Existence and Ulam-Hyers stability of fixed point problem of generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operators

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Abstract:

The aim of this paper is to introduce a class of generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operators and to prove the existence of fixed point and common fixed point of such operators in the setup of b-metric spaces. Some examples are presented to support the results proved herein. Our results generalize and extend various results in the existing literature. An estimate of Hausdorff distance between the fixed point sets of two generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operators is obtained. We further investigate the Ulam-Hyers stability of fixed point problem of operators considered herein in the framework of b-metric spaces.

Keywords: b-metric space, multivalued mapping, fixed point problems, stability, well-posedness:

2000 Mathematics Subject Classification: 47H10, 47H04, 47H07.

1 Introduction and preliminaries

The concept of a distance between two objects plays a vital role in mathematics and its related disciplines. Consequently, there have been several generalizations of a metric function.

By replacing the triangular inequality with a more weaker axiom, Czerwik [9] initiated a notion of a b-metric as follows:

Definition 1.1 *[9] Let X be a nonempty set and $b \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if for any $x, y, z \in X$, the following conditions hold:*

$\mathbf{b}_1$- $d(x, y) = 0$ *if and only if* $x = y$;

$\mathbf{b}_2$- $d(x, y) = d(y, x)$;

$\mathbf{b}_3$- $d(x, y) \leq b\,(d(x, z) + d(z, y))\,.$

The pair (X, d) is called a b-metric space with parameter $b \geq 1$.

If $b = 1$, then b-metric space is a metric space. However, the converse does not hold in general (compare [2, Example 3.9] and [8, 9, 20]).

Let (X, d) be a b-metric space and $P(X)$, $Cl(X)$, $B(X)$, $CB(X)$ and $K(X)$, the family of all subsets of X, the family of all closed subsets of X, the family of all bounded subsets of X, the family of all closed and bounded subsets of X and the family of all compact subsets of X, respectively. Let $A, B \in P(X)$.

The gap functional D induced by the b-metric d on X is given by

$$D(A, B) = \begin{cases} \inf\limits_{a \in A, b \in B} d(a, b), \text{ if } A \neq B \neq \emptyset, \\ 0, \text{ if } A = B = \emptyset, \\ +\infty, \text{ otherwise.} \end{cases}$$

The excess generalized function ρ induced by the b-metric d on X is given by

$$\rho(A,B)=\begin{cases}\sup\limits_{a\in A} D(a,B), \text{ if } A\neq B\neq\emptyset,\\ 0, \text{ if } A=\emptyset,\\ +\infty, \text{ if } B=\emptyset,\ A\neq\emptyset.\end{cases}$$

Pompeiu-Hausdorff generalized functional H induced by the b-metric d on X is defined as

$$H(A,B)=\begin{cases}\max\{\rho(A,B),\rho(B,A)\}, \text{ if } A\neq B\neq\emptyset,\\ 0, \text{ if } A=B=\emptyset,\\ +\infty, \text{ otherwise.}\end{cases}$$

The functional δ induced by the b-metric d on X is given by

$$\delta(A,B)=\begin{cases}\sup\limits_{a\in A,b\in B} d(a,b), \text{ if } A\neq B\neq\emptyset,\\ 0, \text{ if } A=B=\emptyset,\\ +\infty, \text{ otherwise.}\end{cases}$$

Following lemma from [9, 10, 20] is crucial to prove the main result of the paper.

Lemma 1.2 *Let (X,d) be a b-metric space, $x,y\in X$ and $A,B\in CB(X)$. The following statements hold:*

c$_1$) $(CB(X),H)$ *is a b-metric space and* $(CB(X),H)$ *is complete whenever* (X,d) *is complete*;

c$_2$) $D(x,B)\leq H(A,B)$ *for all* $x\in A$;

c$_3$) $D(x,A)\leq b\,(d(x,y)+D(y,A))$;

c$_4$) *For* $h>1$ *and* $\acute{a}\in A$, *there is a* $\acute{b}\in B$ *such that* $d(\acute{a},\acute{b})\leq hH(A,B)$;

c$_5$) *For every* $h>0$ *and* $\acute{a}\in A$, *there is a* $\acute{b}\in B$ *such that* $d(\acute{a},\acute{b})\leq H(A,B)+h$;

c$_6$) $D(x,A)=0$ *if and only if* $x\in\bar{A}=A$;

c$_7$) *For* $\{x_n\}\subseteq X$, $d(x_0,x_n)\leq bd(x_0,x_1)+...+b^{n-1}d(x_{n-2},x_{n-1})+b^{n-1}d(x_{n-1},x_n)$;

Definition 1.3 *Let (X,d) be a b-metric space. A sequence $\{x_n\}$ in X is called:*

c$_8$- Cauchy if and only if for $\epsilon>0$, there exists $n(\epsilon)\in N$ such that for each $n,m\geq n(\epsilon)$ we have $d(x_n,x_m)<\epsilon$;

c$_9$- Convergent if and only if there exists $x\in X$ such that for all $\epsilon>0$ there exists $n(\epsilon)\in N$ such that for all $n\geq n(\epsilon)$ we have $d(x_n,x)<\epsilon$. In this case we write $\lim_{n\to\infty}x_n=x$.

It is known that a sequence $\{x_n\}$ in b-metric space X is Cauchy if and only if $\lim\limits_{n\to\infty} d(x_n,x_{n+p})=0$ for all $p\in N$. A sequence $\{x_n\}$ is convergent to $x\in X$ if and only if $\lim_{n\to\infty}d(x_n,x)=0$. A subset $Y\subset X$ is closed if and only if for each sequence $\{x_n\}$ in Y that converges to an element x, we have $x\in Y$. A subset $Y\subset X$ is bounded if $diam(Y)$ is finite, where $diam(Y)=\sup\{d(a,b),a,b\in Y\}$. A b-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent in X.

An et al. [2] studied the topological properties of b-metric spaces and stated the following assertions.

c$_{10}$- b-metric is not necessarily continuous in each variable;

c$_{11}$- If b-metric is continuous in one variable then it is continuous in other variable;

c$_{12}$- A ball in b-metric space (X,d) is not necessarily an open set. An open ball is open if d is continuous in one variable.

Lemma 1.4 *(compare [17]) Let (X,d) be a b-metric space, $A, B \in P(X)$. If there exists a $\lambda > 0$ such that (i) for each $\tilde{a} \in A$, there exists a $\tilde{b} \in B$ such that $d(\tilde{a},\tilde{b}) \leq \lambda$, (ii) for each $\tilde{b} \in B$, there exists an $\tilde{a} \in A$ such that $d(\tilde{a},\tilde{b}) \leq \lambda$, then $H(A,B) \leq \lambda$.*

Let $T : X \to P(X)$ and $f : X \to X$. We call (f,T) a hybrid pair of mappings. The fixed point problem of T is to find an $x \in X$ such that

$$x \in Tx. \tag{1}$$

A point $x \in X$ is called:

1. a fixed point of T if it solves the fixed point inclusion (1);
2. coincidence point of (f,T) if $fx \in Tx$;
3. common fixed point of (f,T) if $x = fx \in Tx$.

The end point problem of T is to find an $x \in X$ such that $Tx = \{x\}$. The fixed point inclusion (1) is said to be generalized Ulam-Hyers stable if there exists a increasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous at $t = 0$, such that for each $\varepsilon > 0$ and for each solution u_* of

$$D(u,Tu) \leq \varepsilon,\ u \in X \tag{2}$$

there exists a solution z_* of (1) such that

$$d(u_*, z_*) \leq \xi(\varepsilon).$$

If $\xi(t) = ct$ with $c > 0$, then the fixed point problem is said to be Ulam-Hyers stable. For more discussion on Ulam-Hyers stability of fixed point problems, we refer to [12, 14, 18, 22].

