Modular properties of higher spin black holes

by

Sipho Sihle Lukhozi

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UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Declaration

I, **Sipho Sihle Lukhozi**, declare that the dissertation, which I hereby submit for the degree **MSc (Physics)** at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature:........................
Date:.............................

Supervisor: Prof. Konstantinos Zoubos
Department of Physics
University of Pretoria,
South Africa.
February 2017

To Her - the one in whom we live, move and have our being.

Abstract

This project concerns the understanding of the thermodynamic properties of higher spin black holes in AdS_3. Through the conjectured Minimal Model/Higher Spin duality, the partition functions of such black holes should be equal to partition functions in the presence of chemical potentials in an appropriate dual $2d$ CFT. In the well-understood spin-2 case, the modular properties of partition functions play a major role in understanding their behaviour at high and low temperatures, and the goal of this dissertation is to obtain an analogous understanding of modular properties in the higher-spin case.

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Contents

1 Introduction

Black hole thermodynamics is a subject that has continued to fascinate physicists ever since the suggestion, by Bekenstein in 1972, that black holes carry entropy, which was confirmed by Hawking's discovery that black holes emit thermal radiation and thus have temperature [1]. The goal is to understand precisely what the micro-states of the black hole are, and why the entropy scales proportionally to the area of the black hole, unlike all other physical systems, where it is proportional to the volume.

The AdS/CFT correspondence [2] provides an avenue to address this question, at least for certain types of black holes residing on Anti-de Sitter (spaces with negative cosmological constant) spaces (AdS). It conjectures that there exists a dual conformal field theory (CFT) living on the boundary of the AdS space which should describe the same information as the black hole geometry but where the degrees of freedom are encoded very differently.

The relationship between gravity on AdS_3 and $2d$ conformal field theory was first exhibited by the work of Brown and Henneaux [4] and used to understand the thermodynamics of the BTZ black hole by A Strominger [5] and S Carlip [6]. This understanding hinges on the modular properties of partition functions in the CFT, which allow a transformation from low to high temperatures which is needed in order to specify the entropy.

Recently, the Brown-Henneaux duality was generalized to Higher Spin theory on AdS_3, which was conjectured to be dual to a specific type of CFT with W_N-algebra symmetry [7]. This conjecture has generated intense interest and has led to a better understanding of the properties of higher spin black holes [9, 10], and matching of their partition functions with corresponding partition functions with chemical potential on the CFT side [11]. Many of the results are the outcome of brute-force calculations of expansions in the chemical potential. In [13] Beccaria and Macorini do these brute force computations up to order $\mathcal{O}(\alpha^{18})$. A better understanding of their modular properties, which is the goal of this project, would perhaps lead to a discovery of a closed formula for these partition functions and thus a better understanding of higher - spin black hole thermodynamics.

In what follows we review some basic techniques of general relativity and make a few comments about (2+1) dimensional gravity, in particular its degrees of freedom. We then proceed into formulating three dimensional gravity in $sl(2,R)$ as a Chern-Simons theory and write the solution of the Einstein equations which is asymptotic AdS_3 for which with certain identifications we find the BTZ black hole. In section 3 and 4 we generalize the Chern-Simons formulation to $sl(N,R)$ and derive a smooth black hole. In section 5 we review the basics of modular invariance and consider an example from minimal models and the Cardy formula. In section 6 we consider an infinite number of higher spins by considering a one-parameter family of higher spin algebras $hs[\lambda]$. Lastly, in section 7 we comment about the modular properties of these higher spin black hole partition functions using the generalized Jacobi forms of Kaneko-Zagier.

2 A brief introduction to gravity

Gravity was understood to be a pulling force or a force of attraction between two bodies as a result of their mass and the distance between them. Newton, though he discovered the classical concept of gravity, did not understand how it was communicated between two bodies. The formal description of this force was published in 1687, which was a milestone in the history of Physics, though the manner in which this force of gravity was communicated between objects was not clear as yet. That task was left to one of the greatest minds of the 20$^{\text{th}}$ century, Albert Einstein. In 1915, Albert Einstein produced a set of equations that predicted how gravity is communicated between objects. Published in 1916, the theory of general relativity suggested that gravity was the result of the curvature of space-time, space-time being described as a manifold which is locally flat, by matter and thus energy. This resulted in a revolutionary understanding that gravity is communicated by the curvature of space-time. This theory, which was first tested and verified in 1919 and 1922, also perfectly described the strange orbit of Mercury which was not closing on itself, a problem that confronted astronomers for decades.

This section begins by describing some of the key concepts of General relativity which are relevant to describing gravity, i.e. curvature. The degrees of freedom of the key parameter which describes curvature in Einstein's equations, the Riemann tensor and its contractions, is also discussed.

2.1 The Einstein equations

The theory of gravity is described by the Einstein-Hilbert action [14]

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\,(R-2\Lambda)+\mathcal{L}_M \tag{2.1}$$

where Λ is the cosmological constant, $\mathcal{L}_M$ represents matter fields and R is the Ricci scalar. In the presence of matter, the Einstein-Hilbert action yields Einstein's field equations [15]

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=8\pi GT_{\mu\nu} \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor (to be defined in the next section) and R is the Ricci scalar; $g_{\mu\nu}$ the metric tensor which gives information about the configuration of space-time. $T_{\mu\nu}$ the energy momentum tensor which contains information about matter and energy in space-time. Equation (2.2) describes the reaction of space-time curvature to the presence of energy-momentum. Since both sides of (2.2) are symmetric two-index tensors, there are ten independent equations (in $4d$) which match the ten unknown functions of the metric components of the Bianchi identity

$$\bigtriangledown^{\mu}G_{\mu\nu}=0 \tag{2.3}$$

which constrains four of the functions of $R_{\mu\nu}$ [15]. This means that there are six independent equations in (2.2). This means that of the 10 degrees of freedom, four are non-physical and have to do with the gauge redundancy of the coordinate system.

2.2 Riemann tensor

The Ricci tensor is the contraction of the Riemann tensor, as already mentioned above. In standard notation, it is given by [14, 15]

$$R^{\rho}_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^{\rho}_{\nu\mu} + \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} \tag{2.4}$$

It associates a tensor value to each point of a manifold; the tensor measures the extent to which each point on a manifold deviates from that of flat space. Γ is the Christoffel symbol, defined in terms of the metric tensor as follows [15]

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}) \tag{2.5}$$

In other literature, the Christoffel symbol is called the connection coefficient because it connects the value of a vector field at one point with the value at another, thus making it possible to define covariant derivatives on a curved manifold [14]. The Ricci tensor is a symmetric $\binom{0}{2}$ tensor which is the trace of the Riemann tensor on the first and third indices,

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = g^{\rho\lambda}R_{\rho\mu\lambda\nu} \tag{2.6}$$

The Ricci scalar or curvature scalar (this means it is the same in all coordinate systems - unlike $R_{\mu\nu}$) is the contraction of the Ricci tensor with the metric tensor,

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.7}$$

The Riemann tensor has another construct, known as the Weyl tensor; given in terms of the Ricci tensor, the Ricci scalar and the metric tensor [14]

$$C_{\rho\lambda\mu\nu} = R_{\rho\lambda\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\lambda} - g_{\lambda[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}Rg_{\rho[\mu}g_{\nu]\lambda} \tag{2.8}$$

where $_{[\mu}R_{\nu]}$ is from the fact that we can symmetrize or anti-symmetrize upper or lower indices of any given tensor. To symmetrize (anti-symmetrize) respectively

$$\begin{aligned} R^{\sigma}_{(\mu_1\mu_2\cdots\mu_n)\rho} &= \frac{1}{n!}(R^{\sigma}_{\mu_1\mu_2\cdots\mu_n\rho} + \quad \text{sum over permutations of indices} \quad \mu_1\cdots\mu_n) \\ R^{\sigma}_{[\mu_1\mu_2\cdots\mu_n]\rho} &= \frac{1}{n!}(R^{\sigma}_{\mu_1\mu_2\cdots\mu_n\rho} + \quad \text{alternating sum over permutations of indices} \quad \mu_1\cdots\mu_n) \end{aligned} \tag{2.9}$$

The Weyl tensor has been constructed such that all contractions of $C_{\rho\sigma\mu\nu}$ vanish, so it is traceless, while retaining the symmetries of the Riemann tensor (more will be said about this later). The Ricci tensor communicates how the volume of the body changes in curved space-time whilst the Weyl tensor, since it is traceless, does not communicate the distortion of the volume but only how the shape of the object is changed by curved space-time. The Weyl tensor is also known as the *conformal tensor* because of its interesting feature of invariance under conformal transformations, i.e. scaling of the metric tensor.

2.3 Symmetries of the Riemann tensor

The Riemann tensor has the following properties [14]

$$\begin{aligned} R_{\rho\lambda\mu\nu} &= -R_{\rho\lambda\nu\mu} \\ R_{\rho\lambda\mu\nu} &= -R_{\lambda\rho\mu\nu} \\ R_{\rho\lambda\mu\nu} &= +R_{\mu\nu\rho\lambda} \\ 3R_{\rho[\lambda\mu\nu]} &= R_{\rho\lambda\mu\nu} + R_{\rho\mu\nu\lambda} + R_{\rho\nu\lambda\mu} = 0 \end{aligned} \tag{2.10}$$

Like any other tensor, the degrees of freedom of the Riemann tensor depend on the space-time dimension of the manifold it is describing. Since the Riemann tensor is describing a 4-dimensional space-time and has rank 4, we can out-rightly deduce that it has $4^4 = 256$ components but it is not all of these components which are independent. Of the 16 combinations of the last two indices, only 6 are independent because 4 of them are zero because of the antisymmetry of the last two indices as showed in the first property. The second property also gives the same number of independent components. This is because an antisymmetric matrix has $\frac{1}{2}d(d-1)$, $4 \times \frac{3}{2} = 6$ independent components. Now we can write

$$R_{\rho\lambda\mu\nu} = R_{AB}$$

where we have replaced the first and the last pair with A and B respectively. We can think of R_{AB} as a symmetric 6×6 matrix which has 21 independent components since a symmetric matrix has $\frac{1}{2}d(d+1)$. The Riemann tensor is not only antisymmetric in its last two indices but also in the first two. It is the exchange of the two pairs which is symmetric. A symmetric matrix has $\frac{1}{2}d(d+1)$ independent components whilst an antisymmetric matrix has $\frac{1}{2}d(d-1)$ independent components. This means that the Riemann tensor has a total of $\frac{1}{12}d^2(d^2-1)$ independent components because an antisymmetric 4-index tensor has $\frac{1}{4!}d(d-1)(d-2)(d-3)$. The last property imposes only one constraint; that $R_{\lambda[\rho\mu\nu]} = -R_{\rho[\lambda\mu\nu]}$, this also holds for other indices. This means that $R_{\rho\lambda\mu\nu}$ has only 20 linearly independent components out of a total of 256 components.

2.4 Black holes in $(2+1)$ dimensional gravity

The phenomenology of the Universe we perceive is evidently described by standard three spatial and one time dimensions (four dimensional space-time). However, the Einstein-Hilbert action and its variants that defines the theory of gravity holds in any dimensions. The three dimensional case (two spatial and one time) is attractive because the quantum theory in this case has been well understood. The behaviour of gravity in three dimensions is fairly simpler than in four or higher dimensions. This is due to the fact that three dimensional gravity has no local degrees of freedom , i.e. it has no local propagating modes - higher spin gauge fields in $3d$ also have no propagating degrees of freedom in the bulk. In the absence of a cosmological constant, a vacuum, or locally (A)dS in the presence of a (negative) positive cosmological constant the Einstein equations fix the metric to be locally flat. The important case for our purpose is the case with negative cosmological constant, i.e. Anti-de Sitter (AdS). The AdS case admits black hole solutions with thermodynamical properties and plays a defining role in the AdS/CFT correspondence. In this section we show how $3d$ gravity has no local degrees of freedom.

The Weyl tensor, the Riemann tensor with all contractions removed, will be a tool for showing how 3d gravity has no degrees of freedom. The Weyl tensor is only defined in three or higher dimensions. It vanishes in three dimensions and is traceless by construction. This means;

$$C_{\rho[\mu\rho\nu]} = 0 \tag{2.11}$$

Since the Weyl tensor is a symmetric matrix its components are restrained by $\frac{1}{2}d(d+1)$. The Riemann tensor has a total of $\frac{1}{12}d^2(d^2-1))$ independent components. Therefore, to get the total number of independent components for the Weyl tensor, which has all the symmetries of the Riemann tensor we take the following difference;

$$\frac{1}{12}d^2(d-1) - \frac{1}{2}d(d+1) \tag{2.12}$$

which can be readily seen that for $d = 3$ the difference yields 0. Meaning the Weyl tensor has no independent components; thus proving that there are no propagating modes since there are zero degrees of freedom. We also need to show that the remaining six independent components of the Riemann tensor in $d = 3$ are fully constrained by the Einstein equations such that the independent components of the Riemann tensor do not constitute propagating modes. We consider the vacuum,$T_{\mu\nu} = 0$, Einstein equation with non-vanishing cosmological constant;

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.13}$$

Taking the trace of this equation, we find that $R = 6\Lambda$ because $g_{\mu\nu}g^{\mu\nu} = 4$. Substituting R back into the Einstein equation, we find that the Ricci tensor becomes

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \tag{2.14}$$

The Ricci tensor and the metric are two index symmetric tensors. This means that the above expression gives six independent equations that serves as constraints of the Riemann tensor as anticipated.

