n-2} = \sum_{i=0}^{n-2}\sum_{j=1}^{i} \binom{n-2}{i}\binom{i}{j} \operatorname{tr}^j(\mathbf{Y}) \det(\mathbf{Y})^{i-j}. \tag{D.65}$$

Result 48 *(Gross and Richards (1989)).*

Suppose that $\mathbf{\Delta}, \mathbf{\Lambda} \in \mathbb{C}_2^{p\times p}$ with eigenvalues $\mu_1 > \mu_2 > ... > \mu_p > 0$ and $\lambda_1 > \lambda_2 > ... > \lambda_p > 0$ respectively. Then:

$${}_0\mathcal{C}F_1^{(p)}(n;\mathbf{\Delta},\mathbf{\Lambda}) = \frac{\det\left({}_0F_1(n-p+1;\mu_i\lambda_j)\right)}{\prod\limits_{k<l}^{p}(\lambda_k-\lambda_l)\prod\limits_{k<l}^{p}(\mu_k-\mu_l)} \frac{\mathcal{C}\Gamma_p(p)\mathcal{C}\Gamma_p(n)}{((n-p)!)^p} \tag{D.66}$$

where ${}_0F_1(\cdot)$ denotes the confluent hypergeometric function of scalar argument (see C.13) and $\mathcal{C}\Gamma_n(\cdot)$ denotes the complex multivariate gamma function (see D.47).

E. Code

Chapter 2

Nonsingular case

Correlated case

```
options nodate ps=5000 ls=125;
proc iml;
sigmasq = 1;
corr = 0.9;
tune = 250;
step = 0.01;
v=5;
rho = do(0,35,5)`;
rho[1] = 1;
*rho = exp(rho/10);
nr = do(2,50,2)`;
sigma = (sigmasq||corr) // (corr||sigmasq);
call eigen(gammval,gammvec,sigma);
gamma = diag(gammval);
gamminv = inv(gamma);
 a1 = gamminv[1,1];
 a2 = gamminv[2,2];
 diff = a2 - a1;
capacity_matrix = j(nrow(nr),nrow(rho),0);
do i = 1 to nrow(rho);
 do j = 1 to nrow(nr);
 nr_value = nr[j];
 rho_value = rho[i]; *rho[i];

 val1 = 0;
 val2 = 0;
 val3 = 0;
 val4 = 0;
 do k = 0.001 to tune by step;
 w = log(1+10**(rho_value/10)*k/2);
 val1 = val1 + step*w*k**(nr_value-1)*(a1*k+v)**(-(nr_value+v));
 val2 = val2 + step*w*k**(nr_value-1)*(a2*k+v)**(-(nr_value+v));
 val3 = val3 + step*w*k**(nr_value-2)*(a1*k+v)**(-(nr_value+v-1));
 val4 = val4 + step*w*k**(nr_value-2)*(a2*k+v)**(-(nr_value+v-1));
 end;
 cap = a1**nr_value * a2 * v**v * gamma(nr_value+v) / (diff*gamma(nr_value)*gamma(v)) * val1
         - a1 * a2**nr_value * v**v * gamma(nr_value+v) / (diff*gamma(nr_value)*gamma(v))
* val2
         - a1**nr_value * v**v * gamma(nr_value+v-1) / (diff*gamma(nr_value-1)*gamma(v)) *
val3
         + a2**nr_value * v**v * gamma(nr_value+v-1) / (diff*gamma(nr_value-1)*gamma(v)) *
val4;
 capacity_matrix[j,i] = cap;

 end; *j;
end; *i;
capacity_matrix = nr || capacity_matrix;
coll = {"nr" "_0db" "_5db" "_10db" "_15db" "_20db" "_25db" "_30db" "_35db"};
print capacity_matrix[c=coll label='Capacity matrix'];
create capacity from capacity_matrix[colname=coll];
append from capacity_matrix;
quit;
goptions reset=all i=join ;
```

```
    symbol1 w=2;
    symbol2 w=2;
    symbol3 w=2;
    symbol4 w=2;
    symbol5 w=2;
    symbol6 w=2;
    symbol7 w=2;
    symbol8 w=2;
    axis1 label=(f="Times New Roman" h=2 a=90 'Capacity (in nats)') order=0 to 21 by 3
    value=(f="Times New Roman" h=1.5) minor=none;
    axis2 label=(f="Times New Roman" h=2 'n' h=1.1 ' r') order=0 to 35 by 5
    value=(f="Times New Roman" h=1.5) minor=none;
    proc gplot data=capacity;
    *title1 f="Times New Roman" h=3 'Capacity vs n' h=1.4 'r' h=3 ' for correlated channels';
    *title2 f="Times New Roman" h=1.5 'for different SNR values';
    *title3 f="Times New Roman" h=1 'for underlying complex matrix variate t distribution';
    plot _0dB*nr _5dB*nr _10dB*nr _15dB*nr _20dB*nr _25db*nr _30dB*nr _35dB*nr / overlay haxis=axis2
vaxis=axis1;
    run;
    quit;
```