We denote $F(T)$, $E(T)$, $C(f,T)$, $F(f,T)$ and U by the set of all of fixed points of T, the set of all end points of T, the set of all coincidence points of (f,T), the set of all common fixed point of (f,T) and the set of all solutions of (2), respectively.

The hybrid pair (f,T) is called $w-$compatible ([1]), if $f(Tx) \subseteq T(fx)$ for all $x \in C(f,T)$. The mapping f is called $T-$weakly commuting at some point $x \in X$ if $f^2(x) \in T(fx)$.

In what follows we assume that a b-metric d is continuous in one variable.

A mapping $T : X \to P(X)$ is called a multivalued weakly Picard operator (MWP operator [17]), if for all $x \in X$ and for some $y \in Tx$, there exists a sequence $\{x_n\}$ satisfying (a$_1$) $x_0 = x$, $x_1 = y$, (a$_2$) $x_{n+1} \in Tx_n$ for all $n \geq 0$, (a$_3$) $\{x_n\}$ converges to some $z \in F(T)$. The sequence $\{x_n\}$ satisfying (a_1) and (a_2) is called a sequence of successive approximations of T starting from (x,y). If T is a single-valued mapping, then we call it Picard operator provided that it satisfies (a$_1$) to (a$_3$).

Let $T : X \longrightarrow P(X)$ be a MWP operator. Define the mapping $T^\infty : G(T) \to P(F(T))$ by

$$T^\infty(x,y) = \{z : \text{ there is a sequence of successive approximations of } T \text{ starting from } (x,y) \text{ converging to } z\}$$

where $G(T) = \{(x,y) : x \in X, y \in Tx\}$ is the graph of T. A mapping $t : X \to X$ is called selection of $T : X \longrightarrow P(X)$ if $tx \in Tx$ for all $x \in X$.

Definition 1.5 *[17] Let (X,d) be a metric space and $c > 0$. A MWP operator $T : X \longrightarrow P(X)$ is called $c-$multivalued weakly Picard (briefly $c-$MWP) operator if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x,y)) \leq cd(x,y)$ holds for all $(x,y) \in G(T)$.*

One of the main result dealing with $c-$MWP operators is the following:

Theorem 1.6 *[16] Let (X,d) be a metric space and $T_1, T_2 : X \to P(X)$. If T_i is a c_i-MWP operator for each $i \in \{1,2\}$ and there exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$ for all $x \in X$. Then $H(F(T_1), F(T_2)) \leq \lambda \max\{c_1, c_2\}$.*

We need the following Lemma in [13] to prove coincidence and common fixed points of (f,T).

Lemma 1.7 *[13] Let X be a nonempty set and $f : X \to X$, there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one.*

Consistent with [4, 5, 6], following definitions and results will be needed in the sequel.

Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Consider the following classes:

$\mathbf{a}_1$- $\Phi = \{\varphi : \varphi$ is nondecreasing and continuous at $t = 0\}$;

$\mathbf{a}_2$- $\Psi_1 = \{\psi : \psi$ is increasing and$\lim_{n\to\infty} \psi^n(t) = 0$, for any $t \geq 0\}$. The function $\psi \in \Psi_1$ is called a comparison function. It is well known that if $\psi \in \Psi_1$, then (i) for any $n \geq 1$, $nth-$ iterate of ψ is also a comparison function (ii) ψ is continuous at $t = 0$ and (iii) $\psi(t) < t$ for any $t > 0$;

$\mathbf{a}_3$- $\Psi_2 = \{\psi : \psi$ is nondecreasing and$\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ $\}$.Clearly, $\Psi_2 \subseteq \Psi_1$;

$\mathbf{a}_4$- $\Psi_3 = \{\psi : \psi$ is increasing and there exists an $n_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative numbers $\sum_{n=1}^{\infty} u_n$ such that for any $t \geq 0$, $\psi^{n+1}(t) \leq a\psi^n(t) + u_n$ for all $n \geq n_0\}$. The function $\psi \in \Psi_3$ is called a $(c)-$comparison function.

$\mathbf{a}_5$- $\Psi_4 = \{\psi : \psi$ is monotone increasing and there exists an $n_0 \in \mathbb{N}$, $a \in (0, 1)$, $b \geq 1$ and a convergent series of nonnegative numbers $\sum_{n=1}^{\infty} u_n$ such that for any $t \geq 0$, $b^{n+1}\psi^{n+1}(t) \leq ab^n\psi^n(t) + u_n$ for all $n \geq n_0\}$. The function $\psi \in \Psi_4$ is called a $(b)-$comparison function. If $b = 1$, then $\Psi_3 = \Psi_4$.

Lemma 1.8 *[4] If ψ is a $(b)-$comparison functions with $b \geq 1$, then the series $\sum_{n=0}^{\infty} b^n\psi^n(t)$ is convergent for all $t \in \mathbb{R}^+$ and the function $r_b(t) = \sum_{n=0}^{\infty} b^n\psi^n(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and continuous at $t = 0$.*

From above lemma, we have $\Psi_4 \subseteq \Psi_1$.

Let X be a nonempty set, $\alpha : X \times X \to \mathbb{R}^+$, $f : X \to X$ and $T : X \to P(X)$. A mapping f is called $\alpha-$admissible if for any $x, y \in X$

$$\alpha(x, y) \geq 1 \text{ implies that } \alpha(fx, fy) \geq 1. \tag{3}$$

Let $A, B \in P(X)$. Define

$$\alpha_*(A, B) = \inf_{a\in A, b\in B} \alpha(a, b).$$

A mapping T is called α_*-admissible if for any $x, y \in X$

$$\alpha(x, y) \geq 1 \text{ implies that } \alpha_*(Tx, Ty) \geq 1. \tag{4}$$

A mapping T is called $(\alpha_*, f)-$admissible if for any $x, y \in X$, $\alpha(fx, fy) \geq 1$ implies that $\alpha_*(Tx, Ty) \geq 1$.

The set X is said to have a property (H), if for any $x, y \in X$ with $x \neq y$, there exists $w \in X$ such that $\alpha(x, w) \geq 1$, $\alpha(y, w) \geq 1$.

Let (X, d) be a b-metric space, $g : X \to X$ and $T : X \to Cl(X)$. We set:

$$\begin{aligned}
&M_{g,T}(x, y) = \max\left\{d(gx, gy), D(gx, Tx), D(gy, Ty), \frac{D(gx, Ty) + D(gy, Tx)}{2b}\right\}\\
&M_T(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2b}\right\}\\
&R_T(x, y) = \min\{d(x, y), D(y, Tx)\} \text{ and}\\
&R_{g,T}(x, y) = \min\{d(gx, gy), D(gy, Tx)\}.
\end{aligned}$$

Motivated by [11, 19, 21], we give the following definition.

Definition 1.9 *Let (X, d) be a b-metric space. A mapping $T : X \to Cl(X)$ is called a generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operator if there exists a $\psi \in \Psi_4$ and $\varphi \in \Phi$ such that for any $x, y \in X$*

$$\frac{1}{2}D(x, Tx) \leq bd(x, y) \tag{5}$$

implies that

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(M_T(x, y)) + \varphi(R_T(x, y)). \tag{6}$$

If in the above definition, $Cl(X)$ is the family of all singletons $\{x\} \subseteq X$, then $T : X \to Cl(X)$ is called a generalized Suzuki type $(\alpha - \psi_\varphi)-$contractive operator.

Definition 1.10 *Let (X,d) be a b-metric space. A mapping $T : X \to Cl(X)$ is called a generalized Suzuki type $(\alpha_*, \psi)-$contractive multivalued operator if there exists a $\psi \in \Psi_4$ such that for any $x, y \in X$*

$$\frac{1}{2}D(x,Tx) \leq bd(x,y)$$

implies that

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi\left(M_T(x,y)\right).$$

If in the above definition, $Cl(X)$ is the family of all singletons $\{x\} \subseteq X$, then $T : X \to Cl(X)$ is called a generalized Suzuki type $(\alpha - \psi)-$contractive operator.