2.5 Motivation for higher spin

In the tensionless limit of string theory, all higher spin $s > 2$ excitations become massless as will be sketched below. Higher spin gravity, which is a theory of interacting massless fields deals with fields with spin $s > 2$. Two words are worth a bit of unpacking in the previous sentence - interacting and massless. The word interacting means the theory is not free, i.e. it has to be coupled to gravity. Massless means that we have gauge symmetry, redundant degrees of freedom in our system. To show that excitations become massless in the tensionless limit of string theory, we consider the mass-squared of string excitations given by

$$m^2 \sim \frac{N}{\alpha'} \tag{2.15}$$

where $\alpha = 1/T$, is defined as the inverse string tension. We normally do string theory in the limit $\alpha' \to 0$, but we see that the m^2 excitations explode unless we set $N \to 0$. Meaning that we only have to focus on the massless stringy state. So in the limit $\alpha' \to 0$ we truncate our string theory to super-gravity (massless modes). We can also consider the limit $\alpha' \to \infty$. It is trivial to see that we will also have massless excitations, some physicists believe that a theory at work here is higher spin gravity theories. So Higher spin gravity theories might actually be string theories.

In the bulk, the gravity theory with spin $s > 3$ is described by Vasiliev higher spin theory, which has been constructed in various dimensions. In four dimensions, Vasiliev theory is described by some master equations, for which AdS_4 is a solution. Expanding the equation around the vacuum, we get an infinite tower of excitations with all spins. The holographic dual is the singlet sector of $O(N)$ vector model as proposed by Klebanov and Polyakov [16]. In three dimensions, the gravity theory possesses all higher spin fields and some scalar fields with mass parameterized by a continuous parameter λ - this is usually referred to as higher spin gravity in AdS_3. The CFT dual is conjectured to be the large N limit of W_N minimal model (more on these in chapter 4) by Gaberdiel and Gopakumar [7].

3 Anti-de Sitter space and the BTZ black hole

Anti-de Sitter space (AdS) and de Sitter space (dS) are solutions to the vacuum Einstein equations (2.13). The Ricci tensor, as shown in (2.14) is proportional to the metric in these spaces. The difference between AdS and dS is determined by the sign of Λ, the cosmological constant. The cosmological constant, which Einstein called his biggest blunder, was added by Einstein to his equation so that they described a static universe. He called it his biggest blunder because it was later shown that the universe was not static but expanding. But the expanding universe does not make the cosmological constant useless in the Einstein equations because it helps account for the unconventional matter which contributes to the measured acceleration of the expanding universe.

3.1 AdS$_3$ in global coordinates

An $(n+1)$ dimensional AdS$_{\mathrm{n+1}}$ can be considered as an embedding in $(n+2)$ dimensional flat space. For AdS$_3$ the embedding equation is [17]

$$-v^2 - u^2 + x^2 + y^2 = -l^2 \tag{3.1}$$

where $-l^2 = 1/\Lambda$ is the measure of the curvature of the space-time, called the radius of curvature, which is always constant for our purposes. It is clear that the corresponding embedded metric is given by

$$-dv^2 - du^2 + dx^2 + dy^2 = ds^2 \tag{3.2}$$

However, the metric of AdS$_3$ can be derived without making reference to a higher dimensional space by making the following substitutions

$$v = l\cosh\mu\cos\lambda$$

$$u = l\cosh\mu\sin\lambda$$

$$l\sinh\mu = \sqrt{x^2+y^2}$$

Converting to polar coordinates using

$$x = l\sinh\mu\cos\theta, \qquad y = l\sinh\mu\sin\theta \tag{3.3}$$

yields the following metric

$$ds^2 = -l^2\cosh^2\mu d\lambda^2 + l^2 d\mu^2 + \sinh^2\mu d\theta^2 \tag{3.4}$$

where λ is the time-like coordinate; it can also be identified by a negative signature. The above transformation (3.4), however, makes λ a periodic angle, $\lambda = \lambda + 2\pi$. Meaning there are closed time-like curves in space-time. To correct this and unidentify λ with $\lambda + 2\pi$; we identify λ with a time variable.

$$\lambda = \frac{t}{l} r = l\sinh\mu \tag{3.5}$$

Substituting back into (3.2) yields the universal covering space (the space we get when a coordinate is unrolled) of AdS$_3$

$$ds^2 = -(1+\frac{r^2}{l^2})dt^2 + \frac{1}{1+\frac{r^2}{l^2}}dr^2 + r^2\theta^2 \tag{3.6}$$

This is the metric in global coordinates which is called AdS, the general solution of the field equations up to coordinate transformations and global identifications.

The metric can also be written in another coordinate system, the Poincaré coordinates also referred to as hyperbolic coordinates (H^3). Poincaré coordinates define a model of hyperbolic space on the upper-half plane. The following substitutions from the embedding metric yields the Poincaré coordinates

$$z = \frac{l}{u+x}, \qquad \beta = \frac{y}{u+x}, \qquad \gamma = \frac{-v}{u+x} \tag{3.7}$$

with metric

$$ds^2 = \frac{l^2}{z^2}(dz^2 + d\beta^2 + d\gamma^2) \tag{3.8}$$

AdS$_3$ is maximally symmetric with isometry group $SO(2,2)$ [18], the group of all orientation and distance preserving linear transformations, rotations. A maximally symmetric space-time is one which has the largest possible number of Killing vectors (to be explained below). $SO(2,2)$ is isomorphic to $SL(2,R)_{\mathrm{L}} \times SL(2,R)_R$, the group of real, invertible matrices with unit determinant. $SL(2,R)$ acts on the fundamental domain, located in the complex upper-half plane with the following representation

$$ad - bc = 1$$

and transforms like

$$\tau \to \frac{a\tau + b}{c\tau + b} \tag{3.9}$$

where τ as a group parameter is used in anticipation of modular transformations which will be discussed later in this paper.

Killing vectors are infinitesimal generators of a manifold's isometries. Moving in the direction of a Killing vector in a manifold preserves the metric. Killing vectors also imply conserved currents, using Noether's theorem, associated with the motion of free particles. Killing vectors are related, by a linear transformation, to group generators since both are infinitesimal generators of a manifold. This linear relationship is the commutation relations between the symmetries' infinitesimal generators, the Lie algebra. AdS$_3$ Killing vectors are linear combinations of the generators of $SL(2,R)_L \times SL(2,R)_R$. We choose an explicit basis for $SL(2,R)$ generators

$$L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.10}$$

and they obey the following commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} \tag{3.11}$$

In [17] the Killing vectors of $SO(2,2)$ in the space-time coordinates are given to be

$$J_{\mu\nu} = x_\nu \frac{\partial}{\partial x^\mu} - x_\mu \frac{\partial}{\partial x^\nu} \tag{3.12}$$

They can also be given in terms of the original embedding coordinates as in [17]

$$\begin{aligned} J_{xy} &= y\partial x - x\partial y, & J_{uy} &= y\partial u + u\partial y \\ J_{ux} &= x\partial u + u\partial x, & J_{vy} &= y\partial v + v\partial y \\ J_{vu} &= v\partial u - u\partial v, & J_{vx} &= x\partial v + v\partial x \end{aligned} \tag{3.13}$$

The relationship with $SL(2,R)$ generators is given for example in [19], giving only the chiral

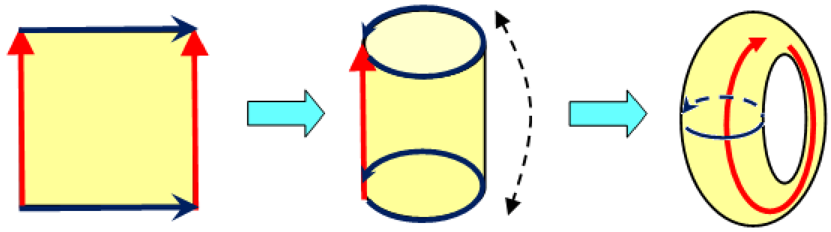

Figure 1: quotient torus

sector[1], to be

$$
\begin{aligned}
L_0 &= \frac{1}{2}(J_{ux} + J_{vy}) \\
L_{-1} &= \frac{1}{2}(J_{xy} - J_{uv} - J_{xv} + J_{yu}) \\
L_1 &= \frac{1}{2}(J_{xy} - J_{uv} + J_{xv} - J_{yu})
\end{aligned} \tag{3.14}
$$

3.2 The BTZ Black hole identifications

To find BTZ (black hole solutions) from AdS_3, certain identifications have to be made. By identification we mean parametrizing the metric patch-way in a certain way and then identifying a coordinate. These identifications are done by considering a Killing vector ξ that defines a one parameter subgroup of isometries of anti de-Sitter space [17, 21]

$$
P \longrightarrow e^{t\xi}P, \qquad t = 0, 2\pi, 4\pi, 6\pi, \cdots \tag{3.15}
$$

It is worth noting that the transformations (3.15) are isometries and that the quotient space obtained by identifying points on a specific orbit of the identification group will inherit from anti-de Sitter space a well defined metric with negative cosmological constant. Therefore the quotient space, formed by topologically identifying points with each other (Figure 1.) remains a solution of the Einstein equations. For the quotient space to preserve causality, these closed geodesics should only be space-like,

$$
\xi \cdot \xi > 0 \tag{3.16}
$$

the above condition (3.16) is not sufficient because some patches of AdS_3 contain regions where the identification Killing vectors are timelike or null. These regions must be excluded from our solution to make the identifications permissible. This also means that the resulting space AdS is 'geodesically incomplete' because there exists geodesics that are $\xi \cdot \xi = 0$ and $\xi \cdot \xi < 0$. However, from the point of view of AdS, removing the timelike and null regions before we make the identifications serves no purpose. But when this is done after we have made the identifications, [17, 21] notes that the frontier of the region $\xi \cdot \xi > 0$, which is the surface $\xi \cdot \xi = 0$ can be identified as a singularity in the causal structure of space-time. Extending beyond it produces closed time-like geodesics.

$$
\xi = \frac{r_+}{l} J_{ux} - \frac{r_-}{l} J_{vy}
$$

[1] A chiral sector is a sector which acts in a parity asymmetric fashion [22]

where $r_\pm$ are parametrizing constants. For the structure of the space-time to be causal, $\xi \cdot \xi > 0$ as mentioned earlier. This means

$$\xi \cdot \xi = \frac{r_+^2}{l^2}(u^2 - x^2) + \frac{r_-^2}{l^2}(v^2 - y^2) \tag{3.17}$$

$$\Rightarrow \xi \cdot \xi = \frac{r_+^2 - r_-^2}{l^2}(u^2 - x^2) + r_-^2 \tag{3.18}$$

$$\Rightarrow -\frac{r_-^2}{l^2} < (u^2 - x^2) < 0 \tag{3.19}$$

The space-like region $\xi \cdot \xi > 0$ can be partitioned into three different types of regions divided by null surfaces $u^2 - x^2 = 0$ or $v^2 - y^2 = l^2 - (u^2 - x^2) = 0$.

These three regions are:
1. Outer regions - $u^2 - x^2 > l^2$, y and u are of definite sign. The norm of the Killing vector obeys $r_+^2 < \xi \cdot \xi < \infty$.
2. Intermediate regions - $0 < u^2 - x^2 < l^2$, u and v are of definite sign. The norm of the Killing vector obeys $r_-^2 < \xi \cdot \xi < r_+^2$.
3. Inner regions - $0 > u^2 - x^2 > -\frac{r_-^2 l^2}{r_+^2 - r_-^2}$, x and v of definite sign. The norm of the Killing vector obeys $0 < \xi \cdot \xi < r_-^2$.

