A practical implementation of XVA in the new normal

by

Christopher Kairinos

Submitted in partial fulfillment of the requirements for the degree

Magister Scientiae

to the Department of Mathematics and Applied Mathematics
in the Faculty of Natural and Agricultural Sciences

University of Pretoria
Pretoria

Supervisor: Prof. E Maré

June 2017

Declaration

I, Christopher Kairinos, declare that this dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Christopher Kairinos
June 2017

Abstract

The Great Financial Crisis (GFC) of 2008 left many financial institutions devastated. Despite the practice of advanced risk management at the time, society witnessed the collapse of the *"too big to fail"* institutions. Gaping holes within the existing risk framework lurked, which both regulators and practitioners failed to detect. This dissertation discusses the symptoms of the crisis that were overlooked and explores the financial engineering implemented post-2008 to avoid the next crisis. The author considers the work of Hull, White, Gregory, Brigo, Kenyon, Green, Morini, Pallavicini, Piterbarg, Burgard, Kjaer, Elouerkhaoui, and Castagna. A literature review is provided for each of the mentioned names to highlight each author's contribution to the field of Total Value Adjustment (XVA) pricing. An in-depth analysis on the funding invariance principle suggested by Elouerkhaoui is provided followed by a model implementation. The core aim of this dissertation is to review XVA valuations from a practitioners perspective using the framework provided by Elouerkhaoui. A secondary aim of the dissertation is to briefly explore the work of Aboura and Maillard on the Cornish-Fisher Transformation (CF). The CF is considered as a parsimonious approach in estimating non-normal distributions, therefore an interesting alternative to price XVA using Monte Carlo (MC) simulation.

Acknowledgments

I would like to thank my supervisor Prof. Eben Maré for the patience he has shown and the invaluable advice he has given me throughout my course. To my friends and family, thank you for providing me with the motivation to complete what I started. To Berenice, your encouragement gave me the will I needed to cross the finish line, I cannot thank you enough.

"To strive, to seek, to find, and not to yield."

Lord Alfred Tennyson

Contents

List of Figures

List of Tables

Acronyms

AIG American International Group
ATM At the Money

bbl Blue Barrel
BOJ Bank of Japan
BS Black-Scholes
BSM Black-Scholes-Merton
BVA Bilateral Value Adjustment

CAPM Capital Asset Pricing Model
CBVA Collateral-Inclusive Bilateral Valuation Adjustment
CCP Central Counterparties
CCVA Collateral-Inclusive Credit Value Adjustment
CDS Credit Default Swaps
CDVA Collateral-Inclusive Debit Value Adjustment
CF Cornish-Fisher Transformation
COLBA Collateral Benefit Adjustment
COLCA Collateral Cost Adjustment
COLVA Collateral Adjustment
CRA Credit Rating Agencies
CRM Comprehensive Risk Measure
CSA Credit Support Annexure
CVA Credit Value Adjustment

DVA Debit Value Adjustment

EAD Exposure at Default
ECB European Central Bank
EE Expected Exposure
EEE Effective expected exposure
EEPE Effective Expected Positive Exposure
EMIR European Market Infrastructure Reform
EONIA Euro Overnight Index Average
EPE Expected Positive Exposure
EURIBOR Euro Interbank Offered Rate
EVA Economic Value adjustment

FAS Financial Accounting Standards
FBA Funding Benefit Adjustment
FCA Funding Cost Adjustment
FVA Funding Value Adjustment
FVO Fair-Value Option

GBM Geometric Brownian Motion
GDP Gross Domestic Product
GFC Great Financial Crisis
GSE Government-Sponsored Enterprises

IAS International Accounting Standards
ICE Inter-Continental Exchange
IM Initial Margin

IMF International Monetary Fund
IMM Internal Model Method
IRC Incremental Risk Charge
ISDA International Swaps and Derivatives Association
ITM In the Money

JIBAR Johannesburg Interbank Agreed Rate

KVA Capital Value Adjustment

LGD Loss Given Default
LIBOR London Interbank Offered Rate

MBS Mortgage-Backed Securities
MC Monte Carlo
MTM Mark-to-Market
MVA Margin Value Adjustment

NEE Negative Expected Exposure

OIS Overnight Index Swap
OTC Over-the-Counter
OTM Out the Money
OU Ornstein-Uhlenbeck Process

PD Probability of Default
PDE Partial Differential Equation
PV Present Value

QE Quantitative Easing
QMC Quasi-Monte Carlo

RWA Risk Weighted Assets

SDE Stochastic Differential Equation
SL Short-Term Variation and Long-Term Dynamic Model
SP Standard and Poor's
SPV Special Purpose Vehicles
SVAR Stressed VAR

TIBOR Tokyo Interbank Offered Rate

UCVA Unilateral Credit Value Adjustment
UDA Unilateral Default Assumption
UDVA Unilateral Debit Valuation Adjustment
US United States

VAR Value-at-Risk
VM Variation Margin

XVA Total Value Adjustment

Notation

$\mathbb{Q}$: risk-neutral measure.

$\mathbb{P}$: real-world measure.

r : constant interest rate.

F(T,x) : value of a derivative at time T given intial value x.

Φ(x) : payoff function with intial value x.

£ : a natural probaility space.

F(t,x) : value of a derivative at time t given intial value x.

$\mathbb{E}^{\mathbb{Q}}{}_{t,x}$: risk-neutral expectation operator taken under a $\mathbb{Q}$-measure given the intial value x at t.

$\Phi(\mathrm{X_T})$: payoff function using underlying X at T.

W : $\mathbb{Q}$-Wiener process.

r_C : constant collateral interest rate.

Φ(S(T)) : payoff function using underlying S at T.

Π : price of a given contingent claim.

Φ : payoff function.

F(t,S(t)) : value of a derivative at time t where $S(t) = s$.

F(t,s) : value of a derivative at time t given intial value s.

$\mathbb{E}^{\mathbb{Q}}{}_{t,s}$: risk-neutral expectation operator taken under a $\mathbb{Q}$-measure given the intial value s at t.

$\mathbb{E}^{\mathbb{Q}}$: risk-neutral expectation operator taken under a $\mathbb{Q}$-measure.

T : valuation at maturity.

λ : hazard rate of default.

R : recovery rate where $R \in [0, 1]$.

r_R : constant repo rate.

r_D : constant dividend rate.

r_V : constant funding interest rate for the internal cash account.

C : amount of collateral posted.

s_F : constant spread between the risk-free rate and the collateral rate.

F_{CSA} : forward contract traded under CSA.

$\mathrm{D_{r+\lambda_B+\lambda_C}(t, u)}$: discount factor where u $\in [t, T]$ and the risky discount rate is given in the subscript.

$\mathbb{E}^{\mathbb{Q}}{}_t$: risk-neutral expectation operator taken under a $\mathbb{Q}$-measure from time t.

$\mathrm{D_k(t, u)}$: generalised discount factor where u $\in [t, T]$ and the risky discount rate is given by k.

M : MTM value at default.

g : closeout function.

ϵ_h : portfolio hedge error.

s_C : constant spread between the risk-free rate and the lenders unsecured funding rate.

$\mathbb{I}$: probability of default within a specified time period.

$\mathrm{D(t, \tau_C)}$: discount factor using the risk-free rate r, where $\tau_C \in [t, T]$ is the time of default for counterparty C.

$\mathrm{D(t, \tau_B)}$: discount factor using the risk-free rate r, where $\tau_B \in [t, T]$ is the time of default for counterparty B.

τ : default times of a specific entity.

$\mathrm{D(t, \tau)}$: discount factor using the risk-free rate r, where $\tau \in [t, T]$ is the time of default.

φ : the fraction of capital used for funding.

$\mathbf{K}$: amount of capital consumed for a trade.

$\gamma_{\mathrm{K}}(\mathrm{t})$: cost of capital at time t.

s_I : constant spread between the risk-free rate and the intial margin funding rate.

$\mathbf{I}$: amount of intial margin posted.

$\mathrm{D^{r}_{t,s}}$: discount factor using the risk-free rate r, where $s \in [t, T]$.

$\mathrm{D^{r*}_{t,s}}$: discount factor using any interest rate, where $s \in [t, T]$.

$\mathrm{D^{r}_{t,\tau}}$: discount factor using the risk-free rate r, where $\tau \in [t, T]$ is the time of first to default.

$\mathrm{D^{r*}_{t,\tau}}$: discount factor using any interest rate, where $\tau \in [t, T]$ is the time of first to default.

$\mathrm{D^{r_v}_{t,\tau}}$: discount factor using the funding rate r_v, where $\tau \in [t, T]$ is the time of first to default.

$\mathrm{D^{r_v}_{t,s}}$: discount factor using the funding rate r_v, where $s \in [t, T]$.

$\mathcal{F}_t$: price process filtration.

Part I

Introduction and Context

Chapter 1

Structure of the Dissertation

The initial objective of this dissertation is to provide perspective on what was considered best practice pre-GFC to date. Consequently, we discuss the events leading up to and during the GFC. We then introduce the reader to the most prominent XVA papers published post-2008 to illustrate how the said events were treated. The field of XVA pricing is relatively new, resulting in a wide selection of what is considered to be best practice in the banking industry. In this dissertation, Elouerkhaoui's work is believed to be the best framework for practical implementation.

The central goal of this dissertation is to elaborate on the mathematics of Elouerkhaoui's framework and demonstrate how it can be implemented using a MC model to be easily understood by practitioners. A side goal of the dissertation is to also explore ways to improve the standard MC model's accuracy and efficiency by implementing Quasi-Monte Carlo (QMC) techniques and the CF. We then analyse the results obtained to draw conclusions on Elouerkhaoui's framework, and to justify whether variations from the standard MC model do indeed improve the pricing model. At the point at which conclusions are drawn, we wish to have provided the reader with a sound knowledge on how pricing has evolved since 2008. Moreover, we aim to provide clarity on what is the most parsimonious approach one could take without sacrificing accuracy. We refer to the work of [Aboura and Maillard 2014] often and demonstrate the ease in which the CF can be used to capture risks typically overlooked when assuming a normal distribution.

We divide the dissertaion into five parts in order to facilitate a progressive understanding of the topic that the dissertation covers. We subdivide each part into chapters. The first part of the dissertation is titled *Introduction and Context*. In the second chapter we introduce the reader to the GFC of 2008, to provide context as to how risk was incorrectly priced leading up to 2008, ultimately resulting in the need for XVAs. Chapter 3 highlights the doubt around the law of one price, a fundamental assumption in the Black-Scholes-Merton (BSM) theory. Chapter 3 introduces all the XVA terms we analyse in this dissertation. We also touch on the key frameworks used to price XVAs and list the academics who favour the different methods. The objective of the abovementioned chapters is to provide context as to why XVA pricing is needed. We discuss the BSM theory as it is central to all derivative pricing theory, therefore some knowledge of it is needed by the reader to fully appreciate the evolution of XVA pricing theory.

The second part of the dissertation is the *Literature Review*. It discusses all the papers published by the authors mentioned in the first part of the dissertation. The objective of part two of the dissertation, is to highlight the most relevant publications available on XVA pricing. Not all of the frameworks discussed in this part of the dissertation are key to the mathematics and model implementation provided in this dissertation, but were included to provide a basic understanding on what practitioners are currently using. In Chapter 4, we discuss the work of renowned professors, Hull and White. We consider their argument as to why including Funding Value Adjustment (FVA) in the pricing of a derivative is not correct. We look at the framework they provide to price for Credit Value Adjustment (CVA) and Debit Value Adjustment (DVA).

In Chapter 5, we begin to unpack the *Pricing by Hedging* framework, one of the two most prominent frameworks used to price XVAs today. We begin with Piterbarg's work to show how the well known BSM Partial Differential Equation (PDE) evolves when we are no longer funding with one key risk-free rate. We then delve into the work of Burgard and Kjaer who expand on Piterbarg's work to include credit risk between the bank and the counterparty it is dealing with. We later consider Burgard and Kjaer's work published to address their inital assumptions that a bank can easily buy its own bonds.

Chapter 6 begins our discussion on the *Pricing by Expectation* framework which relaxes the assumption that the price of a derivative can be determined by creating a replication portfolio. The work of Brigo, Pallavicini, and Morini is examined to determine how to include CVA, DVA, Collateral-Inclusive Bilateral Valuation Adjustment (CBVA) and FVA into the final price of a derivative. Chapters 5 and 6 introduce the reader to the frameworks used immediately post-2008, the last two chapters of part 2 includes the latest additions of the XVA pricing theory. Chapter 7 looks at the inclusion of Margin Value Adjustment (MVA) and Capital Value Adjustment (KVA) within the replication framework discussed in Chapter 5. The work we review is provided by Kenyon and Green. Kenyon and Green modify the framework of Burgard and Kjaer to cater for the new funding and capital costs practitioners face today.

Chapter 8 introduces the reader to the *Funding Invariance Principle* by Elouerkhaoui, the key focus of this dissertation. We begin with his inital work on the funding equation with the inclusion of CVA and DVA. We also include his latest work with the addition of KVA and MVA to the funding equation to provide us with a master pricing equation. The objective of Chapter 8 is to familiarise the reader with the basic principles of Elouerkhaoui's two publications, [Elouerkhaoui 2016a] and [Elouerkhaoui 2016b]. Our goal for part two of the dissertation is to provide the reader with a sound footing in the field if XVA pricing.

Where part 2 introduced us to the theorems and definitions of Elouerkhaoui's work, part 3, titled *Mathematical Preliminaries*, works through the derivation of some of the core pricing formulas we wish to implement later. Chapter 9 provides the derivation of the funding equation, funding invariance principle, the master equation, as well as the funded margined CVA formula. Chapter 10 uses similar arguments used in Chapter 9 to show the derivation of the same formulas mentioned in the chapter, however, inclusive of MVA and KVA in this instance. We also include the regulatory capital formulas banks are required to price for as published in Basel II, 2.5, and III. Chapter 11 is a short summary of how we will account for Probability of Default (PD) in our pricing models. The aforesaid chapters seek to provide more detail on the pricing formulas raised in Chapter 8. We include a brief description on how one can imply PD from market variables for completeness.

Part 4 is titled *Pricing Model Implementation.* Chapter 12, provides the reader with the foundation of how to value a derivative using MC and QMC techniques without any consideration of XVA. There are two main objectives for Chapter 12:

1. To establish a basic MC framework which forms the base of all pricing in the dissertation; also to implement QMC to illustrate how one can improve on efficiency; and
2. To implement three different models to be applied in the MC and QMC framework; Short-Term Variation and Long-Term Dynamic Model (SL), Geometric Brownian Motion (GBM) and Ornstein-Uhlenbeck Process (OU).

This approach shows how efficient pricing can be achieved and that pricing changes significantly when using different models. We choose ICE Brent crude oil data to calibrate our model. The objective of Chapter 13 is to address how one can use the CF to account for skewness and kurtosis when pricing a derivative. Standard BSM theory assumes a normal distribution, which implies the distribution will have a skewness of 0 and a kurtosis of 3. This is not an accurate assumption when one considers that most empirical distributions are non-normal. Typically practitioners default to jump diffusion models to model a non-normal distribution, however, we find the CF approach a more natural approximation as highlighted by Aboura and Maillard's work. Chapters 12 and 13 provide context as to how a derivative was priced pre-2008; Chapter 14 elaborates on how we incorporate XVAs into our model to ensure we are correctly pricing our derivative in the new normal. The XVAs accounted for in this chapter are FVA, CVA, DVA, and COLVA using a normal and a non-normal distirubtion to generate our scenarios. Chapter 15 concludes our pricing model by applying MVA and KVA to a risky derivative price. Our aim in Chapter 14 and 15 is to show how the pricing models implemented in Chapter 12 and 13 can be expanded within the Elouerkhaoui framework to include all XVAs. We achieve this by tabulating our pricing results with references to the pricing formulas listed throughout the dissertation. Chapter 16 concludes the dissertation with a summary of our findings as well as a list for the reader to pursue further research on this topic.

Part 5 of the dissertation contains the appendices provided to supplement the understanding of the main text as well as pseudocode of the model, thereafter the bibliography can be found.

Chapter Summary

The aims and objectives of each chapter are listed below:

Chapter 2 - Introduction

- To describe the events of the GFC and provide background as to how and why derivative pricing needed to evolve.

Chapter 3 - The Law of One Price

- To discuss the divergence between interbank rates and Overnight Index Swap (OIS) rates.
- To highlight the assumption of using one risk-free rate in the BSM model, and to discuss how it no longer holds.
- To introduce each of the XVA terms, as well as the two mainstream pricing frameworks used in the bank industry.
- To illustrate the simplicity of valuing a derivative pre-2008 in comparison to valuing a derivative post-2008 to date.

Chapter 4 - Hull and White

- To review the work of Hull and White on XVA pricing.
- To discuss the FVA debate between Hull and White, and practitioners with a particular focus on Castagna's publications.
- To discuss Hull and White's version of the Black-Scholes (BS) PDE derived to price XVAs.

Chapter 5 - Pricing by Hedging - Piterbarg, Kjaer, and Burgard

- To review the XVA pricing by hedging framework presented by Piterbarg, Kjaer, and Burgard.
- To define the key differences between Merton's pricing by replication approach, and pricing by expectation.

- To unpack Piterbarg's version of the BS PDE derived to account for multiple funding rates.

- To unpack Burgard and Kjaer's version of the BS PDE, which is similar to Piterbarg's PDE but for a risky counterparty. This entails the inclusion of the CVA, FVA, and DVA terms. This chapter also explores the skepticism around the model assumption that markets are complete.

Chapter 6 - Pricing by Expectation - Brigo, Morini, and Pallavicini

- To review the XVA pricing by expectation framework presented by Brigo, Morini, and Pallavicini.

- To clearly demonstrate how Brigo, Morini and Pallavicini's pricing formulas evolve as you include various XVAs.

Chapter 7 - Including MVA and KVA in the Semi-Replication Framework - Kenyon and Green

- To review the work of Kenyon and Green, which involves their adaptation of Burgard and Kjaer's work.

- To discuss the pricing formulas Kenyon and Green derived to include MVA and KVA using the pricing by replication framework.

Chapter 8 - The Funding Invariance Principle with XVA - Elouerkhaoui

- To review the intial work of Elouerkhaoui, which is the funding invariance principle with and without counterparty risk. We focus on how one can price a derivative with XVAs without the complexity of using numerous discount rates. This is the most practical approach one can find in current XVA papers.

- To review the latest work of Elouerkhaoui, which shows how to include the MVA and KVA terms into the funding invariance framework.

Chapter 9 - The Funding Invariance Principle - Elouerkhaoui

- To show and explain the derivation of the most relevant pricing formulas used in Elouerkhaoui's funding invariance principle. This includes mathematical proofs with explanations.

Chapter 10 - From FVA to KVA - Elouerkhaoui

- To show and explain the proofs of the pricing formulas Elouerkhaoui derived which include MVA and KVA terms.

Chapter 11 - The Poisson Process

- To briefly explain how we account for PD calculations in the dissertation.

Chapter 12 - Pricing Commodities without XVA

- To introduce the Schwartz and Smith two factor model for commodity derivative valuations.

- To implement a MC model calibrated to current data.

- To implement a QMC model to improve model efficiency

Chapter 13 - Accounting for a Non-Normal World

- To introduce the CF as a means to account for non-normal skewness and kurtosis in a model. This is done to increase accuracy when using the MC or QMC model.

- To create a working example and provide results for comparison between pricing under the assumption of a normal random number generator and a non-normal random number generator.

Chapter 14 - Adding CVA, DVA, COLVA, and FVA to the Batch

- To apply the pricing framework introduced in Chapter 12 and 13 to demonstrate how one can price for FVA, COLVA, DVA, and CVA.

- To tabulate the results and discuss how values change for different derivatives, under different funding circumstances, and under different collateral agreements.

Chapter 15 - Finishing off with MVA and KVA

- To include MVA and KVA terms in the pricing exercises done in Chapter 14. The exact same framework implemented in previous chapters is applied here.

- To tabulate the results and discuss the trade-off that exists between MVA and KVA. This is dependent on how much Initial Margin (IM) should be placed at inception.

Chapter 16 - Conclusion

- To conclude our findings and provide an assessment on how the pricing of derivatives has evolved since 2008.

- To suggest a number of research topics to be considered in order to further the field of XVA pricing.

Chapter 2

Introduction

"Those who do not learn from history are doomed to repeat it." - George Santayana

Eight years have passed since the collapse of the *"too big to fail"* [1] institutions, yet we still find our world leaders scrambling to stimulate economic growth. The effects of the GFC [2] of 2008 can still be felt by today's society. Global growth struggles to gain momentum despite easy monetary policies put in place far longer than what is deemed comfortable. The United States (US), under the guidance of the Federal Reserve Bank, is struggling to sustain required Gross Domestic Product (GDP) growth despite three rounds of Quantitative Easing (QE) with the Federal funds rate[3] at 0% for almost a decade. The majority of the developed world has followed suit, as the European Central Bank (ECB) and the Bank of Japan (BOJ) implement their own QE to bolster economic growth.

Much of the blame of the crisis has been placed on financial derivatives, but some context is needed. The incorrect use of derivatives was the problem, not the derivatives themselves. Like any powerful man-made tool, if abused can be extremely dangerous. The invention of the BSM [Black and Scholes 1973], [Merton 1973] option-pricing model was a break through in finance and enabled cash flows to be deployed in very innovative and useful ways. However, the abuse of the ingenious formula

[1] *Too big to fail* typically refers to financial institutions that are so large and interconnected that their collapse would be catastrophic to the broader economy

[2] Considered to be the worst financial crisis since the Great Depression during the 1930s

[3] Is the interest rate at which depository institutions lend reserve balances to other depository institutions, source: https://en.wikipedia.org/wiki/Federal_funds_rate

led to a 23% collapse of US stocks on 19 October 1987, known as Black Monday. Despite this, Byron Scholes and Robert Merton[4] were celebrated by winning the Nobel Prize for Economic Science in 1997. As the British mathematician Ian Stewart correctly stated “the equation itself wasn’t the real problem,” it was “one ingredient in a rich stew of financial irresponsibility, political ineptitude, perverse incentives, and lax regulation.”[5]

In more recent times, we once again witnessed the collapse of financial markets due to excess lending to subprime mortgage applicants. Very much like the Black Monday crash of 1987, it is very easy to pin the blame on derivatives, but one needs to indentify the real causes. Banks and Government-Sponsored Enterprises (GSE), Fannie Mae and Freddie Mac [6], used clever financial engineering to package the subprime mortgages into Special Purpose Vehicles (SPV) known as Mortgage-Backed Securities (MBS). The rationale for this was to make mortgages tradable and inject liquidty into mortgage lending. At the time, MBSs were rated highly by Moodys, Standard and Poor’s (SP), as well as Fitch, which implied they were low risk. Moreover, the triple-A rating given to the securities allowed pension funds and money markets to allocate large amounts of cash to MBSs.

MBSs were a great way for investment banks and other investors to gain attractive returns, specifically on subprime mortgages. The high ratings of the MBSs made them appear to be safe investments with attractive returns because of the low interest rate environment leading up to the GFC. When times were good, liquidity did exist in this opaque market, however, when times were bad there simply were no buyers to offload the risk to. To add another layer of complexity, banks could hedge their credit exposure to the subprime mortgages by buying Credit Default Swaps (CDS) [7] from monoline insurers [8]. This was incentivised by two factors:

---

[4] Fisher Black had passed away in 1995 and therefore was ineligible for the prize

[5] Source: https://www.theguardian.com/science/2012/feb/12/black-scholes-equation-credit-crunch

[6] GSEs with the sole purpose of expanding the secondary mortgage market by securitising mortgages held primarily by retail banks

[7] A CDS is an insurance contract on a bond or a name where the buyer pays a periodic fee in exchange for protection from the seller

[8] Monoline insurers’ sole line of business was to provide insurance to banks on debt, mortgage debt in particular

1. Basel II [9] capital relief; and
2. To mitigate credit risk on the trade book.

The monoline insurers were also rated highly by the Credit Rating Agencies (CRA). This created a moral hazard as the banks saw no reason to place limits on CDS exposure to their monoline counterparties. Moreover, collateral requirements were contingent on the insurers losing their triple-A rating instead of the Mark-to-Market (MTM) of the CDSs. In the latter case, the size of the CDS market might have been constrained by the amount of cash available to the insurers to be used as collateral. The CRAs methodology was backward looking and spurious, resulting in a $62 trillion build in CDSs in the Over-the-Counter (OTC) market[10]. A build of this magnitude is often referred to as "herd instinct trade where market traders copy other market traders."

The excessive inflows of cash into the mortgage market were the symptoms of an imbalanced global economy, the mortgage market itself was not the problem. The US economy had problems that were not being properly addressed. This consequently led to huge amounts of cash being invested in what seemed to be low risk investments which promised good returns. The financial engineering used to design the MBSs backed by subprime mortgages amplified these imbalances into a global problem. Years of cheap debt and poor underwriting standards were the true problems leading to the crisis[11]. The field of financial engineering considers the said symptoms and how to best address them. Easily accessible cash will often find the path of least resistance. The excess cash in the system pre-GFC, followed a path directly to the mortgage market, which was based on an extremely shaky foundation over which layers upon layers of complex derivatives were built.

By the end of 2007 the mortgage crisis was in full swing and the US was entering recession. The US politicians and central bank governors did not manage the economic imbalances correctly. In addition, regulators and ratings agencies had simply

[9] Basel II is a set of banking regulations put forth by the Basel Committee on Bank Supervision, which regulates finance and banking internationally, source: www.investopedia.com/terms/b/baselii.asp

[10] A decentralised market where two counterparties trade directly with each other

[11] These problems are beyond the scope of this dissertation and would be best suited in an economic/political research paper

overlooked the symptoms until it was too late. The failure of the CRA and the regulators, to correctly assess the risk associated with MBSs, led to severe mispricing of risk for investors. Banks were eager to build positions in MBSs without assessing their risk correctly. Spurious credit ratings were largely to blame, which ultimately resulted in a severe build up of wrong-way risk [12]associated with the monoline insurers. Simply put, complacency was coaxed by the CRAs poor ratings assessments, incentivising huge concentration risk [13] poorly monitored by banks. These oversights ultimately would bring down a number of institutions throughout 2008 and the years to follow.

In March 2008 Bear Sterns was purchased by JP Morgan Chase at $2 a share when just a month previously it had been trading at $93 per share. A clear indication that counterparty risk was not being priced correctly. By September 2008, Fannie Mae and Freddie Mac were placed into conservatorship[14] by the US Treasury. The two entities accounted for "over half of the outstanding US mortgages", therefore posed as a sizable systemic risk to the US economy. The peak of the crisis hit on 8 September 2008, when the fourth largest bank in the US, known as Lehman Brothers, filed for Chapter 11 bankruptcy protection, see [Gregory 2012] page 5. The credit derivative market did not have a default of this proportion priced nor did the CRAs anticipate it as Lehman Brothers still held an A rating at default. Over $400 billion worth of CDSs had been written on the Lehamn Brothers entity, meaning a default of this size carried huge contagion risk. Lehman's failure was not the last of the systemic failures. Merrill Lynch, another large US investment bank, would have suffered a similar fate as Lehman Brothers if Bank of America had not agreed to provide a $50 billion lifeline. Further to this, four fifths of American International Group (AIG)[15] was bought out by the US government. AIG, like the monoline insurers, had an excellent credit rating prior to 2008, however, AIG was considered *"too big to fail"* and could not be left to default. The toxic combination of CDSs and MBSs across insurers, GSEs, and banks contained huge amounts of uncaptured

[12]Wrong-way risk is defined by the International Swaps and Derivatives Association (ISDA) as the risk that occurs when "exposure to a counterparty is adversely correlated with the credit quality of that counterparty".[Travers and Schwob 2009]

[13]Concentration risk can arise from uneven distribution of exposures (or loan) to its borrowers, source: http://www.businessdictionary.com/definition/concentration-risk.html

[14]Short term nationalisation

[15]A multinational insurance corporation who provided insurance in the mortgage market

counterparty and wrong-way risk.

The contagion was not only limited to multinational institutions but spread to countries within Europe. By November 2008 Iceland had received a $4.6 billion bailout from the International Monetary Fund (IMF) and fellow European countries. Greece was soon to follow in 2010 by receiving a €110 billion bailout from the IMF. Spain, Ireland, Cyprus, and Portugal all joined the ranks of countries in need of a bailout. All these countries were considered to be *"too big to fail"* and were naively thought to have very low counterparty risk. Post 2008, the notion of risk-free entities was proven to be completely false. Counterparty risk had to be taken seriously and accounted for correctly. By 2009 regulators were publishing papers to best address the three issues pertinent to the GFC:

1. Volatility of CVA;
2. Wrong-way risk; and
3. Collateral management.

These papers took form as Basel III global regulatory standard, designed to strengthen bank capital bases and establish new liquidity and leverage requirements; the US Dodd-Frank Wall Street Reform, the Consumer Protection Act, as well as European Market Infrastructure Reform (EMIR) were aimed at ensuring stability in the OTC market, see [Gregory 2012] page 7. The regulators intended to increase pressure to account for CVA risk correctly, however, the new regulations had an adverse effect on the liquidity in the CDS OTC market. CDSs were the only instruments available for banks to hedge their CVA exposure, but the wrong-way risk inherent in CDS derivatives made them scarcely available in the market. The only alternative was for OTC trading to be moved toward central clearing. This required the development of Central Counterparties (CCP) for the CDS market, a notion not to be realised anytime soon. A CCP raises the question, what would happen if it were to collapse? A CCP could certainly be beneficial in mitigating credit risk, but a CCP itself would then become a *"too big to fail"* entity.

Chapter 3

The Law of One Price

"Change is the law of life. And those who look only to the past or present are certain to miss the future." - John F.Kennedy

During the build up to the demise of Lehman Brothers, the market witnessed divergences in lending rates that were not common in markets at the time. The most referred to anomaly was the divergence of the OIS rate and London Interbank Offered Rate (LIBOR), a clear sign that banks were in distress. Banks used the LIBOR rate to discount future cash flows, which were assumed to be *risk-free* pre-2008. The divergence between the OIS rate and LIBOR violated this assumption and indicated banks do indeed carry significant credit risk. This too revealed a major flaw in the BSM theory, which was widely used to price financial derivatives.

The BSM model assumes that banks can lend to each other using the risk-free rate, see [Black and Scholes 1973]. The said rate was any of the interbank rates depending on the region the derivative was traded

1. Euro Interbank Offered Rate (EURIBOR);
2. Tokyo Interbank Offered Rate (TIBOR);
3. LIBOR; and
4. Johannesburg Interbank Agreed Rate (JIBAR).

Pre-2008, it was widely accepted that the fair-price of any derivative can be derived from the expected cash flows discounted by the perceived risk-free rate. The notion of a risk-free rate remained the same, however, it was not certain as to what existing benchmark could represent it. [Gregory 2012] introduces the Credit Support Annexure (CSA)[1], as well as elaborates on the many other techniques of how to mitigate credit risk. The CSA has become pivotal in the pricing of bilateral derivative trades. Any trade under a standardised CSA requires both parties to post collateral, typically in USD, if the MTM is not in their favour. The payer of the collateral would be compensated at the Federal Reserve funds rate, yet the payer incurs a cost of funding the USD at their own internal cost of funding. Interest rate benchmarks set by central banks are widely considered to be the risk-free benchmark.

Typically the Fed Funds rate is lower than a bank's internal funding rate. A funding gap manifests that needs to be recouped by the payer of collateral, this funding gap is now widely known as the FVA[2]. To be clear, this was never an issue pre-GFC as all banks could raise funding at LIBOR, and collateralised trades were the exception, not the rule. The FVA can further be split up into two components:

1. Funding Benefit Adjustment (FBA); and
2. Funding Cost Adjustment (FCA).

The former is the benefit earned by a bank when they receive collateral from a trade entered into under CSA. The latter refers to the opposite situation, whereby the bank needs to fund collateral postings with their counterparty. Naturally, if the client trade is done under CSA with the same clauses as the CSA relating to the corresponding hedge, then there certainly would not be any form of FVA. FVA is one of the many new valuation adjustments that is not entirely accepted by academics. [Hull and White 2012a] strongly argue against the FVA adjustment as they claim it violates the *one price* argument which is fundamental to the BSM model.

[1] An appendage to the ISDA Master Agreement, it stipulates the collateral posting arrangment between two parties pertaining to a specifc trade

[2] Adjusting the derivative price to include the dealer's cost of funding

Figure 3.1: The spread between three month LIBOR and OIS during the 2008 crisis. Source Bloomberg, April 1 2016

In contrast, CVA is far more accepted than FVA. CVA is the cost most banks charge corporate clients to trade OTC derivatives under no CSA. It reflects the expected loss as a result of counterparty default. Prior to 2008, Basel II ensured that banks held capital for credit risk pertaining to default risk. The credit risk inherent in a derivative trade due to the volatility of CVA was not a core focus for regulators. This rationale proved to play a major role in the GFC as it was mainly the MTM of the CVA that caused the liquidity squeeze amongst insurers, not the actual defaults of the distressed entities. A sizeable portion of the Basel III [3] document introduces a CVA capital charge to ensure banks hold a buffer to withstand periods of pronounced CVA volatility. For their trading books, banks need to use VAR to account for any unexpected losses incurred by CVA losses. Much of the literature we consider on CVA, as per [Gregory 2012] page 242 is derived from the work of Sorensen and Bollier; Jarrow and Turnbull; Duffine and Huang, as well as Brigo and Masetti. The aforementioned published their work throughout the 90s, illustrating that CVA is not a new concept. Figure 2.2 elaborates on the increasing importance of CVA through time within financial markets. The measure of VAR is done under a $\mathbb{P}$ measure[4].

[3] Basel III: A global regulatory framework for more resilient banks and banking systems

[4] Historical movements of market variables within the real word not under fair value assumption.