Uncorrelated case options nodate ps=5000 ls=125;

```
    proc iml;
    sigmasq = 1;
    corr = 0;
    tune = 250;
    step = 0.01;
    v=5;
    rho = do(0,35,5)`;
    rho[1] = 1;
    nr = do(2,50,2)`;
    sigma = (sigmasq||corr) // (corr||sigmasq);
    call eigen(gammval,gammvec,sigma);
    gamma = diag(gammval);
    capacity_matrix = j(nrow(nr),nrow(rho),0);
    do i = 1 to nrow(rho);
     do j = 1 to nrow(nr);
     nr_value = nr[j];
     rho_value = rho[i]; *rho[i];

     val1 = 0;
     val2 = 0;
     val3 = 0;
     do k = 0.001 to tune by step;
     val1 = val1 + step*log(1+10**(rho_value/10)*k/2)*k**(nr_value)*(k/sigmasq+v)**(-(nr_value+v+1));
     val2 = val2 + step*log(1+10**(rho_value/10)*k/2)*k**(nr_value-1)*(k/sigmasq+v)**(-(nr_-
value+v));
     val3 = val3 + step*log(1+10**(rho_value/10)*k/2)*k**(nr_value-2)*(k/sigmasq+v)**(-(nr_-
value+v-1));
     end;
     cap = v**v * gamma(nr_value+v+1) * val1 / (sigmasq*gamma(nr_value)*gamma(v))
         - 2* v**v * gamma(nr_value+v) * val2 / (gamma(nr_value-1)*gamma(v))
            + v**v * gamma(nr_value+v-1) * gamma(nr_value+1) * sigmasq * val3
    / (gamma(nr_value)*gamma(nr_value-1)*gamma(v)) ;
     capacity_matrix[j,i] = cap;

     end; *j;
    end; *i;
```

```
capacity_matrix = nr || capacity_matrix;
coll = {"nr" "0dB" "5dB" "10dB" "15dB" "20dB" "25dB" "30dB" "35dB" };
print capacity_matrix[c=coll label='Capacity matrix'];
create capacity from capacity_matrix[colname=coll];
append from capacity_matrix;
quit;
goptions reset=all i=join ;
symbol1 w=2;
symbol2 w=2;
symbol3 w=2;
symbol4 w=2;
symbol5 w=2;
symbol6 w=2;
symbol7 w=2;
symbol8 w=2;
axis1 label=(f="Times New Roman" h=2 a=90 'Capacity (in nats)') order=0 to 24 by 3 value=(f="Times New Roman" h=1.5) minor=none;
axis2 label=(f="Times New Roman" h=2 'n' h=1.1 ' r') order=0 to 35 by 5 value=(f="Times New Roman" h=1.5) minor=none;
proc gplot data=capacity;
*title1 f="Times New Roman" h=3 'Capacity vs n' h=1.4 'r' h=3 ' for uncorrelated channels';
*title2 f="Times New Roman" h=1.5 'for different SNR values';
*title3 f="Times New Roman" h=1 'for underlying complex matrix variate t distribution';
plot _0dB*nr _5dB*nr _10dB*nr _15dB*nr _20dB*nr _25dB*nr _30dB*nr _35dB*nr/ overlay haxis=axis2 vaxis=axis1;
run;
quit;
```