We parametrize the regions by (t, r, ϕ) in the following way:
1. $r_+ < r$

$$\begin{aligned} u &= \sqrt{\mathcal{A}(r)} \cosh \chi(t, \phi) \\ x &= \sqrt{\mathcal{A}(r)} \sinh \chi(t, \phi) \\ y &= \sqrt{\mathcal{B}(r)} \cosh \kappa(t, \phi) \\ v &= \sqrt{\mathcal{B}(r)} \sinh \kappa(t, \phi) \end{aligned} \tag{3.20}$$

2. $r_- < r < r_+$

$$\begin{aligned} u &= \sqrt{\mathcal{A}(r)} \cosh \chi(t, \phi) \\ x &= \sqrt{\mathcal{A}(r)} \sinh \chi(t, \phi) \\ y &= -\sqrt{-\mathcal{B}(r)} \cosh \kappa(t, \phi) \\ v &= -\sqrt{-\mathcal{B}(r)} \sinh \kappa(t, \phi) \end{aligned} \tag{3.21}$$

3. $0 < r < r_-$

$$\begin{aligned} u &= \sqrt{-\mathcal{A}(r)} \cosh \chi(t, \phi) \\ x &= \sqrt{-\mathcal{A}(r)} \sinh \chi(t, \phi) \\ y &= -\sqrt{-\mathcal{B}(r)} \cosh \kappa(t, \phi) \\ v &= -\sqrt{-\mathcal{B}(r)} \sinh \kappa(t, \phi) \end{aligned} \tag{3.22}$$

where

$$\mathcal{A}(r) = l^2 \frac{r^2 - r_-^2}{r_+^2 - r_-^2}, \qquad \mathcal{B}(r) = l^2 \frac{r^2 - r_+^2}{r_+^2 - r_+^2} \tag{3.23}$$

$$\chi = \frac{1}{l}\left(\frac{-r_- t}{l} + r_+ \phi\right), \qquad \kappa = \frac{1}{l}\left(\frac{r_+^2 t}{l}\right) \tag{3.24}$$

$$y = -\sqrt{-\mathcal{B}(r)} \cosh \kappa(t, \phi), \qquad v = -\sqrt{-\mathcal{B}(r)} \sinh \kappa(t, \phi) \tag{3.25}$$

These three regions give us the same metric

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2} dt^2 + \frac{l^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 (d\phi - \frac{r_+ r_-}{l r^2} dt)^2 \tag{3.26}$$

where $r \in (0, \infty)$, $t \in (-\infty, \infty)$, and $\phi \in (-\infty, \infty)$. Which gives us a space-time with the metric in the following form

$$-N^2(r) dt^2 + N^{-2}(r) dr^2 + r^2 (d\phi + N^\phi(r) dt)^2 \tag{3.27}$$

which is a rotating black hole solution with singularities at $r = r_+$ and $r = r_-$. To honestly get the black hole metric as claimed above, the non-periodic coordinate ϕ must be identified as

$$\phi = \phi + 2\pi \tag{3.28}$$

Otherwise, we have a portion of anti de-Sitter space and the horizon that of an accelerated observer. To test whether our black hole singularity is not a coordinate singularity, we do a coordinate transformation and observing whether the smoothness of the spacetime is preserved by the new coordinates. Another test would be observing the behaviour of curvature invariant scalars, because they are coordinate independent they show when there are singularities in the curvature. One such scalar, following [18], is the Kretschmann scalar

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \tag{3.29}$$

The Kretschman scalar for the BTZ black hole is given by

$$K = \frac{12}{l^4} \tag{3.30}$$

which readily shows no curvature singularity. Unlike the Schwarzschild black hole metric

$$ds^2 = (1 - \frac{r_s}{r}) dt^2 - \frac{1}{(1 - \frac{r_s}{r})} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{3.31}$$

Its Kretschmann scalar is given by

$$K = \frac{48 G^2 M^2}{c^4 r^6} \tag{3.32}$$

which shows a curvature singularity for $r \to 0$. So we have seen that our black hole has no curvature singularity. BTZ is still a black hole, but with singularity in its causality rather than its curvature. Causal curves that go through the null surfaces at $r = r_\pm$ which separate the three regions can never return. This feature, known as causality singularity is equivalent to a black hole.

3.3 BTZ black hole in Euclidean signature

To obtain BTZ in Euclidean signature we make the following transformations

$$t \to it, \qquad r_- \to ir_-, \tag{3.33}$$

$$M \to M_E, \qquad J \to iJ_E$$

with the corresponding metric

$$ds^2 = \frac{(r^2 - r_+^2)(r^2 + r_-^2)}{l^2 r^2} dt^2 + \frac{l^2 r^2}{(r^2 - r_+^2)(r^2 + r_-^2)} dr^2 + r^2 (d\phi + \frac{r_+ r_-}{l r^2} dt)^2 \tag{3.34}$$

As we can see, this metric is well posed only for the region $r > r_+$, where r_- becomes purely imaginary. To express the temperature of the black hole in terms of the energy we demand that the horizon is smooth and that the coordinate possesses no conical singularities in metric formulation. We examine the near-outer horizon geometry following [23] by considering the following transformations

$$t_E^{'} = r_+ t_E + r_- \phi, \qquad \phi^{'} = r_+ \phi - r_- t_E, \qquad r^{'2} = \frac{r^2 - r_+^2}{r_+^2 + r_-^2} \tag{3.35}$$

which after substituting in (3.34) give

$$ds_E^2 = \frac{r^{'2}}{l^2} dt_E^{'2} + \frac{l^2 dr^2}{1 + r^{'2}} + (1 + r^{'2}) d\phi^{'2} \tag{3.36}$$

Taking the near horizon limit, $r^{'} \to 0$, or $r \to r_+$, the non-angular part becomes the metric of a plane

$$ds_E^2 = r^{'2} dt_E^{'2} + dr^{'2} \tag{3.37}$$

For $t_E^{'}$ not to be singular, we need to identify as angular, $t \backsimeq t_E^{'} + 2\pi$. Considering the fact that ϕ does not have to be periodic, we substitute back into the original coordinates we find that if β and Φ are the periodicities in the t_E and ϕ direction respectively, then

$$\beta = \frac{2\pi l r_+}{r_+^2 + r_-^2}, \qquad \Phi = \frac{2\pi l^2 r_+}{r_+^2 + r_-^2} \tag{3.38}$$

Now we have identified two cycles, in the ϕ and the t_E directions, and a radial component - which is simply a topology of a solid torus. Thus, the Euclidean BTZ black hole has a solid torus topology with a non-contractible cycle in the ϕ direction, and a contractible cycle in the Euclidean time-cycle, t_E. The Hawking temperature of the black hole is given by

$$T = \frac{1}{\beta} = \frac{r_+^2 + r_-^2}{2\pi l r_+} \tag{3.39}$$

The Bekenstein-Hawking entropy being

$$S = \frac{A}{4G} = \frac{2\pi r_+}{4G} \tag{3.40}$$

where A is the area of the black hole's event horizon. The metric (3.36) can be cast as a quotient of the three dimensional hyperbolic plane H^3. In [37] the metric in Poincaré coordinates is given by

$$ds^2 = \frac{l^2}{z^2}(dw d\bar{w} + dz^2), z > 0 \tag{3.41}$$

with

$$\begin{aligned} w &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{\left\{\frac{r_+}{l}\phi + \frac{ir_-}{l}\tau\right\}} \\ z &= \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}} e^{\left\{\frac{r_+}{l}\phi + \frac{ir_-}{l}\tau\right\}} \end{aligned} \tag{3.42}$$

Euclidean AdS_3 can be written as a matrix element of $SL(2, C)$ as in [37]

$$\mathcal{M} = \begin{pmatrix} z + \frac{w\bar{w}}{z} & \frac{w}{z} \\ \frac{\bar{w}}{z} & \frac{1}{z} \end{pmatrix} \tag{3.43}$$

and the metric is given by

$$ds^2 = \frac{l^2}{2} Tr(\mathcal{M}^{-1} d\mathcal{M} \mathcal{M}^{-1} d\mathcal{M}) \tag{3.44}$$

which is simply (3.41).

3.4 Chern-Simons formulation of 3d AdS gravity

Three dimensional gravity can be formulated in an alternative way known as Chern-Simons theory. A Chern-Simons theory is a demo quantum model of 3d gravity that only computes topological invariants. Witten [26], Achucarro and Townsend [27] were the first to attempt to find a quantum theory of gravity by interpreting (2+1) dimensional gravity as a Chern-Simons theory with a level k.

The BTZ solution is a black hole with inner/outer radius horizons at $r = r_\pm$. We further make the following identifications to ensure smoothness of the Euclidean signature metric at $r = r_+$

$$w \cong w + 2\pi, \qquad \tau = \frac{il}{r_+ + r_-} \tag{3.45}$$

where $w = \phi + \frac{it}{l}$; which makes the conformal boundary of the BTZ black hole to be identified as a torus of modular parameter τ. To write the BTZ solution in Chern-Simons form we need to first write it in the Fefferman-Graham form [25], which also makes it easier to read off the stress tensor; to go to the Fefferman-Graham form we require that (3.34) be written as

$$ds^2 = d\rho^2 + g_{ij}(x,\rho)dx^i dx^j \tag{3.46}$$

with

$$g_{ij}(x,\rho) = e^{2\rho/l} g_{ij}^{(0)} + g_{ij}^{2} + \cdots, \quad \rho \to \infty \tag{3.47}$$

where $g_{ij}^{(0)}$ is the metric at the conformal boundary, $g_{ij}^{(2)}$ will turn out to be related to the stress tensor as will be clear later. We focus on the first term contribution to the metric

$$g_{ij}^{(0)} \mathrm{d}x^i \mathrm{d}x^j = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 = \mathrm{d}x^+ \mathrm{d}x^-$$

$$\mathrm{x}^+ = \omega = \phi + \frac{i\tau}{l} = \mathrm{x}^1 - \mathrm{x}^0$$

$$\mathrm{x}^- = \bar{\omega} = \phi - \frac{i\tau}{l} = \mathrm{x}^1 + \mathrm{x}^0$$

The metric in Fefferman-Graham form is given by

$$ds^2 = d\rho^2 + 8\pi Gl(\mathcal{L}dw^2 + \bar{\mathcal{L}}d\bar{w}^2) + (l^2 e^{\frac{2\rho}{l}} + (8\pi G)^2 \mathcal{L}\bar{\mathcal{L}} e^{\frac{-2\rho}{l}})dw d\bar{w} \tag{3.48}$$

where

$$r^2 = l^2 e^{2\rho} + \frac{e^{-2\rho}}{4l^2}(r_+^2 - r_-^2)^2 + \frac{r_+^2 + r_-^2}{2} \tag{3.49}$$

The constants $\mathcal{L}$ and $\bar{\mathcal{L}}$, are the rescaled Virasoro[2] zero modes related to the horizon $r_\pm$ as

$$\mathcal{L} = \frac{1}{2\pi} L_0, \qquad \bar{\mathcal{L}} = \frac{1}{2\pi} \bar{L}_0 \tag{3.51}$$

[2]the Virasoro algebra is the algebra formed by the conserved charges (generators in a quantum field theory),L_n, in the symmetry of conformal field theories in two-dimensional conformal field theories. The generators obey the following commutation relations

$$\begin{aligned} [L_m, Ln] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\ [L_m, \bar{L}_m] &= 0 \end{aligned} \tag{3.50}$$

where $\bar{L}_m$ is the anti-holomorphic part and c is the central charge of the CFT.

$$L_0 = \frac{(r_+ - r_-)^2}{16Gl}, \qquad \bar{L}_0 = \frac{(r_+ + r_-)^2}{16Gl} \tag{3.52}$$

The area law can be calculated to be $S = \frac{A}{4G}$ (Appendix A), the black hole entropy (Bekeinstein-Hawking)

$$S = 2\pi\sqrt{\frac{c}{6}L_0} + 2\pi\sqrt{\frac{c}{6}\bar{L}_0} \tag{3.53}$$

The Chern-Simons action is given as

$$S_{CS}[A] = \frac{k}{4\pi}\int Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{3.54}$$

where A is a 1-form taking values in the theory's Lie algebra of $SL(2,R)$, Tr is the matrix trace in $SL(2,R)$ and k is the Chern-Simons level, upon equating the Chern-Simons action to the Einstein-Hilbert action we find k to be

$$k = \frac{l}{4G} \tag{3.55}$$

For computations, however, we consider the difference between Chern-Simons functionals

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \tag{3.56}$$

Varying (3.54) with respect to A

$$\delta S_{CS}[A] = \frac{k}{4\pi}\int \delta Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{3.57}$$

gives us the equations of motion which can be seen to correspond to vanishing field strengths

$$F = dA + A \wedge A = 0 \tag{3.58}$$

which is the holomorphic part. F is also known as the curvature 2-form; therefore if F vanishes that is indicative of the flatness of the 1-forms (A). To relate 3D gravity to Chern-Simons theory we need to realize that a Chern-Simons theory is completely described by its topology and global symmetry group in the same way that 3D gravity is since it has no local degrees of freedom. We start by writing the geometry of the manifold in a coordinate dependent basis because we are trying to describe curvature. This basis, for our purposes is the vielbein formalism (frame formulation) - which assigns for every point on the manifold a set of orthonormal basis vectors such that the metric, in vielbein formalism can be written as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{3.59}$$

it can be seen from (3.59) that the veilbein e (dreibein in 3d) is the square root of the metric tensor. Whilst η_{ab} is the Minkowski metric (flat space). We also define differentiation in the new non-coordinate basis formalism by introducing what is called the spin connection given by

$$\nabla_\mu X^a_b = \partial_\mu X^a_b + \omega^a_{\mu c}X^c_b - X^a_c \omega^c_{\mu b} \tag{3.60}$$

The Latin and the Greek indices denote the coordinate independent basis and the coordinate dependent, the veilbein formalism, coordinate. The relationship between the spin connection ω, the coordinate independent basis differentiation, and Γ, the coordinate dependent basis differentiation comes as a natural question and is given by

$$\omega^a_{\mu b} = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda \tag{3.61}$$