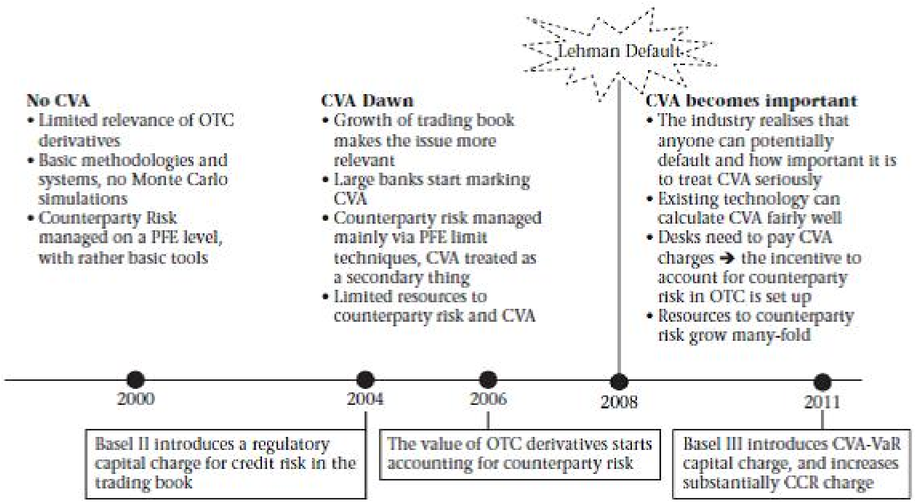

Figure 3.2: The history of counterparty credit risk in financial institution [Ruiz 2015], page 127

With the misconception that banks were risk-free, corporates never considered what counterparty risk they were exposed to by the banks. This brought DVA into the limelight. Like FVA, the topic introduced controversy into the world of derivative valuing. DVA is the CVA for a corporate client with respect to the counterparty risk they are exposed to when trading with a bank. When considering CVA and DVA together, a trade's valuation is then considered to be symmetric and two default-risky parties can agree on the economically correct price of a deal, see [Brigo, Morini and Pallavicini 2013] pages 254 and 255.

It is interesting to note that Basel III does not allow for the capital relief associated with DVA. The aim of the Basel accords is to incentivise prudent behaviour by banks. Since a bank realises profits on DVA when their credit quality deteriorates, it does not make any sense for regulators to recognise DVA. In contrast, Financial Accounting Standards (FAS) 157 and International Accounting Standards (IAS) 39 state that DVA should be fully accounted for in financial reporting. The logic to include DVA is underpinned by a fundamental principle associated with accounting: financial assets and liabilities should be accounted at *fair value*. This implies that a deal is only fair valued if Bilateral Value Adjustment (BVA) is applied to the derivative. Moreover, the hedging mechanics for DVA are not intuitive, this will become clear in later chapters.

A slightly newer adjustment introduced to the finance world is KVA. This involves the adjustment of a derivative's price to account for cost of capital throughout its existence. We mentioned earlier that CVA generates a VAR value which in turn consumes capital for a bank. These costs need to be recouped via transfer pricing to the client at hand. KVA calculations are now used to quantify this cost. We consider the work of [Elouerkhaoui 2016a] and [Kenyon, Green and Dennis 2016]. Both papers consider how to include KVA into the price of a derivative but by leveraging off of different frameworks.

[Kenyon 2012], page 138, points out that there are mainly two schools of thought to price derivatives:

1. Pricing by expectation; and
2. Pricing by hedging (portfolio replication).

[Kenyon, Green and Dennis 2016] leverage their work off the latter framework. With respect to the first method, we consider the $\mathbb{Q}$[5] measure in order to obtain the fair value of a derivative. Pricing via expectation is a framework central to the discussions by [Brigo, Morini and Pallavicini 2013].

The second method assumes markets are complete, which makes it fundamentally different to the first method. We draw on the work of [Burgard and Kjaer 2011], [Burgard and Kjaer 2012a], [Burgard and Kjaer 2012b], [Burgard and Kjaer 2012c] and [Piterbarg 2010] to demonstrate how to price a derivative via portfolio replication. Central to this approach is the very well known Feynman - Kač formula, developed by Richard Feynman and Mark Kač, [Björk 2009] page 69, which illustrates how today's derivative price is the discounted expected value of tomorrow's

[5] Typically used in pricing where the expected value is taken under the $\mathbb{Q}$ measure. Probabilites under $\mathbb{Q}$ are inferred from market prices

derivative price. It also demonstrates that the derivative price is risk-neutral if the expected value is taken under the $\mathbb{Q}$ measure and discounted using the *risk-free* rate.

Theorem 3.1. ***(Feynman-Kač)*** *Assume that F is a solution to a boundary value problem*

$$\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - rF = 0, \tag{3.1}$$

$$F(T,x) = \Phi(\mathrm{x}). \tag{3.2}$$

Assume furthermore that the process $e^{-rs}\sigma\frac{\partial F}{\partial x} \in \pounds^2$, where X is defined as below. Then F has the representation

$$F(t,x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}{}_{t,x}\left[\Phi(\mathrm{X_T})\right], \tag{3.3}$$

*where X satisfies the Stochastic Differential Equation (*SDE*)*

$$dX_s = \mu ds + \sigma dW_s, \tag{3.4}$$

$$X_t = x. \tag{3.5}$$

The Feynman and Kač formula is pivotal to the classical approach of pricing by hedging, refer to [Björk 2009] page 74 for further detail. We will later explore how it is applied by [Piterbarg 2010] to transform the Black-Scholes PDE to account for collateral correctly. The most significant critique for Burgard and Kjaer's approach, is that in order to create a replication portfolio, an entity needs to be able to trade their own bonds freely. It is not quite clear how this would work practically as bonds are issued by a bank to acquire funding, to buy one's bonds back would consume the funding required in the first place.

The latest adjustment to be included into derivative pricing is MVA. We have previously mentioned that client derivatives traded ex-CSA are typically hedged by derivatives under CSA. This generates a funding differential for the desk at hand once a MTM is generated. In the case of MVA, IM is placed upfront and thus a funding liability exists from inception which needs to be funded from treasury. This adjustment is currently a growing concern for traders as most of their markets migrate from OTC to CCP, thus a consistent framework is becoming ever more necessary. Two papers that stand out in this relatively new aspect of derivative pricing are [Elouerkhaoui 2016a] and [Kenyon and Green 2015].

Not all trades require an XVA for it to reflect a fair market value. In order to understand this, we must elaborate on the term *risk-free*. It is now broadly accepted that the Fed Funds rate or the Euro Overnight Index Average (EONIA) rate is the closest real representative for the fictitious BSM risk-free rate. For the purpose of this dissertation, let us assume this to be the case and that a trade under CSA is perfectly collateralised whereby rehypothecation[6] is permitted. Then, the trade is risk-free and is therefore discounted using r_C. This leads us to the below theorem taken from [Björk 2009], page 99

Theorem 3.2. *The arbitrage free price of the claim of* $\Phi(\mathrm{S(T)})$ *is given by* $\Pi(t;\Phi) = F(t, S(t))$*, where F is given by the formula*

$$F(t, s) = e^{-r_c(T-t)} \mathbb{E}^{\mathbb{Q}}{}_{t,s} \left[\Phi(\mathrm{S(T)}) \right]. \tag{3.6}$$

Assume Theorem 3.2 is the starting piece in the puzzle. If we trade a derivative under CSA and we are able to hedge the derivative using a derivative governed by the exact same CSA, then there would be no need for any XVA[7]. Let us make a few assumptions to build on this point:

1. $\Pi(t;\Phi)$ is the price of a risk-free claim at t;
2. The price of the a contingent claim $X = \Phi(S(T))$ at time T where S is the underlying stochastic variable; and
3. In order for us to avoid arbitrage $\Pi(T;X) = X$.

then the MTM at t under CSA would be

[6] The use of assets, posted as collateral, by a bank for their own purposes.

[7] Provided the Loss Given Default (LGD) and the CDS of the counterparty are not excessively large, otherwise KVA will need to be considered.

$$\text{Riskless MTM}= e^{-r_c(T-t)}\mathbb{E}^{\mathbb{Q}}{}_{t,s}\left[\Pi(T;X)\right].$$

The above definition of a riskless MTM implies there is no need for any form of adjustment as the client trade and the corresponding hedges are symmetrical. Let us consider a trade with a typical corporate client who is not setup to post daily margin. The trade will be done OTC whereby the client will have some degree of credit risk that the bank will need to consider. To hedge our market risk, we would need to hedge either in the OTC market or on an exchange. The former implies a standard CSA trade, the latter implies IM as well as Variation Margin (VM). The asymmetrical nature of our client trade and its hedge makes what was considered a relatively simple transaction pre-2008 rather complicated today. From the above description we now are dealing with a fairly risky trade and the MTM, in a very simplistic definition, has evolved as follows

$$\text{Risky MTM}= \text{Riskless MTM}+\text{CVA}+\text{DVA}+\text{KVA}+\text{FVA}+\text{MVA}.$$[8]

It can get more complicated if you consider that collateral is not always posted as cash, however, regulators are working hard to standardise what can and cannot be posted. The below diagrams illustrate the degree of complexity a correct derivative valuation entails.

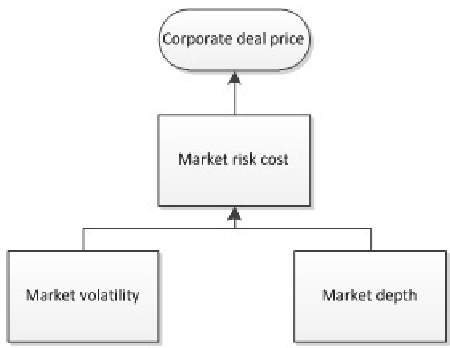

Figure 3.3: A simplistic view of the considerations required to price a derivative pre-2008 accurately.

[8]If the client leg is hedged on an exchange. We will see in part II that this simplistic equation is no longer correct, when funding costs are included, as the FVA term makes the equation non-linear and recursive.

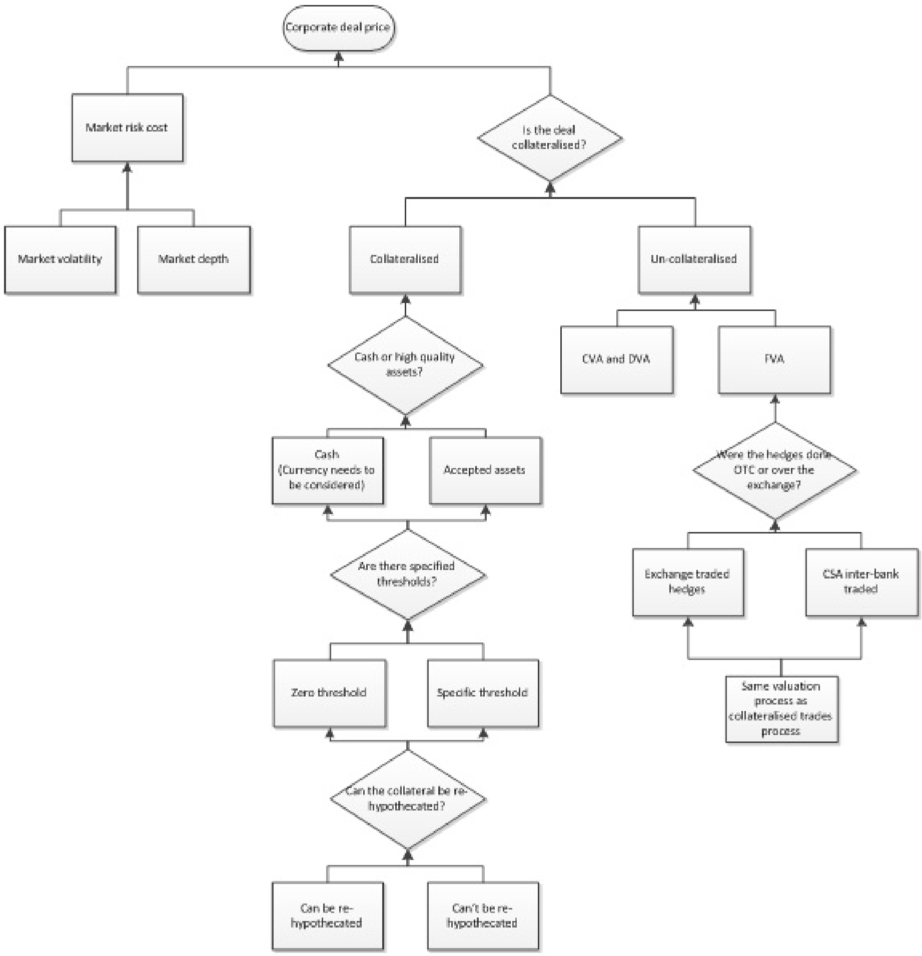

Figure 3.4: A complicated overview of the many considerations required to accurately price a derivative post-2008 accurately.

Part II

Literature Review

Chapter 4

Hull and White

"If you can find something everyone agrees on, it's wrong." - *Mo Udall*

In 2012, Hull and White were the most outspoken against the inclusion of FVA in the valuation of derivatives. Their argument is underpinned by the risk-neutral valuation paradigm, which makes it a philosophical one in the world of financial derivatives. They claim that "we discount at the risk-free rate because this is required by the risk-neutral valuation principle" [Hull and White 2012a]. Moreover, they also state that pricing should be kept separate from funding. From a practitioner's perspective it is difficult to accept this simply because a funding cost can only be recouped via transfer pricing. Therefore the funding rate is essential in determining what discount rate should be applied in derivative pricing.

Hull and White elaborate on their definition of FVA and its relationship with DVA. They define DVA_1 as the risk of a dealer defaulting on his derivative portfolio and DVA_2 as the risk he might default on any other liabilites, both increase in value if the dealer's default risk rises. They define FVA=$\Delta(\text{DVA}_2)$, where $\Delta(\text{DVA}_2)$ is an increase in DVA_2 resulting from the funding requirments of a derivatives portfolio. Hull and White define CVA and DVA as Economic Value adjustment (EVA) required to bring a transaction closer to its true economic value, which supports their view of CVA and DVA as acceptable adjustments in the BSM world of one price. In contrast to this, FVA is classified as an "anti-EVA", moving the derivative's price away from its economic value toward a model price. The inclusion of FVA implies each bank will have their own uniqe price for the same derivative. The two main themes of their paper is that the risk of a derivative, or a project in the corporate finance space, should determine the rate of return the bank/dealer should earn, not the funding cost it is subject to. The second is that the derivative desk should not be charged a funding cost from their funding centre, as this will incentivise them to transfer price clients. They claim it would be better for the funding desk to charge the derivatives desk the risk-free rate for funding. Practically, this theory does not account for the cost of funding consumed when trading a derivative.

The response from practitioners to [Hull and White 2012a] was profound as their concern is one of how their bottom line is impacted, not whether the BSM theory is sound in the new normal. [Laughton and Vaisbrot 2012] provide some practical counter arguments toward Hull and White's academic treatment of FVA. They claim that the BSM theory is based on assumptions that do not hold true in the real world. The most prominent assumptions are that markets are complete and all banks can borrow at the risk-free rate. The former suggests that all risk factors can be hedged and the latter that traders should fund trades at the risk-free rate only. [Laughton and Vaisbrot 2012] make it clear that since real markets are indeed incomplete, risk preferences are reintroduced into valuations, therefore the law of one price is simply not valid. They also state that funding is a real cost to traders that is certainly no longer set at the risk-free rate. Three central points put forward by [Laughton and Vaisbrot 2012] are:

1. The BSM theory needs to be modified to be useful to traders;
2. The cost of borrowing is exogenous and is unaffected by a single trade; and
3. Market-makers give no value to their expected profit or loss upon default.

All these points support the FVA being inculded in the derivative price.

Further to the above reponse, [Antonio Castagna 2012] provides a direct response to each of the statements made in [Hull and White 2012a]. Castagna says that it is correct to discount at the risk-free rate because it is required by the risk-neutrality argument. However, this is only valid if we are dealing in a Black and Scholes economy. Black and Scholes made very specific assumptions [1] in order for portfolio replication to be possible or for the portfolio to earn the risk-free rate. Castagna

[1] Refer to [Antonio Castagna 2012] page 2 for detail

refers to the work of [Rubenstein and Cox 1985] to support his point that the existence of lending and borrowing rates different to the risk-free rate does not impede the replication argument. He claims if one considers what the borrowing rate is; whether they are a buyer or a seller of the option, and whether it is a call or a put; one can use a closed-form formula to price the option. The "risk-neutral" price is still obtained using the replication strategy and a rate different to the risk-free rate, it simply may not be the Black-Scholes risk-free price. Castagna does not discard the BSM model to price options, but suggests it must be modified to include the real world costs banks now face.

Castagna also raises some interesting points regarding the comparison between DVA and FVA. He claims that DVA is the FVA for debt contracts, however, for derivatives he offers a slightly different definition. Castagna states that DVA is the compensation that a risky entity has to pay to their counterparty in order to compensate them for the default risk they bear. The remuneration amount is measured as a CVA on behalf of the counterparty. The FVA related to a derivative's contract is the sum of the funding costs that a counterparty needs to pay in order to pay for the borrowed money required to fund the contract. Therefore, for a derivative contract, FVA is not DVA. Castagna elaborates on the unproved notion of a shareholder's value increasing as a result of a bank's default. He draws on the work of [Merton 1974] pages 11 and 12 to disprove this notion. It must be noted Castagna considers DVA at T, and concludes that indeed it is difficult to monetise the value of DVA. However, at time t where $t \in [0; T]$, DVA does increase in value due to an increase in credit spreads, albeit purely a theoretical value it still does exist as a positive MTM in favour of the bank.

In [Hull and White 2012b] the authors provide an academic response to a practitioners problem. They argue that [Laughton and Vaisbrot 2012] are incorrect in assuming the existence of a single arbitrage-free price is dependent on the market being complete. They state that the valuation arguments of the portfolio replication strategy used in [Merton 1973] yield the same result as the Capital Asset Pricing Model (CAPM) method used in [Black and Scholes 1973] without the assumption of any risk-free borrowers. They also state that when considering the credit risk of the dealer, the bank should still fund at the risk-free rate, and the spread above the risk-free rate should be compensation for the expected cost of the dealer's possible default. This point reverts back to their case in [Hull and White 2012a], that FVA=Δ(DVA_2), hence the inclusion of FVA implies double counting. Hull and White draw on the logic that if a dealer hedges, they reduce their risk and will therefore enjoy lower funding costs. In reality, a point Hull and White concede on, is that the feedback loop of a single trade will not impact the desk's cost of funding immediately as the theory suggests. They believe the FVA argument is underpinned by one's understanding of DVA_1 and DVA_2, and how they relate to FVA. Hull and White claim that if DVA_2 is allocated to the funding desk then the funding desk would recoup the cost of lending to the trading desk, making an FVA charge unnecessary.

[Hull and White 2012c] provide a framework which includes the cost of DVA and CVA. The paper shows a heuristic case as to why including FVA in the replication argument does not make sense. The paper leverages off the orginal framework of Black and Scholes using the CAPM as well as Merton's work on no-arbitrage portfolio replication. Let us assume that we have two entities, bank B and counterparty C. Let us further assume that the derivative will be priced from B's perspective and that λ as well as R are constant. A positive MTM for the derivative will be $F^+ = max(F, 0)$ and $F^- = min(F, 0)$ for the negative part.

Theorem 4.1. *The risk-free price of a derivative that satisfies the Black-Scholes boundary problem is F,*

$$\frac{\partial F}{\partial t} + rx\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - rF = 0, \tag{4.1}$$

$$F(T, x) = \Phi(x). \tag{4.2}$$

The credit risky price of a derivative that satisfies the following boundary problem is $\hat{F}$*,*

$$\frac{\partial \hat{F}}{\partial t} + rx\frac{\partial \hat{F}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \hat{F}}{\partial x^2} - r\hat{F} = \lambda_C(1 - R_C)F^+ + \lambda_B(1 - R_B)F^-, \tag{4.3}$$

$$\hat{F}(T, x) = F(T, x). \tag{4.4}$$

In [Hull and White 2012c], the second boundary problem in Theorem 4.1 reflects the compensation bank B would require, as well as provide, in order to facilitate a trade with a risky counterparty. By including DVA, bank B concedes they too are a risky counterparty. From the above, Hull and White define $CVA = \lambda_c(1 - R_c)F^+$ and $DVA = \lambda_b(1 - R_b)F^-$.

Chapter 5

Pricing by Hedging - Piterbarg, Kjaer, and Burgard

"You want a valve that doesn't leak and you try everything possible to develop one. But the real world provides you with a leaky valve. You have to determine how much leaking you can tolerate." - Obituary of Arthur Rudolph, in The New York Times, January 3, 1996.

Piterbarg, Kjaer, and Burgard based their work on portfolio replication, a classical approach to price derivatives. It is the very same approach discussed in [Merton 1973]. They adapted the fundamental work provided by Merton to accommodate for the additional risks associated with derivatives. This enabled traders to price their risk correctly. Pricing by replication assumes two key points:

1. Markets are complete; and
2. Markets are arbitrage free.

The first point assumes that there exists a tradable instrument to hedge the contingent claim. Moreover, there will be a uniqe price for the contingent claim if the claim is hedgable. The second point assumes that the contingent claim can only ever earn the chosen rate of return, any different return would be considered an arbitrage opportunity. For completeness, consider the below propositions[1].

[1] see [Björk 2009] page 31

Proposition 5.1. *Consider a given claim X. In order to avoid arbitrage, X must then be priced according to the formula*

$$\Pi(0, X) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[X\right], \tag{5.1}$$

where $\mathbb{Q}$ is a martingale measure for the underlying market.

Proposition 5.2. *Consider a contingent claim X. If there exists a portfolio h, based on the underlying assets, such that*

$$V_T^h = X, \tag{5.2}$$

then we say that X is replicated, or hedged by h. If every contingent claim can be replicated, we can then say the market is complete.

[Piterbarg 2010] shows how the Black-Scholes PDE can be extended to accommodate for multiple interest rate curves. He accurately addresses the question of what funding rate is required when trading under CSA with a zero threshold agreement. Piterbarg only seeks to address the question of funding post-2008 with regard to the Black-Scholes PDE, he ignores the possibility of default for either counterparty. The result of his work is shown in the proposition below.

Proposition 5.3. *F is the price of a derivative that satisfies the below boundary problem*

$$\frac{\partial F}{\partial t} + (r_R - r_D)x\tfrac{\partial F}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \tfrac{\partial^2 F}{\partial x^2} = r_V F - (r_V - r_C)C, \tag{5.3}$$

F_T is the payoff of the claim at T.

Proposition 5.3 assumes F $\neq C$, therefore there exists a funding differential $s_F = r_V$ - r_C. However, if we assume that C = F, then the above PDE reduces to the Black-Scholes boundary problem with the risk-free rate being r_C. This point plays a pivotal part in understanding how FVA manifests within derivative pricing as it is very seldom the case where C = F. Further to the funding differential, Piterbarg also elaborates on the effects of convexity for CSA valued derivatives. Much like an exchange traded derivative, cash flows occur between trade initation and T. This generates a similar convexity adjustment witnessed between futures and forward derivatives. The key difference between F_{CSA} and a future is that collateral placed with an exchange does not earn interest, where F_{CSA} earns the rate r_C.

[Piterbarg 2010] provides us with a starting point for pricing derivatives within the new paradigm financial markets find themselves in. The paper provides us with a solution to the Black-Scholes PDE inclusive of funding differentials and convexity adjustments. We now look at the work of [Burgard and Kjaer 2011] to provide a solution to the PDE when a risky counterparty is involved. Burgard and Kjaer show two modifed versions of the Black-Scholes PDE determined by the closeout rules used at default:

1. The recovery rate is applied to the risky value $\hat{F}$; or
2. It is applied to the counterparty-riskless value F.

In the former scenario, the resulting PDE is non-linear and can be solved by solving a non-linear integral equation. In the latter case, the PDE is linear and can be shown in a Feynman-Kač representation. [Gregory and German] provides an in-depth discussion on which exposure to use on closeout. They find that a risky closeout tends to be more supported by [ISDA 2009] as the documentation states that a counterparty "may take into account the creditworthiness of the Determining Party". This means that the surviving entity may include their DVA value in the settlement amount upon the default of the other counterparty. [Gregory 2012] page 278 mentions that there is no obvious best solution to the choice in closeout value. The risk-free choice is the simplest from a theoretical perspective, but causes discontinuities in valuations at default as the DVA term falls away. The more natural yet somewhat complex proxy for a closeout value would be the risky value. A risky value closeout causes our PDE to be non-linear and recursive in nature, however, we no longer have a discontinuity in value at default. Below Burgard and Kjaer distinguish between the counterparty risk-free value; the FVA and the bilateral CVA for both a risky and risk-free value closeout. Once again, let us assume a derivative is traded between counterparty C and bank B.

Theorem 5.4. *The credit risky price of a derivative that satisfies the following boundary problem is given by* $\hat{F}$ *and the* MTM *at default is the riskless price F,* *then*

$$\frac{\partial \hat{F}}{\partial t} - r_R x \frac{\partial \hat{F}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \hat{F}}{\partial x^2} - (r + \lambda_B + \lambda_C)\hat{F} = -\lambda_B(F^+ + R_B F^-) - \lambda_C(F^- + R_C F^+) + s_F F^+. \tag{5.4}$$

However, if we assume a risky closeout, whereby the MTM *at default is the risky value* $\hat{F}$*, then*

$$\frac{\partial \hat{F}}{\partial t} - r_R x \frac{\partial \hat{F}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \hat{F}}{\partial x^2} - r\hat{F} = \lambda_C(1 - R_C)\hat{F}^+ + \lambda_B(1 - R_B)\hat{F}^- + s_F \hat{F}^+, \tag{5.5}$$

$$\hat{F}(T, x) = F(T, x). \tag{5.6}$$

For equation 5.4 and 5.5, the authors assume that the above replication strategy will generate the cash required to fund the repurchase of bank B's own bonds. This is required to hedge B's own credit risk, however, this strategy only works if the market is complete. The idea of a bank buying back their own bonds is counterintuitive as it defeats the purpose of intially issuing the bonds. The three terms on the right of equation 5.4 can be interpreted as DVA, CVA, and FVA respectively. Let us consider a few scenarios:

1. If F is In the Money (ITM) for B, and we assume B defaults then B will receive the full value of F from C;
2. If F is ITM for B, and we assume C defaults then B will receive $R_C F$; lastly
3. The first two points are equal, but opposite for C.

[Burgard and Kjaer 2012c] expands on Theorem 5.4 by exploring the relationship between FCA and the balance sheet. They demonstrate how FCA can be eliminated through two strategies discussed in the paper. Burgard and Kjaer claim that despite FCA and DVA being related to the credit position of bank B, including both terms in the pricing formula is not double counting. To illustrate this clearly, we provide the following definition from [Burgard and Kjaer 2012c].

Definition 5.5. *U is the adjustment term for a claim on F, by applying Feynman-Kač to Theorem 5.4 we have*

$$U(t,s) = CVA + DVA + FCA, \tag{5.7}$$

with

$$CVA = -(1-R_C)\int_t^T \lambda_C(u)\mathrm{D}_{\mathrm{r}+\lambda_\mathrm{B}+\lambda_\mathrm{C}}(\mathrm{t},\mathrm{u})\mathbb{E}^{\mathbb{Q}}{}_t F^+(u,S(u))du, \tag{5.8}$$

$$DVA = -(1-R_B)\int_t^T \lambda_B(u)\mathrm{D}_{\mathrm{r}+\lambda_\mathrm{B}+\lambda_\mathrm{C}}(\mathrm{t},\mathrm{u})\mathbb{E}^{\mathbb{Q}}{}_t F^-(u,S(u))du, \tag{5.9}$$

$$FCA = -\int_t^T s_F(u)\mathrm{D}_{\mathrm{r}+\lambda_\mathrm{B}+\lambda_\mathrm{C}}(\mathrm{t},\mathrm{u})\mathbb{E}^{\mathbb{Q}}{}_t F^+(u,S(u))du, \tag{5.10}$$

where $\mathrm{D_k(t,u)} = e^{-\int_t^u k(v)dv}$ *is the required discount factor.*

It is clear that the FCA and DVA term reflect opposite signs, therefore DVA can be seen as an FBA for bank B. By using a simple balance sheet model, they demonstrate that through proper accounting for derivative assets on the balance sheet one can reduce the funding spread to zero therefore mitigating any need for an FCA. This would make the valuation between bank B and counterparty C symmetrical. However, this model is somewhat indirect, thus making it complicated to allocate the hedge benefit back to the trading desk. [Burgard and Kjaer 2012c] discusses two other approaches which are more direct in protecting the balance sheet from derivative induced funding costs.

Burgard and Kjaer propose using the derivative as collateral. In theory this would be an eloquent solution to the funding question of the derivative, however, a derivative repo market is uncommon in practice much unlike bond repo markets. Moreover, a risk-free derivative initially has a zero value with the MTM constantly changing, making it very difficult to determine how much cash can be advanced against it. This would lead to a large haircut imposed to the value of the derivative. Their second proposal involves balance sheet management which uses a similar framework which provided us with Theorem 5.4 taken from [Burgard and Kjaer 2011]. For this proposal to work, our entity must be able to freely trade two of it's own bonds with different seniority, the entity would need to issue senior bonds and repurchase its junior bonds. This strategy monetises any windfall due to the bondholders while the issuer is still solvent simultaneously reducing s_F to zero. The practicality of this hedge strategy is low, which is why it would be unlikely to reduce s_F to zero.

[Burgard and Kjaer 2012b] encompasses most of Burgard and Kjaer's findings on BVA, FCA, and Collateral Adjustment (COLVA). Their derivation in this paper is one of semi-replication, a step away from their previous papers, which assumes full replication without collateral. Their model makes weak assumptions on the contentious topic of the issuers own bonds availability as well as the freedom the issuer has in trading them. One of the key differences between [Burgard and Kjaer 2012a] and [Burgard and Kjaer 2012b] is that in the latter they call the boundary condition "general" and provide an array of closeout cases to consider, refer to page 3 of their paper.

Let us consider the PDE proposed by [Burgard and Kjaer 2012b] with a bilateral closeout with collateral where M is not specified explicitly. We define g as the close-out function for each of the counterparties.

Theorem 5.6. *We define ϵ_{h}, g_B and g_C as follows*

$$g_B(M,C) = C + (M_B - C)^+ + R_B(M_B - C)^-, \tag{5.11}$$

$$g_C(M,C) = C + (M_C - C)^- + R_C(M_C - C)^+, \tag{5.12}$$

and

$$\epsilon_{\mathrm{h}} = g_B + P_B^D - C, \tag{5.13}$$

where ϵ_{h} denotes the hedge error between the hedge portfolio and the derivative at default with P_B^D defined as the post issuer default value of counterparty B's bond position.

$\hat{F}$ satisfies the following boundary problem

$$\frac{\partial \hat{F}}{\partial t} - r_R x \tfrac{\partial \hat{F}}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \tfrac{\partial^2 \hat{F}}{\partial x^2} - (r + \lambda_B + \lambda_C)\hat{F} = -\lambda_B g_B - \lambda_C g_C + s_C C + \lambda_B \epsilon_{\mathrm{h}}, \quad (5.14)$$

$$\hat{F}(T, x) = F(T, x). \quad (5.15)$$

Consider the right-hand side of equation 5.14. $\lambda_B \epsilon_{\mathrm{h}}$ is the FCA term and can be interpreted as the expected value of the issuer's hedge error at own default. FCA is clearly a by-product of the semi-replication assumption made in [Burgard and Kjaer 2012b]. When $\lambda_B = 1$ there is a windfall of ϵ_{h} to the issuers' bondholders, however, whilst B is solvent, B will incur a cost of $\lambda_B \epsilon_{\mathrm{h}}$ which is transfer priced to counterparty C. Equation 5.14 also introduces a COLVA adjusment which exists so long as $s_C \neq 0$. The CVA and DVA term are denoted by $\lambda_C \mathrm{g}_C$ and $\lambda_B \mathrm{g}_B$, respectively. Both are the expected values of the closeout values for each counterparty with the collateral impact accounted for in the closeout function. Applying (Feynman-Kač) to equation 5.14 provides us with the following.

Definition 5.7. *U is the adjustment for the risk-free derivative F, where M=F,*

$$U(t, s) = CVA + DVA + FCA + COLVA, \quad (5.16)$$

where

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t F(u) - g_C(F(u), C(u)) du, \quad (5.17)$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t F(u) - g_B(F(u), C(u)) du, \quad (5.18)$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t \epsilon_{\mathrm{h}}(u) du, \quad (5.19)$$

$$COLVA = -\int_t^T s_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t C(u) du, \quad (5.20)$$

where $D_k(t, u) = e^{-\int_t^u k(v)dv}$ is the required discount factor.

Burgard and Kjaer summarise the general form of Theorem 5.6 into three different examples by varying the selection of bond portfolios in each instance. In the first instance, they assume that one can trade their own senior and junior bonds freely and can therefore hedge any windfall/shortfall of the derivative at time of default.

This reduces equation 5.14 to the PDE active balance sheet management strategy discussed in [Burgard and Kjaer 2012c]. It also relates to the case put forward in [Hull and White 2012a] as equation 5.14 is equivalent to equation 4.3 if $\epsilon_{\mathrm{h}}=0$ and if we ignore collateral funding. The second model is equivalent to Theorem 5.4 taken from [Burgard and Kjaer 2011]. In this model FCA is not negated by a replication strategy, therefore bondholders would still enjoy a windfall if B were to default ITM. The third model is an extension of [Piterbarg 2010] whereby the authors assume one issuer bond exists for balance sheet management.

Chapter 6

Pricing by Expectation - Brigo, Morini, and Pallavicini

"Likeness to truth is not the same thing as truth." - Socrate

The framework proposed by Brigo, Morini, and Pallavicini does not assume that markets are complete. The authors determine the price of a derivative by taking the expected future cash flows under $\mathbb{E}^{\mathbb{Q}}$ and discounting them to today using a suitable r. The resulting price will be quite different to the method used in Chapter 5 when the market is not complete, [Kenyon 2012] page 145. For much of the work done on XVA pricing via expectation, we draw on [Brigo, Morini and Pallavicini 2013], who have collated much of the relevant papers within the publication. This framework is found to be more flexible than the framework proposed in Chapter 5, at least from a practitioners perspective. The authors of [Brigo, Morini and Pallavicini 2013] extend their framework to different asset classes, which again is testament to its adaptability to the typical products and asset classes traded in the dealing room.