2 Fixed points of generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operators

In this section, we obtain fixed point results for generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operators in complete b-metric spaces. We start with the following result.

Theorem 2.1 *Let (X,d) be a complete b-metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to Cl(X)$. Assume that*

d$_1$- *T is a generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive multivalued operator;*

d$_2$- *T is an α_*-admissible mapping;*

d$_3$- *there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

d$_4$- *if there is a sequence $\{x_n\}$ in X such that $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}^*$.*

Then T is a MWP operator.

Proof. Let x_0 and x_1 be two given points in X such that $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$, then $x_0 \in Tx_0$. Define a sequence $\{x_n\}$ by $x_n = x_1 = x_0$, $n \in \mathbb{N}^*$. Clearly, $x_n \in Tx_n$ and $\{x_n\}$ converges to $x = x_0 \in F(T)$. Hence T is a MWP operator. Suppose that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. Obviously, $\alpha(x_0, x_1) \geq 1$ implies that $\alpha_*(Tx_0, Tx_1) \geq 1$. Also,

$$\frac{1}{2}D(x_0,Tx_0) \leq d(x_0,x_1) \leq bd(x_0,x_1). \tag{7}$$

Thus we have

$$\begin{aligned}
0 &< D(x_1,Tx_1) \leq H(Tx_0,Tx_1) < \alpha_*(Tx_0,Tx_1)H(Tx_0,Tx_1) \\
&\leq \psi\left(M_T(x_0,x_1)\right) + \varphi\left(R_T(x_0,x_1)\right) \\
&= \begin{array}{l} \psi\left(\max\left\{d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \dfrac{D(x_0,Tx_1)+D(x_1,Tx_0)}{2b}\right\}\right) \\ +\varphi\left(\min\left\{d(x_0,x_1), D(x_1,Tx_0)\right\}\right) \end{array} \\
&\leq \begin{array}{l} \psi\left(\max\left\{d(x_0,x_1), d(x_0,x_1), D(x_1,Tx_1), \dfrac{d(x_0,x_1)+D(x_1,Tx_1)}{2}\right\}\right) \\ +\varphi\left(\min\left\{d(x_0,x_1), d(x_1,x_1)\right\}\right) \end{array} \\
&\leq \psi\left(\max\left\{d(x_0,x_1), D(x_1,Tx_1)\right\}\right).
\end{aligned}$$

That is

$$0 < D(x_1,Tx_1) \leq \psi\left(\max\left\{d(x_0,x_1), D(x_1,Tx_1)\right\}\right). \tag{8}$$

If $\max\left\{d(x_0,x_1), D(x_1,Tx_1)\right\} = D(x_1,Tx_1)$, then we have

$$0 < D(x_1,Tx_1) \leq \psi\left(D(x_1,Tx_1)\right). \tag{9}$$

As $D(x_1, Tx_1) > 0$ and $\psi \in \Psi_4$,

$$0 < D(x_1, Tx_1) < D(x_1, Tx_1),$$

gives a contradiction. Hence

$$0 < D(x_1, Tx_1) \leq \psi\left(d(x_0, x_1)\right). \tag{10}$$

We may choose $x_2 \in Tx_1$ and $q > 1$ such that

$$0 < D(x_1, Tx_1) \leq d(x_1, x_2) < qD(x_1, Tx_1).$$

Thus

$$0 < d(x_1, x_2) < qD(x_1, Tx_1) \leq q\psi\left(d(x_0, x_1)\right) = q\psi\left(c_0\right), \tag{11}$$

where $c_0 = d(x_0, x_1)$. Note that $x_1 \neq x_2$, and $\alpha(x_1, x_2) \geq \alpha_*(Tx_0, Tx_1) \geq 1$. Thus $\alpha(x_1, x_2) \geq 1$ and hence $\alpha_*(Tx_1, Tx_2) \geq 1$. As $\psi \in \Psi_4$, $\psi\left(d(x_1, x_2)\right) < \psi\left(q\psi\left(c_0\right)\right)$. If we set $q_1 = \dfrac{\psi\left(q\psi\left(c_0\right)\right)}{\psi\left(d(x_1, x_2)\right)}$, then $q_1 > 1$. Now if $x_2 \in Tx_2$ then the proof is finished. Let $x_2 \notin Tx_2$, then by similar process we obtain

$$0 < D(x_2, Tx_2) \leq \psi\left(d(x_1, x_2)\right) \tag{12}$$

and $x_3 \in Tx_2$ such that

$$0 < d(x_2, x_3) < q_1 D(x_2, Tx_2) \leq q_1 \psi\left(d(x_1, x_2)\right) = \psi\left(q\psi\left(c_0\right)\right). \tag{13}$$

Note that $x_2 \neq x_3$ and $\alpha(x_2, x_3) \geq \alpha_*(Tx_1, Tx_2) \geq 1$. Thus $\alpha(x_2, x_3) \geq 1$ implies that $\alpha(Tx_2, Tx_3) \geq 1$. By (13) we have

$$\psi\left(d(x_2, x_3)\right) < \psi^2\left(q\psi\left(c_0\right)\right). \tag{14}$$

If $q_2 = \dfrac{\psi^2\left(q\psi\left(c_0\right)\right)}{\psi\left(d(x_2, x_3)\right)}$, then $q_2 > 1$. Now if $x_3 \in Tx_3$ then the proof is finished. Let $x_3 \notin Tx_3$. Continuing this way, we obtain a sequence $\{x_n\}$ in X and it satisfies: $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$, $\alpha\left(x_{n+1}, x_{n+2}\right) \geq 1$,

$$0 < D(x_{n+1}, Tx_{n+1}) \leq \psi\left(d(x_n, x_{n+1})\right) \tag{15}$$

and

$$0 < d(x_{n+1}, x_{n+2}) < \psi^n\left(q\psi\left(c_0\right)\right). \tag{16}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X. Note that

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq bd(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + \ldots + b^{p-1} d(x_{n+p-2}, x_{n+p-1}) + b^{p-1} d(x_{n+p-1}, x_{n+p}) \\
&\leq b\psi^{n-1}\left(q\psi\left(c_0\right)\right) + b^2\psi^n\left(q\psi\left(c_0\right)\right) + \ldots + b^{p-1}\psi^{n+p-3}\left(q\psi\left(c_0\right)\right) + b^{p-1}\psi^{n+p-2}\left(q\psi\left(c_0\right)\right) \\
&= \frac{1}{b^{n-2}}\left(b^{n-1}\psi^{n-1}\left(q\psi\left(c_0\right)\right) + b^n\psi^n\left(q\psi\left(c_0\right)\right) + \ldots + b^{n+p-2}\psi^{n+p-2}\left(q\psi\left(c_0\right)\right)\right) \\
&= \frac{1}{b^{n-2}}\sum_{i=n-1}^{n+p-2} b^i\psi^i\left(q\psi\left(c_0\right)\right) = \frac{1}{b^{n-2}}\left(\sum_{i=1}^{n+p-2} b^i\psi^i\left(q\psi\left(c_0\right)\right) - \sum_{i=1}^{n-2} b^i\psi^i\left(q\psi\left(c_0\right)\right)\right).
\end{aligned}$$

That is

$$d(x_n, x_{n+p}) \leq \frac{1}{b^{n-2}}\left(\sum_{i=0}^{n+p-2} b^i\psi^i\left(q\psi\left(c_0\right)\right) - \sum_{i=0}^{n-2} b^i\psi^i\left(q\psi\left(c_0\right)\right)\right). \tag{17}$$

We set $S_n = \sum\limits_{i=0}^{n} b^i\psi^i\left(q\psi\left(c_0\right)\right)$. Then, we have

$$d(x_n, x_{n+p}) \leq \frac{1}{b^{n-2}}\left(S_{n-p-2} - S_{n-1}\right). \tag{18}$$