It is straightforward, using wedge products and exterior derivatives, to show that the Riemann tensor in vielbein formulation is given by

$$R = d\omega + \omega \wedge \omega \tag{3.62}$$

3.4.1 Chern-Simons action equivalence with Einstein-Hilbert action

The Einstein Hilbert action in three dimensions in metric formulation is given by

$$S_{EH} = \frac{1}{16G}\int \sqrt{-g}\left(R + \frac{2}{l^2}\right) \tag{3.63}$$

where l is the radius of curvature. In the veilbein formulation, (3.63) is given by

$$S_{EH} = \frac{1}{16\pi G}\int (e^a \wedge (2d\omega_a + \epsilon_{abc}\omega^b \wedge \omega^c) + \frac{\Lambda}{3}\epsilon_{abc}e^a \wedge e^b \wedge e^c) \tag{3.64}$$

To relate the Einstein-Hilbert action's equation of motion, the Einstein equations, and the Chern-Simons equations of motion we set

$$A = \omega + \frac{e}{l}, \qquad \bar{A} = \omega - \frac{e}{l} \tag{3.65}$$

$\Rightarrow$

$$e = \frac{1}{2}(A - \bar{A}) \tag{3.66}$$

setting $l = 1$ and writing A in index notation

$$A = (\omega^a_\mu + e^a_\mu)L_a dx^\mu, \qquad \bar{A} = (\omega^a_\mu - e^a_\mu)L_a dx^\mu \tag{3.67}$$

$\Rightarrow$

$$S_{CS} = S[A] - S[\bar{A}] \tag{3.68}$$

$$\begin{aligned} S_{CS} = 2(e^a_\mu L_a \wedge d\omega^b_\mu L_b + \omega^a_\mu L_a \wedge de^b_\mu L_b) - \frac{4}{3}e^a_\mu L_a \wedge e^b L_b \wedge e^c L_c \\ + \frac{2}{3}(\omega^a_\mu L_a \wedge \omega^b_\mu L_b \wedge e^c L_c + \omega^a_\mu L_a \wedge e^b L_b \wedge \omega^c_\mu L_c + e^a L_a \wedge \omega^b L_b \wedge \omega^c L_c) \end{aligned} \tag{3.69}$$

We follow [6, 26] by using the following identities

$$\begin{aligned} Tr\int A \wedge dA &= Tr(L_a L_b)\int A^a \wedge dA^b \\ Tr(L_a L_b) &= \frac{1}{2}\delta_{ab} \\ A^a \wedge B^b &= \frac{1}{2}[A^a, B^b] \\ [L_a, L_b] &= \epsilon_{abc}L^c \end{aligned} \tag{3.70}$$

and the fact that the spin ω connection can be given by

$$\omega^a_\mu = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc} \tag{3.71}$$

to simplify (3.69). After some algebra we get the following expression for the Chern-Simons action in veilbein notation

$$S_{CS} = \frac{1}{4\pi}\int \frac{e^a}{l} \wedge (2dw^a + \epsilon_{abc}\omega^b \wedge \omega^c - \frac{1}{3l^2}\epsilon_{abc}e^b \wedge e^c) \tag{3.72}$$

which can be seen to give the Einstein-Hilbert action in veilbein notation (3.64). To see this explicitly one can set $l = 1$ in (3.72) and substitute $\frac{1}{-l^2} = \Lambda$. Similar computations can be performed for the vanishing field strengths

$$F = dA + A \wedge A, \qquad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} \tag{3.73}$$

of Chern-Simons and the Einstein equations of motion. It is important to note that the wedge of the gauge fields and the exterior derivatives of these A's which are 1-forms do not vanish because they are algebra valued. These show that Chern-Simons theory is a demo for 3D gravity, meaning that we should be able to define the BTZ black hole also in the Chern-Simons formulation. The Virasoro zero modes given by (3.51) can also be expressed in terms of mass and angular momentum in the following way

$$\mathcal{L} + \bar{\mathcal{L}} = \frac{Ml}{8\pi G}$$

and

$$\mathcal{L} - \bar{\mathcal{L}} = \frac{J}{8\pi G}$$

where M is the mass of the black hole and J the angular momentum. We also recall the $SL(2,R)$ generators as given by (3.10)

$$L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which, already mentioned, obey

$$[L_m, L_n] = (m-n)L_{m+n} \tag{3.74}$$

We readily identify e_ρ with L_0. Using (3.65) we write the two gauge fields $(A, \bar{A})$ in terms of the two copies of the $SL(2,R)$ generators

$$\begin{aligned} A &= (e^\rho L_1 - \frac{2\pi\mathcal{L}}{k}e^{-\rho}L_{-1})dz + L_0 d\rho \\ \bar{A} &= (e^\rho L_{-1} - \frac{2\pi\bar{\mathcal{L}}}{k}e^{-\rho}L_1)dz - L_0 d\rho \end{aligned} \tag{3.75}$$

It is worth noting that the dependence of the connection on the functions $(\mathcal{L}, \bar{\mathcal{L}})$, the BTZ charges arising from the asymptotic symmetry (as explained below), arises through sub-leading terms at large ρ. Which is what we mean by a connection which is asymptotically AdS_3, it follows (3.75) charge dependence. From (3.75) we can write the BTZ in metric formulation by defining the metric in a bi-linear form

$$\begin{aligned} ds^2 &= \frac{1}{2}\langle A - \bar{A}, A - \bar{A}\rangle \\ &= \frac{4}{2}\langle L_0, L_0\rangle d\rho^2 + \frac{1}{2}\langle L_{+1}, L_{-1}\rangle\left(-2e^{2\rho}dzd\bar{z} - 2\mathcal{L}(dz)^2 - 2\bar{\mathcal{L}}d\bar{z}^2\right) - \mathcal{L}\bar{\mathcal{L}}e^{-2\rho}dzd\bar{z} \end{aligned} \tag{3.76}$$

After some normalization
$\Rightarrow$

$$ds^2 = d\rho^2 + \frac{2\pi}{k}\left(\mathcal{L}dz^2 + \bar{\mathcal{L}}d\bar{z}^2\right) - \left(e^{2\rho} + \left(\frac{2\pi}{k}\right)^2 \mathcal{L}\bar{\mathcal{L}}e^{-2\rho}\right)dzd\bar{z} \tag{3.77}$$

Such asymptotically AdS_3 connections can be cast in gauge form, using gauge freedom to make gauge transformations, as follows [10]

$$A = b^{-1}a(z)b + b^{-1}db, \qquad \bar{A} = b\bar{a}(\bar{z})b^{-1} + bdb^{-1} \tag{3.78}$$

where the radial dependence is accounted for by b

$$b = e^{\rho L_0}$$

This radial dependence can be removed by making use of the freedom of gauge transformations. We, however, want to extract $a(z)$ which contains data on the boundary since the bulk has no information because of no propagating degrees of freedom. We now write down $(a, \bar{a})$ and we can see why the barred version of (3.77) takes that particular form so that a and $\bar{a}$ can have the same form

$$\begin{aligned} a(z) &= (L_1 - \frac{2\pi}{k}\mathcal{L}(z)L_{-1})dz \\ \bar{a}(\bar{z}) &= (L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}(\bar{z})L_1)d\bar{z} \end{aligned} \tag{3.79}$$

where the coefficient functions $\mathcal{L}(z)$ and $\bar{\mathcal{L}}(\bar{z})$ are components of the boundary stress tensor, a measure of how the theory responds to changes in space-time

$$T_{zz} = \mathcal{L}, \qquad T_{\bar{z}\bar{z}} = \bar{\mathcal{L}} \tag{3.80}$$

which should exhibit Brown-Henneaux [4] central charge since these two copies of the Virasoro algebra arise from the asymptotic symmetries of three-dimensional gravity with a negative cosmological constant

$$c = \frac{3l}{2G} \tag{3.81}$$

We give a proof of this in Appendix A. To derive the equations of motion, the field strength which should vanish on-shell is given by

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] = 0 \tag{3.82}$$

$\Rightarrow$

$$F_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z + [a_z, a_{\bar{z}}] = 0$$

It is straightforward, since the first term and the last term vanish, to show that the equations of motion take the following form

$$\begin{aligned} \partial_{\bar{z}}\mathcal{L} &= 0 \\ \partial_z\bar{\mathcal{L}} &= 0 \end{aligned} \tag{3.83}$$

These are the holographic Ward identities, which shows that the holomorphic and the anti-holomorphic charges $(\mathcal{L}, \bar{\mathcal{L}})$ are conserved, which is what we should get in a two-dimensional CFT.

3.4.2 Black hole entropy in Chern-Simons formulation

Since we now know that we can define a black hole in Chern-Simons theory, the natural thing to pursue next is most probably to examine the properties, such as the entropy of such black holes. We do this by defining the black hole partition function as

$$Z(\tau, \bar{\tau}) = Tr\{e^{4\pi^2 i\tau\hat{\mathcal{L}} - 4\pi^2 i\bar{\tau}\hat{\bar{\mathcal{L}}}}\} \tag{3.84}$$

The saddle point approximation is given by [11]

$$Z(\tau, \bar{\tau}) = e^{S + 4\pi^2 i\tau\hat{\mathcal{L}} - 4\pi^2 i\bar{\tau}\hat{\bar{\mathcal{L}}}} \tag{3.85}$$

$\Rightarrow$

$$\hat{\mathcal{L}} = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \tau}, \qquad \hat{\bar{\mathcal{L}}} = \frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \bar{\tau}} \tag{3.86}$$

It follows from (3.85) that

$$S = \ln Z - 4\pi^2 i\tau\hat{\mathcal{L}} + 4\pi^2 i\bar{\tau}\hat{\bar{\mathcal{L}}} \tag{3.87}$$

We now turn towards generalizing our partition function by increasing the number of charges. Following [11], we consider a charge Q with corresponding chemical potential α. Writing the partition function in the grand canonical ensemble

$$Z(\tau, \alpha) = Tr\{e^{4\pi^2 i(\tau\hat{\mathcal{L}}+\alpha\hat{Q})}\} \tag{3.88}$$

The modified saddle point approximation is now given by

$$Z(\tau, \alpha) = e^{S+4\pi^2 i(\tau\mathcal{L}+\alpha Q)} \tag{3.89}$$

where the barred parameters have been suppressed to make it clear how the additional charge and its potential is accounted for in the partition function. So we show the holomorphic part. To determine Z and S, the entropy, we need to compute the charges, $\mathcal{L}$ and Q as functions of τ and α respectively. To achieve the said dependence on τ and α we need to impose certain specific smoothness conditions at the horizon. These smoothness conditions, also known as integrability conditions are found by partially integrating the natural logarithm of the partition function (3.87) with respect to the spin-2 and the spin-3 potentials

$$\mathcal{L} = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \tau}, \qquad Q = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \alpha} \tag{3.90}$$

$\Rightarrow$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial Q}{\partial \tau} \tag{3.91}$$

These integrability conditions ensure the existence of the partition function Z. These are the conditions that fixes various terms in the connection including sub-leading terms as we will see later on this paper. For now we briefly turn to the laws of thermodynamics since it was shown in [28] that black holes are thermodynamic systems.

3.5 Black hole thermodynamics

Hawking, in 1974, discovered that black holes release what is called 'black-body radiation' due to quantum effects close to the event horizon. Thus confirming that black holes are thermodynamic systems. We examine, in this subsection, the analogues of the laws of thermodynamics for black holes.

The Zeroth Law

A standard introductory text on university physics would express the zeroth law in the following way: If bodies A and B are each in thermal equilibrium with the third body C, then A and B are in thermal equilibrium with each other. Which simply means that the temperature does not change in a system which is in thermal equilibrium. The Zeroth Law of black hole thermodynamics states [28] that the surface gravity, κ, of a stationary black hole is constant over the event horizon. Meaning that the surface gravity of a black hole acts as temperature. The BTZ surface gravity is given by [28]

$$\kappa = \frac{8G\sqrt{-\frac{J^2}{l^2}+M^2}}{r_+} = \frac{r_+^2 - r_-^2}{l^2 r_+} \tag{3.92}$$

The relationship between the black hole's temperature, the Hawking temperature, and surface gravity is given by

$$T_H = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+} \tag{3.93}$$

The First Law

The first law of black hole thermodynamics gives the relationship between the changes of the angular momentum J, the surface area A, and the mass M of the black hole. We start from the first law of thermodynamics

$$dE = TdS - pdV \tag{3.94}$$

where E is the internal energy, S the entropy, p the pressure and V the volume. We can also write dE as

$$dE = \frac{\partial E}{\partial S}dS + \frac{\partial E}{\partial V}dV \tag{3.95}$$

The area $A(M, J)$ of the black hole is given by [28]

$$A = 4\pi \left[\frac{2G^2M^2}{c^4} + \frac{2GM}{c^2}\sqrt{\frac{G^2M^2}{c^4} + \frac{J^2}{M^2c^2}} \right]^{1/2} \tag{3.96}$$

Whilst $M(A, J)$ is given by

$$M = \sqrt{\frac{\pi}{A}} \left[\frac{c^4}{G^2}(\frac{A}{4\pi})^2 + \frac{4J^2}{c^2} \right]^{1/2} \tag{3.97}$$

$\Rightarrow$

$$dM = \frac{\partial M}{\partial A}dA + \frac{\partial M}{\partial J}dJ \tag{3.98}$$

Thus giving the first law of black hole thermodynamics to be

$$dE = \frac{\kappa}{8\pi G}dA + \frac{J}{2r_+^2}dJ \tag{3.99}$$

where c has been set to unity.