[Brigo, Morini and Pallavicini 2013] define CVA as the difference between a credit risk-free derivative and a credit risky derivative. They provide a general Unilateral Credit Value Adjustment (UCVA) pricing formula, page 95. We assume rehypothecation is possible as well as no further credit risk will be induced by rehypothecation. Let us assume we have two entities, bank B and counterparty C, with the following theorems written from bank B's perspective.

Theorem 6.1. *The risk-free valuation of a derivative at time t is given by F, then the credit risky valution of a derivative is given by $\hat{F}$, where*

$$\hat{F}_B(t,T) = F_B(t,T) - UCVA_B(t,T), \tag{6.1}$$

with

$$\begin{aligned} UCVA_B(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[(1-R_C)\mathbb{I}_{\{t<\tau_C\leq T\}}\mathrm{D(t,tau_C)}F_B(t,\tau_C)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[(1-R_C)\mathbb{I}_{\{t<\tau_C\leq T\}}\mathrm{D(t,tau_B)}EAD_B\right]. \end{aligned} \tag{6.2}$$

Expectation is under the risk-neutral measure $\mathbb{Q}$ *and* $F_B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_B(t,T))\right]$.

Theorem 6.1 is provided under the assumption that one entity is risk-free whilst the other is credit risky, this we call the Unilateral Default Assumption (UDA). A discount of F_B by the amount of UCVA should incentivise B to trade with C as opposed to trading with a risk-free entity at value F. Under this assumption, the framework is in its simplest form. [Brigo, Morini and Pallavicini 2013] introduces us to the other side of the same coin, Unilateral Debit Valuation Adjustment (UDVA). In this instance we again assume one entity to be credit risk-free whilst the other counterparty is credit risky. In Theorem 6.1, entity C was risky; in the next theorem we assume C to be risk-free.

Theorem 6.2. *The risk-free valuation of a derivative at time t is given by F, then the credit risky valution is given by* $\hat{F}$ *where*

$$\hat{F}_B(t,T) = F_B(t,T) + UDVA_B(t,T), \tag{6.3}$$

with

$$\begin{aligned} UDVA_B(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[(1-R_B)\mathbb{I}_{\{t<\tau_B\leq T\}}D(t,\tau_B)\left(-F_B(t,\tau_B)\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[(1-R_B)\mathbb{I}_{\{t<\tau_B\leq T\}}D(t,\tau_B)\left(-EAD_B\right)\right]. \end{aligned} \tag{6.4}$$

Expectation is under the risk-neutral measure $\mathbb{Q}$ *and* $F_B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_B(t,T))\right]$.

In order for B to trade with credit risk-free entity C, B would have to increase F_B by UDVA to incentivise C to accept a trade with B over a risk-free entity. Both of these theorems consider the adjustments required to make the trade economically viable from each entity's perspective. To make the trade fair and realistic, both CVA and DVA need to be considered for the trade to reflect its true economic value. We then make the assumption that neither B or C are risk-free entities, therefore we require a symmetric general pricing formula. We provide a few definitions and assumptions before providing a general pricing formula for BVA. [Brigo, Morini and Pallavicini 2013] assumes that there is no possibility of simultaneous default, $\mathbb{Q}(\tau_B = \tau_C) = 0$. The authors also assume that if one of the entities were to default, then evaluation stops at $\tau = min(\tau_B, \tau_C)$, otherwise stopping time would be T.

Definition 6.3. *Let us define the following events ordering the potential default times:*

$$I_1 = \{\tau_B < \tau_C < T\}, \quad I_2 = \{\tau_B < T \leq \tau_C\}, \quad I_3 = \{\tau_C \leq \tau_B < T\}$$
$$I_4 = \{\tau_C < T \leq \tau_B\}, \quad I_5 = \{T \leq \tau_B < \tau_C\}, \quad I_6 = \{T \leq \tau_C \leq \tau_B\}$$

Definition 6.4. *Let us define* BVA *as the positive additive adjustment,*

$$\mathrm{BVA}(t,T) = \mathrm{DVA}(t,T) - \mathrm{CVA}(t,T), \tag{6.5}$$

to the risk-free price F, allowing us to define the risky price $\hat{F}$ *as*

$$\hat{F}(t,T) = F(t,T) + \mathrm{BVA}(t,T). \tag{6.6}$$

Given the above, we now provide a general bilateral counterparty risk pricing formula.

Theorem 6.5. *The risk-free valuation of a derivative at time t is given by F, then the bilateral credit risky valution is given by* $\hat{F}$ *where*

$$\hat{F}_B(t,T) = F_B(t,T) + BVA_B(t,T), \tag{6.7}$$

with

$$\begin{aligned} BVA_B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[(1-R_B)\mathbb{I}_{\{I_1 \cup I_2\}}\mathrm{D(t,tau_B)}(-F_B(t,\tau_B)^+\right] \\ - \mathbb{E}^{\mathbb{Q}}\left[(1-R_C)\mathbb{I}_{\{I_3 \cup I_4\}}\mathrm{D(t,tau_C)}F_B(t,\tau_C)^+\right]. \end{aligned} \tag{6.8}$$

Expectation is under the risk-neutral measure $\mathbb{Q}$ *and* $F_B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_B(t,T))\right]$.

Let us consider the right-hand side of Equation 6.7. $F_B(t,T)$ is the risk-free value of the claim, with the second and third term being the DVA and CVA. Theorem 6.5 has been formulated with the assumption of a risk-free closeout.
[Brigo, Morini and Pallavicini 2013] page 312 furthers the above general pricing formula to include collateralisation as well as the impact of a replacement closeout on pricing. To use the *on-default exposure* for the closeout amount is to use the replacement closeout methodology and is inline with what the ISDA Market Review of OTC Derivative Bilateral Collaterisation Practices (2010). [Brigo, Morini and Pallavicini 2013] abbreviate the bilateral counterparty risk adjustment inclusive of collateralisation as CBVA. The CBVA general formula is provided by the following theorem:

Theorem 6.6. *Let F be the risk-free valuation of a derivative at time t, then the* CBVA *credit risky valution is given by* $\hat{F}$ *where*

$$\hat{F}_B(t,T) = F_B(t,T) + CBVA_B(t,T;C), \tag{6.9}$$

with

$$\begin{aligned} CBVA_B(t,T;C) = & -\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\{\tau<T\}}D(t,\tau)(F_\tau - \mathbb{I}_{\{\tau=\tau_C\}}M_{B,\tau} - \mathbb{I}_{\{\tau=\tau_B\}}M_{C,\tau}\right] \\ & -\mathbb{E}^{\mathbb{Q}}\left[(1-R_B)\mathbb{I}_{\{\tau=\tau_B<T\}}\mathrm{D(t,tau)}(M_{B,\tau}-C_\tau)^-\right] \\ & -\mathbb{E}^{\mathbb{Q}}\left[(1-R_C)\mathbb{I}_{\{\tau=\tau_C<T\}}\mathrm{D(t,tau)}(M_{C,\tau}-C_\tau)^+\right]. \end{aligned} \tag{6.10}$$

Expectation is under the risk-neutral measure $\mathbb{Q}$ *with* M_B *and* M_C *being the on-default exposures.*

The first term represents the mismatch in calculating the risk-free exposure and the on-default exposures. The second and third terms are the Collateral-Inclusive Credit Value Adjustment (CCVA) and Collateral-Inclusive Debit Value Adjustment (CDVA) respectively from the point of view of bank B. If we were to assume a risk-free closeout and no collateral posting, then Theorem 6.6 would reduce to Theorem 6.5. As it is presented above, the first term accounts for the replacment cost either C or B would have to endure depending on which is first to default.

[Brigo, Morini and Pallavicini 2013] Chapter 16 page 351 considers the addition of margining costs to the master theorem we have so far shown to be Theorem 6.6. This cost is equivalent to what most practitioners call COLVA. The adjustment is necessary to cater for the differential between r_C and r. In their derivation of the CBVA general formula with margining costs, [Brigo, Morini and Pallavicini 2013] assume discrete collateral posting intervals whereby any cash flows owing to collateral costs or accruing interest are included in the flows between B and C.

Theorem 6.7. *Let F be the risk-free valuation of a derivative at time t, then the credit risky valution with margining costs is given by $\hat{F}$ where*

$$\hat{F}_B(t,T) = F_B(t,T) + CBVA_B(t,T;C) + \Gamma_B(t,T\wedge\tau;C), \tag{6.11}$$

where $\Gamma_B(t,T\wedge\tau;C)$ is the sum of the discrete margining cash flows defined below

$$\Gamma_B(t,T\wedge\tau;C) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=1}^{n-1}\mathbb{I}_{\{t_k<T\wedge\tau\}}D(t,t_k)\left(C_k - C_k^+\frac{D(t_k,t_{k+1})}{D^{r_C^+}(t_k,t_{k+1})} - C_k^-\frac{D(t_k,t_{k+1})}{D^{r_C^-}(t_k,t_{k+1})}\right)\right], \tag{6.12}$$

where we define $T_C := \{t_1,....,t_n\}$ to be the fixed time grid in which collateral can be posted. C_k^+ indicates bank B to be receiving collateral from corporate C. The opposite situation is indicated by C_k^-. Further, r_C^+ assumes that B will earn a different rate than what it would pay, given by r_C^-.

Recall that $\tau = min(\tau_B,\tau_C)$, therefore if a default has occured, collateral payments will no longer take place, the $\mathbb{I}_{\{t_k<T\wedge\tau\}}$ assures us of this.

The final adjustment [Brigo, Morini and Pallavicini 2013] add to their expectation framework is the FVA term. When a trading desk requires funding for its daily operations, it needs to source it from the treasury or the market. As a result, in both cases a funding cost needs to be included in the pricing of the derivative. When we include FVA in Theorem 6.7, our pricing formula is no longer linear and is now of a recursive form. To solve this, the authors suggest using least-square Monte Carlo techniques, [Longstaff and Schwartz 2001]. The authors elaborate on two models when dealing with funding costs, one being a single-deal (micro) model and the second being a homogenous (macro) model. The latter is more common in practice but it is more difficult to implement with the absence of arbitrage. In addition, the authors prefer to remain as general as possible and therefore assume a micro view. The below theorem provides a formula for pricing in funding costs generated by acquiring funding from the treasury only.

Theorem 6.8. *Let F be the risk-free valuation of a derivative at time t, then the credit risky valution with margining costs and funding costs is given by* $\hat{F}$ *where*

$$\hat{F}_B(t,T) = F_B(t,T) + CBVA_B(t,T;C) + \Gamma_B(t,T\wedge\tau;C) + V_B(t,T\wedge\tau;C), \quad (6.13)$$

where $R_B(t,T\wedge\tau;C)$ *is the sum of the discrete funding cash flows defined below*

$$V_B(t,T\wedge\tau;C) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{m-1}\mathbb{I}_{\{t_j<T\wedge\tau\}}D(t,t_j)\left(V_j - V_j^+\frac{D(t_j,t_{j+1})}{D^{r_V^+}(t_j,t_{j+1})} - V_j^-\frac{D(t_j,t_{j+1})}{D^{r_V^-}(t_i,t_{i+1})}\right)\right], \quad (6.14)$$

where we define $T_V := \{t_1, ..., t_m\}$ *to be the fixed time grid in which funding is required. The time grid for margining is different to the time grid where funding is required,* $T_V \neq T_C$. V_j^+ *indicates bank B to be have a surplus of cash in the trade. The opposite situation is indicated by* V_j^-*, whereby B will need funding to perform its operations. Further,* r_V^+ *assumes that B will earn a different rate to what it would pay, given by* r_V^-.

[Brigo, Morini and Pallavicini 2013] discuss the hedging strategy of the cash account R_t when rehypothecation is permitted and when it is not. For the former, they define R_t as

$$V_t = \hat{F}(C,V) - H_t,$$

where the derivative's risky price is a function of the collateral account C and the cash account V. H_t is a portfolio of hedging instruments for $\hat{F}$. For the latter we have

$$V_t = \hat{F}(C,V) - H_t - C_t,$$

where the bank at hand has access to the collateral assets for funding purposes. They highlight that the value of $\hat{F}$ at t is dependent on the funding strategy V after t. Similarly, V after t will depend on the value of $\hat{F}$ at proceeding points in time. Theorem 6.8 provides a general pricing formula pivotal to the practisioner when trading with certain counterparties.

Chapter 7

Including MVA and KVA in the Semi-Replication Framework - Kenyon and Green

"If I have seen further, it is by standing on the shoulders of giants" - Sir Isaac Newton

Throughout much of the literature, we consider the work published during the period of 2010 to 2012, the first wave of pricing solutions to accommodate changes in the regulatory environment post-GFC. The papers discussed in this chapter draw on the work of authors who furthered these pricing formulas to include MVA and KVA. The papers reviewed were published during 2014-2016, and are the latest pieces to address pricing for MVA and KVA in the framework described in Chapter 5.

[Kenyon, Green and Dennis 2016] extend the formulas presented by Kjaer, Burgard and Piterbarg in [Burgard and Kjaer 2012b]. They adjust the Black-Scholes PDE to include pricing for KVA via portfolio semi-replication. The authors highlight that KVA differs from all other XVAs because the hedges implemented for KVA generate capital themselves and that capital can be used for funding. Basel III limits an entity to fully fund certain derivatives through the issuing of capital. We build onto Theorem 5.6 to include capital costs using a semi-replication approach. Once again g_C, and g_B are the closeout functions with M being the closeout value at default.

Theorem 7.1. *We define ϵ_{h}, g_B and g_C to be the value of the derivative at default for counterparty C and B respectively,*

$$g_B(M, C) = C + (M_B - C)^+ + R_B(M_B - C)^-, \tag{7.1}$$

$$g_C(M, C) = C + (M_C - C)^- + R_C(M_C - C)^+, \tag{7.2}$$

and

$$\begin{aligned} \epsilon_{\mathrm{h}} &= g_B - C + P_B^D - \varphi K, \\ &= \epsilon_{h_0} + \epsilon_{h_K}, \end{aligned} \tag{7.3}$$

where ϵ_{h} denotes the hedge error between the hedge portfolio and the derivative at default; P_B^D is defined as the post issuer default value of counterparty B's bond position; and φK is the amount of capital available for funding with $\varphi \in [0, 1]$.

The credit risky price of a derivative that satisfies the following boundary problem is given by $\hat{F}$

$$\frac{\partial \hat{F}}{\partial t} - r_R x \frac{\partial \hat{F}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \hat{F}}{\partial x^2} - (r + \lambda_B + \lambda_C)\hat{F} = -\lambda_B g_B - \lambda_C g_C + s_C C + \lambda_B \epsilon_{\mathrm{h}} + \gamma_{\mathrm{K}}(\mathrm{t})K - r\varphi K, \tag{7.4}$$

$$\hat{F}(T, x) = F(T, x). \tag{7.5}$$

We analyse the right-hand side of the pricing formula in Theorem 7.1. The second last term, $\gamma_{\mathrm{K}}(\mathrm{t})K$, is the cost of the return shareholders expect for putting their capital at risk. Most banks have a hurdle rate set for the desks to benchmark how profitable a particular trade will be. The last term, $r\varphi K$ is the portion of capital that can be used to fund the trade. The two terms combined make up the KVA portion of the price. If $\varphi = 0$ and $\gamma_{\mathrm{K}}(\mathrm{t}) = 0$, the pricing formula would revert to Equation 5.14. By applying the Feynman-Kač theorem to the above PDE, we obtain the following results.

Definition 7.2. *The adjustment for the risk-free derivative F is given by U, where M=F,*

$$U(t, s) = CVA + DVA + FCA + COLVA + KVA, \tag{7.6}$$

where

$$CVA = -\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}^{\mathbb{Q}}{}_t F(u) - g_C(F(u),C(u))du, \tag{7.7}$$

$$DVA = -\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}^{\mathbb{Q}}{}_t F(u) - g_B(F(u),C(u))du, \tag{7.8}$$

$$FCA = -\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}^{\mathbb{Q}}{}_t \epsilon_{\mathrm{h}}(u)du, \tag{7.9}$$

$$COLVA = -\int_t^T s_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}^{\mathbb{Q}}{}_t C(u)du, \tag{7.10}$$

$$KVA = -\int_t^T D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}^{\mathbb{Q}}{}_t(\gamma_K(u) - r(u)\varphi)K(u) + \lambda_B\epsilon_{h_K}du, \tag{7.11}$$

where $D_k(t,u) = e^{-\int_t^u k(v)dv}$ *is the required discount factor.*

The final adjustment provided in the portfolio semi-replication framework is provided by [Kenyon and Green 2015], this is the MVA term. Kenyon and Green seek to expand on Equation 7.4 to include the costs associated with IM. Typically this is not a feature in standardised CSAs, but if a hedge were to be executed through a CCP, one would certainly need to consider the cost of posting IM. Kenyon and Green also note that under the Basel proposal for bilateral IM all non-cleared derivatives will be required to post IM by 2019, further emphasising the importance of understanding how to price for such a requirement. The authors assume that IM cannot be rehypothecated and is funded through the issuance of bonds. In the below theorem, we add the MVA term directly to Equation 7.4 from Theorem 7.1.

Theorem 7.3. *We define* ϵ_{h}, g_B *and* g_C *as follows*

$$g_B(M,C) = C + (M_B - C)^+ + R_B(M_B - C)^-, \tag{7.12}$$

$$g_C(M,C) = C + (M_C - C)^- + R_C(M_C - C)^+, \tag{7.13}$$

and

$$\begin{aligned}\epsilon_{\mathrm{h}} &= g_B - C + P_B^D - \varphi K \\ &= \epsilon_{h_0} + \epsilon_{h_K}\end{aligned} \tag{7.14}$$

where $\epsilon_{\rm h}$ *denotes the hedge error between the hedge portfolio and the derivative at default;* P_B^D *is defined as the post issuer default value of counterparty B's bond position; and* φK *is the amount of capital available for funding with* $\varphi \in [0, 1]$.

The credit risky price of a derivative that satisfies the following boundary problem is given by $\hat{F}$,

$$\frac{\partial \hat{F}}{\partial t} - r_R x \tfrac{\partial \hat{F}}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \tfrac{\partial^2 \hat{F}}{\partial x^2} - (r + \lambda_B + \lambda_C)\hat{F} = -\lambda_B g_B - \lambda_C g_C + s_C C + \lambda_B \epsilon_{\rm h} + \gamma_{\rm K}(\mathrm{t})K - r\varphi K - s_I I, \tag{7.15}$$

$$\hat{F}(T, x) = F(T, x). \tag{7.16}$$

Equation 7.15 now includes the MVA term s_II, which is the cost bank B needs to include in the derivative price to the client. If we were to apply the Feynman-Kač theorem to Equation 7.15, we would get the same result as we got in Definition 7.2 with exception of an adjustment to the COLVA term. Kenyon and Green adjust this term to include the additional cost of IM. For good measure we define all of the adjustments:

Definition 7.4. *The adjustment for the risk-free derivative F is given by U, where M=F,*

$$U(t, s) = CVA + DVA + FCA + COLVA + KVA, \tag{7.17}$$

where

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t F(u) - g_C(F(u), C(u)) du, \tag{7.18}$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t F(u) - g_B(F(u), C(u)) du, \tag{7.19}$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}^{\mathbb{Q}}{}_t \epsilon_{\rm h}(u) du, \tag{7.20}$$

$$
\begin{aligned}
COLVA = &-\int_t^T s_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}^{\mathbb{Q}}{}_t C(u) du \\
&+ \int_t^T s_I(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}^{\mathbb{Q}}{}_t I(u) du,
\end{aligned} \tag{7.21}
$$

$$
KVA = -\int_t^T D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}^{\mathbb{Q}}{}_t (\gamma_K(u) - r(u)\varphi) K(u) + \lambda_B \epsilon_{h_K} du, \tag{7.22}
$$

where $D_k(t,u) = e^{-\int_t^u k(v)dv}$ *is the required discount factor.*

Theorem 7.3 and Definition 7.4 provide the most comprehensive pricing formula within the semi-replication framework to date. We do not provide proofs for this framework as the dissertation focus is on the work of Elouerkhaoui. However, we have provided a fair portion of the literature review to the said framework for the benefit of the reader to understand the benefits and downfalls to each pricing method.

Chapter 8

The Funding Invariance Principle with XVA - Elouerkhaoui

"The role of genius is not to complicate the simple, but to simplify the complicated."
- Criss Jami

[Elouerkhaoui 2016a] provides us with the funding invariance principle. He states that if all the funding cash flows are included in the master equation, then the choice of discounting rate is irrelevant. He seeks to formalise this theory by proving a universal funding invariance principle which he uses to work out the FVA term. His approach is to not change the BSM pricing theory, but to adapt it to price a more complex payoff. For consistency within this dissertation we diverge slightly from the notation used in [Elouerkhaoui 2016a]. Elouerkhaoui defines the risk-free value of a derivative as follows:

Definition 8.1. *We define F to be the value of the risk-free derivative without default risk and funding, then*

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathrm{D}^{\mathrm{r}}_{\mathrm{t,s}} dX_s \right], \tag{8.1}$$

where X is the cumulative dividend process of a generic derivative contract.

Before providing the funding invariance principle, we provide the following proposition.

Proposition 8.2. *(The funding equation) The value of the risk-free derivative with cash flows from treasury and CSA is given by F,*

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D^r_{t,s} dX_s + \int_t^T D^r_{t,s}(r - r_V)V_s ds + \int_t^T D^r_{t,s}(r - r_C)C_s ds \right], \quad (8.2)$$

where the cash account is given by $V_t = F_t - C_t$.

The following theorem introduces the funding invariance principle which displays the irrelevance of choosing a discount rate.

Theorem 8.3. *Let r_* be any interest rate process, then the funding equation can be written equivalently using the discounting with r_* process,*

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathrm{D}^{\mathrm{r}_*}_{\mathrm{t,s}} dX_s + \int_t^T D^{r_*}_{t,s}(r_* - r_V)V_s ds + \int_t^T D^{r_*}_{t,s}(r_* - r_C)C_s ds \right]. \quad (8.3)$$

Theorem 8.3 builds off of the risk-free case defined in Defintion 8.1 by adding the CSA collateral cash flows represented by the third term as well as the treasury funding cash flows shown by the second term. This is very similar to the method shown in [Piterbarg 2010], however, Elouerkhaoui uses an expectation approach over a PDE approach. As we have mentioned, the above funding equation does not consider default risk, the author caters for this in the following proposition:

Proposition 8.4. *(The master equation) The value of the risky derivative with cash flows from treasury, CSA, and the recovery payments at $\tau = min(\tau_B, \tau_C)$ is given by $\hat{F}$,*

$$\begin{aligned} \hat{F}_t = {} & \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^r_{t,s} dX_s \right] \\ & + \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^r_{t,s}(r - r_V)\hat{V}_s ds + \int_t^T \mathbb{I}_{\{\tau > s\}} D^r_{t,s}(r - r_C)C_s ds \right] \\ & + \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathrm{D}^{\mathrm{r}}_{\mathrm{t},\tau} C_\tau + \mathbb{I}_{\{\tau \le T\}} D^r_{t,\tau}(\hat{V}^R_\tau - \hat{V}_\tau) + \mathbb{I}_{\{\tau \le T\}} D^r_{t,\tau} \hat{F}^R_\tau \right], \end{aligned} \quad (8.4)$$

where the recovery payoff post collateral netting is

$$\begin{aligned} \hat{F}^R_\tau = {} & \mathbb{I}_{\{\tau = \tau_C\}} \left(R_C (F_\tau - C_\tau)^+ + (F_\tau - C_\tau)^- \right) \\ & + \mathbb{I}_{\{\tau = \tau_B\}} \left((F_\tau - C_\tau)^+ + R_B (F_\tau - C_\tau)^- \right), \end{aligned} \quad (8.5)$$

and the recovery payoff of the funding from treasury is

$$\hat{V}^R_\tau = \hat{V}^-_\tau + \hat{V}^+_\tau \mathbb{I}_{\{\tau = \tau_C\}} + R^V_B \hat{V}^+_\tau \mathbb{I}_{\{\tau = \tau_B\}}. \quad (8.6)$$

R^V_B is the recovery rate on the funding B receives, if B were to default and the derivative was in the money for B.

Let us consider the right-hand side of Equation 8.4. The first term represents the derivatives' risk-free cash flows as per the ISDA contract; the second and the third term represent the CSA funding and the treasury funding respectively. The fourth, fifth, and the sixth terms represent the credit risky portion of the master equation. More specifically, the fourth term represents the exchange of margin C_τ when a default occurs and the sixth term represents the CVA and DVA for the trading desk. The closeout function represented by $\hat{V}_\tau^R$ is identical to the closeout function g shown in [Burgard and Kjaer 2012b] for the risk-free closeout case. The fifth term represents the Fair-Value Option (FVO) debt CVA for the treasury desk, which is essentially the DVA term for the treasury desk on their debt issuances.

Proposition 8.4 is the master equation for the entire bank which can be split up into two parts for the trading desk and the treasury. Elouerkhaoui shows us which portions of Equation 8.4 are relevant to each division:

Definition 8.5. *(The master equation) The value of the risky derivative on the banks balance sheet defined as $\hat{F}$,*

$$\hat{F}_t = \hat{F}_t^{Desk} + \hat{F}_t^{Treasury}, \tag{8.7}$$

where $\hat{F}_t^{Desk}$ is given by

$$\begin{aligned}\hat{F}_t^{Desk} = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^r dX_s\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^r (r - r_C) C_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^r (r - r_V) \hat{V}_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^r C_\tau + \mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^r \hat{F}_\tau^R\right],\end{aligned} \tag{8.8}$$

and $\hat{F}_t^{Treasury}$ is given by

$$\begin{aligned}\hat{F}_t^{Treasury} = &- \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^r (r - r_V) \hat{V}_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^r (r - r_V) \hat{V}_s ds + \mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^r (\hat{V}_\tau^R - \hat{V}_\tau)\right].\end{aligned} \tag{8.9}$$

R_B^V is the recovery rate on the funding B receives if B were to default and the derivative was in the money for B.

It is F_t^{Desk} we are most interested in. The following theorem provides the funding invariance principle with default risk :

Theorem 8.6. *Let r_* be any interest rate process, then the master funding eqaution with default risk can be written equivalently using the discounting with r_* process,*

$$\begin{aligned}\hat{F}_t^{Desk} &= \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_*} dX_s\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_*}(r_* - r_C) C_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_*}(r_* - r_V) \hat{V}_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathrm{D}_{\mathrm{t},\tau}^{\mathrm{r}_*} C_\tau + \mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_*} \hat{F}_\tau^R\right],\end{aligned} \tag{8.10}$$

where the funding account is given by $\hat{V}_t = \hat{F}_t^{Desk} - C_t$.

Elouerkhaoui shows that by assuming a funded Present Value (PV), the funding invariance principle reduces to the following:

Theorem 8.7. *The value of the trade with default risk, margining, CSA funding, and unsecured treasury funding is provided by*

$$\hat{F}_t^{Desk} = F_t - CVA_t - DVA_t, \tag{8.11}$$

$$CVA_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}}\mathbb{I}_{\{\tau=\tau_C\}} \mathrm{D}_{\mathrm{t},\tau}^{\mathrm{r_V}}(1 - R_C)(F_\tau - C_\tau)^+\right], \tag{8.12}$$

$$DVA_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}}\mathbb{I}_{\{\tau=\tau_B\}} \mathrm{D}_{\mathrm{t},\tau}^{\mathrm{r_V}}(1 - R_B)(F_\tau - C_\tau)^-\right], \tag{8.13}$$

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T D_{t,s}^{r_V} dX_s + \int_t^T \mathrm{D}_{\mathrm{t,s}}^{\mathrm{r_V}}(r_V - r_C) C_s ds\right], \tag{8.14}$$

where the default-free PV *of the trade V_t is the solution to the funding equation.*

Elouerkhaoui uses a symmetric treasury funding rate and argues against the use of asymmetric funding rates. Elouerkhaoui claims that generally asymmetric funding rates make sense for short-term transactions. The long-term funding rates get charged at a fixed rate determined by the issuance level at the time of the desks funding requirements. Both [Brigo, Morini and Pallavicini 2013] and [Burgard and Kjaer 2012b] choose to use asymmetric funding rates. The author provides insight to FVO debt

CVA and compares it to FVA. He draws the conclusion that we can treat FVO debt CVA in a similar fashion to what [Hull and White 2012b] calls DVA2. The key difference between [Hull and White 2012b] and
[Elouerkhaoui 2016a] on this topic is that DVA2 or FVO debt CVA does not completely offset FVA, therefore we cannot choose to ignore FVA on the premise provided by [Hull and White 2012b]. Elouerkhaoui proceeds to show that FVO debt CVA simply marks the bonds back to market and not back to face value.

[Elouerkhaoui 2016b] expands the master pricing equation by including the MVA and KVA terms. He then derives the funding invariance principle within this extended framework. The following proposition builds onto the master equation presented in Proposition 8.4. Elouerkhaoui includes the cash flows associated with IM for a CSA or a CCP traded derivative. [Elouerkhaoui 2016b] mainly considers the value of the derivative from the trading desk's perspective and not treasury. It must also be noted that the framework to be presented is applicable to banks following the Internal Model Method (IMM).

Proposition 8.8. *(The master equation) If we are posting or recieving IM on a derivative, then the value of the risky derivative with cash flows from treasury, CSA, and the recovery payments at $\tau = min(\tau_B, \tau_C)$ is given by $\hat{F}$,*

$$\begin{aligned}\hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} dX_s + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r - r_C)(C_s + I_s)ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r - r_V)\hat{V}_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau\leq T\}} D^r_{t,\tau}(C_\tau + I_\tau) + \mathbb{I}_{\{\tau\leq T\}} D^r_{t,\tau}\hat{F}^R_\tau\right],\end{aligned} \tag{8.15}$$

where the funding account is given by

$$\hat{V}_t = \hat{F}_t - (C_t + I_t), \quad I_t = I^b_t + I^C_t, \tag{8.16}$$

and $I^B_t \leq 0$, $I^C_t \geq 0 \quad \forall t < \tau$. The recovery payoff, post collateral, and margin netting is given by $\hat{F}$,

$$\begin{aligned}\hat{F}^R_\tau = \mathbb{I}_{\{\tau=\tau_C\}} &\left(R_C(\alpha_\tau)^+ + (\alpha_\tau)^-\right) \\ &+ \mathbb{I}_{\{\tau=\tau_B\}} \left((\alpha_\tau)^+ + R_B(\alpha_\tau)^-\right),\end{aligned} \tag{8.17}$$

where $\alpha_\tau = F_\tau - (C_\tau + I_\tau)$.

The final adjustment made to the master equation is the addition of the KVA term. This ensures that the trading desk covers the lifetime cost of capital in the economic value of the derivative.

Proposition 8.9. *If we include the lifetime cost of regulatory capital in the master pricing equation, then the total value of the trade becomes*

$$\begin{aligned}\hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} dX_s + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_C)(C_s+I_s)ds\right] \\ + \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_V)\hat{V}_s ds + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_K)K_s ds\right] \\ + \mathbb{E}^{\mathbb{Q}}{}_t &\left[\mathbb{I}_{\{\tau\leq T\}} D^r_{t,\tau}(C_\tau+I_\tau) + \mathbb{I}_{\{\tau\leq T\}} D^r_{t,\tau}\hat{F}^R_\tau\right],\end{aligned} \tag{8.18}$$

where the funding account is given by

$$\hat{V}_t = \hat{F}_t - (C_t + I_t) - K_t, \quad I_t = I^b_t + I^C_t, \tag{8.19}$$

and $I^B_t \leq 0,\ I^C_t \geq 0 \quad \forall t < \tau$, *and the Basel III regulatory capital is*

$$K_t = K^{MR}_t + K^{CCR}_t + K^{CVA_{VAR}}_t. \tag{8.20}$$

Elouerkhaoui shows K_t to encompass all aspects of the banks risk capital charges. MR = *Market Risk* charge and CCR = *Counterparty Credit Risk* charge, and recently added by Basel regulation, the CVA_{VAR} charge. It must also be noted that the above proposition highlights that debt funding from treasury is now accompanied by capital funding from shareholders. By replacing all of the r terms above with r_* terms and discounting, using r_* brings us to the funding invariance principle inclusive of MVA and KVA. We provide a proof of this result in Part III of the dissertation. The solution to the master equation is presented in the following proposition. Once again we replace r with r_V which results in the removal of the FVA term.

Definition 8.10. *The value of the derivative with default risk, margining, CSA funding, and unsecured funding sourced from treasury, IM, as well as regulatory capital is defined as*

$$\hat{F}_t = F_t - CVA_t - DVA_t - MVA_t - KVA_t, \tag{8.21}$$

where

$$CVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \leq T\}} \mathbb{I}_{\{\tau = \tau_C\}} D^{r_V}_{t,\tau} (1 - R_C)(\hat{F}_\tau - C_\tau - I_\tau)^+ \right], \tag{8.22}$$

$$DVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \leq T\}} \mathbb{I}_{\{\tau = \tau_B\}} D^{r_V}_{t,\tau} (1 - R_B)(\hat{F}_\tau - C_\tau - I_\tau)^- \right], \tag{8.23}$$

$$MVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_C) I_s ds \right], \tag{8.24}$$

$$KVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_K) K_s ds \right], \tag{8.25}$$

and the default-free present value of the trade is F_t *which is the solution of the following equation*

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D^{r_V}_{t,s} dX_s + \int_t^T D^{r_V}_{t,s} (r_V - r_C) C_s ds \right]. \tag{8.26}$$

As we saw in Definition 7.4, [Kenyon, Green and Dennis 2016] provide a very similar KVA solution to what Elouerkhaoui presents, they simply leave the adjustment as an FVA term as opposed to altering the discount factor. Elouerkhaoui shows how to compute the KVA expectation in its three different forms, $K_t = K_t^{MR} + K_t^{CCR} + K_t^{CVA_{VAR}}$. We find [Kenyon, Green and Dennis 2016] and [Kenyon and Green 2015] to be two of the more practical papers of the vast amount of literature in XVA pricing and have chosen to explore the theory and proofs of Elouerkhaoui's work in Part III of this dissertation.