It follows from 1.8 that $\sum_{i=0}^{\infty} b^i\psi^i\left(t\right)$ converges for any $t \geq 0$. Thus $\lim\limits_{n\to\infty} S_{n-2} = S$ for some $S \in [0, \infty)$. From (18), we obtain that $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p \in \mathbb{N}$ and $\{x_n\}$ is a Cauchy sequence in X. Consequently, there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$ and hence $\alpha(x_n, z) \geq 1$. Now we show that $z \in F(T)$. Assume on contrary that $z \notin Tz$, that is, $D(z, Tz) > 0$. We claim that either

$$\frac{1}{2}D(x_n, Tx_n) \leq bd(x_n, z) \tag{19}$$

or

$$\frac{1}{2}D(x_{n+1}, Tx_{n+1}) \leq bd(x_{n+1}, z) \tag{20}$$

holds for all $n \in \mathbb{N}^*$. If not, there exists an $n_0 \in \mathbb{N}_0$ such that

$$\frac{1}{2}D(x_{n_0}, Tx_{n_0}) > bd(x_{n_0}, z) \tag{21}$$

and

$$\frac{1}{2}D(x_{n_0+1}, Tx_{n_0+1}) > bd(x_{n_0+1}, z). \tag{22}$$

Now by (15), (21) and (22), we have

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) &\leq bd(x_{n_0}, z) + bd(z, x_{n_0+1}) \\ &< \frac{1}{2}D(x_{n_0}, Tx_{n_0}) + \frac{1}{2}D(x_{n_0+1}, Tx_{n_0+1}) \\ &\leq \frac{1}{2}d(x_{n_0}, x_{n_0+1}) + \frac{1}{2}\psi\left(d(x_{n_0}, x_{n_0+1})\right) \\ &< \frac{1}{2}d(x_{n_0}, x_{n_0+1}) + \frac{1}{2}d(x_{n_0}, x_{n_0+1}) \\ &= d(x_{n_0}, x_{n_0+1}), \end{aligned}$$

a contradiction. Hence either (19) or (20) holds for an infinite subset $\mathbb{N}_1$ of $\mathbb{N}^*$. As $\alpha(x_n, z) \geq 1$ and T is an α_*-admissible, $\alpha_*(Tx_n, Tz) \geq 1$. Now if (19) holds for all $n \in \mathbb{N}_1$, then we have

$$\begin{aligned} D(x_{n+1}, Tz) &\leq H(Tx_n, Tz) < \alpha_*(Tx_n, Tz)H(Tx_n, Tz) \\ &\leq \psi\left(M_T(x_n, z)\right) + \varphi\left(R_T(x_n, z)\right) \\ &= \begin{array}{l} \psi\left(\max\left\{d(x_n, z), D(x_n, Tx_n), D(z, Tz), \dfrac{D(x_n, Tz) + D(z, Tx_n)}{2b}\right\}\right) \\ +\varphi\left(\min\left\{d(x_n, z), D(z, Tx_n)\right\}\right) \end{array} \\ &\leq \begin{array}{l} \psi\left(\max\left\{d(x_n, z), d(x_n, x_{n+1}), D(z, Tz), \dfrac{D(x_n, Tz) + d(z, x_{n+1})}{2b}\right\}\right) \\ +\varphi\left(\min\left\{d(x_n, z), d(z, x_{n+1})\right\}\right). \end{array} \end{aligned}$$

On taking limit as n tends to infinity on both sides of above inequality, we have

$$D(z, Tz) \leq \psi\left(D(z, Tz)\right) < D(z, Tz)$$

a contradiction. Consequently $z \in Tz$. Similarly, when (20) holds for an infinite subset $\mathbb{N}_1$ of $\mathbb{N}_0$, we obtain that $z \in Tz$. ■

Corollary 2.2 *Let (X, d) be a complete b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to Cl(X)$ an α_*-admissible mapping. Suppose that for any $x, y \in X$*

$$\frac{1}{2}D(x, Tx) \leq bd(x, y) \text{ implies that } \alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(M_T(x, y)) + LR_T(x, y)$$

where $\psi \in \Psi_4$ and $L \geq 0$. Then T is a MWP operator provided that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$ and for any sequence $\{x_n\}$ in X with $x_n \to x$ for some x in X implies that $\alpha(x_n, x) \geq 1$, $n \in \mathbb{N}_0$.

Proof. Set $\varphi(t) = Lt$ for all $t \in \mathbb{R}^+$ in Theorem 2.1. ■

Corollary 2.3 *Let (X,d) be a complete b-metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to Cl(X)$ an α_*-admissible mapping. Suppose that for any $x, y \in X$*

$$\frac{1}{2}D(x,Tx) \leq bd(x,y) \text{ implies that } \alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi(M_T(x,y))$$

where $\psi \in \Psi_4$. Then T is a MWP operator provided that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$ and for any sequence $\{x_n\}$ in X with $x_n \to x$ for some x in X implies that $\alpha(x_n,x) \geq 1$, $n \in \mathbb{N}_0$.

Proof. Put $\varphi(t) = 0$ for all $t \in \mathbb{R}^+$ in Theorem 2.1. ∎

Corollary 2.3 generalize and extend [7, Theorem 1], [3, Theorem 2.1], [15, Theorem 3.1], and [19, Theorem 2.2].

Corollary 2.4 *Let (X,d) be a complete b-metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to Cl(X)$ an α_*-admissible mapping. Suppose that for any $x, y \in X$*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi(M_T(x,y))$$

where $\psi \in \Psi_4$. Then T is a MWP operator provided that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$ and for any sequence $\{x_n\}$ in X with $x_n \to x$ for some x in X implies that $\alpha(x_n,x) \geq 1$, $n \in \mathbb{N}_0$.

Corollary 2.5 *Let (X,d) be a complete b-metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to Cl(X)$ an α_*-admissible mapping. Suppose that for any $x, y \in X$*

$$\frac{1}{2}D(x,Tx) \leq bd(x,y) \text{ implies that } \alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi(d(x,y)) + \varphi(d(x,y))$$

where $\psi \in \Psi_4$ and $\varphi \in \Phi$. Then T is a MWP operator provided that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$ and for any sequence $\{x_n\}$ in X with $x_n \to x$ for some x in X implies that $\alpha(x_n,x) \geq 1$, $n \in \mathbb{N}_0$.

Example 2.6 *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. Define the mapping $d : X \times X \to \mathbb{R}^+$ by*

$$\begin{aligned}
&d(x_2,x_5) = d(x_3,x_4) = d(x_3,x_5) = d(x_2,x_4) = 5,\\
&d(x_2,x_3) = d(x_1,x_4) = d(x_1,x_5) = 9,\\
&d(x_1,x_2) = d(x_1,x_3) = 4,\\
&d(x_4,x_5) = 1,\\
&d(x,x) = 0 \text{ and } d(x,y) = d(y,x) \text{ for all } x,y \in X.
\end{aligned}$$

As $9 = d(x_2,x_3) \nleq d(x_2,x_1) + d(x_1,x_3) = 8$, d is not a metric on X. Clearly, (X,d) is a complete b-metric space with parameter $b \geq \frac{9}{8} > 1$. Define a mapping $T : X \longrightarrow Cl(X)$ by

$$Tx = \begin{cases} \{x_1\} \text{ if } x = x_1, x_2, x_3, \\ \{x_2\} \text{ if } x = x_4, \\ \{x_3\} \text{ if } x = x_5. \end{cases}$$