The Second Law

The second law of thermodynamics states that the entropy of a closed system increases for irreversible processes and remains constant for reversible processes. Simply put, the entropy of an isolated system always increases, $\Delta S \geq 0$. For black hole thermodynamics this law is usually referred to as the area theorem, stating that the area of the event horizon of the black hole never decreases with time, $\delta A \geq 0$. The relationship between the area of the black hole and its entropy is given by what is known as the Bekenstein-Hawking equation

$$S_{BH} = \frac{A}{4G} \tag{3.100}$$

For the BTZ black hole, the entropy is

$$S_{BTZ} = \frac{r_+\pi}{2G} \tag{3.101}$$

It is worth noting that the entropy S scales like an area not with the volume and is thus one dimension less than normal.

The Third Law

The third law of thermodynamics is about constant entropy at absolute zero temperature. The analogous black hole law is that it is impossible to reduce the surface gravity, κ of a black hole to zero by a finite sequence of operations [28]. When $\kappa = 0$ we refer to the black hole as extremal since $T_{BH} = 0$. Otherwise it is referred to as non-extremal. Meaning that the third law is a statement about an impossibility to move from a non-extremal to an extremal black hole.

4 Black holes in $SL(N,R) \oplus SL(N,R)$ higher spin gravity

Thus far, we have successfully recast Einstein gravity with a negative cosmological constant as a $SL(2,R) \oplus SL(2,R)$ Chern-Simons theory. It turns out that the Chern-Simons formulation of Einstein gravity with a negative cosmological constant can be generalized to include fields of spin N. This is one of the reasons Chern-Simons formulation is preferred over the metric formulation. The extension from $SL(2,R)$ to $SL(N,R)$, for $N \geq 3$ describes a gravity theory where the graviton is supplemented by a tower of higher spin fields with spin $s = 3, 4, \cdots, N$. It was shown in [29] that a $SL(N,R) \oplus SL(N,R)$ describes Einstein gravity coupled to $N-2$ symmetric higher spin fields of spin $s = 3, 4, \cdots, N$. For our purposes, we concentrate on $N = 3$ which describes Einstein gravity coupled to a spin-3 field.

4.1 $SL(3,R) \oplus SL(3,R)$ higher spin gravity and $\mathcal{W}_3$ symmetry

We proceed in the same way as in the $SL(2,R)$ Chern-Simons formulation of Einstein gravity. The connections A and $\bar{A}$ now take values in the $SL(3,R)$ Lie algebra and satisfy vanishing field strengths, flatness conditions. We only show the holomorphic

$$F = dA + A \wedge A \tag{4.1}$$

Choosing the following basis

$$\begin{gathered}
L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \\
W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad W_0 = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{gathered} \tag{4.2}$$

Satisfying the following commutation relations

$$\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j}, \\
[L_i, W_m] &= (2i-m)W_{i+m}, \\
[W_m, W_n] &= -\frac{1}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}
\end{aligned} \tag{4.3}$$

and the trace relations

$$Tr(L_0L_0) = 2, \qquad Tr(L_1L_{-1}) = -4$$

$$Tr(W_0W_0) = -\frac{8}{3}, \qquad Tr(W_1W_{-1}) = 4, \qquad Tr(W_2W_{-2}) = -16$$

Other traces involving a product of two generators vanish. The $SL(2,R)$ generators form a sub-algebra of $SL(3,R)$. We see that $SL(2,R)$ has been embedded into $SL(3,R)$. This kind of embedding called the principal embedding (realising $SL(N,R)$ generators in terms of $SL(2,R)$ generators) is not the only available inequivalent embedding of $SL(2,R)$ into $SL(3,R)$ [10]. There are other possibilities, e.g. the $SL(2,R)$ sub-algebra generated by $(W_2/4, L_0/2, -W_{-2}/4)$, the diagonal embedding, instead of $(L_{\pm 1}, L_0)$. However, the principal embedding is the one that gives rise to a higher spin theory. The spacetime fields, found by expanding the vielbein e and the spin connection ω are identified in [10] as

$$g_{\mu\nu} = \frac{1}{2}Tr(e_\mu e_\nu), \qquad \varphi_{\mu\nu\gamma} = \frac{1}{3!}Tr\left(e_{(\mu}e_\nu e_{\gamma)}\right) \tag{4.4}$$

where φ the spin-3 field is totally symmetric as can be seen from the vielbein indices. We consider the following $SL(3,R)$ connections

$$\begin{aligned} A(z) &= \left(e^\rho L_1 - \frac{2\pi}{k}e^{-\rho}\mathcal{L}L_{-1} - \frac{\pi}{2k}e^{-2\rho}\mathcal{W}(z)W_{-2}\right)dz + L_0 d\rho \\ \bar{A}(\bar{z}) &= \left(e^\rho L_{-1} - \frac{2\pi}{k}e^{-\rho}\bar{\mathcal{L}}(\bar{z})L_1 - \frac{\pi}{2k}e^{-2\rho}\bar{\mathcal{W}}(\bar{z})W_2\right)d\bar{z} - L_0 d\rho \end{aligned} \tag{4.5}$$

where the coefficients $\mathcal{W}(z)$ and $\bar{\mathcal{W}}$ are the spin-3 currents. In [11], an asymptotically AdS connection A is one that obeys $A_{\bar{z}} = 0$, $A_\rho = L_0$ and

$$A - A_{AdS} \sim \mathcal{O}(1)$$

as

$$\rho \to \infty$$

Similar conditions hold for the anti-holomorphic connection $\bar{A}$. Upon using the available gauge freedom for gauge transformations as in [11], the connections can be written as

$$\begin{aligned} a(z) &= \left(L_1 - \frac{2\pi}{k}\mathcal{L}(z)L_{-1} - \frac{\pi}{2k}\mathcal{W}(z)W_{-2}\right)dz \\ \bar{a}(\bar{z}) &= \left(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}(\bar{z})L_1 - \frac{\pi}{2k}\bar{\mathcal{W}}(\bar{z})W_2\right)d\bar{z} \end{aligned} \tag{4.6}$$

We now look at the interaction of the two operators, the stress energy tensor $T(z)$ and $W(z)$ a primary of conformal weight 3 as we will see shortly. The mode expansion of these operators are given by

$$T(z) = \sum L_n z^{-n-2}, \qquad W(z) = \sum W_n z^{-n-3} \tag{4.7}$$

The asymptotic symmetry algebra is obtained by finding the most general gauge transformation (3.78) of the connection (4.5) satisfying the asymptotic conditions. The functions $(\mathcal{L}, \bar{\mathcal{L}})$ and $(\mathcal{W}, \bar{\mathcal{W}})$ transform under gauge transformations. If we expand in modes (4.7), we get two copies of the classical W_3 algebra. The symmetry currents (T, W) have the following operator product expansions

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \\ T(z)W(w) &= \frac{3W(w)}{(z-w)^2)} + \frac{\partial_w W}{z-w} + \cdots \\ W(z)W(w) &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial_w T(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\left(2\beta\Lambda(w) + \frac{3}{10}\partial_w^2 T(w)\right) \\ &\quad + \frac{1}{z-w}\left(\beta\partial\Lambda(w) + \frac{1}{15}\partial_w^2 T(w)\right) \end{aligned} \tag{4.8}$$

where the parameters β and Λ are given by

$$\begin{aligned} \beta &= \frac{16}{22+5c} \\ \Lambda &=: T(w)T(w) : -\frac{3}{10}\partial_w^2 T(w) \end{aligned} \tag{4.9}$$

The OPE's of the stress tensor and the spin-3 currents

$$\begin{aligned}
T(z)T(0) &\sim \frac{3k}{z^4}+\frac{2}{z^2}T(0)+\frac{1}{z}\partial T(0)\\
T(z)\mathcal{W}(0) &\sim \frac{3}{z^2}\mathcal{W}(0)+\frac{1}{z}\partial\mathcal{W}(0)\\
\mathcal{W}(z)\mathcal{W}(0) &\sim -\frac{5k}{\pi^2}\frac{1}{z^6}-\frac{5}{\pi^2}\frac{1}{z^4}T(0)-\frac{5}{2\pi^2}\frac{1}{z^3}\partial T(0)-\frac{3}{4\pi^2}\frac{1}{z^2}\partial^2T(0)\\
&\quad -\frac{1}{6\pi^2}\frac{1}{z}\partial^3T(0)-\frac{8}{3k\pi^2}\frac{1}{z}T(0)\partial T(0)-\frac{8T(0)^2}{3k\pi^2}
\end{aligned}\tag{4.10}$$

We can read off the central charge from the OPE to be

$$c=6k=\frac{3l}{2G}\tag{4.11}$$

The OPE between $T(z)$ and $\mathcal{W}(0)$ identifies $\mathcal{W}$ as a spin-3 current because of the conformal weight of 3 which can be read off from the OPE. The above OPE's proves that $SL(3,R)\oplus SL(3,R)$ Chern-Simons gives rise to W_3 algebra [31]. The same holds for the anti-holomorphic connection, giving rise to an anti-holomorphic W_3 algebra. By performing a gauge transformation, [30] has shown that the connection a can always take the following form, showing the holomorphic part

$$\begin{aligned}
a &= \left(L_1-\frac{2\pi}{k}\mathcal{L}L_{-1}-\frac{\pi}{2k}\mathcal{W}W_{-2}\right)dz\\
&\quad -\left(\mu W_2+w_1W_1+w_0W_0+w_{-1}W_{-1}+w_{-2}W_{-2}+w_{-3}L_{-1}\right)d\bar{z}
\end{aligned}\tag{4.12}$$

where the spin-3 chemical potential $\mu(z,\bar{z})$ is given by the $-\mu(z,\bar{z})W_2d\bar{z}$ and the coefficients w_i determined by the flatness conditions ($da+a\wedge a=0$) and found to be [10]

$$\begin{aligned}
w_1 &= -\partial_+\mu,\quad w_0=\frac{1}{2}\partial_+^2\mu-\frac{4\pi}{k}\mathcal{L}\mu\\
w_{-1} &= -\frac{1}{6}\partial_+^3\mu+\frac{4\pi}{3k}\partial_+\mathcal{L}\mu+\frac{10\pi}{3k}\mathcal{L}\partial_+\mu,\quad l=\frac{4\pi}{k}\mathcal{W}\mu\\
w_{-2} &= \frac{1}{24}\partial_+^4\mu-\frac{4\pi}{3k}\mathcal{L}\partial_+^2\mu-\frac{7\pi}{6k}\partial_+\mathcal{L}\partial_+\mu-\frac{\pi}{3k}\partial_+^2\mathcal{L}\mu+\frac{4\pi^2}{k^2}\mathcal{L}^2\mu
\end{aligned}\tag{4.13}$$

with $\mathcal{L}$ and $\mathcal{W}$

$$\begin{aligned}
\partial_-\mathcal{L} &= -3\mathcal{W}\partial_+\mu-2\partial_+\mathcal{W}\mu\\
\partial_-\mathcal{W} &= \frac{\sigma k}{12\pi}\partial_+^5\mu-\frac{2\sigma}{3}\partial_+^3\mathcal{L}\mu-3\sigma\partial_+^2\mathcal{L}\partial_+\mu-5\sigma\partial_+\mathcal{L}\partial_+^2\mu-\frac{10\sigma}{3}\mathcal{L}\partial_+^3\mu\\
&\quad +\frac{64\pi\sigma}{3k}\left(\mathcal{L}\partial_+L\mu+\mathcal{L}^2\partial_+\mu\right)
\end{aligned}\tag{4.14}$$

These are equivalent to the flatness conditions since any solution of (4.14) solves the flatness conditions also. We also note that adding the chemical potential $\mu(z,\bar{z})$ creates a $\bar{z}$ dependence for the stress tensor due to the singular terms in the $T\mathcal{W}$ OPE. We look at this more closely by computing Ward identities

$$\partial_{\bar{z}}\langle T(z,\bar{z})\rangle_\mu,\qquad \partial_{\bar{z}}\langle\mathcal{W}(z,\bar{z})\rangle_\mu\tag{4.15}$$

which are found to yield [32]

$$\begin{aligned}
-\partial_{\bar{z}}\mathcal{W}(z) &= \frac{k}{12\pi}\partial_z^5\mu-\frac{10}{3}\partial_z^3\mu\mathcal{L}-5\partial_z^2\mu\partial_z^2\mu\partial_z\mathcal{L}-3\partial_z\mu\partial_z^2\mathcal{L}-\frac{2}{3}\mu\partial_z^3\mathcal{L}\\
&\quad +\frac{64\pi}{3k}\mu\mathcal{L}\partial_z\mathcal{L}+\frac{64\pi}{3k}\partial_z\mu\mathcal{L}^2
\end{aligned}\tag{4.16}$$

Identifying $\mathcal{L} = -\frac{1}{2\pi}T$ in (4.15), $\mathcal{L}$ and $\mathcal{W}$ are subject to

$$\begin{aligned}\partial_{\bar{z}}\mathcal{L} &= 3\mathcal{W}\partial_z\mu + 2\partial_z\mathcal{W}\mu \\ \partial_{\bar{z}}\mathcal{W} &= \frac{k}{12\pi}\partial_z^5\mu - \frac{2}{3}\partial_z^3\mathcal{L}\mu - 3\partial_z^2\mathcal{L}\partial_z\mu - 5\partial_z\mathcal{L}\partial_z^2\mu - \frac{10}{3}\mathcal{L}\partial_z^3\mu \\ &\quad + \frac{64\pi}{3k}\mathcal{L}\mu\partial_z\mathcal{L} + \frac{64\pi}{3k}\mathcal{L}^2\partial_z\mu\end{aligned} \tag{4.17}$$

Showing that the ward identities (for $\sigma = 1$) can be derived from the equations of motion of spin-3 gravity. This as mentioned in [32] justifies *a posteriori* the AdS/CFT dictionary for the incorporation of the spin-3 chemical potential μ in the gauge connections.