Part III

Mathematical Preliminaries

Chapter 9

The Funding Invariance Principle

"No human investigation can claim to be scientific if it doesn't pass the test of mathematical proof." - Leonardo da Vinci

In this chapter we refer to the work done in [Elouerkhaoui 2016a]. We start by defining the principals associated with the funding invariance principle:

1. There is no need for a new arbitrage-free pricing theory;
2. The same risk-neutral measure and the same money market account numeraire is applicable;
3. The final payoff is now more complicated and must be articulated correctly; and
4. All default contingent legs and funding legs are included in the pricing formula.

We define the cash flows associated with a derivative trade. The three contracts that determine a derivative's cash flows are:

1. ISDA contract to govern the standard cash flows associated with the derivative;
2. CSA agreement to govern the usually daily collateral payments or receipts; and
3. The internal agreement between treasury and the trading desk.

To fully appreciate the framework of the funding invariance principle, one must be aware of all the cash flows associated with a derivative trade within a banks infrastructure. With regard to the cash flows governed by the ISDA agreement, the desk receives $F(t)$ at t and pays $F(T)$ at T. The desk will deposit $F(t)$ with the money market and will receive $F(t)(1 + rdt)$ at $t + dt$. If the trade has a CSA agreement and considering the cash flows of $F(t)$, then the desk will pay $C(t)$ at t and receive $c(t)(1+r_C dt)$ at $t+dt$. The desk would receive $C(t)$ from the money market and pay back $C(t)(1 + rdt)$ at $t + dt$. The differential between $F(t)$ and $C(t)$ will determine the role of treasury in the cash flows of the derivative. The desk pays $(F(t) - C(t))$ to treasury at t and receives $(F(t) - C(t))(1 + r_F dt)$ at $t + dt$. Simultaneously, the desk is funded $(F(t) - C(t))$ at t and pay $(F(t) - C(t))(1 + rdt)$. Considering the money market cash flows at time t, we have

$$-F(t) + C(t) + (F(t) - C(t)) = 0. \tag{9.1}$$

At time $t + dt$ we then have

$$F(t)(1 + rdt) - C(t)(1 + rdt) - (F(t) - C(t))(1 + rdt) = 0. \tag{9.2}$$

Clearly $1+rdt$ falls away, leaving us only with the funding and the funding rates paid or received by the CSA and treasury. Recall Elouerkhaoui assumes that both bank B and counterparty C cannot default at the same time, therefore $\tau = min(\tau_B, \tau_C)$. Further to this, another assumption we recall is that the closeout value is denoted by $\Theta_\tau = F_\tau$ and is considered to be the value of the contract without counterparty risk but including funding costs at τ. We first consider the funding equation and the dymanics of the cash accounts that make up its components. We recall the funding equation from [Elouerkhaoui 2016a]:

Proposition 9.1. *(The funding equation) The value of the risk-free derivative with cash flows from treasury and CSA is given by F,*

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D^r_{t,s} dX_s + \int_t^T D^r_{t,s}(r - r_V)V_s ds + \int_t^T D^r_{t,s}(r - r_C)C_s ds \right], \tag{9.3}$$

where the cash account is given by $V_t = F_t - C_t$.

We start with the derivation of term three in Equation 9.3, the CSA funding component. We refer to [Elouerkhaoui 2014] for the proof.

Proof. In order for us to show the derivation of the funding equation we need to define the cash account and its dynamics. We define the collateral account as the sum of variation margins and interest paid on the cash in the account

$$A_t^C = C_t + \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r d\eta_s^C \right], \tag{9.4}$$

where we have the following boundary problem

$$\begin{aligned} d\eta_t^C &= dC_t - r_C C_t dt, \\ C_T &= 0. \end{aligned} \tag{9.5}$$

By substituting $d\eta_t^C = dC_t - r_t C_t dt$ into $\mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r d\eta_s^C \right]$ we get

$$A_t^C = C_t + \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r dC_s - \int_t^T r_C D_{t,s}^r C_s ds \right], \tag{9.6}$$

where we solve $\int_t^T D_{t,s}^r dC_s$ using integration by parts. The result is as follows

$$\begin{aligned} \int_t^T D_{t,s}^r dC_s =& -C_t + D_{t,T}^r C_T - \int_t^T C_s dD_{t,s}^r, \\ =& -C_t + \int_t^T r C_s D_{t,s}^r ds. \end{aligned} \tag{9.7}$$

Lastly, we substitute the result of Equation 9.7 into Equation 9.6 which gives us the CSA funding term shown as the third term in funding Equation 9.3,

$$A_t^C = \int_t^T D_{t,s}^r (r - r_C) C_s ds. \tag{9.8}$$

We now consider term two of Equation 9.3, the treasury funding component. We define the treasury funding account as

$$A_t^V = V_t + \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r d\eta_s^V \right], \tag{9.9}$$

where we have the following boundary problem

$$\begin{aligned} d\eta_t^V &= dV_t - r_V V_t dt, \\ V_T &= 0, \\ V_t &= F_t - C_t, \quad \forall \quad t < T. \end{aligned} \tag{9.10}$$

Following the same reasoning used in the CSA funding proof, we obtain the below result

$$A_t^V = \int_t^T D_{t,s}^r (r - r_V) V_s ds. \tag{9.11}$$

□

The first term is defined as the value of the risk-free derivative as per Definition 8.1 in Chapter 8. Which concludes our derivation of Elouerkhaoui's funding equation. We now look to the funding invariance principle. We work through the proof provided in [Elouerkhaoui 2014]. Let us first define the risk neutral valuation formula taken from [Björk 2009] Theorem 10.19 as well as Definition 2.4.

Definition 9.2. *A probability measure $\mathbb{Q}$ is called a martingale measure if the following condition holds*

$$\Pi(t, X) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} X \| \mathcal{F}_t \right], \tag{9.12}$$

with the money market account $D_{t,T}^r = e^{-\int_t^T r(s)ds}$ as the numeraire.

We provide the funding invariance principle theorem with proof.

Theorem 9.3. *Let r_* be any interest rate process, then the funding equation can be written equivalently using the discounting with r_* process,*

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^{r_*} dX_s + \int_t^T D_{t,s}^{r_*} (r_* - r_V) V_s ds + \int_t^T D_{t,s}^{r_*} (r_* - r_C) C_s ds \right], \tag{9.13}$$

where the funding account is given by $V_t = F_t - C_t$.

Proof. To begin we define $d\tilde{X}_s^r$ as the sum of all the cash flows associated with the derivative.

$$d\tilde{X}_s^r = dX_s + (r - r_C)C_s ds + (r - r_V)V_s ds. \tag{9.14}$$

By substituting process dX with $d\tilde{X}^r$, we get

$$d\tilde{X}_s^r = dX_s + (r - r_C)C_s ds + (r - r_V)V_s ds, \tag{9.15}$$

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r d\tilde{X}_s^r \right]. \tag{9.16}$$

Which becomes the funding equation if we expand $d\tilde{X}^r$. We define process A_t^r as

$$A_t^r = D_{0,t}^r F_t + \int_0^t D_{0,s}^r d\tilde{X}_s^r. \tag{9.17}$$

We wish to show that A_t^r is a martingale. By substituting Equation 9.16 into 9.17 we get the following,

$$\begin{aligned} A_t^r &= D_{0,t}^r \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r d\tilde{X}_s^r \right] + \int_0^t D_{0,s}^r d\tilde{X}_s^r \\ &= \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_0^T D_{0,s}^r d\tilde{X}_s^r \right] \\ &= \mathbb{E}^{\mathbb{Q}}{}_t \left[A_T^r \right]. \end{aligned} \tag{9.18}$$

Therefore in accordance with the condition provided in Definition 9.2, $\mathbb{Q}$ is a martingale measure and A_t^r is a martingale. We now write A_t^r as a differential,

$$dA_t^r = D_{0,t}^r \left[-rF_t dt + dF_t + d\tilde{X}_t^r \right]. \tag{9.19}$$

Given the result of dA_t^r, we can introduce $dA_t^{r_*}$ as

$$dA_t^{r_*} = D_{0,t}^{r_*} \left[-r_* F_t dt + dF_t + d\tilde{X}_t^{r_*} \right]. \tag{9.20}$$

Recall our definition of $d\tilde{X}_t^r$, similarly we define $d\tilde{X}_t^{r_*}$ as

$$d\tilde{X}_t^{r_*} = dX_s + (r_* - r_C)C_t dt + (r_* - r_V)V_t dt, \tag{9.21}$$

by subtracting $d\tilde{X}_t^{r}$ from both sides we get the following result

$$d\tilde{X}_t^{r_*} = d\tilde{X}_t^{r} + (r_* - r)C_t dt + (r_* - r)V_t dt. \tag{9.22}$$

Lastly we are required to show $A_t^{r_*}$ is a martingale $\forall\ t < T$

$$\begin{aligned} dA_t^{r_*} &= D_{0,t}^{r_*}\left[-r_* F_t dt + dF_t + d\tilde{X}_t^{r_*}\right] \\ &= D_{0,t}^{r_*}\left[-r_* F_t dt + dF_t + d\tilde{X}_t^{r} + (r_* - r)C_t dt + (r_* - r)V_t dt\right] \\ &= D_{0,t}^{r_*}\left[-r_* F_t dt + dF_t + \left[d\tilde{X}_t^{r} + (r_* - r)(C_t + V_t)dt\right]\right], \end{aligned} \tag{9.23}$$

we now add $(r - r)F_t dt$ to the equation

$$\begin{aligned} dA_t^{r_*} &= D_{0,t}^{r_*}\left[-r_* F_t dt + (r - r)F_t dt + dF_t + \left[d\tilde{X}_t^{r} + (r_* - r)(C_t + V_t)dt\right]\right] \\ &= D_{0,t}^{r_*}\left[-r F_t dt + (r - r_*)F_t dt + dF_t + \left[d\tilde{X}_t^{r} + (r_* - r)(C_t + V_t)dt\right]\right], \end{aligned} \tag{9.24}$$

by collecting like terms we have the following result

$$dA_t^{r_*} = D_{0,t}^{r_*}\left[(r - r_*)\left[F_t - C_t - V_t\right]dt - rF_t dt + dF_t + d\tilde{X}_t^{r}\right]. \tag{9.25}$$

Recall $V_t = F_t - C_t$ hence $F_t - C_t - V_t$ reduces to 0. In addition, earlier we defined $dA_t^r = D_{0,t}^r\left[-rF_t dt + dF_t + d\tilde{X}_t^r\right]$, therefore

$$\begin{aligned} dA_t^{r_*} &= D_{0,t}^{r_*}\left[(r - r_*)0 - \frac{dA_t^r}{D_{0,t}^r}\right] \\ &= \frac{D_{0,t}^{r_*}}{D_{0,t}^r} dA_t^r. \end{aligned} \tag{9.26}$$

The above result shows $A_t^{r_*}$ is also a martingale. Using Definition 9.2, $A_t^{r_*}$ can be written as

$$\begin{aligned} A_t^{r_*} &= \mathbb{E}^{\mathbb{Q}}{}_t\left[A_T^{r_*}\right] \\ &= \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_0^T D_{0,s}^{r_*} d\tilde{X}_s^{r_*}\right]. \end{aligned} \tag{9.27}$$

□

The funding invariance principle we have provided and proved in this chapter only considers the risk-free price and the funding implications of a derivative under CSA including funding costs. We now provide the master equation which builds off of the funding equation from Proposition 9.1.

Proposition 9.4. *(The master equation) The value of a credit risky derivative with cash flows from treasury, CSA, and the recovery payments at* $\tau = min(\tau_B, \tau_C)$ *is given by* $\hat{F}$*, then*

$$\begin{aligned} \hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} dX_s\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_V)\hat{V}_s ds + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_C)C_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau} C_\tau + \mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau}(\hat{V}^R_\tau - \hat{V}_\tau) + \mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau}\hat{F}^R_\tau\right], \end{aligned} \tag{9.28}$$

where the recovery payoff post collateral netting is

$$\begin{aligned} \hat{F}^R_\tau = \mathbb{I}_{\{\tau=\tau_C\}} &\left(R_C (F_\tau - C_\tau)^+ + (F_\tau - C_\tau)^-\right) \\ &+ \mathbb{I}_{\{\tau=\tau_B\}} \left((F_\tau - C_\tau)^+ + R_B (F_\tau - C_\tau)^-\right), \end{aligned} \tag{9.29}$$

and the recovery payoff of the funding from treasury is

$$\hat{V}^R_\tau = \hat{V}^-_\tau + \hat{V}^+_\tau \mathbb{I}_{\{\tau=\tau_C\}} + R^V_B \hat{V}^+_\tau \mathbb{I}_{\{\tau=\tau_B\}}. \tag{9.30}$$

R^V_B *is the recovery rate on the funding B receives if B were to default and the derivative was in the money for B.*

Recall $\hat{F}_t = \hat{F}^{Desk}_t + \hat{F}^{Treasury}_t$, where $\hat{F}^{Desk}_t$ is given by

$$\begin{aligned} \hat{F}^{Desk}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} dX_s\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_C)C_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r-r_V)\hat{V}_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau} C_\tau + \mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau}\hat{F}^R_\tau\right], \end{aligned} \tag{9.31}$$

We refer to [Elouerkhaoui 2014] for the proof of $\hat{F}^{Desk}_t$, but for the sake of simplicity we refer to $\hat{F}^{Desk}_t$ as $\hat{F}_t$ going forward. Further to this, we assume default of both counterparties is independent and they cannot default simultaneously.

Theorem 9.5. *Let r_* be any interest rate process, then the master funding equation with default risk can be written equivalently using the discounting with r_* process,*

$$\begin{aligned}\hat{F}_t = {\mathbb{E}^{\mathbb{Q}}}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s} dX_s\right] \\ &+ {\mathbb{E}^{\mathbb{Q}}}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s}(r_* - r_C) C_s ds\right] \\ &+ {\mathbb{E}^{\mathbb{Q}}}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s}(r_* - r_V) \hat{V}_s ds\right] \\ &+ {\mathbb{E}^{\mathbb{Q}}}_t \left[\mathbb{I}_{\{\tau\leq T\}} D^{r_*}_{t,\tau} C_\tau + \mathbb{I}_{\{\tau\leq T\}} D^{r_*}_{t,\tau} \hat{F}^R_\tau\right],\end{aligned} \tag{9.32}$$

where the funding account is given by $\hat{V}_t = \hat{F}_t - C_t$.

Proof. Since we are working with a risky derivative, we need to consider what the payoff will be at default. We define ξ_τ as

$$\xi_\tau = C_\tau + \hat{F}^R_\tau, \tag{9.33}$$

where C_τ is the margin account and $\hat{F}^R_\tau$ is the recovery payoff after netting with the margin. Similar to our proof of the invariance funding principle, we use $d\tilde{X}^r_s$ to represent all the cash flows associated with the derivative,

$$d\tilde{X}^r_s = dX_s + (r - r_C)C_s ds + (r - r_V)V_s ds. \tag{9.34}$$

If we join Equation 9.33 and 9.34 we get the value of the risky derivative $\hat{F}_t$ at time t as

$$\hat{F}_t = {\mathbb{E}^{\mathbb{Q}}}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} d\tilde{X}^r_s\right] + {\mathbb{E}^{\mathbb{Q}}}_t \left[\int_t^T D^r_{t,s} \xi_s dP^\tau_s\right]. \tag{9.35}$$

The first term of the right-hand side of Equation 9.35 consists of the risk-free and funding components of the value of F_t. The second term is the credit risky portion of the derivative where we define dP^τ as the instantaneous default probability process where there is no default up until τ.
We define process $\hat{A}^r_t$ as

$$\hat{A}^r_t = D^r_{0,t}\hat{F}_t + \int_0^t \mathbb{I}_{\{\tau>s\}} D^r_{0,s} d\tilde{X}^r_s + \int_0^t D^r_{0,s}\xi_s dP^\tau_s. \tag{9.36}$$

We wish to show that $\hat{A}_t^r$ is a martingale. We start by substituting Equation 9.35 into 9.36.

$$\begin{aligned}
\hat{A}_t^r &= D_{0,t}^r \left[\mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D_{t,s}^r d\tilde{X}_s^r \right] + \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^r \xi_s dP_s^\tau \right] \right] \\
&\quad + \int_0^t \mathbb{I}_{\{\tau > s\}} D_{0,s}^r d\tilde{X}_s^r + \int_0^t D_{0,s}^r \xi_s dP_s^\tau \\
&= \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_0^T \mathbb{I}_{\{\tau > s\}} D_{0,s}^r d\tilde{X}_s^r + \int_0^T D_{0,s}^r \xi_s dP_s^\tau \right]. \\
&= \mathbb{E}^{\mathbb{Q}}{}_t \left[\hat{A}_T^r \right]
\end{aligned} \tag{9.37}$$

Once again we refer to Definition 9.2 to conclude $\mathbb{Q}$ is a martingale measure and therefore $\hat{A}_t^r$ is a martingale. We differentiate $\hat{A}_t^r$ with respect to t

$$dA_t^r = D_{0,t}^r \left[-rF_t dt + dF_t + d\tilde{X}_t^r \right]. \tag{9.38}$$

Given the result of dA_t^r, we can introduce $dA_t^{r_*}$ as

$$dA_t^{r_*} = D_{0,t}^{r_*} \left[-r_* F_t dt + dF_t + d\tilde{X}_t^{r_*} \right]. \tag{9.39}$$

Recall our definition of $d\tilde{X}_t^r$, similarly we define $d\tilde{X}_t^{r_*}$ as

$$d\tilde{X}_t^{r_*} = dX_s + (r_* - r_C)C_t dt + (r_* - r_V)V_t dt, \tag{9.40}$$

by subtracting $d\tilde{X}_t^r$ from both sides we get the following result

$$d\tilde{X}_t^{r_*} = d\tilde{X}_t^r + (r_* - r)C_t dt + (r_* - r)V_t dt. \tag{9.41}$$

Lastly we are required to show $A_t^{r_*}$ is a martingale $\forall\ t < T$

$$\begin{aligned}
dA_t^{r_*} &= D_{0,t}^{r_*} \left[-r_* F_t dt + dF_t + d\tilde{X}_t^{r_*} \right] \\
&= D_{0,t}^{r_*} \left[-r_* F_t dt + dF_t + d\tilde{X}_t^r + (r_* - r)C_t dt + (r_* - r)V_t dt \right] \\
&= D_{0,t}^{r_*} \left[-r_* F_t dt + dF_t + \left[d\tilde{X}_t^r + (r_* - r)(C_t + V_t)dt \right] \right],
\end{aligned} \tag{9.42}$$

we now add $(r - r)F_t dt$ to the equation,

$$
\begin{aligned}
dA_t^{r_*} &= D_{0,t}^{r_*}\left[-r_*F_tdt + (r-r)F_tdt + dF_t + \left[d\tilde{X}_t^r + (r_*-r)(C_t+V_t)dt\right]\right] \\
&= D_{0,t}^{r_*}\left[-rF_tdt + (r-r_*)F_tdt + dF_t + \left[d\tilde{X}_t^r + (r_*-r)(C_t+V_t)dt\right]\right],
\end{aligned} \tag{9.43}
$$

by collecting like terms we have the following result

$$
dA_t^{r_*} = D_{0,t}^{r_*}\left[(r-r_*)\left[F_t - C_t - V_t\right]dt - rF_tdt + dF_t + d\tilde{X}_t^r\right]. \tag{9.44}
$$

Recall $V_t = F_t - C_t$ hence $F_t - C_t - V_t$ reduces to 0. In addition, earlier we defined $dA_t^r = D_{0,t}^r\left[-rF_tdt + dF_t + d\tilde{X}_t^r\right]$, therefore

$$
\begin{aligned}
dA_t^{r_*} &= D_{0,t}^{r_*}\left[(r-r_*)0 - \frac{dA_t^r}{D_{0,t}^r}\right] \\
&= \frac{D_{0,t}^{r_*}}{D_{0,t}^r}dA_t^r.
\end{aligned} \tag{9.45}
$$

The above result shows $A_t^{r_*}$ is a martingale. Using Definition 9.2, $A_t^{r_*}$ can be written as

$$
\begin{aligned}
A_t^{r_*} &= \mathbb{E}^{\mathbb{Q}}{}_t\left[A_T^{r_*}\right] \\
&= \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_0^T \mathbb{I}_{\{\tau>s\}}D_{0,s}^{r_*}d\tilde{X}_s^{r_*} + \int_0^T D_{0,s}^{r_*}\xi_s dP_s^\tau\right].
\end{aligned} \tag{9.46}
$$

□

We have provided the proof for the funding invariance principle with and without default risk. We now use the funding invariance principle to provide the solution to the master CVA equation with funding. The master CVA equation with funding is equivalent to the value of a default risky derivative with funding from the trading desk's perspective. We refer to [Elouerkhaoui 2014] for most of our workings. We start by splitting the trading desk's portion of the funding invariance principle into two components and substitute r_* for r_V. Which provides us with

Risk-free funded (base) PV

$$
F_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T D_{t,s}^{r_V}dX_s\right] + \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T D_{t,s}^{r_V}(r_V - r_C)C_sds\right]. \tag{9.47}
$$

Clearly we no longer have a funding adjustment term and the base PV focuses on the default free portion of the derivative. We now add the CVA portion to the funded base PV to get

Risky funded (margined) PV

$$\begin{aligned}\hat{F}_t = {\mathbb{E}^{\mathbb{Q}}}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_V} dX_s\right] + {\mathbb{E}^{\mathbb{Q}}}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_V}(r_V - r_C) C_s ds\right] \\ + {\mathbb{E}^{\mathbb{Q}}}_t\left[\mathbb{I}_{\{\tau\le T\}} D_{t,\tau}^{r_V} C_\tau + \mathbb{I}_{\{\tau\le T\}} D_{t,\tau}^{r_V} \hat{F}_\tau^R\right],\end{aligned} \tag{9.48}$$

which is referred to as the funded margined CVA in [Elouerkhaoui 2016a] and provides the price for a funded risky derivative with collateral from the desk's perspective. This leads us to the following theorem with proof.

Theorem 9.6. *(Funded margined CVA) The price of a credit risky derivative for the trading desk with margining, CSA funding, and unsecured funding from treasury is given by* $\hat{F}_t$

$$\hat{F}_t = F_t - CVA_t - DVA_t, \tag{9.49}$$

$$CVA_t = {\mathbb{E}^{\mathbb{Q}}}_t\left[\mathbb{I}_{\{\tau\le T\}}\mathbb{I}_{\{\tau=\tau_C\}} D_{t,\tau}^{r_V}(1-R_C)(F_\tau - C_\tau)^+\right], \tag{9.50}$$

$$DVA_t = {\mathbb{E}^{\mathbb{Q}}}_t\left[\mathbb{I}_{\{\tau\le T\}}\mathbb{I}_{\{\tau=\tau_B\}} D_{t,\tau}^{r_V}(1-R_B)(F_\tau - C_\tau)^-\right], \tag{9.51}$$

where F_t *is the solution to the default-free funded base PV equation,*

$$F_t = {\mathbb{E}^{\mathbb{Q}}}_t\left[\int_t^T D_{t,s}^{r_V} dX_s\right] + {\mathbb{E}^{\mathbb{Q}}}_t\left[\int_t^T D_{t,s}^{r_V}(r_V - r_C) C_s ds\right]. \tag{9.52}$$

Proof. Define $F_t^{r_V}$ and $F_t^{(r_V - r_C)}$ as the value of a derivative without CSA funding and with CSA funding respectively. Assume both derivative to not carry any default risk. We then have

$$F_t^{r_V} = {\mathbb{E}^{\mathbb{Q}}}_t\left[\int_t^T D_{t,s}^{r_V} dX_s\right], \tag{9.53}$$

and

$$F_t^{(r_V - r_C)} = {\mathbb{E}^{\mathbb{Q}}}_t\left[\int_t^T D_{t,s}^{r_V}(r_V - r_C) C_s ds\right]. \tag{9.54}$$

Then the value of the funded PV without default risk is given by

$$F_t = F_t^{r_V} + F_t^{(r_V - r_C)}. \tag{9.55}$$

We now include the survival indicator in equations 9.53 and 9.54 to get

$$\mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_V} dX_s\right] = F_t^{r_V} - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} F_\tau^{r_V}\right], \tag{9.56}$$

and

$$\mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_V} (r_V - r_C) C_s ds\right] = F_t^{(r_V - r_C)} - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} F_\tau^{(r_v - r_c)}\right]. \tag{9.57}$$

Sum equation 9.56 and 9.57 together to get

$$\mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_V} dX_s\right] + \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D_{t,s}^{r_V} (r_V - r_C) C_s ds\right] = F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} F_\tau\right]. \tag{9.58}$$

We substitute equation 9.58 into equation 9.48 to get

$$\begin{aligned}\hat{F}_t &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} F_\tau\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} C_\tau + \mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} \hat{F}_\tau^R\right] \\ &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} (F_\tau - C_\tau)\right] + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} \hat{F}_\tau^R\right].\end{aligned} \tag{9.59}$$

Recall, $\hat{F}_\tau^R$ represents the recovery payoff for the risky derivative and is represented by Equation 9.29. We substitue Equation 9.29 into 9.59

$$\begin{aligned}\hat{F}_t &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} (F_\tau - C_\tau)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_C\}} D_{t,\tau}^{r_V} \left(R_C (F_\tau - C_\tau)^+ + (F_\tau - C_\tau)^-\right)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_B\}} D_{t,\tau}^{r_V} \left((F_\tau - C_\tau)^+ + R_B (F_\tau - C_\tau)^-\right)\right],\end{aligned} \tag{9.60}$$

through re-arranging and knowing that $(F_\tau - C_\tau)^- = (F_\tau - C_\tau) - (F_\tau - C_\tau)^+$, we get

$$\begin{aligned}\hat{F}_t &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} (F_\tau - C_\tau)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_C\}} D_{t,\tau}^{r_V} \left(R_C (F_\tau - C_\tau)^+ + (F_\tau - C_\tau) - (F_\tau - C_\tau)^+\right)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_B\}} D_{t,\tau}^{r_V} \left((F_\tau - C_\tau) - (F_\tau - C_\tau)^- + R_B (F_\tau - C_\tau)^-\right)\right] \\ &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} D_{t,\tau}^{r_V} \left((F_\tau - C_\tau) - \mathbb{I}_{\{\tau=\tau_C\}}(F_\tau - C_\tau) - \mathbb{I}_{\{\tau=\tau_B\}}(F_\tau - C_\tau)\right)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_C\}} D_{t,\tau}^{r_V} \left(R_C (F_\tau - C_\tau)^+ - (F_\tau - C_\tau)^+\right)\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_B\}} D_{t,\tau}^{r_V} \left(-(F_\tau - C_\tau)^- + R_B (F_\tau - C_\tau)^-\right)\right] \\ &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_C\}} D_{t,\tau}^{r_V} \left((1 - R_C)(F_\tau - C_\tau)^+\right)\right] \\ &\quad - \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau\leq T\}} \mathbb{I}_{\{\tau=\tau_B\}} D_{t,\tau}^{r_V} \left((1 - R_B)(F_\tau - C_\tau)^-\right)\right],\end{aligned}$$

(9.61)

which gives the result

$$\hat{F}_t = F_t - CVA_t - DVA_t. \tag{9.62}$$

□

Theorem 9.6 provides us with with a practical pricing equation for pricing a risky derivative. The assumptions one must consider when using this formula are:

1. No wrong-way risk is considered;

2. Bank B and counterparty C cannot default simultaneously;

3. The closeout amount at default is the risk-free value; and

4. Default between the two entities is completely independent.

If we had to consider a funded PV for closeout using each bank's respective cost of funding, as per [ISDA 2009], then the solution from Theorem 9.6, would be

$$CVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_C\}} D_{t,\tau}^{r_{V,B}} (1 - R_C)(F_\tau^B - C_\tau)^+ \right], \tag{9.63}$$

$$\begin{aligned} DVA_t = {} & \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_B\}} D_{t,\tau}^{r_{V,C}} (1 - R_B)(F_\tau^C - C_\tau)^- \right] \\ & + \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_B\}} D_{t,\tau}^{r_{V,C}} (F_\tau^B - F_\tau^C)^- \right] \end{aligned}, \tag{9.64}$$

with the risk-free funded PVs as

$$F_t^B = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^{r_{V,B}} dX_s + \int_t^T D_{t,s}^{r_{V,B}} (r_{V,B} - r_C) C_s ds \right], \tag{9.65}$$

$$F_t^C = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D_{t,s}^{r_{V,C}} dX_s + \int_t^T D_{t,s}^{r_{V,C}} (r_{V,C} - r_C) C_s ds \right]. \tag{9.66}$$

Chapter 10

From FVA to KVA

"Small minds are concerned with the extraordinary, great minds with the ordinary."
- Blaise Pascal

We refer to [Elouerkhaoui 2016b] throughout this chapter for the required propositions, theorems, and proofs. Before we provide a proof for the invariance principle with IM and capital costs, let us revaluate some of the propositions that build up to the invariance principle formula. The following proposition is the master equation with IM.

Proposition 10.1. *(The master equation) If we are posting or receiving IM on a derivative, then the value of a credit risky derivative with cash flows from treasury, CSA and the recovery payments at $\tau = min(\tau_B, \tau_C)$, is given by $\hat{F}$*

$$\begin{aligned} \hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} dX_s + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} (r - r_C)(C_s + I_s) ds \right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} (r - r_V) \hat{V}_s ds \right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau\leq T\}} D^r_{t,\tau} (C_\tau + I_\tau) + \mathbb{I}_{\{\tau\leq T\}} D^r_{t,\tau} \hat{F}^R_\tau \right], \end{aligned} \tag{10.1}$$

where the funding account is given by

$$\hat{V}_t = \hat{F}_t - (C_t + I_t), \quad I_t = I^B_t + I^C_t, \tag{10.2}$$

and $I^B_t \leq 0$, $I^C_t \geq 0 \quad \forall t < \tau$. The recovery payoff, post collateral, and margin netting is given by

$$\begin{aligned} \hat{F}^R_\tau :=\ & \mathbb{I}_{\{\tau=\tau_C\}} \left(R_C (\alpha_\tau)^+ + (\alpha_\tau)^- \right) \\ &+ \mathbb{I}_{\{\tau=\tau_B\}} \left((\alpha_\tau)^+ + R_B (\alpha_\tau)^- \right), \end{aligned} \tag{10.3}$$

where $\alpha_\tau = F_\tau - (C_\tau + I_\tau)$.

Proposition 10.1 is identical to Proposition 9.4 with the exception of replacing all C_t terms with $C_t + I_t$ where $I_t = I_t^B + I_t^C$. This only stands for the case where there is no segregation of both IM and VM. Next we consider how the master equation evolves when the cost of regulatory capital is considered

Proposition 10.2. *If we include the lifetime cost of regulatory capital in the master pricing equation, then the total value of the trade becomes*

$$\begin{aligned}\hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s} dX_s + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r - r_C)(C_s + I_s) ds\right] \\ + \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r - r_V)\hat{V}_s ds + \int_t^T \mathbb{I}_{\{\tau>s\}} D^r_{t,s}(r - r_K) K_s ds\right] \\ + \mathbb{E}^{\mathbb{Q}}{}_t &\left[\mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau}(C_\tau + I_\tau) + \mathbb{I}_{\{\tau\le T\}} D^r_{t,\tau}\hat{F}^R_\tau\right],\end{aligned} \quad (10.4)$$

where the funding account is given by

$$\hat{V}_t = \hat{F}_t - (C_t + I_t) - K_t, \qquad I_t = I_t^b + I_t^C, \quad (10.5)$$

and $I_t^B \le 0$, $I_t^C \ge 0 \quad \forall t < \tau$, *and the Basel III regulatory capital is*

$$K_t = K_t^{MR} + K_t^{CCR} + K_t^{CVA_{VAR}}. \quad (10.6)$$

The above propositions provide the basis for the invariance principle for funding, CVA and cost of capital. The theorem is provided as per [Elouerkhaoui 2016b].

Theorem 10.3. *Let* r_* *be any interest rate process, then the master funding equation with default risk, IM, and the cost of capital can be written equivalently using the discounting with* r_* *process,*

$$\begin{aligned}\hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s} dX_s + \int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s}(r_* - r_C)(C_s + I_s) ds\right] \\ + \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s}(r_* - r_V)\hat{V}_s ds + \int_t^T \mathbb{I}_{\{\tau>s\}} D^{r_*}_{t,s}(r_* - r_K) K_s ds\right] \\ + \mathbb{E}^{\mathbb{Q}}{}_t &\left[\mathbb{I}_{\{\tau\le T\}} D^{r_*}_{t,\tau}(C_\tau + I_\tau) + \mathbb{I}_{\{\tau\le T\}} D^{r_*}_{t,\tau}\hat{F}^R_\tau\right],\end{aligned} \quad (10.7)$$

where the funding account is given by

$$\hat{V}_t = \hat{F}_t - (C_t + I_t) - K_t, \qquad I_t = I_t^b + I_t^C, \quad (10.8)$$

and $I_t^B \leq 0$, $I_t^C \geq 0$ $\quad \forall t < \tau$, *and the Basel III regulatory capital is*

$$K_t = K_t^{MR} + K_t^{CCR} + K_t^{CVA_{VAR}}. \tag{10.9}$$

Proof. The proof for Theorem 10.3 shares much of the same reasoning as the proof we provided for Theorem 9.5. However, in this instance we are required to use $C_t + I_t$ for each C_t shown in the proof for Theorem 9.5. Further to this, to accomodate for the addition of regulatory capital we need to redefine $d\tilde{X}_s^r$ as

$$d\tilde{X}_s^r = dX_s^r + (r - r_C)(C_s + I_t)ds + (r - r_V)V_s ds + (r - r_K)K_s ds. \tag{10.10}$$

By following the same steps as the proof provided for Theorem 9.5, together with the newly defined $d\tilde{X}_s^r$ and the use of $C_t + I_t$, we get the required result

$$\begin{aligned} A_t^{r_*} &= \mathbb{E}^{\mathbb{Q}}{}_t \left[A_T^{r_*}\right] \\ &= \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_0^T \mathbb{I}_{\{\tau > s\}} D_{0,s}^{r_*} d\tilde{X}_s^{r_*} + \int_0^T D_{0,s}^{r_*} \xi_s dP_s^{\tau}\right]. \end{aligned} \tag{10.11}$$

□

Recall the *Risk-free funded (base) PV* and the *Risky funded (margined) PV* given by Equation 9.47 and 9.48. We reintroduce *Risky funded (margined) PV* with IM and capital as

$$\begin{aligned} \hat{F}_t = \mathbb{E}^{\mathbb{Q}}{}_t &\left[\int_t^T \mathbb{I}_{\{\tau > s\}} D_{t,s}^{r_V} dX_s + \int_t^T \mathbb{I}_{\{\tau > s\}} D_{t,s}^{r_V} (r_V - r_C)(C_s + I_s) ds + \int_t^T \mathbb{I}_{\{\tau > s\}} D_{t,s}^{r_V} (r_V - r_K) K_s ds\right] \\ &+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \leq T\}} D_{t,\tau}^{r_V} (C_\tau + I_\tau) + \mathbb{I}_{\{\tau \leq T\}} D_{t,\tau}^{r_V} \hat{F}_\tau^R\right]. \end{aligned} \tag{10.12}$$

Note that that the treasury funding term $(r - r_*)\hat{V}_t$ has fallen away since we are using r_V to discount our cash flows. Considering the *Risk-free funded (base) PV* and the newly defined *Risky funded (margined) PV*, we provide the solution to the invariance principle with proof.