For fixed ρ

```
options nodate ps=5000 ls=125;
proc iml;
sigmasq = 1;
corr = 0;
tune = 300;
step = 0.01;
v=5; *degrees of freedom for t distribution;
nr = do(2,30,2)`;
*specify specific RHO value for t and normal comparison;
rho = 20;
sigma = (sigmasq||corr) // (corr||sigmasq);
call eigen(gammval,gammvec,sigma);
gamma = diag(gammval);
*****************NORMAL DISTRIBUTION STARTS;
capacity_matrix_nor = j(nrow(nr),nrow(rho),0);
do i = 1 to nrow(rho);
 do j = 1 to nrow(nr);
 nr_value = nr[j];
 rho_value = rho[i]; *rho[i];

 val1 = 0;
 val2 = 0;
 val3 = 0;
 do k = 0.001 to tune by step;
 w = log(1+10**(rho_value/10)*k/2);
 val1 = val1 + step*w*k**(nr_value)*exp(-k/sigmasq);
 val2 = val2 + step*w*k**(nr_value-1)*exp(-k/sigmasq);
 val3 = val3 + step*w*k**(nr_value-2)*exp(-k/sigmasq);
 end;
 cap = sigmasq**(-nr_value-1) / gamma(nr_value) * val1
```

```
         - 2 * sigmasq**(-nr_value) / gamma(nr_value-1) * val2
        + gamma(nr_value+1) * sigmasq**(-nr_value+1)
/ (gamma(nr_value)*gamma(nr_value-1)) * val3;
 capacity_matrix_nor[j,i] = cap;

 end; *j;
end; *i;
*****************NORMAL DISTRIBUTION ENDS;
*****************T DISTRIBUTION STARTS;
capacity_matrix_t = j(nrow(nr),nrow(rho),0);
do i = 1 to nrow(rho);
 do j = 1 to nrow(nr);
 nr_value = nr[j];
 rho_value = rho[i]; *rho[i];

 val1 = 0;
 val2 = 0;
 val3 = 0;
 do k = 0.001 to tune by step;
 w = log(1+10**(rho_value/10)*k/2);
 val1 = val1 + step*w*k**(nr_value)*(k/sigmasq+v)**(-(nr_value+v+1));
 val2 = val2 + step*w*k**(nr_value-1)*(k/sigmasq+v)**(-(nr_value+v));
 val3 = val3 + step*w*k**(nr_value-2)*(k/sigmasq+v)**(-(nr_value+v-1));
 end;
 cap = v**v * gamma(nr_value+v+1) * val1 / (sigmasq*gamma(nr_value)*gamma(v))
     - 2* v**v * gamma(nr_value+v) * val2 / (gamma(nr_value-1)*gamma(v))
        + v**v * gamma(nr_value+v-1) * gamma(nr_value+1) * sigmasq * val3
/ (gamma(nr_value)*gamma(nr_value-1)*gamma(v)) ;
 capacity_matrix_t[j,i] = cap;

 end; *j;
end; *i;
*************************************T distribution ends;
capacity_matrix = nr || capacity_matrix_nor || capacity_matrix_t;
coll = {"nr" "_db_nor" "_db_t"};
create capacity from capacity_matrix[colname=coll];
append from capacity_matrix;
quit;
goptions reset=all i=join;
symbol1 color=black w=2 ;
symbol2 color=grey w=2 ;
axis1 label=(f="Times New Roman" h=2 a=90 'Capacity (in nats)')
value=(f="Times New Roman" h=1.5) minor=none;
axis2 label=(f="Times New Roman" h=2 'n' h=1.1 ' r') order=0 to 30 by 5
value=(f="Times New Roman" h=1.5) minor=none;
legend1 label=(f="Times New Roman" h=1.5 "Legend")
value=(f="Times New Roman" h=1.5 "Normal" "t");
proc gplot data=capacity;
*title1 f="Times New Roman" h=3 'Capacity vs nr for uncorrelated channels';
*title2 f="Times New Roman" h=1.5 'for underlying normal and t distribution';
*title3 f="Times New Roman" h=1 'for fixed SNR value';
plot _dB_nor*nr _dB_t*nr / overlay haxis=axis2 vaxis=axis1 legend=legend1;
run;
quit;
```