4.2 Black holes with higher spin charge

For black holes with higher spin charge within $SL(3,R) \oplus SL(3,R)$ Chern-Simons theory, the following solution was proposed by [30]

$$\begin{aligned}a &= \left(L_1 - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-2}\right)dz \\ &\quad - \mu\left(W_2 - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}W_{-2} + \frac{4\pi\mathcal{W}}{k}L_{-1}\right)d\bar{z} \\ \bar{a} &= -\left(L_{-1} - \frac{2\pi}{k}\mathcal{L}L_1 - \frac{\pi}{2k}\mathcal{W}W_2\right)d\bar{z} \\ &\quad - \bar{\mu}\left(W_{-2} - \frac{4\pi\bar{\mathcal{L}}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}W_2 + \frac{4\pi\bar{\mathcal{W}}}{k}L_1\right)dz\end{aligned} \tag{4.18}$$

Corresponding to the following connections

$$\begin{aligned}A &= \left(e^{\rho}L_1 - \frac{2\pi}{k}\mathcal{L}e^{-\rho}L_{-1} - \frac{\pi}{2k}\mathcal{W}e^{-2\rho}W_{-2}\right)dz \\ &\quad - \mu\left(e^{2\rho}W_2 - \frac{4\pi\mathcal{L}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}e^{-2\rho}W_{-2} + \frac{4\pi\mathcal{W}}{k}e^{-\rho}L_{-1}\right)d\bar{z} \; + L_0 d\rho \\ \bar{A} &= -\left(e^{\rho}L_{-1} - \frac{2\pi}{k}\mathcal{L}e^{-\rho}L_1 - \frac{\pi}{2k}\mathcal{W}e^{-2\rho}W_2\right)d\bar{z} \\ &\quad - \bar{\mu}\left(e^{2\rho}W_{-2} - \frac{4\pi\bar{\mathcal{L}}}{k}W_0 + \frac{4\pi^2\mathcal{L}^2}{k^2}e^{-2\rho}W_2 + \frac{4\pi\bar{\mathcal{W}}}{k}e^{-\rho}L_1\right)dz - L_0 d\rho\end{aligned} \tag{4.19}$$

All thermodynamic potentials have conjugate pairs. For black holes, the energy and the spin-3 current have conjugate thermodynamic potentials - which are temperature and spin-3 chemical potential. The temperature is accounted for by the periodicity of imaginary time, $t \cong t + i\beta$ whilst the spin is accounted for by the μW_2 term in $a_{\bar{z}}$. The remaining terms in $a_{\bar{z}}$ are said to be fixed by the equations of motion in [10]. It turns out, [10], that we can write the $a_{\bar{z}}$ component in the following form

$$a_{\bar{z}} = -2\mu\left[(a_z)^2 - \frac{1}{3}Tr(a_z^2)\right] \tag{4.20}$$

To solve the equations of motion, the flatness conditions , we need only show the following commutation relation as we have done in (3.82) for the $SL(2,R)$ case

$$[a_z, a_{\bar{z}}] = 0 \tag{4.21}$$

It is straightforward to show that (4.20) satisfies (4.23) (section 3.3.1). We consider the non rotating case by setting

$$\bar{\mathcal{L}} = \mathcal{L}, \qquad \bar{\mathcal{W}} = -\mathcal{W}, \qquad \bar{\mu} = -\mu \tag{4.22}$$

The non-rotating solution is thought of as depending on four parameters, $(\mathcal{L}, \mathcal{W}, \mu)$ and the inverse temperature β, corresponding to the periodicity of imaginary time, $t \simeq t + i\beta$. However, we expect that there should only be a two parameter space of physically admissible solutions. If one specifies the temperature and chemical potential then the conjugates can be determined thermodynamically as the function of the modular parameter τ of the torus given by

$$\tau = \frac{i\beta}{2\pi} \tag{4.23}$$

and some potential α related to the chemical potential μ by

$$\alpha = \bar{\tau}\mu, \qquad \bar{\alpha} = \tau\bar{\mu} \tag{4.24}$$

Now we would like to constrain the functions $\mathcal{L}$ and $\mathcal{W}$, this procedure is a bit subtle. Unlike in the $SL(2,R)$ case for the uncharged BTZ where the relation between the energy stress tensor and the temperature is obtained by demanding the absence of a conical singularity at the event horizon in Euclidean signature, for the $SL(3,R)$ case we proceed by imposing the following conditions as suggested in [10]

1. The smoothness of the Euclidean geometry and the non-singularity of the spin-3 field at the event horizon. By τ in the BTZ solution, $\mathcal{L}$ is obtained by demanding that the Euclidean geometry close off smoothly at the location of the event horizon.

2. In the limit $\mu \to 0$ the solution should smoothly recover BTZ; $\mathcal{W} \to 0$ in particular.

3. The charge assignment $\mathcal{L} = \mathcal{L}(\tau, \alpha)$ and $\mathcal{W} = \mathcal{W}(\tau, \alpha)$ should arise from an underlying partition function and should therefore obey the integrability condition

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial \mathcal{W}}{\partial \tau} \tag{4.25}$$

The integrability condition needs to be satisfied in order that the thermodynamic quantities assigned to the black hole will obey the first law of thermodynamics. We proceed as in [10] to investigate the above conditions. The metric (4.4) associated with the gauge connection (4.20) is given by

$$ds^2 = d\rho^2 - \mathcal{F}(\rho)dt^2 + \mathcal{G}(\rho)d\phi^2 \tag{4.26}$$

where

$$\mathcal{F} = \left(2\mu e^{2\rho} + \frac{\pi}{k}\mathcal{W}e^{-2\rho} - \frac{8\pi^2}{k^2}\mu\mathcal{L}^2 e^{-2\rho}\right)^2 + \left(e^{\rho} - \frac{2\pi}{k}\mathcal{L}e^{-\rho} + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho}\right)^2 \tag{4.27}$$

$$\begin{aligned}\mathcal{G}(\rho) = &\left(e^{\rho} + \frac{2\pi}{k}\mathcal{L}e^{-\rho} + \frac{4\pi}{k}\mu\mathcal{W}e^{-\rho}\right)^2 + 4\left(\mu e^{2\rho} + \frac{\pi}{2k}\mathcal{W}e^{-2\rho} + \frac{4\pi^2}{k^2}\mu\mathcal{L}^2 e^{-2\rho}\right)^2 \\ &+ \frac{4}{3}\left(\frac{4\pi}{k}\right)^2 \mu^2\mathcal{L}^2\end{aligned} \tag{4.28}$$

Setting $\mathcal{W} = \mu = 0$, the uncharged limit, the metric (4.26) recovers BTZ in Fefferman-Graham coordinates

$$ds^2 = d\rho^2 - \left(e^{\rho} - \frac{2\pi}{k}\mathcal{L}e^{-\rho}\right)^2 dt^2 + \left(e^{\rho} + \frac{2\pi}{k}\mathcal{L}e^{-\rho}\right)^2 d\phi^2 \tag{4.29}$$

The subtleness is with generic charge, for which $\mathcal{F}(\rho)$ and $\mathcal{G}(\rho)$ are positive definite quantities. The horizon occurs where g_{tt} vanishes, and since the radial dependence of the metric is simply $d\rho^2$, g_{tt} must have a double zero at the event horizon for the Euclidean time circle to smoothly pinch off. One solution is achieved by demanding $\mathcal{W} = 0$ and $\mu = 0$, a second solution is guaranteed by

$$k + 32\mu^2\pi\left(\mu\mathcal{W} - \mathcal{L}\right) = 0 \tag{4.30}$$

The temperature is found by requiring the absence of a conical singularity at the event horizon, located at

$$e^{\rho_+} = \sqrt{\frac{2\pi\mathcal{L} - 4\pi\mu\mathcal{W}}{k}} \tag{4.31}$$

$\rightarrow$

$$\beta = \frac{2\sqrt{2}\pi k}{\sqrt{\left(3k - 32\pi\mathcal{L}\mu^2\right)\left(16\pi\mathcal{L}\mu^2 - k\right)}} \tag{4.32}$$

From these two equations (4.30) and (4.32) we can determine the charge assignments $\mathcal{L}(\tau, \alpha)$ and $\mathcal{W}(\tau, \alpha)$ to be

$$\mathcal{L} = \frac{k}{64\pi}\frac{\beta^2}{\alpha^2}\left(\sqrt{1 - 64\pi^2\alpha^2/\beta^4} - 5\right) \tag{4.33}$$

$$\mathcal{W} = -\frac{k}{64\pi}\frac{\beta^3}{\alpha^3}\left(\sqrt{1 - 64\pi^2\alpha^2/\beta^4} - 3\right) \tag{4.34}$$

These charges, however, do not satisfy the condition (2) and (3). But are the only one for which the geometry of the connection (4.20) has an event horizon. For any other charge assignment, $\mathcal{F}(\rho)$ does not vanish, meaning there is no event horizon because the geometry possesses a globally defined Killing vector. It is noted in [10] that at large positive and negative ρ, $\mathcal{F}$ and $\mathcal{G}$ have leading behaviour $e^{4\rho}$, corresponding to an AdS_3 with a radius of $l = \frac{1}{2}$, which means that the metric (4.27) and (4.28) describes a traversable wormhole connecting two asymptotic AdS_3 geometries. [10] notes that the metric of the spin-3 theory is not preserved under higher spin gauge transformation and the connection (4.20) after a suitable gauge transformation represents a smooth black hole whose black charge assignment satisfies condition (2) and (3). To find this charge assignment we consider Wilson lines, given by the holonomy (an indication of how curved a connection is), ω around the time circle. For contractible cycles (no singularity), the holonomy should be trivial, i.e. the holonomy will give an identity matrix or a Casimir of the group of $a(z)$. Generally, the holonomy of a is given by

$$Hol_\tau(a) = e^\omega \tag{4.35}$$

where ω is defined as

$$\omega = 2\pi\left(\tau a_z + \bar{\tau} a_{\bar{z}}\right) \tag{4.36}$$

With the following eigenvalue conditions, calculated from the determinants of the ω's

$$Tr(\omega^2) = -8\pi^2, \qquad Tr(\omega^3) = 0 \tag{4.37}$$

In the following sections we show that the above holonomy condition satisfies the integrability condition and the smoothness condition.

4.3 Holonomy and the integrability conditions

The holonomy condition for the connection (4.20) is explicitly given by

$$2048\pi^2\mu^3\mathcal{L}^3 - 576\pi k\mu\mathcal{L}^2 + 864\pi k\mu^2\mathcal{W}\mathcal{L} - 864\pi k\mu^3\mathcal{W}^2 + 27k^2\mathcal{W} = 0 \tag{4.38}$$

$$2546\pi^2\mu^2\mathcal{L}^2 - 24\pi k\mathcal{L} - 72\pi k\mu\mathcal{W} + \frac{3k^2}{\tau^2} = 0$$

the anti-holomorphic part takes a similar form. From the second equation of (4.38) we write $\mathcal{W}$ and $\frac{\partial \mathcal{W}}{\partial \tau}$ in terms of $\frac{\partial \mathcal{L}}{\partial \tau}$

$$\mathcal{W} = \frac{256\mu^2\pi^2\tau^2\mathcal{L}^2 + 24k\mu\pi\tau^2\mathcal{L} + 3k^2}{72k\mu\pi\tau^2} \tag{4.39}$$

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \tau} &= \frac{512\mu^2\pi^2\tau^2\mathcal{L}(\tau,\alpha)\left(\frac{\partial \mathcal{L}}{\partial \tau}\right) + 24k\mu\pi\tau^2\frac{\partial \mathcal{L}}{\partial \tau} + 512\mu^2\pi^2\tau\mathcal{L}^2 + 48k\mu\pi\tau\mathcal{L}}{72k\mu\pi\tau^2} \\ &- \frac{256\mu^2\pi^2\tau^2\mathcal{L}^2 + 24k\mu\pi\tau^2\mathcal{L} + 3k^2}{36k\mu\pi\tau^3}\end{aligned} \tag{4.40}$$

We then substitute $\mathcal{W}$ into the first equation of (4.38) and write $\frac{\partial \mathcal{L}}{\partial \alpha}$ and $\frac{\partial \mathcal{L}}{\partial \tau}$. We finally substitute the latter into $\frac{\partial \mathcal{W}}{\partial \tau}$ and see that the integrability condition holds for these charge assignments. We do not explicitly show these computations because they are too long to display.