Theorem 10.4. *(Funded margined CVA)* $\hat{F}_t$ *is the price of a risky derivative for the trading desk with margining, CSA funding, and unsecured funding from treasury, IM and cost of capital, and is defined as*

$$\hat{F}_t = F_t - CVA_t - DVA_t - MVA_t - KVA_t, \tag{10.13}$$

$$CVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_C\}} D^{r_V}_{t,\tau} (1 - R_C)(F_\tau - C_\tau - I_\tau)^+ \right], \tag{10.14}$$

$$DVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_B\}} D^{r_V}_{t,\tau} (1 - R_B)(F_\tau - C_\tau - I_\tau)^- \right], \tag{10.15}$$

$$KVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_K) K_s ds \right], \tag{10.16}$$

$$MVA_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_C) I_s ds \right], \tag{10.17}$$

where F_t is the solution to the default-free funded base PV equation,

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D^{r_V}_{t,s} dX_s \right] + \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T D^{r_V}_{t,s} (r_V - r_C) C_s ds \right]. \tag{10.18}$$

Proof. The mechanics for this proof are identical to the proof provided for Theorem 9.6, except now we include the MVA and the KVA terms. We use the result of Equation 9.58 to substitute into Equation 10.12, providing us with

$$\begin{aligned}
\hat{F}_t = F_t &- \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} D^{r_V}_{t,\tau} F_\tau \right] \\
&- \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_C) I_s ds \right] \\
&- \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_K) K_s ds \right] \\
&+ \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} D^{r_V}_{t,\tau} (C_\tau + I_\tau) \mathbb{I}_{\{\tau \le T\}} D^{r_V}_{t,\tau} \hat{F}^R_\tau \right] \\
= F_t &- \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} D^{r_V}_{t,\tau} (F_\tau - C_\tau - I_\tau) \right] - \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} D^{r_V}_{t,\tau} \hat{F}^R_\tau \right] \\
&- \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_C) I_s ds \right] \\
&- \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_K) K_s ds \right],
\end{aligned} \tag{10.19}$$

where

$$\begin{aligned}
\hat{F}^R_\tau = \mathbb{I}_{\{\tau = \tau_C\}} &\left(R_C (\alpha_\tau)^+ + (\alpha_\tau)^- \right) \\
+ \mathbb{I}_{\{\tau = \tau_B\}} &\left((\alpha_\tau)^+ + R_B (\alpha_\tau)^- \right),
\end{aligned} \tag{10.20}$$

where $\alpha_\tau = F_\tau - (C_\tau + I_\tau)$. By following the same re-arranging and logic used in

Equation 9.61, we get

$$\begin{aligned}
\hat{F}_t &= F_t - \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_C\}} D^{r_V}_{t,\tau} \left((1-R_C)(\alpha_\tau)^+ \right) \right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}{}_t \left[\mathbb{I}_{\{\tau \le T\}} \mathbb{I}_{\{\tau = \tau_B\}} D^{r_V}_{t,\tau} \left((1-R_B)(\alpha_\tau)^- \right) \right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_C) I_s ds \right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}{}_t \left[\int_t^T \mathbb{I}_{\{\tau > s\}} D^{r_V}_{t,s} (r_V - r_K) K_s ds \right] \\
&= F_t - CVA_t - DVA_t - MVA_t - KVA_t.
\end{aligned} \tag{10.21}$$

□

Recall $K_t = K_t^{MR} + K_t^{CCR} + K_t^{CVA_{VAR}}$, the most prominenet of the three is KVA_t^{CCR} as the other two are dynamically hedged and are generally a smaller exposure to the bank. We refer to the regulatory capital formulas for IMM banks under Basel III provided in [Elouerkhaoui 2016b].

K^{MR}, as per Basel II and Basel 2.5, is the sum of VAR, SVAR, and the ratio of the Incremental Risk Charge (IRC) and the Comprehensive Risk Measure (CRM). The formula is given by

$$K_t^{MR} = VAR_t + SVAR_t + \frac{IRC_t}{CRM_t}, \tag{10.22}$$

where VAR is defined the 99the percentile loss over a ten-day period taken from the latest sample of data and SVAR is the 99th percentile loss over a ten-day period taken from a specific period of data where market stress was abnormally high. Both IRC and CRM were introduced in March 2008 by the Basel Committee to address the shortcomings of the current 99% ten-day VAR framework for the trading book. IRC was specifically introduced to handle default and migartion risks for non-securitised products. CRM is an incremental charge for correlation trading portfolio. Both are based on 99.9% loss over a one-year horizon. The dramatically increased horizon of one year in comparison to the ten-day horizon is to address the risks associated with illiquid products held on the trading book [1].

$K^{CVA_{VAR}}$, as per Basel III, is the product of the Risk Weighted Assets (RWA) and the capital ratio defined as $\alpha_{Capital}$ set to 8 or 10%. We define the RWA as

[1] The description on the IRC and CRM terms are taken from [BNP Paribas - GRM Risk IM]

$$RWA_t^{CVA_{VAR}} = 12.5 \times 3 \times (CVA_{VAR_t} + SCVA_{VAR_t}), \tag{10.23}$$

where $SCVA_{VAR}$ is stressed CVA_{VAR}. The sum of the two are grossed up by a factor of 3 and by a capital ratio of 12.5. $K_t^{CVA_{VAR}}$ is then given by

$$K_t^{CVA_{VAR}} = \alpha_{Capital} RWA_t^{CVA_{VAR}}, \tag{10.24}$$

where $\alpha_{Capital}$ can be set to 8% minimum. VAR and SVAR is approximated by the standard Gaussian approximation

$$VAR = \left[\sum_i RCS01_i \times \sigma_i^2 + 2 \sum_{i<j} RCS01_i RCS01_j \times \rho_{i,j}\sigma_i\sigma_j \right]^{\frac{1}{2}} \times \sqrt{10} \times \sqrt{\Phi^-1(0.99)}, \tag{10.25}$$

where

$$RCS01_t = 0.00001 \times t_i \times e^{-\frac{s_{t,i} \times t_i}{LGD_{mkt}}} \times \frac{EE_{t,i}D_{t,i-1} - EE_{t,i+1}D_{t,i+1}}{2}, \tag{10.26}$$

with $\sigma_{i,j}$ relating to each CDS's volatility; $\rho_{i,j}$ the correlation of the CDSs belonging to entity i and j and $D_{t,i}$ the risk-free discount factor in equation 10.26. Further detail on equation 10.26 can be found in the [Basel III] page 32.

Expected Exposure (EE) is defined as the expected value over the postive future values of the derivative at a point in time and can be defined as

$$EE = \mu\Phi(\frac{\mu}{\sigma}) + \sigma\varphi(\frac{\mu}{\sigma}), \tag{10.27}$$

as seen in [Gregory 2012, Appendix 8]. With μ is the drift of the derivative; σ the volatility; Φ the cumulative normal distirbution and ϕ the normal distribution function. Equation 10.25 is scaled to ten-day VAR and is considered for the 99th percentile of a normal distribution. Aggregation occurs at the counterparty level using Equation 10.25 for the IMM approach.

Lastly we consider the regulatory formula for K^{CCR}, as per Basel III. This can be considered as a more stringent capital value to $K^{CVA_{VAR}}$ as we derive our RWA_t^{CCR} value using the greater of stressed EAD and base EAD. By doing this we are able to capture a jump-to-default RWA. This is defined as

$$RWA_t^{CCR} = 1.06 \times 12.5 \times \omega \times (EAD_t - CVA_t), \tag{10.28}$$

with

$$K_t^{CCR} = \alpha_{Capital} RWA_t^{CCR}. \tag{10.29}$$

The CVA_t term for the capital charge is slightly different to the CVA formula provided by Equation 10.14, in that for the said term we do not conisder the suvival of bank B, only the probability of default for counterparty C is considered. This is evident in the CVA formula below provided by [Basel III], page 31:

$$\begin{aligned} CVA = (LGD_C)\,.\sum_{i=1}^{T} Max\left(0; exp\left(-\frac{s_{i-1}.t_{i-1}}{LGD_C}\right) - exp\left(-\frac{s_i.t_i}{LGD_C}\right)\right) \\ .\left(\frac{EE_{i-1}.D_{i-1} + EE_i.D_i}{2}\right). \end{aligned} \tag{10.30}$$

We get the value of EAD in Equation 10.28 by using the following IMM based computation

$$EAD_t = 1.4EEPE_t, \tag{10.31}$$

where Effective expected exposure (EEE) is a non-decreasing EE and effective Expected Positive Exposure (EPE) is the average EE over time. As per [Elouerkhaoui 2016b], Effective Expected Positive Exposure (EEPE) is defined as

$$\begin{aligned} EEPE_t &= max(EEPE_{Base_t}, EEPE_{Stressed_t}) \\ &= \sum_{t_k=0}^{1Y} EEE_t(t+t_k)\Delta t_k, \end{aligned} \tag{10.32}$$

with

$$EEE_t(t+t_k) = max(EEE_t(t+t_{k-1}), EPE_t(t+t_k)) \tag{10.33}$$

and we define EPE under $\mathbb{P}$ without wrong-way risk

$$EPE_t^{\mathbb{P}}(s) = \mathbb{E}_t^{\mathbb{P}}[(F_s - C_s - I_s^C)^+]. \tag{10.34}$$

Thus concluding the regulatory capital formulas that can be used by IMM approved banks. Pricing regulatory capital for a bank using the standard method would be a very different matter and is beyond the scope of this dissertation .

Chapter 11

The Poisson Process

"The probability of an event is the reason we have to believe that it has taken place, or that it will take place." - Siméon-Denis Poisson

A Poisson process calibrated to a firm's CDS spread is used to best fit the probabiltiy of default. We refer to [Gregory 2012, Appendix 10] to define the default function. The cumulative default probability is provided by the definition below.

Definition 11.1. *Define $G(t)$ to be a function that describes the default process for a specific entity*

$$G_t = 1 - e^{-\lambda t}, \tag{11.1}$$

with the instantaneous default probability given by

$$\frac{dG_t}{dt} = \lambda e^{-\lambda t}. \tag{11.2}$$

λ is the intensity of default otherwise known as the hazard rate for a specifc entity.

Recall that $e^{-\lambda t}$ provides the probability of survival for an entity, therefore we have $1-e^{-\lambda t}$ giving us the probability of default. It is useful to understand the relationship between the hazard rate and the CDS spread relating to a particular entity.

Definition 11.2. *Let $S(s) = 1 - e^{-\lambda s}$, then we define a continuous risky cash flow stream as*

$$\int_0^T D^r_{(s)} S_s ds. \tag{11.3}$$

Again referring to [Gregory 2012, Appendix 10], we see that the value of a CDS is given by

$$(1-R)\int_t^T D_s^r dG_s, \tag{11.4}$$

by substituting 10.2 into 10.4 we get

$$(1-R)\int_t^T D_s^r dG_s = (1-R)\lambda \int_0^T D_{(s)}^r S_s ds, \tag{11.5}$$

and by rearranging Equation 10.5, leads us to

$$(1-R)\lambda = \frac{(1-R)\int_t^T D_s^r dG_s}{\int_0^T D_{(s)}^r S_s ds}, \tag{11.6}$$

with

$$CDS_{Spread} = (1-R)\lambda. \tag{11.7}$$

This affirms the explanation provided in Chapter 4, regarding the windfall to bondholders. The protection seller, or the bondholder, needs to be compensated by an amount equivalent to the CDS_{Spread}. They forego an amount equivalent to the LGD to the protection buyer or bond issuer on default of the issuing or buying entity. For simplicity, and the purpose of this paper, we assume the the hazard rate to be deterministic despite the volatility associated with the said spread being very high at times. We will refer to this chapter later on in the dissertation as we include our XVAs.

Part IV

Pricing Model Implementation

Chapter 12

Pricing Commodities without XVA

"The price of a commodity will never go to zero. When you invest in commodities futures, you're not buying a piece of paper that says you own an intangible piece of company that can go bankrupt." - Jim Rogers

For each of the various chapters in this part of the dissertation, we will draw on the work of Youssef Elourkhaoui to provide results of derivative prices with the various adjustments. We will begin by pricing a commodity derivative with no credit risk assumed. Here we will explore the different methods and pricing algorithms used over the years to best describe the price of a derivative. We have chosen to demonstrate our pricing example through a European call option, however, the theory can be applied to puts too.

For this chapter, we draw on the work of [Schwartz and Smith 2000] to provide us with the mechanics to calculate the EAD for a commodity derivative. The authors develop a two-factor model that allows for mean reversion of short-term prices to their more stable long-term equilibrium prices, we will refer to this model as the SL model going forward. The rationale used for this model is that short-term spikes in commodity prices, although rather large, usually dissipate with time as prices converge to long-term break-even prices. Typically, a mean reversion model would be calibrated to the break-even price of a particular commodity, depending on the subjective analysis from a cost curve publisher. In this case, the authors use the long-term futures or forward prices to provide us with objective information on what the break-even can be. This is possible because producers usually use longer dated contracts to lock in prices at which they can sell, allowing the back-end of the curve to be a good proxy for consensus break-even levels. Although the back-end of the curve is far less volatile than the front-end, it does change more often than break-even cost curves get published, allowing the SL model to be calibrated more accurately.

The SL model combines two well known processes to describe the short-term variations and long-term dynamics of the commodity at hand. For the short-term process, [Schwartz and Smith 2000] use an OU [1] process. For simplicity, we will use the same notation from [Schwartz and Smith 2000]. The OU process models the difference between the spot price and the forward equilibrium price and is calibrated to capture short disruptions in the near dated future contracts. The GBM process models the long-term equilibrium prices and is calibrated using further dated future contracts to capture fundamental changes that persist in the market. We now define the risk-neutral process in [Schwartz and Smith 2000] as the model we will use to obtain our results.

Definition 12.1. *Let $ln(S_t) = \chi_t + \xi_t$, with the short-term process dynamics given by*

$$d\chi_t = -(\kappa\chi_t + \lambda_\chi)dt + \sigma_\chi dz_\chi, \tag{12.1}$$

and the long-term process dynamics given by

$$d\xi_t = (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dz_\xi. \tag{12.2}$$

We can see from the above definition that ξ_t reverts to $-\frac{\lambda_\chi}{\kappa}$, which is the ratio of the reduction in drift and the reversion speed of spot versus long-term equilibrium prices. The reduction in drift is needed to allow us to calibrate our model to risk-neutral market prices. Both dz_χ and dz_ξ are correlated increments of standard Brownian motion processes. The drift term, μ_ξ, in equation 12.2 is adjusted by λ_ξ in order to fit the drift term required to price the current market futures denoted by $F_{T,0}$. [Schwartz and Smith 2000] refers to two methods of calibration, the first they refer to is a Kalman filter procedure which fits the model's state variables and parameters to historical future's prices, see [Schwartz and Smith 2000] page 901. The second

[1] See [Héylette Geman 2005], page 64 for more detail.

method is more applicable for pricing as it is forward looking and suits the risk-neutral approach used throughout this dissertation. The latter approach implies the required state variables as well as the required parameters by using the model to match the observable futures and quoted volatility. In the risk-neutral framework we have the future's price equal to the expected future spot price, as defined below

$$\begin{aligned} ln\left(F_{T,0}\right) &= ln\left(\mathbb{E}^{\mathbb{Q}}\left[S_T\right]\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left[ln(S_T)\right] + \frac{1}{2}Var\left[ln(S_T)\right] \\ &= e^{-\kappa T}\chi_0 + \xi_0 + A(T). \end{aligned} \tag{12.3}$$

where

$$\begin{aligned} A(T) &= (\mu_\xi - \lambda_\xi)T - (1-e^{-\kappa T})\frac{\lambda_\chi}{\kappa} \\ &+ \frac{1}{2}\left((1-e^{-2\kappa T})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2(1-e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right). \end{aligned} \tag{12.4}$$

Equation 12.3 shows the relationship of the observable future's prices with the model's parameters and initial state variables. The above equation will be used to match our implied future prices to the observed future prices. To further our model's accuracy we can leverage off of observable volatility quotes. We define the volatility for $lnF_{T,0}$ as $\sigma_\phi(t,T)$ given below

$$\begin{aligned} \sigma_\phi^2(t,T) &= Var[ln(F_{T,t})] \\ &= e^{-2\kappa(T-t)}Var[\chi_t] + Var[\xi_t] + 2e^{-\kappa(T-t)}cov(\chi_t,\xi_t) \\ &= e^{-2\kappa(T-t)}(1-e^{-2\kappa t})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 t + 2e^{-\kappa(T-t)}(1-e^{-\kappa t})\frac{\rho_{\chi,\xi}\sigma_\chi\sigma_\xi}{\kappa}, \end{aligned} \tag{12.5}$$

where t is the maturity of the option and T is the maturity of the underlying future of the option, in most cases we assume that the option and the future maturity coincide.

Appendix A.1 shows the algorithm coded in *MATLAB*[2] to estimate the required parameters using the Kalman filter method. Appendix A.2 shows the implied parameters method discussed in detail in [Schwartz and Smith 2000]. Appendix A.3

[2]MATLAB is a technical computing language developed by the MathWorks, Inc. for programming algorithms; data visualisation and analysis; and numerical computation.

Parameters	Processes		
	SL	GBM	OU
S0	55.67	55.67	55.67
X	56.00	56.00	56.00
LambdaX	5.07%		
LambdaE	1.00%		
T	X	X	X
Sigmax	25.05%	26.11%	31%
Sigmae	15.00%		
k	0.490		0.286
Correlation	20%		
Mue	1.0%	0.28%	58.49

Table 12.1: Parameters used in the MC Simulation.

Processes / Tenor	Price US$ per bbl				
	1 Yr	2 Yr	3 Yr	4 Yr	5 Yr
OU	6.18	7.52	8.25	8.35	8.29
GBM	5.77	8.09	10.15	11.72	12.98
SL	6.26	7.40	7.84	8.33	8.77

Table 12.2: Summary of European call prices for three different processes using the MC Simulation.

and A.4 provides us with a visual comparison between the more complicated SL model to the more standard GBM and OU models. All three attempt to estimate the daily Inter-Continental Exchange (ICE) Brent crude front month price over time, it is clear from the graph that the SL model is most successful at fitting estimated to actual prices.

It is important for us to show the difference in derivative prices of a standard GBM model as opposed to a more fitting model. Table 12.2 reflects the difference in value of a simple European call and put option using the different models, the parameters used to price the options can be seen in Table 12.1[3].

As can be seen in Table 12.1, each process uses a different number of parameters. The GBM process being the most parsimonious and the SL model being the least. The number of parameters is directly proportionate to the accuracy of each model,

[3]The parameters used in this example were calibrated to the ICE Brent futures curve and ATM volatility curve as of the 26 February 2017

measured by the sum of squared errors each produces during calibration. Mue refers to the drift term for the GBM model and the long-term mean for the OU process. Recall, as per the literature for the SL model [Schwartz and Smith 2000], the SL model is a combination of the GBM process for long-term dynamics and OU for the short-term price dynamics.

Clearly, each process provides significantly different results with the difference increasing as we increase tenor. This can lead to vastly different results in our XVA pricing. If we assume GBM pricing, then clearly we must be comfortable in accepting the impact of a drift term, a well known characteristic of the GBM process.

Along with highlighting the differences in price from using different processes, we must also elaborate on the form of MC we are using. Table 12.2's results were produced using an MC process whilst applying the antithetic variate method to reduce variance and thus speed up convergence, [Glasserman 2003], page 205. 50,000 paths were used to generate the sample prices. The graph below shows the price of a one-year European call option with exactly the same parameters used to obtain Table 12.2's results. In this case we deploy the QMC technique to achieve our price. More, specifically we used a low-discrepancy method called the Halton's sequence to generate our random numbers. Low-discrepancy methods can improve the rate of convergence from $O\left(\frac{1}{\sqrt{n}}\right)$ to $O\left(\frac{1}{n}\right)$ where n is the number of paths or points generated, readers are referred to [Glasserman 2003] pg. 303 for further detail on Halton's sequence. The graph below clearly illustrates the superior rate of convergence and stability associated with the QMC in contrast to the standard MC.

The reader is referred to Appendix B.1 for the scenario generation code, these scenarios are then used to value the call option discussed above.

In the next chapter we explore what can be done to capture non-normal distribution behaviour within our derivative price.

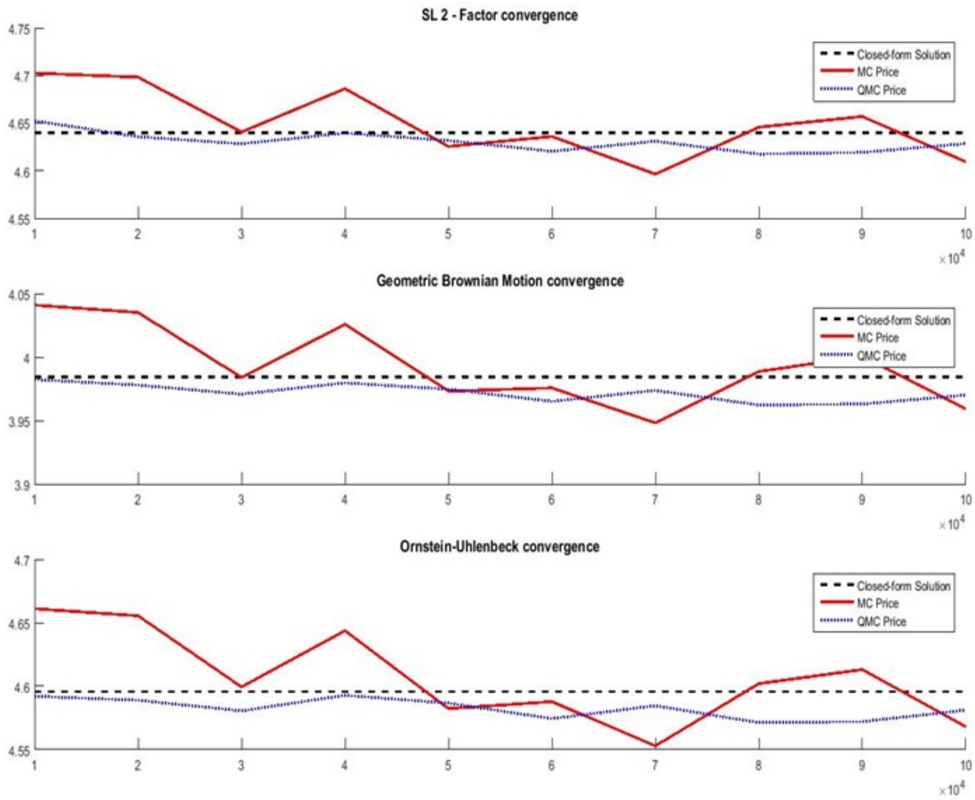

Figure 12.1: Comparison of MC vs QMC techniques when pricing a ICE Brent call option using T = 0.5 and the same parameters as listed in Table 12.1.

Chapter 13

Accounting for a Non-Normal World

"If you hear a "prominent" economist using the word 'equilibrium', or 'normal distribution', do not argue with him; just ignore him, or try to put a rat down his shirt."
- Nassim Nicholas Taleb

Throughout this chapter we refer to [Aboura and Maillard 2014], a unique finding in the field of pricing derivatives outside of the normal distribution setup. Traditionally most practitioners would use stochastic volatility models or jump-diffusion models to include the impact of heavy tails. In our opinion, and as per the findings of [Aboura and Maillard 2014], the CF approach is just as effective and is a far more parsimonious approach than the latter, drastically reducing model risk. Our pricing example in the previous chapter only considered an ATM volatility curve for the generated scenarios, in this chapter we consider what impact a volatility skew can have on pricing. Readers are refered to [Gatheral 2006] for detail and volatility models, and calibration to implied volatility skews.

In order to appreciate the work of [Aboura and Maillard 2014], we must understand that the market implied volatility surface is nothing more than a forward looking guess of where the market sees At the Money (ATM) volatility levels and Out the Money (OTM) levels. The ATM volatility levels tell us what the second moment of the distribution is, whilst the OTM volatility levels inform us as to what the third and fourth moments of the distribution are. The CF allows us to effectively capture the impact of skewness and excess kurtosis in our option price allowing for heavy tails. Naturally, we would need to estimate what these moments are if we are to use them in the pricing framework, this can either be done using historical data, under $\mathbb{P}$, or they can be estimated under $\mathbb{Q}$ using quoted volatility levels, where available.

Definition 13.1. *Let z be a normally distributed variable with mean and variance as N(0,1). We then define the Cornish-Fisher transformation polynomial as* $\boldsymbol{Z}$*,*

$$\boldsymbol{Z} = z + (z^2 - 1)\frac{s}{6} + (z^3 - 3z)\frac{k}{24} - (2z^3 - 5z)\frac{s^2}{36}, \tag{13.1}$$

with s representing the skewness measure and k the excess kurtosis measure , taken either under $\mathbb{Q}$ *or* $\mathbb{P}$ *.*

If s and k were set to zero, then the **Z** would reduce to the normally distributed variable z. The reader is referred to [Aboura and Maillard 2014] pages 8 and 9 to view how the volatility surface is manipulated as s and k vary. The performance of the CF approach over time is not in the scope of this paper, but can be found in [Aboura and Maillard 2014]. We use the CF within our QMC and MC framework to reflect how s and k can impact option prices; the code can be found in Appendix B.1. We calibrate using a market implied volatility surface under $\mathbb{Q}$ measure.

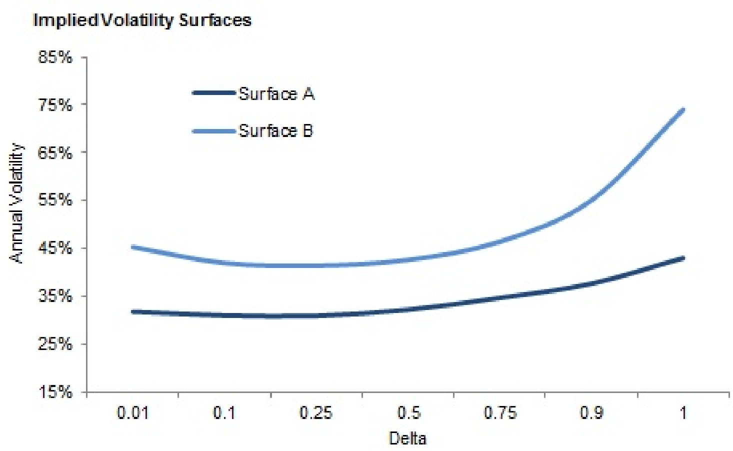

Figure 13.1: Surface A implies k=0.57 and s=0.21; surface B implies k=4.2 and s=0.96

Surface B demonstrates a volatility surface in a distressed market. The distribution associated with surface B can be described as a leptokurtic distribution skewed to the right. The impact of s and k on pricing can be seen below:

Strike Price	Model	Cornish-Fisher: k=0.21 s=0.57		Cornish-Fisher: k=4.2 s=0.96		Standard Normal Distribution		
USD per bbl		MC	QMC	MC	QMC	MC	QMC	Closed-Form
X=56	GBM	3.9821	3.9501	4.2391	4.1720	4.0258	3.9747	3.9842
	OU	4.5914	4.5604	4.8931	4.8209	4.6446	4.5867	4.5959
	SL	4.6093	4.6567	4.8510	4.9141	4.6827	4.6317	4.6396
X=57	GBM	3.5150	3.5197	3.8273	3.7453	3.5840	3.5401	3.5496
	OU	4.1135	4.1241	4.4760	4.3888	4.1957	4.1462	4.1557
	SL	4.1321	4.2079	4.4526	4.4579	4.2387	4.1871	4.1953
X=58	GBM	3.1721	3.1261	3.4389	3.3539	3.1449	3.1420	3.1521
	OU	3.7722	3.7211	4.0831	3.9884	3.7399	3.7382	3.7486
	SL	3.7704	3.7931	4.0023	4.0035	3.7758	3.7752	3.7844
X=59	GBM	2.8960	2.7681	3.0308	2.9964	2.7694	2.7790	2.7901
	OU	3.4989	3.3498	3.6560	3.6186	3.3500	3.3617	3.3735
	SL	3.4653	3.4103	3.5641	3.6436	3.3853	3.3951	3.4056
X=60	GBM	2.4936	2.4428	2.7592	2.6706	2.4228	2.4500	2.4619
	OU	3.0647	3.0087	3.3766	3.2773	2.9855	3.0165	3.0290
	SL	3.0552	3.0582	3.2484	3.2819	3.0134	3.0465	3.0577

Table 13.1: Summary of European call prices using MC and QMC techniques across different strikes and changes in k and s where indicated. Parameters from Table 12.1 were used with T=0.5.

In the instance where k becomes large, we notice prices begin to tend away from standard Black-Scholes formula pricing, this is clearly explained by the change in distribution shape. When obtaining Exposure at Default (EAD) values, these factors become important in order to capture the risk associated with a non-normal asset class. Readers are urged to look at [Aboura and Maillard 2014] pages 5 and 6, as wel as pages 12 to 15 to further understand the CF shortfalls and specifically with its performance as kurtosis becomes larger and skewness moves significantly away from zero.

Chapter 14

Adding CVA, DVA, COLVA, and FVA to the Batch

"Valuation is an art not a science." - Mohandas Pai

Using the framework provided in Chapters 12 and 13, we implement the theory provided in Chapters 9, 10 and 11 to adjust our pre-GFC pricing in Table 13.1 to correctly capture typical risks that concern banks and regulators today.

Using the notation provided in Chapter 9, we refer to the equation in Theorem 9.6 to price our risky derivative, $\hat{F}_t$. To maintain some form of simplicity, we will only use the QMC framework to price our risky price on the premise that it provides prices closest to the analytic prices shown in Chapter 13. Recall there are two portions of the equation we need to solve for, using QMC; one is the default-free portion,

$$F_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T D^r_{t,s} dX_s\right] + \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T D^r_{t,s}(r - r_C)C_s ds\right] + \mathbb{E}^{\mathbb{Q}}{}_t\left[\int_t^T D^r_{t,s}(r - r_V)V_s ds\right].$$

The first portion of the equation will be the MTM of the derivative considering the cash flows of the derivative itself and the funding component only. The second and third terms are the COLVA and the FVA respectively.
The risky portion of our derivative is given by

$$CVA_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau \leq T\}}\mathbb{I}_{\{\tau = \tau_C\}} D^r_{t,\tau}(1 - R_C)(F_\tau - C_\tau)^+\right],$$

$$DVA_t = \mathbb{E}^{\mathbb{Q}}{}_t\left[\mathbb{I}_{\{\tau \leq T\}}\mathbb{I}_{\{\tau = \tau_B\}} D^r_{t,\tau}(1 - R_B)(F_\tau - C_\tau)^-\right].$$

The two portions combined provides us with $\hat{F}_t = F_t - CVA_t - DVA_t$, with the COLVA and the FVA included in F_t. Note, we price in accordance to the assumptions made on page 69 of this dissertation.
To calculate our PD, we refer to the theory provided in Chapter 11. We use the cumulative distribution function for the exponential distribution,

$$PD_t = 1 - exp(-\lambda t), \tag{14.1}$$

to generate a PD. One could create default simulations by making t the subject of the formula in the above equation, this leaves us with

$$\tau = \frac{-ln(1 - PD_t)}{\lambda}, \tag{14.2}$$

where we let $U = (1 - PD_t)$ with U being a random number $\in [0, 1]$. For the results of this paper, this is not required.

Some futher assumptions we make in our pricing:

1. Bank B and counterparty C do not pay margin to each other;
2. Bank B does pay and receive margin to and from the interbank market, a by-product of trading under CSA for a hedge with another bank;
3. The CSA is a zero threshold CSA with cash being the only source of collateral;
4. Collateral cash and cash generated from the trade itself are denominated in the same currency; and
5. Bank B acquires cash from treasury to fund the margin payments.

Based on the above assumptions, we deviate slightly from [Elouerkhaoui 2016a], not from the framework itself, but from the rationale of the various cash flows. In the practical world, if a desk does a corporate hedge, typically no margin is paid between the bank and the counterparty. However, the bank will always need to hedge their market risk and will do so either on an exchange or with an interbank counterparty; in both cases margin will be posted or received at some point in time.

Therefore there is hardly ever a case where we would have an uncollateralised trade that does not result in indirect margin payments. Uncollateralised refers to an uncollateralised trade by a client with a consequent collateralised hedge under a zero threshold CSA. Thus even in the uncollateralised example, COLVA and FVA will be required. In the instance where we have a collateralised trade between the bank and the corporate client, then most of the XVAs would reduce drastically. Refer to Figure 3.4 to better understand the decision tree one must consider.