Singular case

```
options nodate ps=5000 ls=125;
    proc iml;
    sigmasq = 1;
    corr = 0;
    tune = 250;
    step = 0.25     ;
    v=5;
    rho = do(0,25,5)`;
    nt = 4;
    sigma = (sigmasq||corr) // (corr||sigmasq);
    call eigen(gammval,gammvec,sigma);
    gamma = diag(gammval);
    capacity_matrix = j(nrow(rho),nrow(nt),0);
    do i = 1 to nrow(nt);
     do j = 1 to nrow(rho);
     nt_value = nt[i];
     rho_value = rho[j]; *rho[i];

     val1 = 0;
     val2 = 0;
     val3 = 0;
     do k = 0.001 to tune by step;
     val1 = val1 + step*log(1+10**(rho_value/10)*k/nt_value)*k**(nt_value)*(k/sigmasq+v)**(-(nt_-
value+v+1));
     val2 = val2 + step*log(1+10**(rho_value/10)*k/nt_value)*k**(nt_value-1)*(k/sigmasq+v)**(-(nt_-
value+v));
     val3 = val3 + step*log(1+10**(rho_value/10)*k/nt_value)*k**(nt_value-2)*(k/sigmasq+v)**(-(nt_-
value+v-1));
     end;
     cap = v**v * gamma(nt_value+v+1) * val1 / ( (sigmasq)**(nt_value+1) *gamma(nt_value)*gamma(v))

         - 2* v**v * gamma(nt_value+v) * val2 / ( (sigmasq)**(nt_value)*gamma(nt_value-1)*gamma(v))

            + v**v * gamma(nt_value+v-1) * gamma(nt_value+1) * (sigmasq)**(nt_value+1) * val3
/ (gamma(nt_value)*gamma(nt_value-1)*gamma(v)) ;
     capacity_matrix[j,i] = cap;

     end; *j;
    end; *i;
    capacity_matrix = rho || capacity_matrix;
    coll = {"rho" "nt4_uncor"};
    print capacity_matrix[c=coll label='Capacity matrix'];
    create capacity_sing_uncorr from capacity_matrix[colname=coll];
    append from capacity_matrix;
    quit;
    goptions reset=all i=join ;
    symbol1 w=2;
    axis1 label=(f=cmr10 h=2 a=90 'Capacity (in nats)') value=(f=cmr10 h=1.5) minor=none;
    axis2 label=(f=cmr10 h=2 'SNR (' f=greek 'r' f=cmr10 ')') order=0 to 25 by 5 value=(f=cmr10
h=1.5) minor=none;
    proc gplot data=capacity_sing_uncorr;
    title1 f=cmr10 h=3 'Capacity vs SNR for uncorrelated channels';
    title2 f=cmr10 h=1.5 'for n' h=1 't' h=1.5 '=4';
    title3 f=cmr10 h=1 'for underlying singular t distribution';
    plot nt4_uncor*rho / overlay haxis=axis2 vaxis=axis1;
    run;
```

```
quit;


data mmerge;
merge capacity_sing_corr capacity_sing_uncorr;
run;
goptions reset=all i=join ;
symbol1 c=grey w=2;
symbol2 c=black w=2;
axis1 label=(f="Times New Roman" h=2 a=90 'Capacity (in nats)') value=(f="Times New Roman" h=1.5) minor=none;
axis2 label=(f="Times New Roman" h=2 'SNR (' f=greek 'r' f="Times New Roman" ')') order=0 to 25 by 5 value=(f="Times New Roman" h=1.5) minor=none;
legend1 label=(f="Times New Roman" h=1.5 "Legend") value=(f="Times New Roman" h=1.5 "correlated case" "uncorrelated case");
proc gplot data=mmerge;
*title1 f="Times New Roman" h=3 'Capacity vs SNR for correlated and uncorrelated channels';
*title2 f="Times New Roman" h=1.5 'for n' h=1 't' h=1.5 '=4, v=5';
*title3 f="Times New Roman" h=1 'for underlying complex singular matrix variate t distribution';
plot nt4_cor*rho nt4_uncor*rho / overlay haxis=axis2 vaxis=axis1 legend=legend1;
run;
quit;
```