To show that the second condition is fulfilled we define dimensionless versions of the charge and the chemical potential as in [10]

$$\zeta = \sqrt{\frac{k}{32\pi\mathcal{L}^3}}\mathcal{W}, \qquad \gamma = \sqrt{\frac{2\pi\mathcal{L}}{k}}\mu \tag{4.41}$$

These are then rewritten in terms of these new quantities as detailed in [10] giving a maximal spin-3 charge $\mathcal{W}$ for a given $\mathcal{L}$ $\left(\mathcal{W}^2_{max} = \frac{128\pi}{27k}\mathcal{L}^3\right)$, which is accessible through holonomy conditions (not through the connections). Perhaps it should be mentioned that to make (4.20) to be a smooth black hole, new connections related to the wormhole gauge are introduced as shown in detail in [32].

4.4 $SL(3,R)$ Black hole thermodynamics

Now that we have a smooth higher spin black hole, we would like to do computations of thermodynamic variables from the partition function, like entropy which should be consistent with the first law of thermodynamics. For black holes in $(2+1)$ gravity we can compute the entropy from the area of the event horizon as given by Bekenstein-Hawking equation. But we cannot in the present context because we have turned on a spin-3 field on $(2+1)$ gravity; so we do not know *a priori* whether the entropy is related to the area of the event horizon. We, therefore, base our entropy calculations by demanding adherence to the first law of black hole thermodynamics as in [32]. Written in terms of $\mathcal{L}$ and $\mathcal{W}$ the entropy (3.87) should satisfy the following (holomorphic) thermodynamic relations

$$\tau = \frac{i}{4\pi^2}\frac{\partial S}{\partial \mathcal{L}}, \qquad \alpha = \frac{i}{4\pi^2}\frac{\partial S}{\partial \mathcal{W}} \tag{4.42}$$

The anti-holomorphic parts take a similar form. In [32], the entropy is conveniently written as

$$S = 2\pi\sqrt{2\pi k\mathcal{L}}f(y) \tag{4.43}$$

where

$$y = \frac{27}{2}\zeta^2 = \frac{27k\mathcal{W}^2}{64\pi\mathcal{L}^3} \tag{4.44}$$

$\zeta^2 \sim \frac{\mathcal{W}^2}{\mathcal{L}^3}$ is a dimensionless ratio. such that $f(0) = 1$ recovers the holomorphic part of the entropy of BTZ. Using (4.42) and (4.44) and plugging into the second line of (4.38) we get the following differential equation

$$36y(2-y)(f^{'})^2 + f^2 - 1 = 0 \tag{4.45}$$

The solution is given by

$$f(y) = \cos\theta, \qquad \theta = \frac{1}{6}\arctan\left(\frac{\sqrt{y(2-y)}}{1-y}\right) \tag{4.46}$$

The range of y is [0,2] and the choice of θ is such that $[0, \frac{\pi}{6}]$. The full expression of the entropy, including both right movers and left movers, is then given by

$$S = 2\pi\sqrt{2\pi k\mathcal{L}}f\left(\frac{27k\mathcal{W}^2}{64\pi\mathcal{L}^3}\right) + 2\pi\sqrt{2\pi k\bar{\mathcal{L}}}f\left(\frac{27k\bar{\mathcal{W}}^2}{64\pi\bar{\mathcal{L}}^3}\right) \tag{4.47}$$

Upon plugging in for ζ as a function of C, $\zeta = \frac{C-1}{C^{3/2}}$, we get

$$\theta = \frac{1}{6}\arctan\left(\mathcal{K}(C)\sqrt{1-\frac{3}{4C}}\right) \tag{4.48}$$

with

$$\begin{aligned} \mathcal{K}(C) &\equiv \frac{6\sqrt{3C}(C-1)(C-3)}{(2C-3)(C^2-12C+9)} \\ f(y) &= \cos\left[\frac{1}{6}\arctan\left(\mathcal{K}(C)\sqrt{1-\frac{3}{4C}}\right)\right] \end{aligned} \tag{4.49}$$

$\Rightarrow$

$$f(y) = \sqrt{1-\frac{3}{4C}} \tag{4.50}$$

Showing that the entropy is expressed in terms of C, the charges $(\mathcal{L}, \mathcal{W})$. We also observe that (3.53) is recovered when the spin-3 field which gives rise to a spin-3 charge has been turned off or $C \to \infty$.

5 Modular invariance

Modular invariance as invariance under the modular group $SL(2, \mathbf{Z})$ is studied in this chapter because it is expected that since black holes in AdS_3 have a dual CFT description, their partition functions should be modular invariant or at least have simple transformation properties under modular transformations. This is expected because modular invariance is a property of 2D CFT. The main interesting feature of conformal field theories is that it is done on the infinite plane which is topologically equivalent to a sphere. A sphere can be defined as a Riemann surface of genus $h = 0$. Generally conformal field theories can be studied on a Riemann surface of arbitrary genus h. The case of a torus, $h = 1$, is equivalent to a plane with periodic boundary conditions in two directions.

5.1 Conformal field theory on the torus

Following [36] we define a torus by specifying two linearly independent lattice vectors (ω_1, ω_2) on the plane and identifying points that vary by integer combination of these vectors. These lattice vectors (ω_1, ω_2), on the complex plane, are complex numbers. The parameter of interest is the ratio $\tau = \frac{\omega_2}{\omega_1}$ known as the modular parameter.

The partition function, Z, describing our system depends on the modular parameter τ. To write the partition function in terms of the Virasoro generators L_0 and $\bar{L}_0$ we must define our space and time directions. These are thought as running along the real and imaginary axes, respectively; it is precisely the orientation of the period relative to the space-time axes that concerns us. Considering the hamiltonian (H) and the total momentum of the theory, [36] gives the operator that translates the system parallel to the period ω_2 over a distance a in Euclidean space-time to be

$$\exp -\frac{a}{|\omega_2|}\left\{HIm(\omega_2) - iPRe(\omega_2)\right\} \tag{5.1}$$

where a is the lattice spacing; (5.1) is a translation that takes us from one row of a lattice to the next, parallel to the period ω_2. A complete period that contains m lattice spacings $(|\omega_2| = ma)$, the partition function which is the sum over all energy states. This is similar to taking the trace of the above translation

$$Z(\omega_1, \omega_2) = Tr \exp -\left\{HIm(\omega_2) - iPRe(\omega_2)\right\} \tag{5.2}$$

If we consider the torus to be a cylinder of finite length with ends glued back together. We can then express H and P in terms of the Virasoro generators L_0 and $\bar{L}_0$. For a cylinder of circumference L (ω_1), the Hamiltonian and the momentum are given by [36]

$$\begin{aligned} H &= \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) \\ P &= \frac{2\pi i}{L}(L_0 - \bar{L}_0) \end{aligned} \tag{5.3}$$

The partition function may be finally written as

$$\begin{aligned} Z(\tau) &= Tr \exp \pi i \left\{(\tau - \bar{\tau})(L_0 + \bar{L}_0 - \frac{c}{12} + (\tau + \bar{\tau})(L_0 - \bar{L}_0)\right\} \\ Z(\tau) &= Tr \exp 2\pi i \left\{\tau(L_0 - \frac{c}{24}) - \bar{\tau}(\bar{L}_0 - \frac{c}{24})\right\} \end{aligned} \tag{5.4}$$

Setting

$$q = \exp 2\pi i\tau \qquad \bar{q} = \exp -2\pi i\bar{\tau} \tag{5.5}$$

Substituting (5.5) into (5.4) we write the partition function as

$$Z(\tau) = Tr\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right) \tag{5.6}$$

It is worth noting, as mentioned earlier, that the partition function depends on the ratio τ of the periods (ω_1, ω_2). The above expression for the partition function involves what is known as characters [3], given by

$$\chi_{c,h}(\tau) = Tr\left(q^{L_0 - \frac{c}{24}}\right) \tag{5.7}$$

where the trace is taken over the Verma module[4] built upon a state of highest weight $|h\rangle$[5]

5.2 Modular invariance

Conformal field theories on a torus are advantageous because of the constraints on the operator content that are imposed by the requirement that the partition function be independent of the choice of periods (ω_1, ω_2).
If $(\omega_1^{'}, \omega_2^{'})$ are two periods describing the same lattice as (ω_1, ω_2). Then $(\omega_1^{'}, \omega_2^{'})$ can be expressed as integer combinations of (ω_1, ω_2)

$$\begin{pmatrix} \omega_1^{'} \\ \omega_2^{'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{5.8}$$

where $ad - bc = 1$,$a, b, c, d \in \mathbf{Z}$. The same relationship holds for (ω_1, ω_2) in terms of $(\omega_1^{'}, {\omega'}_2)$, implying that the above matrix is invertible with integer components. It is noted in [36] that the unit cell of the lattice should have the same area regardless of the choice of periods; the determinant of such a matrix should be unity. This means we must consider the group of integer, invertible matrices with unit determinant, denoted as $SL(2, \mathbf{Z})$.

Under the change of period in the above matrix, the modular parameter τ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad ad - bc = 1 \tag{5.9}$$

The symmetry we are interested in is the one that preserves these modular transformations; this is the modular group $SL(2, \mathbf{Z})/\mathbf{Z}_2$ or $PSL(2, \mathbf{Z})$. Modular properties are checked by applying these modular transformations (5.9) to our partition function $Z(\tau)$. If Γ is the action of the modular group, a fundamental domain (F_0) of Γ is a domain of the upper half-plane such that no pair of points within it can be reached through a modular transformations denoted by the shaded part of Figure 2. The modular group keeps τ on the upper half-plane. This can be seen by considering the following transformations

$$\begin{aligned} \mathcal{T} : \tau \to \tau + 1 \quad &\text{or} \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \mathcal{S} : \tau \to -\frac{1}{\tau} \quad &\text{or} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{5.10}$$

These two $(\mathcal{T}, \mathcal{S})$ generate the whole modular group and they satisfy [36]

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = 1 \tag{5.11}$$

[3]Characters are generating functions, $\chi(c, h)$, associated to a Verma module generated by the Virasoro generators L_{-n} acting on the highest weight state $|h\rangle$

[4]the representation space of all states created by acting on the highest weight state with the Virasoro generator L_n

[5]We denote by $|h\rangle$ the highest weight state with eigenvalue h of L_0: $L_0|h\rangle = h|h\rangle$

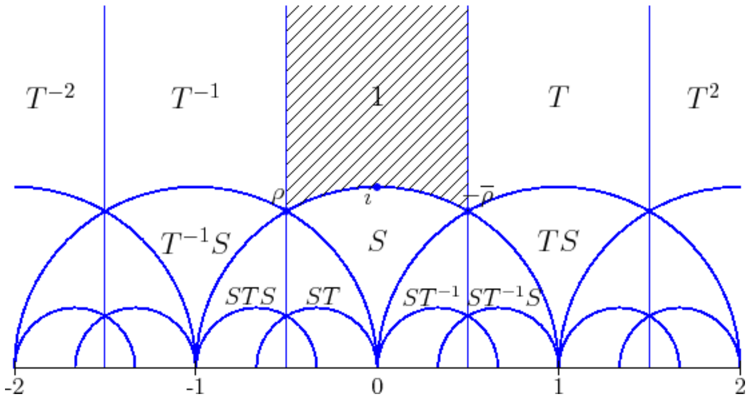

Figure 2: fundamental domain

5.2.1 An example: Minimal Models - Modular transformations of the Characters

A Minimal model is a special conformal field theory whose spectrum is built from finitely many irreducible representations of the Virasoro algebra. These models are usually parametrized by two integers p, p'. The characters of minimal models with central charge [36]

$$c(p,p^{'}) = 1 - 6\frac{(p-p^{'})^2}{pp^{'}} \tag{5.12}$$

can be written in the form

$$\chi_{r,s}(\tau) \equiv \chi_{\lambda_{r,s}}(\tau) = K_{\lambda_{r,s}}(\tau) - K_{\lambda_{r,-s}}(\tau) \tag{5.13}$$

where

$$\lambda_{r,s} = pr - p^{'}s \qquad \lambda_{r,-s} = pr + p^{'}s \tag{5.14}$$

the Kac indices (r,s) in the range

$$\begin{aligned} 1 \leq r \leq p^{'} - 1 \\ 1 \leq s \leq p - 1 \\ p^{'}s < pr \end{aligned} \tag{5.15}$$

and

$$K_{\lambda}(\tau) = \frac{1}{\eta(\tau)} \sum\nolimits_{n\in\mathbf{Z}} q^{(Nn+\lambda)^2/2N} \tag{5.16}$$

with

$$N = 2pp^{'} \tag{5.17}$$

and the Dedekind function given by (also in Appendix C)

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$$

To do the modular transformations, starting with the $\mathcal{T} : \tau \to \tau + 1$ transformation we need to use the $\mathcal{T}$ transformation of η given by