Let us consider the case of an uncollateralised client trade with a collateralised hedge. In this instance, we pay and receive margin with the bank providing a hedge, therefore all COLVA and FVA costs associated with the trade will be incurred between bank B and the hedge bank. The corporate, counterparty C, does not exchange cash flows with bank B before maturity. The only change we make to the framework [Elouerkhaoui 2016a] is to state $V_t = -C_t$ and $V_t = F_{t-lag}$. This then allows us to pay margin to the bank providing a CSA market risk hedge and earn r_c, whilst paying r_f to the treasury within the [Elouerkhaoui 2016a] setup.

Strike Price	Model	Standard Normal Distribution - Call Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	3.8755	3.7884	0.0256	-	(0.0640)	-	-	(0.0487)
	OU	4.4824	4.3787	0.0305	-	(0.0762)	-	-	(0.0580)
	SL	4.5127	4.4090	0.0305	-	(0.0762)	-	-	(0.0580)
X=57	GBM	3.4552	3.3822	0.0215	-	(0.0537)	-	-	(0.0408)
	OU	4.0525	3.9632	0.0262	-	(0.0656)	-	-	(0.0499)
	SL	4.0792	3.9901	0.0262	-	(0.0655)	-	-	(0.0498)
X=58	GBM	3.0635	3.0024	0.0180	-	(0.0450)	-	-	(0.0341)
	OU	3.6544	3.5778	0.0226	-	(0.0564)	-	-	(0.0428)
	SL	3.6770	3.6005	0.0225	-	(0.0563)	-	-	(0.0427)
X=59	GBM	2.7096	2.6585	0.0150	-	(0.0376)	-	-	(0.0285)
	OU	3.2869	3.2212	0.0194	-	(0.0484)	-	-	(0.0367)
	SL	3.3071	3.2415	0.0193	-	(0.0483)	-	-	(0.0366)
X=60	GBM	2.3887	2.3461	0.0125	-	(0.0314)	-	-	(0.0237)
	OU	2.9499	2.8935	0.0166	-	(0.0415)	-	-	(0.0315)
	SL	2.9671	2.9110	0.0166	-	(0.0414)	-	-	(0.0313)

Table 14.1: Summary of European call prices across different strikes. A normal distribution was assumed for pricing with $R_C = 0.3$; $R_B = 0.4$; $CDS_B = 400$ bps; $CDS_C = 700$ bps; $r_f = 0.05$; $r_c = 0.01$ and $r_v = 0.1$. Bank B sells to counterparty C.

Table 14.1 and Table 14.2 contain the prices of both the risky and risk-free ATM

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Call Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	4.0678	3.9771	0.0267	-	(0.0667)	-	-	(0.0507)
	OU	4.7111	4.6031	0.0318	-	(0.0794)	-	-	(0.0604)
	SL	4.7953	4.6855	0.0323	-	(0.0807)	-	-	(0.0614)
X=57	GBM	3.6517	3.5745	0.0227	-	(0.0568)	-	-	(0.0431)
	OU	4.2895	4.1954	0.0277	-	(0.0692)	-	-	(0.0526)
	SL	4.3501	4.2554	0.0279	-	(0.0697)	-	-	(0.0529)
X=58	GBM	3.2700	3.2047	0.0193	-	(0.0480)	-	-	(0.0366)
	OU	3.8987	3.8167	0.0241	-	(0.0603)	-	-	(0.0458)
	SL	3.9371	3.8556	0.0240	-	(0.0600)	-	-	(0.0455)
X=59	GBM	2.9216	2.8659	0.0164	-	(0.0410)	-	-	(0.0311)
	OU	3.5378	3.4663	0.0210	-	(0.0526)	-	-	(0.0399)
	SL	3.5547	3.4846	0.0206	-	(0.0516)	-	-	(0.0391)
X=60	GBM	2.6038	2.5564	0.0139	-	(0.0349)	-	-	(0.0264)
	OU	3.2046	3.1424	0.0183	-	(0.0458)	-	-	(0.0347)
	SL	3.2019	3.1417	0.0177	-	(0.0443)	-	-	(0.0336)

Table 14.2: Summary of European call prices across different strikes using the CF transformation with bank B selling to counterparty C.

call options. The tables then split the prices per model used to generate the values. The aforementioned prices are to be read as USD per Blue Barrel (BBL) in the first two columns of the table with the subsequent columns showing the various XVAs that get added to the risky price in column two. Notice Table 14.2 shows a price considerably higher than Table 14.1, this is to be expected given the larger tails associated with a leptokurtic and positively skewed distribution. This behaviour follows through in the XVA calculations, the reasoning follows the same argument we use for pricing of risk-free options using the CF trasnformation. In both tables, the seller of the call, bank B, receives premium upfront from counterparty C. Bank B will immediatly buy back the option exposure from the hedge bank, thus eliminating any market risk associated with the trade. The hedge, as mentioned earlier is done under a CSA agreement, implying bank B will receive collateral as soon as the call option is in ITM for counterparty C and consequently ITM for bank B on the hedge. Bank B receives free funding on which he will earn r_v as we assume rehypothecation of the collateral. Bank B will need to reimburse the hedge bank using r_c for the collateral they have posted to bank B. These cash flows generate a funding benefit, FBA, for bank B and a collateral cost, Collateral Cost Adjustment (COLCA).

Both tables only show a DVA term, this is because of the asymmetric nature of the option. If bank B is the seller, it will only ever be exposed to a DVA adjustment, provided the buyer posts collateral upfront, which in our example, is the case.

Consider Tables 14.3 and 14.4 below, where we make bank B the buyer of the call option. Again we assume bank B hedges by selling the option to the interbank market under CSA. Bank B is now faced with managing a CVA term, as well as a FCA and Collateral Benefit Adjustment (COLBA) term for which it should transfer price to counterparty C.

Strike Price	Model	Standard Normal Distribution - Call Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	3.8755	3.9992	-	(0.0256)	-	0.0640	0.0853	-
	OU	4.4824	4.6296	-	(0.0305)	-	0.0762	0.1015	-
	SL	4.5127	4.6598	-	(0.0305)	-	0.0762	0.1014	-
X=57	GBM	3.4552	3.5588	-	(0.0215)	-	0.0537	0.0714	-
	OU	4.0525	4.1792	-	(0.0262)	-	0.0656	0.0873	-
	SL	4.0792	4.2057	-	(0.0262)	-	0.0655	0.0872	-
X=58	GBM	3.0635	3.1502	-	(0.0180)	-	0.0450	0.0597	-
	OU	3.6544	3.7631	-	(0.0226)	-	0.0564	0.0749	-
	SL	3.6770	3.7856	-	(0.0225)	-	0.0563	0.0748	-
X=59	GBM	2.7096	2.7820	-	(0.0150)	-	0.0376	0.0498	-
	OU	3.2869	3.3802	-	(0.0194)	-	0.0484	0.0643	-
	SL	3.3071	3.4002	-	(0.0193)	-	0.0483	0.0641	-
X=60	GBM	2.3887	2.4491	-	(0.0125)	-	0.0314	0.0415	-
	OU	2.9499	3.0299	-	(0.0166)	-	0.0415	0.0551	-
	SL	2.9671	3.0468	-	(0.0166)	-	0.0414	0.0549	-

Table 14.3: Summary of European call prices across different strikes using a normal distribution with bank B buying from counterparty C.

The large differences between each of the model prices, for both risk-free and risky can be explained by the model calibration. The GBM model has a drift very close to 0, for the pricing of the call option, this implies that the spot rate does not drift away from the strike as we move through time. For both the SL and the OU model, their mean reversion characteristic pulls the spot rate higher as time passes, causing the call to be further ITM as time passes. The ICE Brent curve was in contango at the time of calibration, explaining why the spot price moves higher over time in our simulations.

Let us consider the above exercises for an uncollateralised put. For this exercise the key component that requires changing in the pricing is the exposure, in this instance

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Call Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	4.0678	4.1958	-	(0.0267)	-	0.0667	0.0880	-
	OU	4.7111	4.8644	-	(0.0318)	-	0.0794	0.1057	-
	SL	4.7953	4.9512	-	(0.0323)	-	0.0807	0.1075	-
X=57	GBM	3.6517	3.7613	-	(0.0227)	-	0.0568	0.0755	-
	OU	4.2895	4.4231	-	(0.0277)	-	0.0692	0.0921	-
	SL	4.3501	4.4845	-	(0.0279)	-	0.0697	0.0926	-
X=58	GBM	3.2700	3.3628	-	(0.0193)	-	0.0480	0.0641	-
	OU	3.8987	4.0151	-	(0.0241)	-	0.0603	0.0802	-
	SL	3.9371	4.0528	-	(0.0240)	-	0.0600	0.0797	-
X=59	GBM	2.9216	3.0006	-	(0.0164)	-	0.0410	0.0544	-
	OU	3.5378	3.6392	-	(0.0210)	-	0.0526	0.0698	-
	SL	3.5547	3.6542	-	(0.0206)	-	0.0516	0.0685	-
X=60	GBM	2.6038	2.6710	-	(0.0139)	-	0.0349	0.0462	-
	OU	3.2046	3.2928	-	(0.0183)	-	0.0458	0.0607	-
	SL	3.2019	3.2872	-	(0.0177)	-	0.0443	0.0587	-

Table 14.4: Summary of European call prices across different strikes using the CF transformation with bank B buying from counterparty C, s=0.97 and k=4.2.

the EAD is calculated using the Negative Expected Exposure (NEE) as an input as opposed to EE. We start with the scenario where bank B is a seller of the put option to counterparty C, again we assume bank B buys the hedge from another bank under CSA. We use the same parameters used to price the calls:

Strike Price	Model	Standard Normal Distribution - Put Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	4.1253	4.0282	0.0285	-	(0.0713)	-	-	(0.0543)
	OU	4.5237	4.4143	0.0321	-	(0.0803)	-	-	(0.0612)
	SL	4.4531	4.3457	0.0316	-	(0.0789)	-	-	(0.0601)
X=55	GBM	3.6110	3.5312	0.0235	-	(0.0587)	-	-	(0.0446)
	OU	4.0113	3.9199	0.0271	-	(0.0670)	-	-	(0.0515)
	SL	3.9447	3.8544	0.0265	-	(0.0663)	-	-	(0.0505)
X=54	GBM	3.1354	3.0703	0.0192	-	(0.0479)	-	-	(0.0364)
	OU	3.5327	3.4557	0.0226	-	(0.0566)	-	-	(0.0430)
	SL	3.4705	3.4151	0.0421	-	(0.0554)	-	-	(0.0421)
X=53	GBM	2.6989	2.6464	0.0155	-	(0.0387)	-	-	(0.0293)
	OU	3.0885	3.0246	0.0188	-	(0.0470)	-	-	(0.0357)
	SL	3.0309	2.9685	0.0183	-	(0.0459)	-	-	(0.0348)
X=52	GBM	2.3005	2.2585	0.0124	-	(0.0310)	-	-	(0.0234)
	OU	2.6790	2.6265	0.0155	-	(0.0387)	-	-	(0.0293)
	SL	2.6256	2.5746	0.0151	-	(0.0376)	-	-	(0.0285)

Table 14.5: Summary of European put prices across different strikes using a normal distribution with bank B selling to counterparty C.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Put Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	4.3291	4.2268	0.0300	-	(0.0751)	-	-	(0.0572)
	OU	4.7446	4.6294	0.0338	-	(0.0846)	-	-	(0.0644)
	SL	4.5309	4.4224	0.0319	-	(0.0797)	-	-	(0.0607)
X=55	GBM	3.8057	3.7213	0.0248	-	(0.0620)	-	-	(0.0472)
	OU	4.2220	4.1247	0.0286	-	(0.0715)	-	-	(0.0544)
	SL	4.0338	3.9417	0.0271	-	(0.0677)	-	-	(0.0515)
X=54	GBM	3.3190	3.2499	0.0203	-	(0.0508)	-	-	(0.0386)
	OU	3.3712	3.2896	0.0240	-	(0.0600)	-	-	(0.0456)
	SL	3.5697	3.4920	0.0228	-	(0.0571)	-	-	(0.0434)
X=53	GBM	2.8703	2.8135	0.0165	-	(0.0412)	-	-	(0.0321)
	OU	3.2762	3.2084	0.0200	-	(0.0499)	-	-	(0.0379)
	SL	3.1385	3.0734	0.0192	-	(0.0479)	-	-	(0.0364)
X=52	GBM	2.4597	2.4148	0.0132	-	(0.0331)	-	-	(0.0250)
	OU	2.8550	2.7991	0.0165	-	(0.0412)	-	-	(0.0312)
	SL	2.7404	2.6862	0.0160	-	(0.0399)	-	-	(0.0303)

Table 14.6: Summary of European put prices across different strikes using the CF transformation with bank B selling to counterparty C, s=0.97 and k=4.2.

The most noticeable change in Table 14.5 and 14.6, in contrast to 14.1 and 14.2, is that the prices of calls are more expensive for similar levels of moneyness[1]. This can be explained by the slight positive drift inherent in the GBM model and the higher than spot reversion level for both the SL and OU models. However, when using the CF, both puts and calls become more expensive as a result of excess kurtosis, yet some of the higher prices on the call is negated by the positive skewness implied by the Brent volatility surface. With bank B, as the buyer of the put options, the pricing changes considerably. Refer to Tables 14.7 and 14.8. Both tables show pricing for put options in USD per bbl per model used, the said prices can be seen in columns one and two, the risky and the risk-free price respectively. As with the earlier tables, the subsequent columns post column two display the various XVAs to be added to the risky price.

[1]Term used to describe the distance of strikes to the forward

Strike Price	Model	Standard Normal Distribution - Put Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	4.1253	4.2631	-	(0.0285)	-	0.0713	0.0950	-
	OU	4.5237	4.6790	-	(0.0321)	-	0.0803	0.1071	-
	SL	4.4531	4.6055	-	(0.0316)	-	0.0789	0.1051	-
X=55	GBM	3.6110	3.7243	-	(0.0235)	-	0.0587	0.0781	-
	OU	4.0113	4.1413	-	(0.0271)	-	0.0670	0.0901	-
	SL	3.9447	4.0728	-	(0.0265)	-	0.0663	0.0883	-
X=54	GBM	3.1354	3.2277	-	(0.0192)	-	0.0479	0.0636	-
	OU	3.5327	3.6420	-	(0.0226)	-	0.0566	0.0753	-
	SL	3.4705	3.5574	-	(0.0421)	-	0.0554	0.0736	-
X=53	GBM	2.6989	2.7734	-	(0.0155)	-	0.0387	0.0513	-
	OU	3.0885	3.1791	-	(0.0188)	-	0.0470	0.0624	-
	SL	3.0309	3.1194	-	(0.0183)	-	0.0459	0.0609	-
X=52	GBM	2.3005	2.3601	-	(0.0124)	-	0.0310	0.0410	-
	OU	2.6790	2.7535	-	(0.0155)	-	0.0387	0.0513	-
	SL	2.6256	2.6980	-	(0.0151)	-	0.0376	0.0499	-

Table 14.7: Summary of European put prices across different strikes using a normal distribution with bank B buying from counterparty C.

Strike Price	Model	djusted Distribution - Put Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	4.3291	4.4742	-	(0.0300)	-	0.0751	0.1000	-
	OU	4.7446	4.9081	-	(0.0338)	-	0.0846	0.1127	-
	SL	4.5309	4.6849	-	(0.0319)	-	0.0797	0.1062	-
X=55	GBM	3.8057	3.9254	-	(0.0248)	-	0.0620	0.0825	-
	OU	4.2220	4.3601	-	(0.0286)	-	0.0715	0.0952	-
	SL	4.0338	4.1645	-	(0.0271)	-	0.0677	0.0901	-
X=54	GBM	3.3190	3.4170	-	(0.0203)	-	0.0508	0.0675	-
	OU	3.3712	3.4870	-	(0.0240)	-	0.0600	0.0798	-
	SL	3.5697	3.6800	-	(0.0228)	-	0.0571	0.0760	-
X=53	GBM	2.8703	2.9497	-	(0.0165)	-	0.0412	0.0547	-
	OU	3.2762	3.3724	-	(0.0200)	-	0.0499	0.0663	-
	SL	3.1385	3.2308	-	(0.0192)	-	0.0479	0.0636	-
X=52	GBM	2.4597	2.5234	-	(0.0132)	-	0.0331	0.0438	-
	OU	2.8550	2.9344	-	(0.0165)	-	0.0412	0.0547	-
	SL	2.7404	2.8172	-	(0.0160)	-	0.0399	0.0529	-

Table 14.8: Summary of European put prices across different strikes using the CF transformation with bank B buying from counterparty C, s=0.97 and k=4.2.

Once again, the CVA factor comes into play for bank B when they are the buyer of an uncollateralised option. The CVA is always larger than the DVA portion of the price becase of the poor credit quality associated with counterparty C relative to bank B. It can become a little overwhelming thinking about what to include and what not to include when pricing an uncollateralised trade, Table 14.9 gives some guidance on what should be included.

Transaction Direction	Costing Summary for Options - Bank B's perspective					
	CVA	DVA	FCA	FBA	ColCA	ColBA
Bank B sells to Counterparty C	-	Yes	-	Yes	Yes	-
Bank B buys from Counterparty C	Yes	-	Yes	-	-	Yes

Table 14.9: Summary of XVAs to apply when pricing uncollateralised options.

Lastly, we look at the pricing of a forward contract. For a forward, both the PEE and the NEE come into play as the exposure is close to symmetrical in comparsion with an option. The tenor for the forward is T=0.5, as per our previous exercises, bank B hedges in the interbank market under CSA:

Strike Price	Model	Standard Normal Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	56.0506	0.0256	(0.0285)	(0.0641)	0.0713	0.0950	(0.0487)
	OU	56.0000	56.0519	0.0304	(0.0321)	(0.0760)	0.0803	0.1071	(0.0578)
	SL	56.0000	56.0488	0.0305	(0.0316)	(0.0762)	0.0789	0.1051	(0.0579)

Table 14.10: Brent crude oil forward price with XVAs calculated using a normal distribution. Bank B sells to counterparty C.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	56.0466	0.0272	(0.0288)	(0.0680)	0.0720	0.0959	(0.0517)
	OU	56.0000	56.0471	0.0322	(0.0324)	(0.0806)	0.0811	0.1081	(0.0613)
	SL	56.0000	56.0522	0.0317	(0.0332)	(0.0793)	0.0829	0.1105	(0.0604)

Table 14.11: Brent crude oil forward price with XVAs calculated using the CF transformation with bank B sellng to counterparty C, s=0.97 and k=4.2.

When we assume bank B to be the buyer of the forward, then the pricing changes slightly.

Strike Price	Model	Standard Normal Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9733	(0.0285)	0.0256	0.0713	(0.0641)	(0.0853)	0.0543
	OU	56.0000	55.9626	(0.0321)	0.0304	0.0803	(0.0760)	(0.1012)	0.0612
	SL	56.0000	55.9603	(0.0316)	0.0305	0.0789	(0.0762)	(0.1014)	0.0601

Table 14.12: Brent crude oil forward price with XVAs calculated using a normal distribution. Bank B buys from counterparty C.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9667	(0.0288)	0.0272	0.0720	(0.0680)	(0.0905)	0.0548
	OU	56.0000	55.9547	(0.0324)	0.0322	0.0811	(0.0806)	(0.1074)	0.0618
	SL	56.0000	55.9596	(0.0332)	0.0317	0.0829	(0.0793)	(0.1057)	0.0632

Table 14.13: Brent crude oil forward price with XVAs calculated using the CF transformation with bank B buying from counterparty C, s=0.97 and k=4.2.

The change in pricing from Tables 14.10 and 14.11 to 14.12 and 14.13, depends on the difference in NEE and PEE, as discussed earlier. We have only priced for one strike as this is usually the only choice made when a trade is concluded. Varying strikes only matter in the situation of closeouts when a forward trade can be ITM or OTM.

The above tables only consider the case when a trade between bank B and the counterparty C is uncollateralised. We now consider the case when all aspects of the trade are collateralised. In this situation, we will see the CVA and DVA terms diminish substantially, depending on the lag between collateral payments and MTM. There is no funding implication in the collateralised case as the collateral paid to or received from counterparty C funds the interbank hedge, and vice versa. We therefore assume that both trades between counterparty C and the interbank market are under the same CSA terms, we also assume no credit risk on the margin paid and received on the hedge and the client trade. The below scenarios assume a one day lag between reported MTM and collateral posted.

Strike Price	Model	Standard Normal Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	56.0023	-	-	-	-	0.0054	(0.0031)
	OU	56.0000	56.0025	-	-	-	-	0.0063	(0.0038)
	SL	56.0000	56.0024	-	-	-	-	0.0062	(0.0038)

Table 14.14: Collateralised Brent crude oil forward price with XVAs calculated using a normal distribution. Bank B sells to counterparty C.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	56.0019	-	-	-	-	0.0047	(0.0028)
	OU	56.0000	56.0021	-	-	-	-	0.0055	(0.0034)
	SL	56.0000	56.0021	-	-	-	-	0.0055	(0.0034)

Table 14.15: Collateralised Brent crude oil forward price with XVAs calculated using the CF transformation with bank B selling to counterparty C, s=0.97 and k=4.2.

Strike Price	Model	Standard Normal Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9977	-	-	-	-	(0.0054)	0.0031
	OU	56.0000	55.9969	-	-	-	-	(0.0067)	0.0036
	SL	56.0000	55.9970	-	-	-	-	(0.0066)	0.0036

Table 14.16: Collateralised Brent crude oil forward price with XVAs calculated using a normal distribution. Bank B buys from counterparty C.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9978	-	-	-	-	(0.0049)	0.0027
	OU	56.0000	55.9971	-	-	-	-	(0.0060)	0.0031
	SL	56.0000	55.9973	-	-	-	-	(0.0059)	0.0032

Table 14.17: Collateralised Brent crude oil forward price with XVAs calculated using the CF transformation with bank B buying from counterparty C, s=0.97 and k=4.2.

Notice how the CVA and the DVA reduce to insignificant quantities. The FVA and COLVA components do not exist as the combination of the client trade and the hedge create a self-financing portfolio. Notice how the CVA and DVA terms start to increase as we increase the time lag on collateral payment.

Strike Price	Model	Standard Normal Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9918	-	-	-	-	(0.0194)	0.0112
	OU	56.0000	55.9894	-	-	-	-	(0.0236)	0.0130
	SL	56.0000	55.9893	-	-	-	-	(0.0235)	0.0128

Table 14.18: Collateralised Brent crude oil forward price with XVAs calculated using a normal distribution. Bank B buys from counterparty C, collateral payment lag is set to 10 days.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9917	-	-	-	-	(0.0198)	0.0115
	OU	56.0000	55.9890	-	-	-	-	(0.0243)	0.0133
	SL	56.0000	55.9886	-	-	-	-	(0.0242)	0.0128

Table 14.19: Collateralised Brent crude oil forward price with XVAs calculated using the CF transformation with bank B buying from counterparty C, s=0.97 and k=4.2, collateral payment lag is set to 10 days.

Strike Price	Model	Standard Normal Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9729	-	-	-	-	(0.0704)	0.0433
	OU	56.0000	55.9650	-	-	-	-	(0.0839)	0.0489
	SL	56.0000	55.9642	-	-	-	-	(0.0839)	0.0481

Table 14.20: Collateralised Brent crude oil forward price with XVAs calculated using a normal distribution. Bank B buys from counterparty C, collateral payment lag is set to 100 days.

Strike Price	Model	Cornish-Fisher Adjusted Distribution - Forward Prices							
USD per bbl		Risk-Free Price	Risky Funded Price	ColCA	ColBA	FBA	FCA	CVA	DVA
X=56	GBM	56.0000	55.9721	-	-	-	-	(0.0734)	0.0455
	OU	56.0000	55.9638	-	-	-	-	(0.0876)	0.0514
	SL	56.0000	55.9599	-	-	-	-	(0.0888)	0.0487

Table 14.21: Collateralised Brent crude oil forward price with XVAs calculated using the CF transformation with bank B buying from counterparty C, s=0.97 and k=4.2, collateral payment lag is set to 100 days.

To summarise the results of Chapter 14, the inclusion of collateral greatly reduces the XVA costs associated with a trade. Provided both the counterparty and the hedge provider are placing collateral on the same terms. We also see that the CF transformation makes a significant change to pricing XVAs in all of the derivatives included in the exercises above. For the case of forwards, unless the credit quality of one counterparty is significantly better than its counterpart, then the BCVA term reduces to zero. Further to this, if the NEE and the PEE for a forward are identical, then we have symmetrical pricing and most of the XVAs listed thus far are irrelevant, contingent on the condition of equal credit quality. The code used to generate these scenarios can be found in Appendix B.2.

Chapter 15

Finishing off with MVA and KVA

"Plans are nothing; planning is everything." - Dwight D. Eisenhower

As we have shown in Chapter 14, the value of a derivative can change substantially when considering all the post-2008 adjustments that came to the fore. Since the late 2008 period, market participants were required to include two more three lettered acronyms, namely MVA and KVA, the former covering the cost of initial margin typically required to trade on an exchange and the latter a necessity to cover capital charges to meet return requirements. We refer to the theory provided in Chapter 10 to implement our MVA and KVA charges, which mostly draws on [Elouerkhaoui 2016b]. We reuse the pramaters used throughout Chapters 12, 13, and 14 to obtain our market value price, as well as the XVA we have discussed so far. For MVA we use the same implied parameters, however, for KVA we need to look at historical real-world data to obtain our VAR and SVAR numbers.

Recall according to [Basel III], capital held for a derivative trade under the IMM approach needs to cover three different elements of risk:

1. Counterpary credit risk K_t^{CCR};
2. CVA VAR $K^{CVA_{VAR}}$; and
3. Market risk K_t^{MR}.

These join to create $K_t = K_t^{MR} + K_t^{CCR} + K_t^{CVA_{VAR}}$, which is plugged into Equation 10.16 to obtain our KVA add-on. To implement all three, we calibrate our model using a $\mathbb{P}$-measure. The data set we use is 2007 to 2008 for the stress period and

2015 to 2016 for a normal market period. The parameters for each model for the different periods are as follows:

Parameters	Processes - 2007/2008 period		
	SL	GBM	OU
S0	55.67	55.67	55.67
X	55.67	55.67	55.67
LambdaX	-6.13%		
LambdaE	0.67%		
T	X	X	X
Sigmax	32.13%		32.06%
Sigmae	26.30%	30.04%	
K	0.51		0.61
Correlation	-7.10%		
Mue	-6.32%	-2.69%	51.66

Table 15.1: Model parameters calibrated to real-world data from 2007 to 2008 to obtain 99th percentile SVAR.

Parameters	Processes - 2015/2016 period		
	SL	GBM	OU
S0	55.67	55.67	55.67
X	55.67	55.67	55.67
LambdaX	16.42%		
LambdaE	0.009%		
T	X	X	X
Sigmax	41.94%		60.25%
Sigmae	28.33%	43.12%	
K	0.74		0.60
Correlation	57.85%		
Mue	-6.60%	9.88%	74.11

Table 15.2: Model parameters calibrated to real-world data from 2015 to 2016 to obtain 99th percentile VAR.

Using equations 10.22, 10.23, and 10.24, originally defined in the [Basel III] paper, we obtain the following results when applied to an oil forward. We assume bank B buys forward from counterparty C. This is the same scenario depicted in Tables 14.12 and 14.13, with the exception of the inclusion of the MVA and KVA terms.

We assume bank B and counterparty C do not post margin and that bank B does not hedge the market risk. The table below shows how significant the charges can be despite the tenor of the trade being less than one year.

Model	Standard Normal Distribution - Forward Prices						
	Risk-Free Pri	Risky Price	CVA	DVA	KVA_MR	KVA_CVA	KVA_CCR
GBM	56.0000	55.4107	(0.0830)	0.0543	(0.1414)	(0.1893)	(0.2299)
OU	56.0000	55.3166	(0.1012)	0.0612	(0.1513)	(0.2279)	(0.2642)
SL	56.0000	55.0988	(0.1077)	0.0600	(0.2272)	(0.2698)	(0.3565)

Table 15.3: Unhedged and uncollateralised forward price, bank B buys from counterparty C. A normal distribution was assumed for pricing with $R_C = 0.3$; $R_B = 0.4$; $CDS_B = 400$ bps; $CDS_C = 700$ bps; $r_f = 0.05$; $r_k = 0.12$; $r_c = 0.01$ and $r_v = 0.1$.

Clearly this is not favourable and unrealistic. It is unfavourable, because bank B would most likely be pricing against other banks to win the deal, therefore to reduce the cost of capital alternative structures need to be considered. It is unrealistic, because most banks would not leave all of the market risk unhedged. We therefore introduce an exact hedge whereby bank B sells the forward to the interbank market under CSA with a zero threshold clause. To mitigate some of the capital costs, the counterparty could either place IM with the bank or trade under CSA. The former is more likely than the latter as we assume the client is a corporate entity not ideally setup to manage daily collateral calls.

The relationship between MVA and KVA is not linear as we will demonstrate in the table to follow. The amount of IM required to negate the KVA cost would in most instances incur an MVA cost greater than the intial KVA value. For the exercise below, we assume there is no segregation between IM and VM, and rehypothecation is possible. Furthermore we assume there is no additional CVA pricing to be done on the IM. Counterparty C is required to post IM to bank B for the purpose of reducing their CVA and KVA charges. We assume the IM to be a predetermined amount, usually calculated using a historical VAR methodology placed on trade initiation.

IM Posted	Model	Standard Normal Distribution - Forward Prices											
USD per bbl		Risk-Free Pri	Risky Price	CVA	DVA	FBA	FCA	ColCA	ColBA	MVA	KVA_MR	KVA_CVA	KVA_CCR
$5	GBM	56.0000	55.7920	(0.0346)	0.0543	0.0713	(0.0641)	(0.0285)	0.0256	0.0488	-	(0.1412)	(0.1396)
	OU	56.0000	55.7396	(0.0469)	0.0612	0.0803	(0.0760)	(0.0321)	0.0304	0.0488	-	(0.1732)	(0.1529)
	SL	56.0000	55.5904	(0.0510)	0.0600	0.0787	(0.0808)	(0.0315)	0.0323	0.0488	-	(0.2244)	(0.2417)
$10	GBM	56.0000	55.9501	(0.0135)	0.0543	0.0713	(0.0641)	(0.0285)	0.0256	0.0976	-	(0.1077)	(0.0849)
	OU	56.0000	55.9417	(0.0212)	0.0612	0.0803	(0.0760)	(0.0321)	0.0304	0.0976	-	(0.1089)	(0.0896)
	SL	56.0000	55.7821	(0.0234)	0.0600	0.0787	(0.0808)	(0.0315)	0.0323	0.0976	-	(0.1857)	(0.1651)
$15	GBM	56.0000	56.0715	(0.0051)	0.0543	0.0713	(0.0641)	(0.0285)	0.0256	0.1464	-	(0.0767)	(0.0517)
	OU	56.0000	56.0545	(0.0093)	0.0612	0.0803	(0.0760)	(0.0321)	0.0304	0.1464	-	(0.0939)	(0.0525)
	SL	56.0000	55.9251	(0.0104)	0.0600	0.0787	(0.0808)	(0.0315)	0.0323	0.1464	-	(0.1523)	(0.1173)
$20	GBM	56.0000	56.1639	(0.0018)	0.0543	0.0713	(0.0641)	(0.0285)	0.0256	0.1952	-	(0.0565)	(0.0316)
	OU	56.0000	56.1566	(0.0040)	0.0612	0.0803	(0.0760)	(0.0321)	0.0304	0.1952	-	(0.0679)	(0.0305)
	SL	56.0000	56.0388	(0.0046)	0.0600	0.0787	(0.0808)	(0.0315)	0.0323	0.1952	-	(0.1251)	(0.0854)

Table 15.4: Hedged forward price with IM, bank B buys from counterparty C. A normal distribution was assumed for pricing with the same parameters as Table 15.3.

Table 15.4 shows how the cost of capital can be transfered into an IM charge, MVA. This then becomes an optimisation problem with the goal to determine how much IM should be placed to decrease KVA without generating a punitive MVA. This is also dependent on what the bank's funding rate is compared to its return on capital rate. Notice the market risk charge has disappeared but has manifested as COLCA and FCA charges. Even if bank B were not to consider the benefit of receiving collateral from the hedge placed in the interbank market, the FCA and COLVA charges are insignificant in comparison to the market risk charge in Table 15.3. The reason for the marginal decrease in KVA as we significantly increase IM is simply to prevent banks from using external funds as opposed to their own capital to fund a trade. One last point to note is the CCR charge is typically much larger than the other charges, yet in some cases it is smaller than the CVA VAR charge. This can be explained by the excessively large CDS set for counterparty C.

For the sake of completeness, we run the same exercise using the CF to generate our exposures, results are shown in Tables 15.5 and 15.6

Model	Cornish-Fisher Adjusted Distribution - Forward Prices						
	Risk-Free Pri	Risky Price	CVA	DVA	KVA_MR	KVA_CVA	KVA_CCR
GBM	56.0000	55.3574	(0.0888)	0.0572	(0.1768)	(0.1938)	(0.2404)
OU	56.0000	55.2670	(0.1055)	0.0644	(0.1917)	(0.2307)	(0.2695)
SL	56.0000	55.0327	(0.1116)	0.0637	(0.2787)	(0.2756)	(0.3651)

Table 15.5: Unhedged and uncollateralised forward price, bank B buys from counterparty C. A CF generated distribution was assumed for pricing using the same parameters as before, s=0.96 and k=4.2.