Define $\alpha : X \times X \to \mathbb{R}^+$ by $\alpha(x_i,x_j) = 1$ for all $i,j \in \{1,2,3,4,5\}$. If we set $\psi(t) = \frac{8}{9}t$ and $\varphi(t) = 0$ for $t \in \mathbb{R}^+$, then $\psi \in \Psi_4$ and $\varphi \in \Phi$. Note that

$$H(Tx,Ty) = 0 \leq \psi(M_T(x,y))$$

holds for any $x, y \in \{x_1, x_2, x_3\}$. We now consider the following cases:
If $x = x_1$ and $y \in \{x_4, x_5\}$ then

$$H(Tx_1,Ty) = 4 \leq 8 = \psi(d(x_1,y)) \leq \psi(M_T(x_1,y)) + \varphi(R_T(x_1,y)).$$

When $x \in \{x_2, x_3\}$ and $y \in \{x_4, x_5\}$. Then for $x \neq y$,

$$H(Tx, Ty) = 4 \leq \frac{40}{9} = \psi(d(x, y)) \leq \psi(M_T(x, y)) + \varphi(R_T(x, y)).$$

Note that for $x = x_4$, $y = x_2$ and $x = x_5$ $y = x_3$

$$\frac{1}{2}D(x, Tx) = \frac{5}{2} > \frac{9}{8} = bd(x, y).$$

Hence, for any $x, y \in X$,

$$\frac{1}{2}D(x, Tx) \leq bd(x, y) \text{ implies that } \alpha(Tx, Ty)H(Tx, Ty) \leq \psi\left(M_T(x, y)\right) + \varphi(R_T(x, y)).$$

Thus all the conditions of Theorem 2.1 are satisfied. On the other hand, if $x = x_4$, $y = x_5$, then we have

$$\begin{aligned}
H(Tx_4, Tx_5) &= d(x_2, x_3) = 9 > \psi\left(M_T(x_4, x_5)\right) \\
&= \psi\left(\max\left\{d(x_4, x_5), D(x_4, Tx_4), D(x_5, Tx_5), \frac{D(x_4, Tx_5) + D(x_5, Tx_4)}{2b}\right\}\right) \\
&= \psi\left(\max\left\{d(x_4, x_5), d(x_4, x_2), d(x_5, x_3), \frac{4\left(d(x_4, x_3) + d(x_5, x_2)\right)}{9}\right\}\right) \\
&= \psi\left(\max\left\{1, 5, 5, \frac{40}{9}\right\}\right) = \psi\left(5\right) = \frac{40}{9}.
\end{aligned}$$

Hence

$$H(Tx_4, Tx_5) = 9 \nleq \frac{40}{9} = \psi\left(M_T(x_4, x_5)\right)$$

Hence [3, Theorem 2.1], [7, Theorem 1], [15, Theorem 3.1], [19, Theorem 2.2] are not applicable in this case.

Corollary 2.7 *Let (X, d) be a complete b-metric space, $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ an $\alpha-$admissible and a generalized Suzuki type $(\alpha, \psi)-$contractive mapping. Let $x_0 \in X$ and $x_1 = fx_0$ be such that $\alpha(x_0, x_1) \geq 1$. Then f has a fixed point provided that for any $\{x_n\}$ in X with $x_n \to x$ for some x in X gives that $\alpha(x_n, x) \geq 1$. Moreover the fixed point of f is unique if X has property (H).*

Proof. By Theorem 2.1, $F(f)$ is nonempty and f is a Picard operator. To prove the uniqueness: Let u and v be two fixed points of f such that $u \neq v$. Due to property (H), there exists a point $z \in X$ such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$. As f is $\alpha-$admissible, we have $\alpha(u, f^n z) \geq 1$ and $\alpha(v, f^n z) \geq 1$, for all $n \geq 1$. Since f is a Picard operator so $\lim_{n \to \infty} f^n z = w \in F(f)$. Clearly

$$\frac{1}{2}d(u, fu) = 0 \leq bd(u, f^{n-1}z).$$

Note that

$$\begin{aligned}
d(u, f^n z) &\leq \alpha(fu, ff^{n-1}z)d(fu, ff^{n-1}z) \leq \psi(M_f(u, f^{n-1}z)) + \varphi(R_f(u, f^{n-1}z)) \\
&\leq \psi\left(\max\left\{d(u, f^{n-1}z), d(u, fu), d(f^{n-1}z, ff^{n-1}z), \frac{d(u, ff^{n-1}z) + d(f^{n-1}z, fu)}{2b}\right\}\right).
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides of above inequality we get

$$d(u, w) \leq \psi\left(d(u, w)\right).$$

This implies that $u = w$. Similarly, we obtain that $w = v$. Hence the result follows. ■

3 Common fixed point results in b-metric spaces

As an application of Theorem 2.1, we obtain the existence of coincidence and common fixed point of generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive hybrid pair of operators in the framework of b-metric spaces.

Definition 3.1 *Let (X, d) be a b-metric space. A hybrid pair (g, T) is called generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive pair of operators if there exists a $\psi \in \Psi_4$ and $\varphi \in \Phi$ such that for any $x, y \in X$*

$$\frac{1}{2}D(gx, Tx) \leq bd(gx, gy) \tag{23}$$

implies

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi\left(M_{g,T}(x, y)\right) + \varphi\left(R_{g,T}(x, y)\right). \tag{24}$$

Definition 3.2 *Let (X, d) be a b-metric space. A hybrid pair (g, T) is called generalized Suzuki type $(\alpha_*, \psi)-$contractive pair of operators if there exists a $\psi \in \Psi_4$ such that for any $x, y \in X$*

$$\frac{1}{2}D(gx, Tx) \leq bd(gx, gy)$$

implies

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi\left(M_{g,T}(x, y)\right).$$

Definition 3.3 *Let (X, d) be a b-metric space. A hybrid pair (g, T) is called Suzuki type $(\alpha_*, \psi)-$contractive pair of operators if there exists a $\psi \in \Psi_4$ such that for any $x, y \in X$*

$$\frac{1}{2}D(gx, Tx) \leq bd(gx, gy)$$

implies

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi\left(d(gx, gy)\right).$$

Definition 3.4 *Let (X, d) be a b-metric space and $f, g : X \to X$. A pair (f, g) is called Suzuki type $(\alpha, \psi)-$contractive pair if there exists a $\psi \in \Psi_4$ such that for any $x, y \in X$*

$$\frac{1}{2}d(gx, fx) \leq bd(gx, gy)$$

implies

$$\alpha(fx, fy)d(fx, fy) \leq \psi\left(d(gx, gy)\right).$$

Theorem 3.5 *Let (X, d) be a b-metric space, (g, T) a generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive pair with $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Suppose that T is $(\alpha_*, g)-$admissible and there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0, gx_1) \geq 1$. Then $C(g, T)$ is nonempty provided that for any sequence $\{x_n\}$ in X with $gx_n \to gx$ implies that $\alpha(gx_n, gx) \geq 1$. Furthermore, the set $F(g, T)$ is nonempty if any of the following conditions hold:*

e$_1$- *The pair (g, T) is $w-$compatible, $\lim_{n\to\infty} g^n(x) = w$ for some $w \in X$ and $x \in C(g, T)$ and g is continuous at w;*

e$_2$- *The mapping g is $T-$weakly commuting at some $x \in C(g, T)$ and $g^2x = gx$;*

e$_3$- *The mapping g is continuous at at some $x \in C(g, T)$ and $\lim_{n\to\infty} g^n(w) = x$ for some $w \in X$.*