Chapter 3

Normal case

```
clear all
    format short
    m = 50000;
    psi = 0.5;
    mu = [0 0];
    Sigma = [1 psi; psi 1];
    n = 3;
    x = zeros(m,1);
    for j = 1:m
     X = mvnrnd(mu,Sigma,n);
     Y = mvnrnd(mu,Sigma,n);
     H = X+Y*1i;
     W = H'*H;
     W = inv(W);
     w_ = diag(W);
     x1 = 1/w_(1);
     x2 = 1/w_(2);
     x(j) = max(x1,x2);
     wb = waitbar(j/m);
    end
    close(wb)
    A = sqrt(mean(x));
    x = x/A;
    x = 10*log10(x);
    data = x;
    [y1,x1] = ecdf(data);
```

t case

```
clear all
    format short
    m = 10000;
    psi = 0.5;
    mu = [0 0];
    Sigma = [1 psi; psi 1];
    v = 15; %degrees of freedom
    n = 3;
    x = zeros(m,1);
    for j = 1:m
     X = mvtrnd(Sigma,v, n);
     Y = mvtrnd(Sigma,v, n);
     H = X+Y*1i;
     W = H'*H;
     W = inv(W);
     w_ = diag(W);
     x1 = 1/w_(1);
     x2 = 1/w_(2);
     x(j) = max(x1,x2);
     wb = waitbar(j/m);
    end
    close(wb)
    A = sqrt(mean(x));
    x = x/A;
    x = 10*log10(x);
    data = x;
    [y1,x1] = ecdf(data);
```

Chapter 5

Normal case

```
clear all
    format short
    n = 2;
    p = n;
    m = 50000;
    a = zeros(n,1);
    b = zeros(p,1);
    sigma = zeros(p,p);
    theta = pi/4;
    for i = 1:n
     a(i) = exp(2.*1i.*(i-1).*pi.*cos(theta));
    end
    for i = 1:p
     b(i) = exp(2.*1i.*(i-1).*pi.*cos(theta));
    end
    mu = a*b';
    for i = 1:p
     for j = 1:p
     sigma(i,j) = exp(-1.*pi.^3.*((i-j).^2)./32);
     end
    end
    x = zeros(m,1);
    muzero = zeros(n,p);
    ident = eye(p);
    for j = 1:m
```

```
 X = mvnrnd(muzero,sigma/2,n);
 Y = mvnrnd(muzero,sigma/2,n);
 H = (X+Y) + mu;
 W = H'*H;
 x(j) = min(eig(W));
 wb = waitbar(j/m);
end
close(wb)
A = ((mean(x)));
x = x*A;
data = x;
[y1,x1] = ecdf(data);
```

t **case**

```
clear all
    format short
    n = 2;
    p = n;
    m = 50000;
    a = zeros(n,1);
    b = zeros(p,1);
    sigma = zeros(p,p);
    theta = pi/4;
    for i = 1:n
     a(i) = exp(2.*1i.*(i-1).*pi.*cos(theta));
    end
    for i = 1:p
     b(i) = exp(2.*1i.*(i-1).*pi.*cos(theta));
    end
    mu = a*b';
    for i = 1:p
     for j = 1:p
     sigma(i,j) = exp(-1.*pi.^3.*((i-j).^2)./32);
     end
    end
    v = 15; %degrees of freedom
    x = zeros(m,1);
    muzero = zeros(n,p);
    ident = eye(p);
    for j = 1:m
     X = mvnrnd(muzero,sigma/2,n); %t matrikse;
     Y = mvnrnd(muzero,sigma/2,n); %t matrikse;
     H = ((X+Y) + mu)/sqrt(chi2rnd(v)/v); %maak 'n komplekse 0 sigma matriks, voeg skuif by;

     W = H'*H;
     x(j) = min(eig(W));
     wb = waitbar(j/m);
    end
    close(wb)
    A = ((mean(x)));
    x = x*A;
    data = x;
    [y1,x1] = ecdf(data);
```

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