IM Posted	Model	Cornish-Fisher Adjusted Distribution - Forward Prices											
USD per bbl		Risk-Free Pri	Risky Price	CVA	DVA	FBA	FCA	ColCA	ColBA	MVA	KVA_MR	KVA_CVA	KVA_CCR
$5	GBM	56.0000	55.7701	(0.0392)	0.0572	0.0751	(0.0667)	(0.0300)	0.0267	0.0488	-	(0.1488)	(0.1529)
	OU	56.0000	55.7208	(0.0526)	0.0644	0.0846	(0.0792)	(0.0338)	0.0317	0.0488	-	(0.1791)	(0.1639)
	SL	56.0000	55.5661	(0.0565)	0.0637	0.0836	(0.0838)	(0.0334)	0.0335	0.0488	-	(0.2327)	(0.2571)
$10	GBM	56.0000	55.9310	(0.0170)	0.0572	0.0751	(0.0667)	(0.0300)	0.0267	0.0976	-	(0.1138)	(0.0980)
	OU	56.0000	55.9004	(0.0259)	0.0644	0.0846	(0.0792)	(0.0338)	0.0317	0.0976	-	(0.1371)	(0.1019)
	SL	56.0000	55.7521	(0.0281)	0.0637	0.0836	(0.0838)	(0.0334)	0.0335	0.0976	-	(0.1957)	(0.1853)
$15	GBM	56.0000	56.2014	(0.0072)	0.0572	0.0751	(0.0667)	(0.0300)	0.0267	0.1464	-	(0.0870)	(0.0632)
	OU	56.0000	56.2014	(0.0126)	0.0644	0.0846	(0.0792)	(0.0338)	0.0317	0.1464	-	(0.1039)	(0.0633)
	SL	56.0000	56.1962	(0.0138)	0.0637	0.0836	(0.0838)	(0.0334)	0.0335	0.1464	-	(0.1645)	(0.1381)
$20	GBM	56.0000	56.2544	(0.0030)	0.0572	0.0751	(0.0667)	(0.0300)	0.0267	0.1952	-	(0.0667)	(0.4100)
	OU	56.0000	56.2567	(0.0061)	0.0644	0.0846	(0.0792)	(0.0338)	0.0317	0.1952	-	(0.0784)	(0.0393)
	SL	56.0000	56.2521	(0.0067)	0.0637	0.0836	(0.0838)	(0.0334)	0.0335	0.1952	-	(0.1383)	(0.1047)

Table 15.6: Hedged forward price with IM, bank B buys from counterparty C. A CF generated distribution was assumed for pricing with the same parameters as Table 15.4.

The results confirm that the CF does capture the additional risk associated with skewness and kurtosis measures different to that of a normal distribution, 0 and 3 respectively.

Chapter 16

Conclusion

"Nature's music is never over; her silence are pauses, not conclusions." - *Mary Webb*

This dissertation provided a high-level view on the most relevant pricing frameworks of XVA as per Burgard, Kjaer, and Piterbarg as well as Brigo, Pallavicini and Morini. This was done with the intention of allowing the reader to have a good knowledge on what is considered good practice in the field and context as to why this is the case.

We use the work of Elouerkhaoui to demonstrate a more current pricing model by including KVA and MVA. More importantly, Elouerkhaoui's publications build off of the pricing by expectations framework, which we favour as a more general approach unburdened by the unrealistic assumption of a complete market. His theory is based on the premise that traditional arbitrage pricing theory can still be applied, all that is required is financial engineering to accommodate for a more complex payoff. We demonstrated how to apply this theory in a practical example in Chapters 15 and 16. The results displayed in those Chapters facilitate our goal by illustrating how pricing has evolved since 2008, a stark contrast to the simplicity associated with the tables in Chapter 12. Figures 3.3 and 3.4 summarise the shift in complexity.

The results shown in Chapter 12 support the notion that QMC and MC are sufficient to implement an XVA model correctly. In particular, we use QMC because of its efficiency over MC with variance reduction. Further, QMC provides an easy platform for us to apply the CF, as we demonstrate in Chapter 13. We show how the CF can easily be calibrated to an implied volatility skew in order to capture non-normal skewness and kurtosis to reflect improved accuracy in the pricing. We demonstrate in Chapter 14 and 15 that the CF can be applied to a derivative with XVA.

To summarise, parts one and two provide the reader with a high-level understanding of what XVA is and why it is necessary to continue considering it. We cover the most prominent frameworks currently used by well known academics and practitioners in the industry. Parts three and four show how the theory of Elouerkhaoui is derived and more importantly how one can implement it to successfully value a derivative with XVA. The last two parts demonstrate how valuing a derivative in 2008 was almost comically simple in comparison to what is correct today.

The field of XVA is an exciting one that is evolving at a rapid pace. This makes it a daunting task for banks, particularly second tier banks, to remain abreast with industry standards. It would be beneficial for a bank to consider models that are parsimonious for the purpose of achieving accuracy without increasing model risk. This dissertation demonstrates a simple framework to address this and attempts to cover many of the pricing concerns practitioners have today.

Further research topics to be considered in this field:

- Valuing a portfolio rather than only one derivative as in this dissertation, with XVA;
- Introducing stochastic hazard rates in order to capture the impact of wrong way risk on a portoflio;
- Determining MVA under a contingent IM agreement;
- Providing an indepth comparison between the CF and various well known non-gaussian models; and
- Determining the value of a derivative with XVA assuming a replacement cost at default instead of a risk-free assumption.

Part V

Appendices

Appendix A

Model Calibration Code and Results

A.1 Estimating State Variables and Parameters using the Kalman Filter and Maximum Likelihood Estimation

```
function Maxln_L = Kalman_Filter(observed_forwards, param_new, dates, dt,_
                                 a0, P0, N, nobs,model)


% Initial parameters

k = param_new(1,1);
sigmast = param_new(2,1);
lambdast = param_new(3,1);
mu = param_new(4,1);
sigmlte = param_new(5,1);
Adjst_mu = param_new(6,1);
pxe = param_new(7,1);
lambdalt = param_new(8,1);

%G applies the mean reversion speed to the long term level for the OU
%and SL process
```

```
G=[exp(-k*dt),0;0,1];


%Calibrating under P-Measure using the Kalman Filter to determine the best
%fit parameters in accordance to Schwartz and Smith 2000.
%SL model parameters calibrated to historical observed futures

if model==1

Sigma11=(1-exp(-2*k*dt))*(sigmast)^2/(2*k);
Sigma12=(1-exp(-k*dt))*pxe*sigmast*sigmalt/k;
Sigma21=(1-exp(-k*dt))*pxe*sigmast*sigmalt/k;
Sigma22=(sigmast)^2*dt;
COV=[Sigma11,Sigma12;Sigma21,Sigma22];
R=eye(size(Q,1));

% Defining mean and variance terms

        Sigma1=(1-exp(-2*k.*dates))*(sigmast)^2/(2*k);
        Sigma2=(sigmalt)^2.*dates;
        Sigma3=2*(1-exp(-k.*dates))*pxe*sigmast*sigmalt/k;
        d=(Adjst_mu).*dates'-(1-exp(-k.*dates'))*lambdast/k+.5.*_
          (sigma1'+sigma2'+sigma3');
        Ft1=exp(-k.*dates);
        Ft2=zeros(size(Z1,2),1)+1;
        Ft=[Ft1' Ft2];


H=diag(s);
H=H(1:end-1,1:end-1);


save_at   = zeros(nobs,N);
save_diff1 = zeros(nobs,N);
```

```
save_diff2 = zeros(nobs,N);
save_at   = zeros(nobs,m);
save_mt    = zeros(nobs,m);
save_Rt    = zeros(nobs,m*m);
save_Ct    = zeros(nobs,m*m);
save_Qt    = zeros(nobs,N*N);
save_dQt   = zeros(nobs,1);
save_vQv   = zeros(nobs,1);

%Initial state variables

Ct = P0;
mt = a0;


    for t = 1:nobs
        Rt   = G*Ct*G'+R*COV*R';
        Qt   = Ft*Rt*Ft'+H;
        dQt  = det(Qt);



        at   = G*mt  + c;
        yt   = observed_forwards(t,:)';

        ft   = Ft*at+d;
        diff1   = yt-ft;

        mt = at + Rt*Ft'*inv(Qt)*(diff1);
        Ct = Rt - Rt*Ft'*inv(Qt)*Ft*Rt;

        ft     = Ft*mt+d;
        diff2  = yt-ft;

        save_ft(t,:) = ft';
        save_diff1(t,:) = diff1';
```

```
        save_diff2(t,:)    = (diff2)';
        save_mt(t,:)   = mt';
        save_Rt(t,:) = [Rt(1,1), Rt(1,2), Rt(2,1), Rt(2,2)];
        save_Ct(t,:)   = [Ct(1,1), Ct(1,2), Ct(2,1), Ct(2,2)];
        save_dQt(t,:)= dQt;
        save_vQv(t,:)   = diff1'*inv(Qt)*diff1;

    end

%GBM model parameters calibrated to historical observed futures

elseif model==2



    Sigma11=0;
Sigma12=0;
Sigma21=0;
Sigma22=(sigma1t)^2*dt;
COV=[Sigma11,Sigma12;Sigma21,Sigma22];
R=eye(size(Q,1));
c=[0;Adjst_mu*dt];

% Defining mean and variance terms



Sigma1=0;
        Sigma2=(sigma1t)^2.*dates;
        Sigma3=0;
        d=(Adjst_mu).*dates+.5.*(sigma1'+sigma2'+sigma3');
        Ft1=zeros(N,1);
        Ft2=Ft1+1;
        Ft=[Ft1 Ft2];

H=diag(s);
```

```
save_at   = zeros(nobs,N);
save_diff1 = zeros(nobs,N);
save_diff2 = zeros(nobs,N);
save_at   = zeros(nobs,m);
save_mt    = zeros(nobs,m);
save_Rt    = zeros(nobs,m*m);
save_Ct    = zeros(nobs,m*m);
save_Qt    = zeros(nobs,N*N);
save_dQt   = zeros(nobs,1);
save_vQv   = zeros(nobs,1);

%Initial state variables

Ct = P0;
mt = a0;


    for t = 1:nobs
        Rt   = G*Ct*G'+R*COV*R';
        Qt   = Ft*Rt*Ft'+H;
        dQt  = det(Qt);


        at   = G*mt  + c;
        yt   = observed_forwards(t,:)';

        ft   = Ft*at+d;
        diff1   = yt-ft;

        mt = at + Rt*Ft'*inv(Qt)*(diff1);
        Ct = Rt - Rt*Ft'*inv(Qt)*Ft*Rt;

        ft     = Ft*mt+d;
        diff2  = yt-ft;
```

```
        save_ft(t,:) = ft';
        save_diff1(t,:) = diff1';
        save_diff2(t,:)    = (diff2)';
        save_mt(t,:)   = mt';
        save_Rt(t,:) = [Rt(1,1), Rt(1,2), Rt(2,1), Rt(2,2)];
        save_Ct(t,:)   = [Ct(1,1), Ct(1,2), Ct(2,1), Ct(2,2)];
        save_dQt(t,:)= dQt;
        save_vQv(t,:)   = diff1'*inv(Qt)*diff1;

    end

%OU model parameters calibrated to historical observed futures

elseif model==3



Sigma11=(1-exp(-2*k*dt))*(sigmast)^2/(2*k);
Sigma12=0;
Sigma21=0;
Sigma22=0;
COV=[Sigma11,Sigma12;Sigma21,Sigma22];
R=eye(size(Q,1));

% Defining mean and variance terms

        Sigma1=(1-exp(-2*k.*dates))*(sigmast)^2/(2*k);
        Sigma2=0;
        Sigma3=0;
        d=(1-exp(-k.*dates'))*(log(mu)-(sigmast^2)/(2*k))+.5.*_
          (sigma1'+sigma2'+sigma3');
        Ft1=exp(-k.*dates);
        Ft2=zeros(size(Z1,2),1)+1;
        Ft=[Ft1' Ft2];
```

```
H=diag(s);


save_at   = zeros(nobs,N);
save_diff1 = zeros(nobs,N);
save_diff2 = zeros(nobs,N);
save_at   = zeros(nobs,m);
save_mt    = zeros(nobs,m);
save_Rt    = zeros(nobs,m*m);
save_Ct    = zeros(nobs,m*m);
save_Qt    = zeros(nobs,N*N);
save_dQt   = zeros(nobs,1);
save_vQv   = zeros(nobs,1);

%Initial state variables

Ct = P0;
mt = a0;


    for t = 1:nobs
        Rt   = G*Ct*G'+R*COV*R';
        Qt   = Ft*Rt*Ft'+H;
        dQt  = det(Qt);



        at   = G*mt  + c;
        yt   = observed_forwards(t,:)';

        ft   = Ft*at+d;
        diff1   = yt-ft;

        mt = at + Rt*Ft'*inv(Qt)*(diff1);
        Ct = Rt - Rt*Ft'*inv(Qt)*Ft*Rt;
```

```
        ft     = Ft*mt+d;
        diff2  = yt-ft;

        save_ft(t,:) = ft';
        save_diff1(t,:) = diff1';
        save_diff2(t,:)    = (diff2)';
        save_mt(t,:)   = mt';
        save_Rt(t,:) = [Rt(1,1), Rt(1,2), Rt(2,1), Rt(2,2)];
        save_Ct(t,:)   = [Ct(1,1), Ct(1,2), Ct(2,1), Ct(2,2)];
        save_dQt(t,:)= dQt;
        save_vQv(t,:)   = diff1'*inv(Qt)*diff1;

    end

end


MaxlnL = -(N*nobs/2)*log(2*pi)-0.5*sum(log(save_dQt))-_
           0.5*sum(save_vQv);
Maxln_L = -MaxlnL;
```

A.2 Implying Parameters from Current Market Data by Minimising Squared Errors

```
function SSE = Implied_Param_State(observed_forwards,
 param_new, dates, observed_vols,model,a0)


%Initial parameters

%a0=zeros(2,1);
k = param_new(1,1);
```

```
sigmalt = param_new(2,1);
lambdast = param_new(3,1);
mu = param_new(4,1);
lambdalt = param_new(4,1);
sigmalt = param_new(5,1);
Adjst_mu = param_new(6,1);
pxe = param_new(7,1);
%a0(1,1)=psi_new(8,1);
%a0(2,1)=psi_new(9,1);
a1=a0(1,1)+a0(2,1);


%SL model parameters calibarted to current observed
%ATM volatility term structure and futures

if model==1


    G1=exp(-k.*dates);
    Sigmast=(1-exp(-2*k.*dates))*(sigmast)^2/(2*k);
    Sigma2=(sigmalt)^2.*dates;
    Sigma3=2*(1-exp(-k.*dates))*pxe*sigmast*sigmalt/k;
    d(:,1)=0.5*(Sigma1+Sigma2+Sigma3);


    yt      = observed_forwards(:,1);
    ft_1   =zeros(size(yt,1),2);
    ft_2   =zeros(size(yt,1),1);
    ft_1(1,:)= [a0(1,1),a0(2,1)];

    ft_1(:,1)  = ytt_1(1,1).*T1-(1-exp(-k.*dates))*
     lambdax/k;  %OU part of E[linS(T)]
    ft_1(:,2)  = ytt_1(1,2)+(rnmu-lambdae).
     *dates;  %GBM part of E[linS(T)]
    ft_2      = exp(ytt_1(:,1) + ytt_1(:,2) + d);
```

```
    diff1     = yt-ft_2;
    Sigmat    = sqrt(yt.^2.*(exp(observed_vols.^2.*dates)-1));
    ft_3    = (ft_1(:,1) + ft_1(:,2))';
    Sigma_1   = sqrt((exp(Sigma1+Sigma2+Sigma3)-1).*_
                exp(2.*ft_3'+Sigma1+Sigma2+Sigma3));
    diff2     = Sigmat-Sigma_1;

elseif model==2 %GBM model parameters calibarted to current
                %observed ATM volatility term structure and futures

    yt      = observed_forwards(:,1);
    ft_1    =zeros(size(yt,1),1);
    ft_1(1) = exp(a1);
    ft_1    = ft_1(1)*exp(Adjst_mu.*dates);
    Sigmat  = sqrt(yt.^2.*(exp(observed_vols.^2.*dates)-1));
    Sigma_1 = sqrt(ytt_1.^2.*(exp(sigmae^2.*dates)-1));
    diff2   = Sigmat-Sigma_1;
    diff1   = yt-ft_1;

elseif model==3 %OU model parameters calibarted to current observed_
                %ATM volatility term structure and futures


    yt      = observed_forwards(:,1);
    ft_1    = zeros(size(yt,1),1);
    Sigma1  = (1-exp(-2*k.*dates))*(sigmast)^2/(2*k);
    ft_3    = a1.*exp(-k.*dates)+(1-exp(-k.*dates))*(log(mu)-(sigmast^2)/(2*k));
    ft_1    = exp(ft_3 + .5*Sigma1);
    diff1   = yt-ft_1;
    Sigmat  = sqrt(yt.^2.*(exp(observed_vols.^2.*dates)-1));
    Sigma_1 = sqrt((exp(p1)-1).*exp(2.*att_2 + Sigma1));
    diff2   = Sigmat-Sigma_1;


end
```

```
SSE = sum(diff1.^2+diff2.^2);
```

A.3 Kalman Filter Best Fit Results

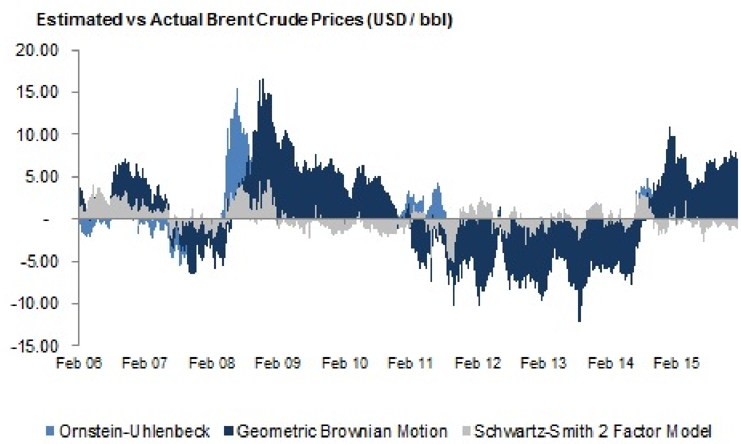

Figure A.1: The difference between the estimated and actual time series of ICE Brent crude oil for each model, 24 February 2006 - 20 January 2016.

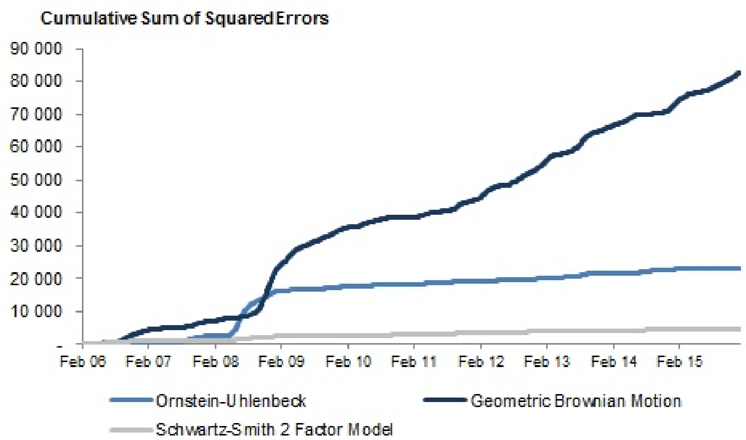

Figure A.2: The cumulative sum of the squared errors for each model produced via the Kalman filter parameter estimate, 24 February 2006 - 20 January 2016.

A.4 Implied Parameter Best Fit Results

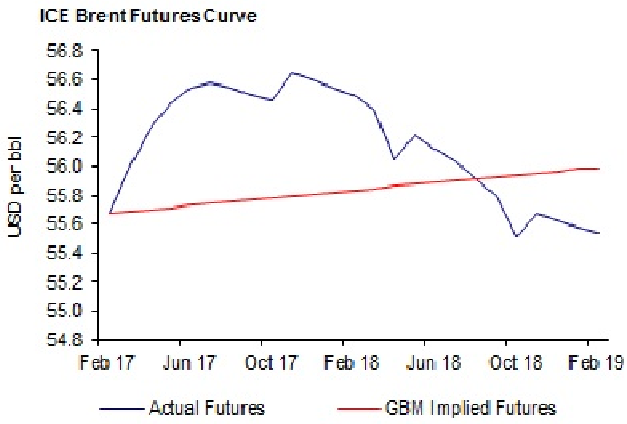

Figure A.3: The estimated GBM ICE Brent curve vs actual.

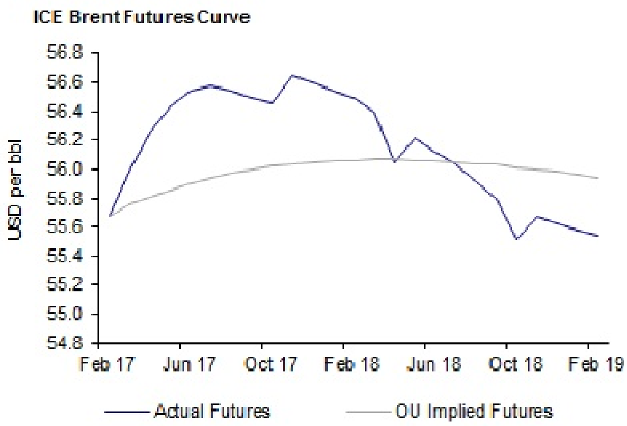

Figure A.4: The estimated OU ICE Brent curve vs actual.

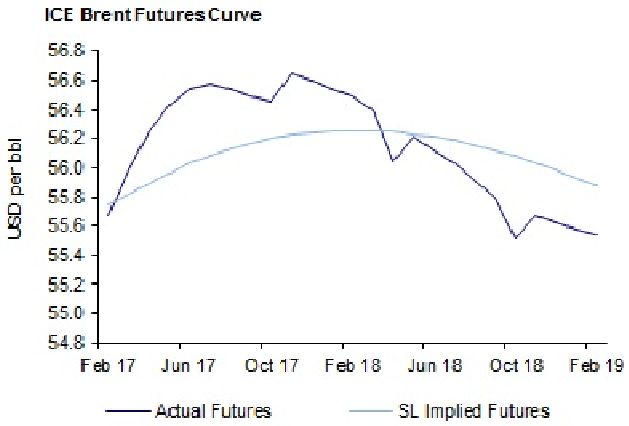

Figure A.5: The estimated SL ICE Brent curve vs actual.

Appendix B

Model Pseudocode

B.1 MC and QMC code for GBM, OU, and SL processes

```
% This code prices our options using both Quasi-MC and standard MC with
% Antithetic Variate
% We begin by generating random numbers using both a psuedo-random number
% generator as well as a quasi-random number generator, Glasserman 2003 pg. 303,
% Hlatons's sequence construction.
% If 1 we use the Cornish-Fisher transformation to generate Z1 numbers to
% account for skewness and kurtosis calibrated off of a volatility
% surface,Aboura and Maillard 2014.


% We begin by generating random numbers using both a psuedo-random number
% generator as well as a quasi-random number generator,
the reader is advised to consider Glasserman 2003 pg. 293

    %Initial set of random numbers

    p = haltonset(Days*T,'Skip',1e4,'Leap',1e2);
    p = scramble(p,'RR2');
```

```
    QuasiRandom_int=p(1:Paths/2,:);
    QAntiThet=1-QuasiRandom_int;
    QuasiRandom=[QuasiRandom_int;QAntiThet];
    PseudoRandom_int=rand(round(Paths)/2,Days*T);
    AntiThet=1-PseudoRandom_int;
    PseudoRandom=[PseudoRandom_int;AntiThet];

    % Second set required for bivariate distribution used in the SL model

     p = haltonset(Days*T,'Skip',1e3,'Leap',1e2);
     p = scramble(p,'RR2');

     QuasiRandom_int2=p(1:Paths/2,:);
     QAntiThet2=1-QuasiRandom_int2;
      QuasiRandom2=[QuasiRandom_int2;QAntiThet2];
      PseudoRandom_int2=rand(round(Paths)/2,Days*T);
    AntiThet2=1-PseudoRandom_int2;
     PseudoRandom2=[PseudoRandom_int2;AntiThet2];


% If Norm is not chosen we use the Cornish-Fisher
  approximation to generate Z3 & Z4 numbers to
% account for skewness and kurtosis calibrated
off  a volatility surface, Aboura and Maillard 2014.

if strcmp(Type,'Norm')

    Z1=norminv(QuasiRandom,0,1);
    Z2=Z1*correl+sqrt(1-correl^2)*norminv(QuasiRandom2,0,1);
    Z3=norminv(PseudoRandom,0,1);
    Z4=Z3*correl+sqrt(1-correl^2)*norminv(PseudoRandom2,0,1);

else

    Z =norminv(QuasiRandom, 0, 1);
```

```
    Z1=(Z + ((Z.^ 2) - 1) * Skewness / 6 + ((Z.^ 3) - 3 * Z) * ExcessKurtosis
     / 24 - (2 * (Z.^ 3) - 5 * Z) * (Skewness ) / 36);
    Z =norminv(QuasiRandom2, 0, 1);
    Z2=Z1*correl+sqrt(1-correl^2) * ((Z + ((Z.^ 2) - 1) * Skewness / 6
       + ((Z.^ 3) - 3 * Z)* ExcessKurtosis / 24 - (2 * (Z.^ 3) - 5 * Z)
       * (Skewness) / 36));
    Z =norminv(PseudoRandom, 0, 1);
    Z3=(Z + ((Z.^ 2) - 1) * Skewness / 6 + ((Z.^ 3) - 3 * Z) *
      ExcessKurtosis / 24 - (2 * (Z.^ 3) - 5 * Z) * (Skewness ) / 36);
    Z =norminv(PseudoRandom2, 0, 1);
    Z4=Z3*correl+sqrt(1-correl^2)*((Z + ((Z.^ 2) - 1) * Skewness / 6 +
      ((Z.^ 3) - 3 * Z) * ExcessKurtosis / 24 - (2 * (Z.^ 3) - 5 * Z)
       * (Skewness ) / 36));
end


for i=1:size(S_GBM_Pseu,2);
%GBM matrix
S_GBM_Pseu(:,i+1)=S_GBM_Pseu(:,i).*exp((Mue_GBM-0.5*Sigma_GBM^2)*dt
+Sigma_GBM*sqrt(dt).*Z3(:,i));
S_GBM_Quasi(:,i+1)=S_GBM_Quasi(:,i).*exp((Mue_GBM-0.5*Sigma_GBM^2)*dt
 +Sigma_GBM*sqrt(dt).*Z1(:,i));
%OU matrix
S_OU_Pseu(:,i+1)=S_OU_Pseu(:,i).^exp(-k_OU*dt).*exp((Mue_OU
 -0.5*Sigma_OU^2/k_OU)*(1-exp(-k_OU*dt))
 +Sigma_OU*sqrt((1-exp(-2*k_OU*dt))/2/k_OU).*Z3(:,i));
S_OU_Quasi(:,i+1)=S_OU_Quasi(:,i).^exp(-k_OU*dt).*exp((Mue_OU
-0.5*Sigma_OU^2/k_OU)*(1-exp(-k_OU*dt))
+Sigma_OU*sqrt((1-exp(-2*k_OU*dt))/2/k_OU).*Z1(:,i));
%SL matrix
S_SLE_Pseu(:,i+1)=S_SLE_Pseu(:,i)+(Mue_SL-LambdaE)*dt
+SigmaE*sqrt(dt).*Z3(:,i);
S_SLE_Quasi(:,i+1)=S_SLE_Quasi(:,i)+(Mue_SL-LambdaE)*dt
 +SigmaE*sqrt(dt).*Z1(:,i);
S_SLX_Pseu(:,i+1)=S_SLX_Pseu(:,i).*exp(-k_SL*dt)-(1-exp(-k_SL*dt))
```

```
  *(LambdaX/k_SL)+SigmaX*sqrt((1-exp(-2*k_SL*dt))
  /2/k_SL).*Z4(:,i);
S_SLX_Quasi(:,i+1)=S_SLX_Quasi(:,i).*exp(-k_SL*dt)-(1-exp(-k_SL*dt))
 *(LambdaX/k_SL)+SigmaX*sqrt((1-exp(-2*k_SL*dt))
 /2/k_SL).*Z2(:,i);
S_SL_Pseu(:,i+1)=exp(S_SLE_Pseu(:,i+1)+S_SLX_Pseu(:,i+1));
S_SL_Quasi(:,i+1)=exp(S_SLE_Quasi(:,i+1)+S_SLX_Quasi(:,i+1));

end


%Fwds at T

GBMfwdP=mean(S_GBM_Pseu(:,end));
GBMfwdQ=mean(S_GBM_Quasi(:,end));
OUfwdP=mean(S_OU_Pseu(:,end));
OUfwdQ=mean(S_OU_Quasi(:,end));
SLfwdP=mean(S_SL_Pseu(:,end));
SLfwdQ=mean(S_SL_Quasi(:,end));

%Risk-free MTM of instruments using both expected values from the
%MC and QMC simulations. Further, we include the closed-form solutions
%where available fr comparison to our scenarios.

if strcmp('Call',Inst)

    %QMC prices

    MTM_rf_GBMP=exp(-rf*T)*mean(max(S_GBM_Pseu(:,end)-X,0));
    MTM_rf_GBMQ=exp(-rf*T)*mean(max(S_GBM_Quasi(:,end)-X,0));
    MTM_rf_OUP=exp(-rf*T)*mean(max(S_OU_Pseu(:,end)-X,0));
    MTM_rf_OUQ=exp(-rf*T)*mean(max(S_OU_Quasi(:,end)-X,0));
    MTM_rf_SLP=exp(-rf*T)*mean(max(S_SL_Pseu(:,end)-X,0));
    MTM_rf_SLQ=exp(-rf*T)*mean(max(S_SL_Quasi(:,end)-X,0));
```

```
    if strcmp(Type,'Norm')

        %GBM closed-form

        d1=(log((S0*exp(Mue_GBM*T))/X)+(.5*Sigma_GBM^2)*T)/Sigma_GBM/sqrt(T);
        d2=d1-Sigma_GBM*sqrt(T);
        BS=exp(-rf*T)*(S0*exp(Mue_GBM*T)*normcdf(d1)-X*normcdf(d2));

        %OU closed-form

        ELnS=exp(-k_OU*T)*log(S0)+(1-exp(-k_OU*T))*(Mue_OU-Sigma_OU^2/2/k_OU);
        VLnS=((1-exp(-2*k_OU*T))*Sigma_OU^2/2/k_OU);
        F=exp(ELnS+0.5*VLnS);
        d1=log(F/X)/sqrt(VLnS)+.5*sqrt(VLnS);
        d2=d1-sqrt(VLnS);
        BS_OU=exp(-rf*T)*(F*normcdf(d1)-X*normcdf(d2));

        %SL closed-form

        ELnS=exp(-k_SL*T)*X0+E0+(Mue_SL-LambdaE)*
          T-(1-exp(-k_SL*T))*(LambdaX/k_SL);
        VLnS=(1-exp(-2*k_SL*T))*SigmaX^2/2/k_SL+SigmaE^2*T+2*(1-exp(-k_SL*T))*
         SigmaX*SigmaE*correl/k_SL;
        F=exp(ELnS+0.5*VLnS);
        d1=log(F/X)/sqrt(VLnS)+.5*sqrt(VLnS);
        d2=d1-sqrt(VLnS);
        BS_SL=exp(-rf*T)*(F*normcdf(d1)-X*normcdf(d2));


    end


elseif strcmp('Put',Inst)

    %QMC prices
```

```
MTM_rf_GBMP=exp(-rf*T)*mean(max(-S_GBM_Pseu(:,end)+X,0));
MTM_rf_GBMQ=exp(-rf*T)*mean(max(-S_GBM_Quasi(:,end)+X,0));
MTM_rf_OUP=exp(-rf*T)*mean(max(-S_OU_Pseu(:,end)+X,0));
MTM_rf_OUQ=exp(-rf*T)*mean(max(-S_OU_Quasi(:,end)+X,0));
MTM_rf_SLP=exp(-rf*T)*mean(max(-S_SL_Pseu(:,end)+X,0));
MTM_rf_SLQ=exp(-rf*T)*mean(max(-S_SL_Quasi(:,end)+X,0));


if strcmp(Type,'Norm')


    %GBM closed-form

    d1=(log((S0*exp(Mue_GBM*T))/X)+(.5*Sigma_GBM^2)*T)/Sigma_GBM/sqrt(T);
    d2=d1-Sigma_GBM*sqrt(T);
    BS=exp(-rf*T)*(-S0*exp(Mue_GBM*T)*normcdf(-d1)+X*normcdf(-d2));

    %OU closed-form

    ELnS=exp(-k_OU*T)*log(S0)+(1-exp(-k_OU*T))*
      (Mue_OU-Sigma_OU^2/2/k_OU);
    VLnS=((1-exp(-2*k_OU*T))*Sigma_OU^2/2/k_OU);
    F=exp(ELnS+0.5*VLnS);
    d1=log(F/X)/sqrt(VLnS)+.5*sqrt(VLnS);
    d2=d1-sqrt(VLnS);
    BS_OU=exp(-rf*T)*(-F*normcdf(-d1)+X*normcdf(-d2));

    %SL closed-form

    ELnS=exp(-k_SL*T)*X0+E0+(Mue_SL-LambdaE)*
      T-(1-exp(-k_SL*T))*(LambdaX/k_SL);
    VLnS=(1-exp(-2*k_SL*T))*SigmaX^2/2/k_SL+SigmaE^2*T+2*(1-exp(-k_SL*T))
      *SigmaX*SigmaE*correl/k_SL;
    F=exp(ELnS+0.5*VLnS);
```

```
        d1=log(F/X)/sqrt(VLnS)+.5*sqrt(VLnS);
        d2=d1-sqrt(VLnS);
        BS_SL=exp(-rf*T)*(-F*normcdf(-d1)+X*normcdf(-d2));

    end


else



    F=S0*exp(Mue_GBM*T);
    MTM_rf_GBM=F-X;



    ELnS=exp(-k_OU*T)*log(S0)+(1-exp(-k_OU*T))*(Mue_OU-Sigma_OU^2/2/k_OU);
    VLnS=((1-exp(-2*k_OU*T))*Sigma_OU^2/2/k_OU);
    F=exp(ELnS+0.5*VLnS);
    MTM_rf_OU=F-X;

    ELnS=exp(-k_SL*T)*X0+E0+(Mue_SL-LambdaE)*T-(1-exp(-k_SL*T))
      *(LambdaX/k_SL);
    VLnS=(1-exp(-2*k_SL*T))*SigmaX^2/2/k_SL+SigmaE^2*T+2*
      (1-exp(-k_SL*T))*SigmaX*SigmaE*correl/k_SL;
    F=exp(ELnS+0.5*VLnS);

    MTM_rf_SL=F-X;

end
```