Proof. By Lemma 1.7, there is a set $E \subseteq X$ such that $g : E \to X$ is one-to-one and $g(E) = g(X)$. Define a mapping $\mathcal{T} : g(E) \to CB(X)$ by $\mathcal{T}gx = Tx$ for all $g(x) \in g(E)$. The mapping $\mathcal{T}$ is well defined because g is one–to-one. As (g, T) is generalized Suzuki type $(\alpha_*, \psi_\varphi)-$contractive pair, for any $x, y \in X$

$$\begin{cases} \frac{1}{2}D(gx, Tx) \leq bd(gx, gy)) \\ \text{implies that} \\ \alpha_*(Tx, Ty)H(Tx, Ty) \\ \leq \psi\left(\max\left\{d(gx, gy), D(gx, Tx), D(gy, Ty), \dfrac{D(gx, Ty) + D(gy, Tx)}{2b}\right\}\right) \\ +\varphi\left(\min\left\{d(gx, gy), D(gy, Tx)\right\}\right) \end{cases}$$

where $\psi \in \Psi_4$ and $\varphi \in \Phi$. Thus

$$\begin{cases} \frac{1}{2}D(gx, \mathcal{T}gx) \leq bd(gx, gy)) \\ \text{implies} \\ \alpha_*(\mathcal{T}gx, \mathcal{T}gy)H(\mathcal{T}gx, \mathcal{T}gy) \\ \leq \psi\left(\max\left\{d(gx, gy), D(gx, \mathcal{T}gx), D(gy, \mathcal{T}gy), \dfrac{D(gx, \mathcal{T}gy) + D(gy, \mathcal{T}gx)}{2b}\right\}\right) \\ +\varphi\left(\min\left\{d(gx, gy), D(gy, \mathcal{T}gx)\right\}\right) \end{cases}$$

for all $gx, gy \in g(E)$. As $g(E)$ is complete so is $g(X)$. By given assumption, there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0, gx_1) \geq 1$. That is, there exists $x_0 \in X$ and $gx_1 \in \mathcal{T}gx_0$ such that $\alpha(gx_0, gx_1) \geq 1$. Since T is $(\alpha_*, g)-$admissible, $\alpha(gx, gy) \geq 1$ gives $\alpha_*(Tx, Ty) \geq 1$, that is, $\alpha_*(\mathcal{T}gx, \mathcal{T}gy) \geq 1$ and hence $\mathcal{T}$ is α_*-admissible. It follows from Theorem 2.1 that the mapping $\mathcal{T}$ on $g(E)$ is a MWP operator. Thus we may choose a point $u \in g(E)$ such that $u \in \mathcal{T}u$. Since $u \in g(E) = g(X)$, there exists $x \in X$ such that $gx = u$. Hence $gx \in \mathcal{T}gx = Tx$, that is, $x \in C(g, T)$. To prove $F(g, T) \neq \emptyset$: Suppose that (e$_1$) holds. Now, $\lim\limits_{n\to\infty} g^n(x) = u$ for some $u \in X$ and the continuity of g at u imply that $gu = u$ and hence $\lim\limits_{n\to\infty} g^n(x) = gu$. This further implies that $\alpha(g^n(x), gu) \geq 1$. As T is $(\alpha_*, g)-$admissible, $\alpha(Tg^{n-1}(x), Tu) \geq 1$. From $w-$compatibility of (g, T), we have $g^n(x) \in T\left(g^n(x)\right)$, that is $g^n(x) \in C(g, T)$ for all $n \in \mathbb{N}$. Clearly,

$$\frac{1}{2}D(g^n(x), T\left(g^{n-1}(x)\right)) \leq d(g^n(x), g^n(x)) = 0 \leq bd(gg^{n-1}(x), gu).$$

Thus, we have

$$\begin{aligned} D(g^n x, Tu) &\leq H(Tg^{n-1}x, Tu) \leq \alpha_*(Tg^{n-1}(x), Tu)H(Tg^{n-1}x, Tu) \\ &\leq \psi \max\left\{d(g^n x, gu), D(g^n x, Tg^{n-1}x), D(gu, Tu), \frac{D(g^n x, Tu) + D(gu, Tg^{n-1}x)}{2b}\right\} \\ &\quad +\varphi\left(\min\{d(g^n x, gu), D(gu, Tg^{n-1}x)\}\right) \\ &\leq \psi \max\left\{d(g^n x, gu), d(g^n x, g^n x), D(gu, Tu), \frac{D(g^n x, Tu) + d(gu, g^n x)}{2b}\right\} \\ &\quad +\varphi\left(\min\{d(g^n x, gu), d(gu, g^n x)\}\right). \end{aligned}$$

On taking limit as $n \to \infty$ on both sides of above inequality, we obtain that

$$D(gu, Tu) \leq \psi\left(D(gu, Tu)\right).$$

Hence $D(gu, Tu) = 0$ implies that $u = gu \in Tu$. Now if (e$_2$) holds, then $g^2x = gx$ for some $x \in C(g, T)$. Also, g is $T-$weakly commuting, $gx = g^2x \in Tgx$. Hence $gx \in F(g, T)$. If (e$_3$) holds , then we have $\lim\limits_{n\to\infty} g^n(u) = x$ for some $u \in X$ and $x \in C(g, T)$. By continuity of g, $x = gx \in Tx$. Hence in all cases, $F(g, T) \neq \emptyset$. ■

Theorem 3.6 *Let (X, d) be a b-metric space, (g, T) a generalized Suzuki type $(\alpha_*, \psi)-$contractive pair with $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Suppose that T is $(\alpha_*, g)-$admissible and there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0, gx_1) \geq 1$. Then $C(g, T)$ is nonempty provided that for any sequence $\{x_n\}$ in X with $gx_n \to gx$ implies that $\alpha(gx_n, gx) \geq 1$. Furthermore, the set $F(g, T)$ is nonempty if any of the following conditions hold:*

e$_4$- *The pair (g,T) is $w-$compatible, $\lim\limits_{n\to\infty} g^n(x) = w$ for some $w \in X$ and $x \in C(g,T)$ and g is continuous at w;*

e$_5$- *The mapping g is $T-$weakly commuting at some $x \in C(g,T)$ and $g^2x = gx$;*

e$_6$- *The mapping g is continuous at at some $x \in C(g,T)$ and $\lim\limits_{n\to\infty} g^n(w) = x$ for some $w \in X$.*

Theorem 3.7 *Let (X,d) be a b-metric space, (g,T) a Suzuki type $(\alpha_*,\psi)-$contractive pair with $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Suppose that T is $(\alpha_*,g)-$admissible and there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0,gx_1) \geq 1$. Then $C(g,T)$ is nonempty provided that for any sequence $\{x_n\}$ in X with $gx_n \to gx$ implies that $\alpha(gx_n,gx) \geq 1$. Furthermore, the set $F(g,T)$ is nonempty if any of the following conditions hold:*

e$_7$- *The pair (g,T) is $w-$compatible, $\lim\limits_{n\to\infty} g^n(x) = w$ for some $w \in X$ and $x \in C(g,T)$ and g is continuous at w;*

e$_8$- *The mapping g is $T-$weakly commuting at some $x \in C(g,T)$ and $g^2x = gx$;*

e$_9$- *The mapping g is continuous at at some $x \in C(g,T)$ and $\lim\limits_{n\to\infty} g^n(w) = x$ for some $w \in X$.*

For a pair of single-valued mappings, we have the following result.

Corollary 3.8 *Let (X,d) be a b-metric space, (f,g) a Suzuki type $(\alpha,\psi)-$contractive pair with $f(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Suppose that f is $(\alpha,g)-$admissible and for $x_0 \in X$ with $gx_1 = fx_0$, we have $\alpha(gx_0,gx_1) \geq 1$. Then $C(g,f)$ is nonempty provided that for any sequence $\{x_n\}$ in X with $gx_n \to gx$ implies that $\alpha(gx_n,gx) \geq 1$. Furthermore, $F(g,f)$ is nonempty if f and g are weakly compatible (that is f and g commute at their coincidence points). Also, g and f have unique point in $F(g,f)$ if $g(X)$ has property (H).*

4 Stability Results in b-metric spaces

In this section, we find an upper bound of Hausdorff distance between the fixed point sets of two generalized Suzuki type $(\alpha_*,\psi_\varphi)-$contractive multivalued operators and then we study the Ulam-Hyers stability of fixed point problem of such mappings in the setup of b-metric spaces.

Theorem 4.1 *Let (X,d) be a complete b-metric space, $\alpha : X \times X \to [0,\infty)$ and $T_1, T_2 : X \to P(X)$. If the following conditions hold:*

d$_1$- *for each $i \in \{1,2\}$,*

$$\frac{1}{2}D(x,T_ix) \leq bd(x,y) \text{ implies that } \alpha_*(T_ix,T_iy)H(T_ix,T_iy) \leq \psi_i(d(x,y)) + \varphi_i(d(x,y)) \tag{25}$$

for any $x,y \in X$, where $\psi_i \in \Psi_4$ and $\varphi_i \in \Phi$;

d$_2$- *T_i is α_*-admissible mapping for each $i \in \{1,2\}$;*

d$_3$- *there exists $x_0 \in X$ and $x_1 \in T_ix_0$ such that $\alpha(x_0,x_1) \geq 1$ for each $i \in \{1,2\}$;*

d$_4$- *if there is a sequence $\{x_n\}$ in X such that $x_n \to x$, then $\alpha(x_n,x) \geq 1$;*

d$_5$- *there exists $\lambda > 0$ such that for all $x \in X$, $H(T_1x,T_2x) \leq \lambda$.*

Then for each $i \in \{1,2\}$, $Fix(T_i)$ is a closed subset of X, T_i is a MWP operator and

$$H(Fix(T_1),Fix(T_2)) \leq b^2 \max\{\lambda_1,\lambda_2\}$$

where $\lambda_i = \sum\limits_{k=0}^{\infty} b^k \psi_i^k(\lambda)$.

Proof. It follows from Corollary 2.5 that $Fix(T_i) \neq \emptyset$ for each $i \in \{1, 2\}$. Let $\{x_n\}$ be a sequence in $Fix(T_1)$ such that $x_n \to z$ as $n \to \infty$. Then $\alpha(x_n, z) \geq 1$. As T_1 is α_*-admissible, $\alpha(Tx_n, Tz) \geq 1$. Clearly

$$\frac{1}{2}D(x_n, T_1x_n) = 0 \leq bd(z, x_n).$$

Hence

$$\begin{aligned} D(z, T_1z) &\leq bd(z, x_n) + bD(x_n, T_1z) \\ &\leq bd(z, x_n) + bH(T_1z, T_1x_n) \leq bd(z, x_n) + b\alpha_*(T_1x_n, T_1z)H(T_1z, T_1x_n) \\ &\leq bd(z, x_n) + b\psi_1(d(z, x_n)) + b\varphi_1(D(x_n, T_1z)) \\ &\leq bd(z, x_n) + b\psi_1(d(z, x_n)) + b\varphi_1(d(x_n, z)). \end{aligned}$$

On taking limit as $n \to \infty$ we obtain that $D(z, T_1z) = 0$, that is, $z \in T_1z$ and hence $F(T_1)$ is closed. Similarly, $F(T_2)$ is a closed subset of X. From Corollary 2.5, we conclude that for each $i \in \{1, 2\}$, T_i is a MWP operator. Following arguments similar to those in the proof of Theorem 2.1 with $x_0 \in F(T_1)$ and $x_1 \in T_2x_0$, we obtain a sequence a Cauchy sequence $\{x_n\}$ such that $x_{n+1} \in T_2x_n$ for all $n \geq 1$ and it satisfies

$$\begin{array}{l} x_{n+1} \neq x_n, \\ \alpha(x_{n+1}, x_{n+2}) \geq 1 \\ 0 < D(x_{n+1}, T_2x_{n+1}) \leq \psi_2(d(x_n, x_{n+1})) \\ 0 < d(x_{n+1}, x_{n+2}) < \psi_2^n(q\psi(c_0)) \end{array} \tag{26}$$

for all $n \geq 1$, where $c_0 = d(x_0, x_1)$. We choose an element u in X such that $x_n \to u$ as $n \to \infty$ and $u \in T_2u$. As in the proof of Theorem 2.1 we obtain

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \frac{1}{b^{n-2}} \sum_{k=n-1}^{n+p-2} b^k\psi_2^k(q\psi_2(c_0)) \leq \frac{1}{b^{n-2}} \sum_{k=n-1}^{n+p-2} b^k\psi_2^k(q\psi_2(\lambda)) \\ &= \frac{1}{b^{n-2}} \left(\sum_{k=0}^{n+p-2} b^k\psi_2^k(q\psi_2(\lambda)) - \sum_{k=0}^{n} b^k\psi_2^k(q\psi_2(\lambda)) + b^n\psi_2^n(q\psi_2(\lambda)) \right). \end{aligned} \tag{27}$$

On taking limit as $p \to \infty$ on both sides of above inequality, we obtain that

$$d(x_n, u) \leq \frac{1}{b^{n-2}} \left(\sum_{k=0}^{\infty} b^k\psi_2^k(q\psi_2(\lambda)) - \sum_{k=0}^{n} b^k\psi_2^k(q\psi_2(\lambda)) + b^n\psi_2^n(q\psi_2(\lambda)) \right) \tag{28}$$

and hence by Lemma 1.8, $\sum_{k=0}^{\infty} b^k\psi_2^k(t)$ converges for any $t \geq 0$. Thus there exists $\lambda_2 \geq 0$ such that $\sum_{k=0}^{\infty} b^k\psi_2^k(\lambda) = \lambda_2$. Then from 28, it follows that

$$d(x_n, u) \leq \frac{1}{b^{n-2}} \left(\lambda_2 - \sum_{k=0}^{n} b^k\psi_2^k(q\psi(\lambda)) + b^n\psi_2^n(q\psi(\lambda)) \right) \tag{29}$$

and for $n = 0$, we get $d(x_0, u) \leq b^2\lambda_2$. Hence for $x_0 \in F(T_1)$, there exists $u \in F(T_2)$ such that $d(x_0, u) \leq b\lambda_2$. Similarly, for each $z_0 \in F(T_2)$, we get $v \in F(T_1)$ and $\lambda_1 \geq 0$ such that $d(z_0, v) \leq b^2\lambda_1$. It follows from Lemma 1.4 that

$$H(F(T_1), F(T_2)) \leq b^2 \max\{\lambda_1, \lambda_2\}.$$

■

Theorem 4.2 *Let (X, d) be a b-metric space and $T : X \to Cl(X)$ be a multivalued mapping as given in Corollary 2.5 hold. Then following statements hold:*

u₁- *The fixed point inclusion (1) is $\zeta_1^{-1}-$generalized Ulam-Hyers stable provided that for $x \in F(T)$, there exists $z \in U$ such that $\alpha(x, z) \geq 1$, where $\zeta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\zeta_1(t) = t - b^2(\psi(t) + \varphi(t))$ is strictly increasing and onto;*

u$_2$- *If $E(T) \neq \emptyset$, then the fixed point inclusion (1) is $\zeta_2^{-1}-$generalized Ulam-Hyers stable provided that for $x \in F(T)$ there exists $z \in U$ such that $\alpha(x,z) \geq 1$, where $\zeta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\zeta_2(t) = t - b(\psi(t) + \varphi(t))$ is strictly increasing and onto;*

u$_3$- *If $S : X \to Cl(X)$ is such that for $x \in F(S)$, there exists $z \in F(T)$ such that $\alpha(x,z) \geq 1$ and for $x \in F(T)$, there exists $z \in F(S)$ such that $\alpha(x,z) \geq 1$, and $H(S(x),T(x)) \leq \lambda$ for all $x \in X$, then $H(F(S),F(T)) \leq \zeta_1^{-1}(b^2\lambda)$;*

u$_4$- *If $S : X \to Cl(X)$ is such that for $x \in F(S)$, there exists $z \in E(T)$ such that $\alpha(x,z) \geq 1$ and for $x \in E(T)$ there exists $z \in F(S)$ such that $\alpha(x,z) \geq 1$, and $H(S(x),T(x)) \leq \lambda$ for all $x \in X$, then $H(F(S),F(T)) \leq \zeta_2^{-1}(b\lambda)$;*