B.2 Generating XVA Numbers for Collateralised and Uncollateralised trades

```
if strcmp('XVA',Depth)

 % Uncollateralised, both IM and VM between counterparty C and bank B
 % We consider costs of CVA, DVA and KVA between C and B. We pass on the
 % cost of VM and IM between bank B and hedges counterparties

 % Stripping PD curves

 Hazard_B=CDS_B/(1-R_B);
 PD_B=1-exp(-Hazard_B.*dt);
 Hazard_C=CDS_C/(1-R_C);
 PD_C=1-exp(-Hazard_C.*dt);

 %Change in PD curves from t to t+1

 Delta_PD_B=[PD_B(1) PD_B(1,2:end)-PD_B(1,1:end-1)];
 Delta_PD_C=[PD_C(1) PD_C(1,2:end)-PD_C(1,1:end-1)];

 %Estimating EE(t) and NEE(t)


     EE_rf_GBMQ=exp(-rf.*dt).*mean(max(S_GBM_Quasi(:,:)-X,0),1);
     EE_rf_OUQ=exp(-rf.*dt).*mean(max(S_OU_Quasi(:,:)-X,0),1);
     EE_rf_SLQ=exp(-rf.*dt).*mean(max(S_SL_Quasi(:,:)-X,0),1);


     NEE_rf_GBMQ=exp(-rf.*dt).*mean(min(S_GBM_Quasi(:,:)-X,0),1);
     NEE_rf_OUQ=exp(-rf.*dt).*mean(min(S_OU_Quasi(:,:)-X,0),1);
     NEE_rf_SLQ=exp(-rf.*dt).*mean(min(S_SL_Quasi(:,:)-X,0),1);


     %Setting up the Funded Base (PV)
     %Calcualte FVA and COLVA portion of the trade
```

```
%Cost of funding from treasury

FCA_GBM=sum(EE_rf_GBMQ*(rf-rv)*dt(1));
FCA_OU=sum(EE_rf_OUQ*(rf-rv)*dt(1));
FCA_SL=sum(EE_rf_SLQ*(rf-rv)*dt(1));

%Benefit from placing with treasury

FBA_GBM=sum(NEE_rf_GBMQ*(rf-rv)*dt(1));
FBA_OU=sum(NEE_rf_OUQ*(rf-rv)*dt(1));
FBA_SL=sum(NEE_rf_SLQ*(rf-rv)*dt(1));

%Cost of funding from CSA

COLVAC_GBM=sum(NEE_rf_GBMQ*(rf-rc)*dt(1));
COLVAC_OU=sum(NEE_rf_OUQ*(rf-rc)*dt(1));
COLVAC_SL=sum(NEE_rf_SLQ*(rf-rc)*dt(1));

%Benefit from placing under CSA

COLVAB_GBM=sum(EE_rf_GBMQ*(rf-rc)*dt(1));
COLVAB_OU=sum(EE_rf_OUQ*(rf-rc)*dt(1));
COLVAB_SL=sum(EE_rf_SLQ*(rf-rc)*dt(1));


%Calcualte CVA and DVA portion of the trade


CVA_GBM=(1-R_C)*dot(EE_rf_GBMQ,(1-[0 PD_B(1,1:end-1)]).*_
        [0 Delta_PD_C(1,1:end-1)]);
CVA_OU=(1-R_C)*dot(EE_rf_OUQ,(1-[0 PD_B(1,1:end-1)]).*_
       [0 Delta_PD_C(1,1:end-1)]);
CVA_SL=(1-R_C)*dot(EE_rf_SLQ,(1-[0 PD_B(1,1:end-1)]).*_
       [0 Delta_PD_C(1,1:end-1)]);
```

```
DVA_GBM=(1-R_B)*dot(NEE_rf_GBMQ,(1-[0 PD_C(1,1:end-1)]).*_
        [0 Delta_PD_B(1,1:end-1)]);
DVA_OU=(1-R_B)*dot(NEE_rf_OUQ,(1-[0 PD_C(1,1:end-1)]).*_
       [0 Delta_PD_B(1,1:end-1)]);
DVA_SL=(1-R_B)*dot(NEE_rf_SLQ,(1-[0 PD_C(1,1:end-1)]).*_
       [0 Delta_PD_B(1,1:end-1)]);


BCVA_GBM=CVA_GBM+DVA_GBM;
BCVA_OU=CVA_OU+DVA_OU;
BCVA_SL=CVA_SL+DVA_SL;

%Calcualte MVA (assume no segregation of initial margin and
%variation margin)

MVAC_GBM=dot((1-[0 PD_B(1,1:end-1)])*(IM_B*(rf-rc)*dt(1)),_
            (1-[0 PD_C(1,1:end-1)])); %Cost of funding from CSA
MVAC_OU=dot((1-[0 PD_B(1,1:end-1)])*(IM_B*(rf-rc)*dt(1)),_
           (1-[0 PD_C(1,1:end-1)]));
MVAC_SL=dot((1-[0 PD_B(1,1:end-1)])*(IM_B*(rf-rc)*dt(1)),_
        (1-[0 PD_C(1,1:end-1)]));
MVAB_GBM=dot((1-[0 PD_B(1,1:end-1)])*(-IM_C*(rf-rc)*dt(1)),_
            (1-[0 PD_C(1,1:end-1)])); %Benefit from placing under CSA
MVAB_OU=dot((1-[0 PD_B(1,1:end-1)])*(-IM_C*(rf-rc)*dt(1)),_
           (1-[0 PD_C(1,1:end-1)]));
MVAB_SL=dot((1-[0 PD_B(1,1:end-1)])*(-IM_C*(rf-rc)*dt(1)),_
           (1-[0 PD_C(1,1:end-1)]));

%Setting up for K calculation for KVA

    %Setup for market risk charge (BAsel II & 2.5)

    GBM_10dVAR=M_Mult*(exp(-rf.*dt(10:end)).*prctile(S_GBM_Quasi_
              (:,10:end)-S_GBM_Quasi(:,1:end-9),MRP));
    OU_10dVAR=M_Mult*(exp(-rf.*dt(10:end)).*prctile(S_OU_Quasi_
```

```
          (:,10:end)-S_OU_Quasi(:,1:end-9),MRP));
SL_10dVAR=M_Mult*(exp(-rf.*dt(10:end)).*prctile(S_SL_Quasi_
          (:,10:end)-S_SL_Quasi(:,1:end-9),MRP));

%Setup for CVA add-on (Basel III)

RCS01_GBM=0.0001*dt.*(1-PD_C).*[(EE_rf_GBMQ(:,1)+EE_rf_GBMQ(:,2))/2_
          (EE_rf_GBMQ(:,1:end-2)+EE_rf_GBMQ(:,3:end))/2 (EE_rf_GBMQ_
          (:,end-1)+EE_rf_GBMQ(:,end))/2];
RCS01_OU=0.0001*dt.*(1-PD_C).*[(EE_rf_OUQ(:,1)+EE_rf_OUQ(:,2))/2_
         (EE_rf_OUQ(:,1:end-2)+EE_rf_OUQ(:,3:end))/2 (EE_rf_OUQ_
         (:,end-1)+EE_rf_OUQ(:,end))/2];
RCS01_SL=0.0001*dt.*(1-PD_C).*[(EE_rf_SLQ(:,1)+EE_rf_SLQ(:,2))/2_
         (EE_rf_SLQ(:,1:end-2)+EE_rf_SLQ(:,3:end))/2 (EE_rf_SLQ_
         (:,end-1)+EE_rf_SLQ(:,end))/2];


%Setup for CCR (Basel III)
%Unilateral CVA portion

CVA_GBM_U=(1-R_C)*EE_rf_GBMQ.*[0 Delta_PD_C(1,1:end-1)];
CVA_OU_U=(1-R_C)*EE_rf_OUQ.*[0 Delta_PD_C(1,1:end-1)];
CVA_SL_U=(1-R_C)*EE_rf_SLQ.*[0 Delta_PD_C(1,1:end-1)];

%EAD portion , 1.4*EEPE
%Calculate the effective expected exposure across time
%and th eexpected positive exposure to get the max of
%the two across all t. Then use the result to get
%time-weighted average.


GBM_EEE=zeros(size(EE_rf_GBMQ,2),1)';
OU_EEE=zeros(size(EE_rf_GBMQ,2),1)';
SL_EEE=zeros(size(EE_rf_GBMQ,2),1)';
```

```
    GBM_EEE(1)=EE_rf_GBMQ(1);
    OU_EEE(1)=EE_rf_OUQ(1);
    SL_EEE(1)=EE_rf_SLQ(1);

    for j=1:size(EE_rf_GBMQ,2)-1
        if GBM_EEE(j)>=EE_rf_GBMQ(j+1)
            GBM_EEE(j+1)=GBM_EEE(j);
        else
            GBM_EEE(j+1)=EE_rf_GBMQ(j+1);
        end
        if OU_EEE(j)>=EE_rf_OUQ(j+1)
            OU_EEE(j+1)=OU_EEE(j);
        else
            OU_EEE(j+1)=EE_rf_OUQ(j+1);
        end
        if SL_EEE(j)>=EE_rf_SLQ(j+1)
            SL_EEE(j+1)=SL_EEE(j);
        else
            SL_EEE(j+1)=EE_rf_SLQ(j+1);
        end

    end


if strcmp(Stress_Period,'Y')

    %K_MR

    GBM_KVA_MR_S=GBM_10dVAR;
    OU_KVA_MR_S=OU_10dVAR;
    SL_KVA_MR_S=SL_10dVAR;

    %K_CVA Add-on

    GBM_KVA_CVAV_S=abs(RCS01_GBM).^0.5*CDS_C*10000*Sigma_GBM*_
```

```
                  (10/Days)^0.5*norminv(MRP/100,0,1);
    OU_KVA_CVAV_S=abs(RCS01_OU).^0.5*Sigma_OU*CDS_C*10000*_
                  (10/Days)^0.5*norminv(MRP/100,0,1);
    SL_KVA_CVAV_S=abs(RCS01_SL).^0.5*SigmaX*10000*CDS_C*_
                  (10/Days)^0.5*norminv(MRP/100,0,1);

    %K_CCR

    GBM_EAD_CCR_S=1.4*GBM_EEE;
    OU_EAD_CCR_S=1.4*OU_EEE;
    SL_EAD_CCR_S=1.4*SL_EEE;

    GBM_CVA_CCR_S=CVA_GBM_U;
    OU_CVA_CCR_S=CVA_OU_U;
    SL_CVA_CCR_S=CVA_SL_U;

elseif strcmp(Stress_Period,'N')

    %K_MR

    GBM_KVA_MR_N=GBM_10dVAR;
    OU_KVA_MR_N=OU_10dVAR;
    SL_KVA_MR_N=SL_10dVAR;

    %K_CVA Add-on

    GBM_KVA_CVAV_N=abs(RCS01_GBM).^0.5*CDS_C*10000*_
                   Sigma_GBM*(10/Days)^0.5*norminv(MRP/100,0,1);
    OU_KVA_CVAV_N=abs(RCS01_OU).^0.5*Sigma_OU*CDS_C*_
                  10000*(10/Days)^0.5*norminv(MRP/100,0,1);
    SL_KVA_CVAV_N=abs(RCS01_SL).^0.5*SigmaX*10000*CDS_C*_
                  (10/Days)^0.5*norminv(MRP/100,0,1);

    %K_CCR
```

```
    GBM_EAD_CCR_N=1.4*GBM_EEE;
    OU_EAD_CCR_N=1.4*OU_EEE;
    SL_EAD_CCR_N=1.4*SL_EEE;

    GBM_CVA_CCR_N=CVA_GBM_U;
    OU_CVA_CCR_N=CVA_OU_U;
    SL_CVA_CCR_N=CVA_SL_U;


end

    %Total MR

    GBM_KVA_MR=dot((1-[0 PD_B(1,10:end-1)]).*(1-[0 PD_C_
               (1,10:end-1)]),(alpha_cap*(GBM_KVA_MR_S+_
                GBM_KVA_MR_N)*(rf-rk)*dt(1)));
    OU_KVA_MR=dot((1-[0 PD_B(1,10:end-1)]).*(1-[0 PD_C_
              (1,10:end-1)]),(alpha_cap*(OU_KVA_MR_S+_
               OU_KVA_MR_N)*(rf-rk)*dt(1)));
    SL_KVA_MR=dot((1-[0 PD_B(1,10:end-1)]).*(1-[0 PD_C_
             (1,10:end-1)]),(alpha_cap*(SL_KVA_MR_S+_
             SL_KVA_MR_N)*(rf-rk)*dt(1)));

    %Total CVA Add-on

    GBM_KVA_CVAV=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0 PD_C(1,1:_
                 end-1)]),((alpha_cap*12.5*3*(GBM_KVA_CVAV_S+_
                 GBM_KVA_CVAV_N))*(rf-rk)*dt(1)));
    OU_KVA_CVAV=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0 PD_C(1,1:_
                end-1)]),((alpha_cap*12.5*3*(OU_KVA_CVAV_S+_
                OU_KVA_CVAV_N))*(rf-rk)*dt(1)));
    SL_KVA_CVAV=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0 PD_C(1,1:_
                end-1)]),((alpha_cap*12.5*3*(SL_KVA_CVAV_S+_
                SL_KVA_CVAV_N))*(rf-rk)*dt(1)));
```

```
    %Total CCR

    GBM_KVA_CCR=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0 PD_C(1,1:_
                end-1)]),((alpha_cap*RW*1.06*12.5*(max_
                (GBM_EAD_CCR_S,GBM_EAD_CCR_N)-CVA_GBM_U))*_
                (rf-rk)*dt(1)));
    OU_KVA_CCR=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0 PD_C(1,1:_
               end-1)]),((alpha_cap*RW*1.06*12.5*max_
               (OU_EAD_CCR_S,OU_EAD_CCR_N)-CVA_OU_U)*_
               (rf-rk)*dt(1)));
    SL_KVA_CCR=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0 PD_C(1,1:_
               end-1)]),((alpha_cap*RW*1.06*12.5*max_
               (SL_EAD_CCR_S,SL_EAD_CCR_N)-CVA_SL_U)*_
               (rf-rk)*dt(1)));

  Funded_Base_GBM=MTM_rf_GBMQ(end)+FCA_GBM+FBA_GBM+_
                COLVAB_GBM+COLVAC_GBM+MVAC_GBM+MVAB_GBM;
  Funded_Base_OU=MTM_rf_OUQ(end)+FCA_OU+FBA_OU+COLVAB_OU+_
               COLVAC_OU+MVAC_OU+MVAB_OU;
  Funded_Base_SL=MTM_rf_SLQ(end)+FCA_SL+FBA_SL+COLVAB_SL_
               +COLVAC_SL+MVAC_SL+MVAB_SL;

  Risky_GBM=Funded_Base_GBM+BCVA_GBM+GBM_KVA_CCR+_
          GBM_KVA_CVAV+GBM_KVA_MR;
  Risky_OU=Funded_Base_OU+BCVA_OU+OU_KVA_CCR+_
         OU_KVA_CVAV+OU_KVA_MR;
  Risky_SL=Funded_Base_SL+BCVA_SL+SL_KVA_CCR+_
         SL_KVA_CVAV+SL_KVA_MR;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The EE and NEE calcualtions incorporate the impact of collateral
% inherent in trades between bank B and counterparty C

if strcmp(Col,'Y')
```

```
%Estimating EE(t) and NEE(t) with 0 threshold CSA
%We reduce EAD by the amount of IM and VM is posted. We assume that
%either bank B or counterparty C needs to fund IM between
%eachother. VM is assuemd to be earned/paid from hedge to
%counterpart. No impact on funding VM



%Create Collateral matrices

%For positive exposure, EE(t)

GBM_F_Q=max(S_GBM_Quasi(:,:)-X,0);       %Risk-free MTM
GBM_C_Q=[zeros(size(GBM_F_Q,1),Col_lag) _
        GBM_F_Q(:,1:end-Col_lag)]; %Collateral matrix
OU_F_Q=max(S_OU_Quasi(:,:)-X,0);         %Risk-free MTM
OU_C_Q=[zeros(size(OU_F_Q,1),Col_lag) _
       OU_F_Q(:,1:end-Col_lag)]; %Collateral matrix
SL_F_Q=max(S_SL_Quasi(:,:)-X,0);         %Risk-free MTM
SL_C_Q=[zeros(size(SL_F_Q,1),Col_lag)_
       SL_F_Q(:,1:end-Col_lag)]; %Collateral matrix

EE_rfM_GBMQ=exp(-rf.*dt).*mean(max(GBM_F_Q-GBM_C_Q-IM_C,0),1);
EE_rfM_OUQ=exp(-rf.*dt).*mean(max(OU_F_Q-OU_C_Q-IM_C,0),1);
EE_rfM_SLQ=exp(-rf.*dt).*mean(max(SL_F_Q-SL_C_Q-IM_C,0),1);

%For negative exposure, NEE(t)

GBM_F_Q=min(S_GBM_Quasi(:,:)-X,0);       %Risk-free MTM
GBM_C_Q=[zeros(size(GBM_F_Q,1),Col_lag)_
        GBM_F_Q(:,1:end-Col_lag)]; %Collateral matrix
OU_F_Q=min(S_OU_Quasi(:,:)-X,0);         %Risk-free MTM
OU_C_Q=[zeros(size(OU_F_Q,1),Col_lag)_
       OU_F_Q(:,1:end-Col_lag)]; %Collateral matrix
SL_F_Q=min(S_SL_Quasi(:,:)-X,0);         %Risk-free MTM
```

```
SL_C_Q=[zeros(size(SL_F_Q,1),Col_lag)_
       SL_F_Q(:,1:end-Col_lag)]; %Collateral matrix

NEE_rfM_GBMQ=exp(-rf.*dt).*mean(min(GBM_F_Q-GBM_C_Q+IM_B,0),1);
NEE_rfM_OUQ=exp(-rf.*dt).*mean(min(OU_F_Q-OU_C_Q+IM_B,0),1);
NEE_rfM_SLQ=exp(-rf.*dt).*mean(min(SL_F_Q-SL_C_Q+IM_B,0),1);


  %Setting up for K calculation for KVA with IM

    %Setup for CVA add-on (Basel III)

    RCS01_GBM=0.0001*dt.*(1-PD_C).*[(EE_rfM_GBMQ(:,1)+_
             EE_rfM_GBMQ(:,2))/2 (EE_rfM_GBMQ(:,1:end-2)+_
             EE_rfM_GBMQ(:,3:end))/2 (EE_rfM_GBMQ(:,end-1)_
             +EE_rfM_GBMQ(:,end))/2];
    RCS01_OU=0.0001*dt.*(1-PD_C).*[(EE_rfM_OUQ(:,1)+_
             EE_rfM_OUQ(:,2))/2 (EE_rfM_OUQ(:,1:end-2)+_
             EE_rfM_OUQ(:,3:end))/2 (EE_rfM_OUQ(:,end-1)_
             +EE_rfM_OUQ(:,end))/2];
    RCS01_SL=0.0001*dt.*(1-PD_C).*[(EE_rfM_SLQ(:,1)+_
             EE_rfM_SLQ(:,2))/2 (EE_rfM_SLQ(:,1:end-2)+_
             EE_rfM_SLQ(:,3:end))/2 (EE_rfM_SLQ(:,end-1)_
             +EE_rfM_SLQ(:,end))/2];


    %Setup for CCR (Basel III)
    %Unilateral CVA portion

    CVA_GBM_U=(1-R_C)*EE_rfM_GBMQ.*[0 Delta_PD_C(1,1:end-1)];
    CVA_OU_U=(1-R_C)*EE_rfM_OUQ.*[0 Delta_PD_C(1,1:end-1)];
    CVA_SL_U=(1-R_C)*EE_rfM_SLQ.*[0 Delta_PD_C(1,1:end-1)];

    %EAD portion , 1.4*EEPE
    %Calculate the effective expected exposure across time
```

```
    %and th eexpected positive exposure to get the max of
    %the two across all t. Then use the result to get
    %time-weighted average.

    GBM_EEE=zeros(size(EE_rfM_GBMQ,2),1)';
    OU_EEE=zeros(size(EE_rfM_GBMQ,2),1)';
    SL_EEE=zeros(size(EE_rfM_GBMQ,2),1)';

    GBM_EEE(1)=EE_rfM_GBMQ(1);
    OU_EEE(1)=EE_rfM_OUQ(1);
    SL_EEE(1)=EE_rfM_SLQ(1);

    for j=1:size(EE_rfM_GBMQ,2)-1
        if GBM_EEE(j)>=EE_rfM_GBMQ(j+1)
            GBM_EEE(j+1)=GBM_EEE(j);
        else
            GBM_EEE(j+1)=EE_rfM_GBMQ(j+1);
        end
        if OU_EEE(j)>=EE_rfM_OUQ(j+1)
            OU_EEE(j+1)=OU_EEE(j);
        else
            OU_EEE(j+1)=EE_rfM_OUQ(j+1);
        end
        if SL_EEE(j)>=EE_rfM_SLQ(j+1)
            SL_EEE(j+1)=SL_EEE(j);
        else
            SL_EEE(j+1)=EE_rfM_SLQ(j+1);
        end

    end

if strcmp(Stress_Period,'Y')


    %K_CVA Add-on
```

```
    GBM_KVA_CVAV_S=abs(RCS01_GBM).^0.5*CDS_C*10000*_
                   Sigma_GBM*(10/Days)^0.5*norminv(MRP/100,0,1);
    OU_KVA_CVAV_S=abs(RCS01_OU).^0.5*Sigma_OU*CDS_C*_
                  10000*(10/Days)^0.5*norminv(MRP/100,0,1);
    SL_KVA_CVAV_S=abs(RCS01_SL).^0.5*SigmaX*10000*_
                  CDS_C*(10/Days)^0.5*norminv(MRP/100,0,1);

    %K_CCR

    GBM_EAD_CCR_S=1.4*GBM_EEE;
    OU_EAD_CCR_S=1.4*OU_EEE;
    SL_EAD_CCR_S=1.4*SL_EEE;

    GBM_CVA_CCR_S=CVA_GBM_U;
    OU_CVA_CCR_S=CVA_OU_U;
    SL_CVA_CCR_S=CVA_SL_U;

elseif strcmp(Stress_Period,'N')


    %K_CVA Add-on

    GBM_KVA_CVAV_N=abs(RCS01_GBM).^0.5*CDS_C*10000*_
                   Sigma_GBM*(10/Days)^0.5*norminv(MRP/100,0,1);
    OU_KVA_CVAV_N=abs(RCS01_OU).^0.5*Sigma_OU*CDS_C_
                  *10000*(10/Days)^0.5*norminv(MRP/100,0,1);
    SL_KVA_CVAV_N=abs(RCS01_SL).^0.5*SigmaX*10000*_
                  CDS_C*(10/Days)^0.5*norminv(MRP/100,0,1);

    %K_CCR

    GBM_EAD_CCR_N=1.4*GBM_EEE;
    OU_EAD_CCR_N=1.4*OU_EEE;
    SL_EAD_CCR_N=1.4*SL_EEE;
```

```
    GBM_CVA_CCR_N=CVA_GBM_U;
    OU_CVA_CCR_N=CVA_OU_U;
    SL_CVA_CCR_N=CVA_SL_U;


end


    %Total CVA Add-on with IM and collateral

    GBM_KVA_CVAV=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0_
                 PD_C(1,1:end-1)]),((alpha_cap*12.5*3*_
                 (GBM_KVA_CVAV_S+GBM_KVA_CVAV_N))*(rf-rk)*dt(1)));
    OU_KVA_CVAV=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0_
                PD_C(1,1:end-1)]),((alpha_cap*12.5*3*_
                (OU_KVA_CVAV_S+OU_KVA_CVAV_N))*(rf-rk)*dt(1)));
    SL_KVA_CVAV=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0_
                PD_C(1,1:end-1)]),((alpha_cap*12.5*3*_
                (SL_KVA_CVAV_S+SL_KVA_CVAV_N))*(rf-rk)*dt(1)));

    %Total CCR with IM and collateral

    GBM_KVA_CCR=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0_
                PD_C(1,1:end-1)]),((alpha_cap*RW*1.06*12.5_
                *(max(GBM_EAD_CCR_S,GBM_EAD_CCR_N)_
                -CVA_GBM_U))*(rf-rk)*dt(1)));
    OU_KVA_CCR=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0_
               PD_C(1,1:end-1)]),((alpha_cap*RW*1.06*12.5_
               *max(OU_EAD_CCR_S,OU_EAD_CCR_N)-CVA_OU_U)_
               *(rf-rk)*dt(1)));
    SL_KVA_CCR=dot((1-[0 PD_B(1,1:end-1)]).*(1-[0_
               PD_C(1,1:end-1)]),((alpha_cap*RW*1.06*12.5_
               *max(SL_EAD_CCR_S,SL_EAD_CCR_N)-CVA_SL_U)_
               *(rf-rk)*dt(1)));
```

```
    %BCVA with IM and collateral


    CVA_GBM=(1-R_C)*dot(EE_rfM_GBMQ,(1-[0 PD_B(1,1:end-1)])_
            .*[0 Delta_PD_C(1,1:end-1)]);
    CVA_OU=(1-R_C)*dot(EE_rfM_OUQ,(1-[0 PD_B(1,1:end-1)])_
            .*[0 Delta_PD_C(1,1:end-1)]);
    CVA_SL=(1-R_C)*dot(EE_rfM_SLQ,(1-[0 PD_B(1,1:end-1)])_
           .*[0 Delta_PD_C(1,1:end-1)]);

    DVA_GBM=(1-R_B)*dot(NEE_rfM_GBMQ,(1-[0 PD_C(1,1:end-1)])_
            .*[0 Delta_PD_B(1,1:end-1)]);
    DVA_OU=(1-R_B)*dot(NEE_rfM_OUQ,(1-[0 PD_C(1,1:end-1)])_
           .*[0 Delta_PD_B(1,1:end-1)]);
    DVA_SL=(1-R_B)*dot(NEE_rfM_SLQ,(1-[0 PD_C(1,1:end-1)])_
           .*[0 Delta_PD_B(1,1:end-1)]);

    BCVA_GBM=CVA_GBM+DVA_GBM;
    BCVA_OU=CVA_OU+DVA_OU;
    BCVA_SL=CVA_SL+DVA_SL;

    Funded_Base_GBM=MTM_rf_GBMQ+MVAC_GBM+MVAB_GBM;
    Funded_Base_OU=MTM_rf_OUQ+MVAC_OU+MVAB_OU;
    Funded_Base_SL=MTM_rf_SLQ+MVAC_SL+MVAB_SL;

    Risky_GBM=Funded_Base_GBM+BCVA_GBM+GBM_KVA_CCR+GBM_KVA_CVAV;
    Risky_OU=Funded_Base_OU+BCVA_OU+OU_KVA_CCR+OU_KVA_CVAV;
    Risky_SL=Funded_Base_SL+BCVA_SL+SL_KVA_CCR+SL_KVA_CVAV;


end
```

Bibliography

[Aboura and Maillard 2014] S. Aboura and D. Maillard, *Option Pricing under Skewness and Kurtosis using a Cornish Fisher Expansion*, SSRN working paper, Université Paris XIII, Amundi Asset, December 2014.

[Antonio Castagna 2012] A. Castagna, *Yes, FVA is a Cost for Derivatives Desks*, SSRN working paper, Iason Ltd., August 2012.

[Basel III] Basel Committee on Banking Supervision, *Basel III: A global regulatory framework for more resilient banks and banking systems*, Bank for International Settlements, Decemebr 2010, available at www.bis.org.

[Björk 2009] T. *Björk*, *Arbitrage Theory in Continuous Time*, Third Edition. Stockholm: Oxford University Press., 2009.

[Black and Scholes 1973] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*. Chicago: The Journal of Political Economey, Vol. 81, No. 3, May - June, 1973, pages 637 - 654.

[BNP Paribas - GRM Risk IM] J. Brunac, *Incremental Risk Capital(IRC) and Comprehensive Risk Measure(CRM): Modelling Challenges in a Bank-wide System*. London, February 2012.

[Brigo, Morini and Pallavicini 2013] D. Brigo, M. Morini and A. Pallavicini, *Counterparty credit risk, collateral and funding*, First Edition. London: Wiley & Sons Ltd., 2013.

[Burgard and Kjaer 2011] C. Burgard and M. Kjaer, *Partial differential equation representations of derivatives with bilateral counterparty risk and funding costs*, The Journal of Credit Risk, Vol. 7, No. 3, 1-19, 2011.

[Burgard and Kjaer 2012a] C. Burgard and M. Kjaer, *CVA and FVA with funding aware close outs*, SSRN working paper, Barclays, London, 2012.

[Burgard and Kjaer 2012b] C. Burgard and M. Kjaer, *Generalised CVA with funding and collateral via semi-replication*, SSRN working paper, Barclays, London, 2012.

[Burgard and Kjaer 2012c] C. Burgard and M. Kjaer, *In the Balance*, SSRN working paper, Barclays, London, 2012.

[Elouerkhaoui 2016a] Y. Elouerkhaoui, *The funding invariance principle*, *Risk.net*, London, February 2016.

[Elouerkhaoui 2016b] Y. Elouerkhaoui, *From FVA to KVA: including cost of capital in derivatives pricing*, *Risk.net*, London, March 2016.

[Elouerkhaoui 2014] Y. Elouerkhaoui, *A Unified Framework for CVA, DVA and FVA Applied to Credit Portfolios*, Citigroup Global Markets Limited, London, 19 March 2014.

[Gatheral 2006] J. Gatheral, *The Volatility Surface: A Practitioner's Guide.* New Jersey: Wiley & Sons Ltd., 2006.

[Glasserman 2003] P. Glasserman, *Monte Carlo Methods in Financial Engineering.* New York: Springer., 2003.

[Gregory 2012, Appendix 10] J. Gregory, *Counterparty credit risk and credit value adjustment - A continuing challenge for global financial markets*, Second Edition. London: Wiley & Sons Ltd., 2012.

[Gregory 2012, Appendix 8] J. Gregory, *Counterparty credit risk and credit value adjustment - A continuing challenge for global financial markets*, Second Edition. London: Wiley & Sons Ltd., 2012.

[Gregory 2012] J. Gregory, *Counterparty credit risk and credit value adjustment - A continuing challenge for global financial markets*, Second Edition. London: Wiley & Sons Ltd., 2012.

[Gregory and German] J. Gregory and I. German, *Closing out the DVA debate*, SSRN working paper, 27 September 2012.

[Héylette Geman 2005] H. Geman, *Commodities and commodity derivatives: Modeling and Pricing for Agriculturals, Metals and Energy.* London: Wiley & Sons Ltd., 2005.

[Hull 2015] J. Hull, *Risk Management and Financial Institutions*, Fourth Edition. New Jersey: Wiley & Sons Ltd., 2015.

[Hull and White 2012a] J. Hull and A. White, *The FVA Debate*, Toronto: Univeristy of Toronto, July 2012.

[Hull and White 2012b] J. Hull and A. White, *The FVA debate continues: Hull and White respond to their critics*, Risk Magazine, October 2012.

[Hull and White 2012c] J. Hull and A. White, *CVA,DVA,FVA and the Black-Scholes-Merton Arguments*, October 2012.

[Hull and White 2014] J. Hull and A. White, *Valuing Derivatives: Funding Value Adjustments and Fair Value*, Toronto: Univeristy of Toronto, March 2014.

[Hull 2003] J. Hull, *Options, Futures and Other Derivatives*, Fifth Edition. New Jersey: Prentice Hall, 2003.

[ISDA 2009] International Swaps and Derivatives Association, *ISDA closeout protocol*, 2009, available at www.isda.org.

[Kenyon 2012] C. Kenyon and R. Stamm, *Discounting, LIBOR, CVA and Funding - Interest Rate and Credit Pricing*. England: Palgrave Macmillan, 2012.

[Kenyon, Green and Dennis 2016] C. Kenyon, A. Green and C. Dennis, *KVA: capital valuation adjustment by replication*, *Risk.net*, London, Decemeber 2014.

[Kenyon and Green 2015] C. Kenyon and A. Green, *MVA: Intial Margin Valuation Adjustment by Replication and Regression*, SSRN working paper, Scotiabank, Loyds Banking Group, London, January 2015.

[Laughton and Vaisbrot 2012] S. Laughton and A. Vaibrot, *In defense of FVA - a response to Hull and White*, Risk Magazine, September 2012.

[Longstaff and Schwartz 2001] F. Longstaff and E. Schwartz *Valuing American options by simulation: A simple least-squares approach*, Review of Financial Studies, vol. 8, no. 2, pages 107 128, 2001.

[Merton 1973] R.C. Merton, *Theory of Rational Option Pricing*. The Bell Journal of Economics and Management Science, Vol. 4, No. 1, Spring, 1973, pages 141 - 183.

[Merton 1974] R.C. Merton, *On the pricing of corporate debt: The risk structure of interest rates*. Journal of Finance, Vol. 29, No. 2, November, 1974, pages 449 - 470.

[Piterbarg 2010] V. Piterbarg, *Funding beyond discounting: Collateral agreements and derivatives pricing*, Risk Magazine, vol.23,no. 2, 2010,pages 97 - 102.

[Rubenstein and Cox 1985] M. Rubenstein and .C. Cox, *Options Market*, First Edition. New York: Prentice Hall, 1985.

[Ruiz 2015] I. Ruiz , *XVA Desks - A New Era for Risk Management*. England: Palgrave Macmillan, 2015.

[Schwartz and Smith 2000] E. Schwartz and J. Smith, *Short-Term Variations and Long-Term Dynamics in Commodity Prices*, Management Science, vol. 46, no. 7, July 2000.

[Travers and Schwob 2009] D. Travers and J. Schwob, *Wrong Way Risk*. Sungard Adaptiv, February 2009.