u$_5$- *If $\{x_n\}$ is a sequence in X and there exists a unique $x^* \in E(T)$ such that $\alpha(x_n,x^*) \geq 1$ and*

$$\lim_{n\to\infty} D(x_n,Tx_n) = 0.$$

Then $\lim_{n\to\infty} d(x_n,x^*) = 0$. In this case, we say an end point problem is well-posed with respect to b-metric d;

u$_6$- *If $\{x_n\}$ is a sequence in X, and there exists a unique $x^* \in E(T)$ such that $\alpha(x_n,x^*) \geq 1$ and $\lim_{n\to\infty} H(\{x_n\},Tx_n) = 0$. Then $\lim_{n\to\infty} d(x_n,x^*) = 0$. In this case, we say an end point problem is well-posed with respect to Hausdorff b-metric H;*

u$_7$- *If $\{x_n\}$ is a sequence in X, and there exists a unique $x^* \in E(T)$ such that $\alpha(x_n,x^*) \geq 1$ and $\lim_{n\to\infty} D(x_n,Tx_n) = 0$, then there exists a sequence of successive approximation y_n such that*

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

This is known as limit shadowing property of the multivalued operators.

Proof. (u$_1$-) It follows from Corollary 2.5 that T is a MWP operator and hence $F(T)$ is nonempty. If $x^* \in F(T)$, then by given condition there exists a $y^* \in U$ such that $\alpha(x^*,y^*) \geq 1$. As T is α_*-admissible, we have $\alpha_*(Tx^*,Ty^*) \geq 1$. Since $y^* \in U$, therefore $D(y^*,Ty^*) \leq \varepsilon$ for any given $\varepsilon > 0$. Clearly

$$\frac{1}{2}D(x^*,Tx^*) = 0 \leq bd(x^*,y^*).$$

Thus

$$\begin{aligned}
d(x^*,y^*) &\leq bD(x^*,Tx^*) + bD(Tx^*,y^*) = bD(Tx^*,y^*) \\
&\leq b^2\left(H(Tx^*,Ty^*) + D(Ty^*,y^*)\right) \\
&\leq b^2\left(\alpha_*(Tx^*,Ty^*)H(Tx^*,Ty^*) + \varepsilon\right) \\
&\leq b^2\left(\psi(d(x^*,y^*)) + \varphi\left(d(x^*,y^*)\right) + \varepsilon\right) \\
&\leq b^2\left(\psi(d(x^*,y^*)) + \varphi\left(d(x^*,y^*)\right) + \varepsilon\right).
\end{aligned}$$

Now, $\zeta_1\left(d(x^*,y^*)\right) = d(x^*,y^*) - b^2\left(\psi(d(x^*,y^*)) + \varphi\left(d(x^*,y^*)\right)\right)$. From the above inequality we have $\zeta_1\left(d(x^*,y^*)\right) \leq b^2\varepsilon$ and hence

$$d(x^*,y^*) \leq \zeta_1^{-1}\left(b^2\varepsilon\right).$$

Consequently the fixed point inclusion (1) is $\xi-$generalized Ulam-Hyers stable, where $\xi = \zeta_1^{-1}$.

(u$_2$-) Let $E(T) \neq \emptyset$, then $d(x^*,y^*) = D(Tx^*,y^*) \leq b\left(H(Tx^*,Ty^*) + D(Ty^*,y^*)\right)$. Following arguments similar to those in the proof of (u$_1$), result follows.

(u$_3$-) Let $x^* \in F(S)$, then there exists a $y^* \in F(T)$ such that $\alpha(x^*,y^*) \geq 1$. By α_*-admissibility of T we obtain that $\alpha_*(Tx^*,Ty^*) \geq 1$. Clearly

$$\frac{1}{2}D(x^*,Tx^*) = 0 \leq bd(x^*,y^*).$$

Thus

$$
\begin{aligned}
d(x^*, y^*) &\leq bD(x^*, Sx^*) + bD(Sx^*, y^*) = bD(Sx^*, y^*) \\
&\leq b^2 \left(H(Sx^*, Tx^*) + D(Tx^*, y^*) \right) \\
&\leq b^2 \left(H(Sx^*, Tx^*) + H(Tx^*, Ty^*) \right) \\
&\leq b^2 \left(\lambda + H(Tx^*, Ty^*) \right) \\
&\leq b^2 \left(\lambda + \alpha_*(Tx^*, Ty^*) H(Tx^*, Ty^*) \right) \\
&\leq b^2 \left(\lambda + \psi(d(x^*, y^*)) + \varphi\left(d(x^*, y^*)\right) \right) \\
&\leq b^2 \left(\lambda + \psi(d(x^*, y^*)) + \varphi\left(d(x^*, y^*)\right) \right).
\end{aligned}
$$

Now, $\zeta_1\left(d(x^*, y^*)\right) = d(x^*, y^*) - b^2\left(\psi(d(x^*, y^*)) + \varphi\left(d(x^*, y^*)\right)\right)$. From the above inequality we obtain that $\zeta_1\left(d(x^*, y^*)\right) \leq b^2\lambda$. Consequently for every $x^* \in F(S)$, there exists a $y^* \in F(T)$ such that $d(x^*, y^*) \leq \zeta_1^{-1}\left(b^2\lambda\right)$. Similarly it can be proved that for every $y^* \in F(T)$, there exists a $x^* \in F(T)$ such that $d(x^*, y^*) \leq \zeta_1^{-1}\left(b^2\lambda\right)$. Hence by Lemma 1.4 we obtain that

$$H(F(S), F(T)) \leq \zeta_1^{-1}\left(b^2\lambda\right).$$

(u_4-) The proof follows using the similar arguments to those in the proof of (u_3) with definition of $E(T)$.

(u_5-) Let $\{x_n\}$ be a sequence in X. By given assumption, there exists a unique $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geq 1$, and $\lim_{n\to\infty} D(x_n, Tx_n) = 0$. Then we obtain a sequence $\{u_n\}$ such that $u_n \in Tx_n$ and $\lim_{n\to\infty} D(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, u_n) = 0$. As T is α_*-admissible, $\alpha_*(Tx_n, Tx^*) \geq 1$. Note that $\frac{1}{2}D(x^*, Tx^*) = 0 \leq bd(x_n, x^*)$. Hence

$$
\begin{aligned}
d(x_n, x^*) &\leq b\left(D(x_n, Tx_n) + D(Tx_n, x^*)\right) \\
&\leq b\left(D(x_n, Tx_n) + H(Tx_n, Tx^*)\right) \\
&\leq b\left(D(x_n, Tx_n) + \alpha_*(Tx_n, Tx^*) H(Tx_n, Tx^*)\right) \\
&\leq b\left(D(x_n, Tx_n) + \psi\left(d(x_n, x^*)\right) + \varphi\left(d(x_n, x^*)\right)\right).
\end{aligned}
$$

This implies that

$$d(x_n, x^*) - b(\psi\left(d(x_n, x^*)\right) + \varphi\left(d(x_n, x^*)\right)) \leq bD(x_n, Tx_n).$$

That is $\zeta_2\left(d(x_n, x^*)\right) \leq bD(x_n, Tx_n)$. Taking limit as n tends to ∞, the result follows. ($\mathbf{u_6}$) follows from (u_3) as $D(x_n, Tx_n) \leq H(\{x_n\}, Tx_n)$. ($u_7$-) From ($u_5$) it is clear that $\lim_{n\to\infty} d(x_n, x^*) = 0$. Since $x^* \in E(T)$, so there exists a sequence of successive approximations defined as $y_n = x^*$ for all n such that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n, x^*) = 0$. ■

Acknowledgement: The authors are grateful to the reviewers for their valuable remarks and suggestions as it helped us to improve the original version of the paper.

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