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau) \tag{5.18}$$

and the relation [36]

$$\frac{(Nn+\lambda)^2}{2N} = \frac{\lambda^2}{2N} \mod 1 \tag{5.19}$$

$\Rightarrow$

$$\begin{aligned} K_\lambda(\tau+1) &= \frac{1}{\eta(\tau+1)}\sum\nolimits_{n\in\mathbf{Z}} q_{\tau\to\tau+1}^{(Nn+\lambda)^2/2N} \\ &= \frac{e^{-\pi i/12}}{\eta(\tau)}\sum\nolimits_{n\in\mathbf{Z}} e^{2\pi i(\tau+1)(Nn+\lambda)^2/2N} \\ &= e^{-\pi i/12}\sum\nolimits_{n\in\mathbf{Z}} e^{2\pi i(Nn+\lambda)^2/2N} e^{2\pi i\tau(Nn+\lambda)^2/2N} \\ &= e^{-\pi i/12}\sum\nolimits_{n\in\mathbf{Z}} e^{2\pi i\left(\frac{\lambda^2}{2N}\right)} q^{(Nn+\lambda)^2/2N} \\ &= e^{2\pi i(\frac{\lambda^2}{2N}-\frac{1}{24})}\sum\nolimits_{n\in\mathbf{Z}} q^{(Nn+\lambda)^2/2N} \\ K_\lambda(\tau+1) &= e^{2\pi i(\frac{\lambda^2}{2N}-\frac{1}{24})} K_\lambda(\tau) \end{aligned} \tag{5.20}$$

The following relation [36]

$$\frac{\lambda_{r,-s}^2}{2N} - \frac{\lambda_{r,s}^2}{2N} = rs = 0 \mod 1 \tag{5.21}$$

makes it possible to write

$$K_{\lambda_{r,-s}}(\tau+1) = e^{2i\pi(\frac{\lambda_{r,s}^2}{2N}-\frac{1}{24})} K_{\lambda_{r,-s}}(\tau) \tag{5.22}$$

Applying these results on the $\mathcal{T}$ transformation on the minimal characters is

$$\chi_{r,s}(\tau+1) = e^{2i\pi(\frac{\lambda_{r,s}^2}{2N}-\frac{1}{24})} \chi_{r,s}(\tau) \tag{5.23}$$

We now turn to the $\mathcal{S} : \tau \to -\frac{1}{\tau}$ transformation. For two relatively prime integers p and p' there is a unique pair (r_0, s_0) in the range (5.15) such that

$$pr_0 - p' s_0 = 1 \tag{5.24}$$

known as the Bezout lemma. We also define

$$\omega_0 = pr_0 + p' s_0 \mod N \tag{5.25}$$

for which

$$\omega_0^2 = 1 \mod 2N \tag{5.26}$$

ω_0 is the integer that is designed to generate the transformation $s \to -s$ in $\lambda_{r,s}$

$$\lambda_{r,-s} = \omega_0 \lambda_{r,s} \mod N \tag{5.27}$$

We then express the minimal characters as

$$\chi_\lambda(\tau) = K_\lambda(\tau) - K_{\omega_0\lambda}(\tau) \tag{5.28}$$

$\Rightarrow$

$$\chi_\lambda = \chi_{\lambda+N} = \chi_{-\lambda} = -\chi_{\omega_0\lambda} \tag{5.29}$$

because of the following symmetries

$$K_{\lambda+N} = K_\lambda = K_{-\lambda} \tag{5.30}$$

To show that the $\mathcal{S}$ transformation acts linearly on K_λ we need to make use of the Poisson re-summation formula

$$\sum_{n\in\mathbf{Z}} \exp(-\pi a n^2 + bn) = \frac{1}{\sqrt{a}} \sum_{k\in\mathbf{Z}} \exp -\frac{\pi}{a}(k + \frac{b}{2\pi i})^2 \tag{5.31}$$

$$\begin{aligned}
K_\lambda(-\frac{1}{\tau}) &= \frac{1}{\sqrt{-i\tau}\eta(\tau)} \sum_{n\in\mathbf{Z}} \exp\left[-\frac{2\pi i}{\tau}\frac{(Nn+\lambda)^2}{2N}\right] \\
&= \frac{1}{\sqrt{-i\tau}\eta(\tau)} \int_{\mathbf{R}} dx \sum_{k\in\mathbf{Z}} \exp 2\pi i \left[kx - \frac{N}{2\tau}(x + \frac{\lambda}{N})^2\right] \\
&= \frac{1}{\sqrt{-i\tau}\eta(\tau)} \int_{\mathbf{R}} dx \sum_{k\in\mathbf{Z}} \exp 2\pi i \left[\frac{\tau}{2N}k^2 - \frac{k\lambda}{N} - \frac{N}{2\tau}(x + \frac{\lambda}{N} - \frac{k\tau}{N})^2\right] \\
&= \frac{1}{\sqrt{-i\tau}\eta(\tau)} \sum_{k\in\mathbf{Z}} \exp 2\pi i \left(\frac{\tau k^2}{2N}\right) \sum_{k\in\mathbf{Z}} \exp\left(-2\pi i \frac{k\lambda}{N}\right) \\
&\sum_{k\in\mathbf{Z}} \int_{\mathbf{R}} dx - \left(\frac{N}{2\tau}(x + (\frac{\lambda}{N} - \frac{k\tau}{N}))^2\right) \\
&= q^{k^2/2N} \sum_{k\in\mathbf{Z}} \exp\left(-2\pi i \frac{k\lambda}{N}\right) \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}} dx \sum_{k\in\mathbf{Z}} - \left(\frac{N}{2\tau}(x + (\frac{\lambda}{N} - \frac{k\tau}{N}))^2\right)
\end{aligned} \tag{5.32}$$

where we have applied the re-summation formula from line 3 to line 4. We apply the re-summation formula again with

$$a = \frac{N}{-i\tau} \qquad b = \frac{2\pi}{-i\tau}(\lambda - k\tau)$$

and evaluating the integral yields

$$K_\lambda(-\frac{1}{\tau}) = \frac{1}{\sqrt{2N}\eta(\tau)} \sum_{k\in\mathbf{Z}} \exp\left(-2\pi i \frac{k\lambda}{N}\right) q^{k^2/2N} \tag{5.33}$$

Setting

$$k = \mu + Nm \quad m \in \mathbf{Z}, \mu \in [0, N-1]$$

we have

$$K_\lambda(-\frac{1}{\tau}) = \sum_{\mu=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi i\lambda\mu/N} K_\mu(\tau) \tag{5.34}$$

Therefore the minimal characters transform as

$$\chi_\lambda(-\frac{1}{\tau}) = \sum_{\mu=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi i\lambda\mu/N} \chi_\mu(\tau) \tag{5.35}$$

It is worth noting though that the range of summation has been tampered with, the sum over μ should be restricted to the fundamental domain. However, for our purposes, we have shown how to perform modular transformations thus testing for modular invariance. Finding that the modular transformed character can be expressed as a linear combination of the untransformed characters. Which means that the partition function, as a particular sum of characters, stands a chance of satisfying modular invariance.

5.3 Partition function and Cardy's behaviour

In this subsection we review the main result of Cardy which is to show the modular invariance of the 2D CFT partition function and use the $\mathcal{S}$ transformation in relating high and low temperature behaviours where the saddle point calculation fails.

We begin with a 2D CFT with the Virasoro algebra

$$
\begin{aligned}
[L_m, Ln] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[L_m, \bar{L}_n] &= 0
\end{aligned}
\tag{5.36}
$$

where c is the central charge. The partition function of a 2d CFT on a torus given by (5.6) can be written as [37]

$$
Z(\tau, \bar{\tau}) = \sum \rho(\triangle, \bar{\triangle}) e^{2\pi i \triangle \tau} e^{-2\pi i \bar{\triangle} \bar{\tau}} \tag{5.37}
$$

where ρ, for a unitary theory, is the number of states with $L_0 = \triangle$, $\bar{L}_0 = \bar{\triangle}$. Whereas for a non-unitary theory ρ is the difference between the positive and negative-norm states with appropriate eigenvalues.

If $(\tau, \bar{\tau})$ are complex variables, and we set $q = e^{2\pi i\tau}$, $\quad \bar{q} = e^{2\pi i\bar{\tau}}$ and $\tau = \tau_1 + i\tau_2$. We can then write

$$
\rho(\triangle, \bar{\triangle}) = \frac{1}{(2\pi i)^2} \int \frac{dq}{q^{\triangle+1}} \frac{d\bar{q}}{\bar{q}^{\triangle+1}} Z(q, \bar{q}) \tag{5.38}
$$

this integral is evaluated at the contours that enclose $(q, \bar{q}) = 0$. Cardy's achievement has to do with how we can handle $Z(q, \bar{q})$; by writing a relationship between the behaviour of the partition function at high temperatures and at low temperatures. To do so, Cardy needed the result that the following quantity is modular invariant

$$
Z(\tau, \bar{\tau}) = e^{\frac{\pi c}{6}\tau_2} Z_0(\tau, \bar{\tau}) \tag{5.39}
$$

where

$$
Z_0(\tau, \bar{\tau}) = Tr\left[e^{2\pi i(L_0 - \frac{c}{24})\tau} e^{-2\pi i(\bar{L}_0 - \frac{c}{24})\bar{\tau}}\right]
$$

We then use the modular invariance of Z_0 to write $Z(q, \bar{q})$ in a form suitable for a saddle point approximation, suppressing $\bar{\tau}$ [38]

$$
\begin{aligned}
Z(\tau) &= e^{\frac{\pi i c}{12}} Z_0(\tau) \\
Z(\tau) = e^{\frac{\pi i c}{12}} Z_0\left(-\frac{1}{\tau}\right) &= e^{\frac{\pi i c}{12}\tau} e^{\frac{\pi i c}{12}\frac{1}{\tau}} Z\left(-\frac{1}{\tau}\right)
\end{aligned}
\tag{5.40}
$$

$\Rightarrow$

$$
\rho(\triangle) = \int d\tau \left\{ e^{-2\pi i \triangle \tau} e^{\frac{\pi i c}{12}\tau} e^{\frac{\pi i c}{12}\frac{1}{\tau}} Z\left(-\frac{1}{\tau}\right) \right\} \tag{5.41}
$$

If we assume that $Z\left(-\frac{1}{\tau}\right)$ varies slowly near the extremum of the phase as in [37]. We need to divide the integrand into a rapidly varying phase and a slow one in order to use a saddle point approximation. For $\triangle \to \infty$, the extremum of the exponent is

$$
\tau \approx i\sqrt{\frac{c}{24\triangle}} \tag{5.42}
$$

Substituting (5.42) into (5.41) yields the Cardy formula

$$
\rho(\triangle) \approx \exp\left\{2\pi\sqrt{\frac{c\triangle}{6}}\right\} Z(i\infty) \tag{5.43}
$$

where

$$
Z(i\infty) = \left(\frac{c}{96\triangle^3}\right)^{1/4} \tag{5.44}
$$

The Cardy formula (5.43) is useful in computing the asymptotic density of states, which in turn can be used to compute the entropy of a BTZ black hole [5]. Using the Brown-Henneaux [4] central charge $c = 3l/2G$ in the Cardy formula

$$\rho(\triangle) \approx \exp\left\{2\pi\sqrt{\frac{3l\triangle}{12G}}\right\}\left(\frac{3l}{192G\triangle^3}\right)^{1/4} \tag{5.45}$$

Taking the logarithm (causing the $Z(i\infty)$ term to be very small) of the above expression, the density of states, gives the entropy [6]

$$\begin{aligned} S &= \ln\rho(\triangle) + \ln\rho(\bar{\triangle}) \\ &= 2\pi\left(\sqrt{\frac{cL_0}{6}} + \sqrt{\frac{c\bar{L}_0}{6}}\right) \\ &= 2\pi\left(\sqrt{\frac{3l}{12G}\frac{(r_+ - r_-)^2}{4Gl}} + \sqrt{\frac{3l}{12G}\frac{(r_+ + r_-)^2}{4Gl}}\right) \\ &= 2\pi\left(\sqrt{\frac{1}{16G^2}}\right)r_+ \\ &= \frac{2\pi r_+}{4G} \end{aligned} \tag{5.46}$$

where in line 2 we took the eigenvalues to be $\triangle = 4L_0$ and used $L_0 = \frac{(r_+ - r_-)^2}{16Gl}$ and $\bar{L}_0 = \frac{(r_+ + r_-)^2}{16Gl}$ in line 3. Thus showing that (5.46) which emerges from relating low and high temperature behaviour via modular invariance matches the entropy found from 2+1 dimensions by computing the area (Bekeinstein-Hawking) (3.40).

The above results show the importance of modular properties of the partition function in the BTZ case. In section 4 we saw that the asymptotic symmetry of higher-spin gravity gives a W algebra in a similar way that asymptotic symmetries of BTZ gives rise to a Virasoro algebra [4]. Which in turn made it possible, because of known modular properties, to derive the Cardy formula and thus get an expression for the entropy of a BTZ black hole [5]. Therefore, it is expected that if we know the modular properties of higher spin partition functions then we can reproduce Cardy's argument and thus be able to compute entropies for higher spin black holes for specific values of λ and thus understand black hole thermodynamics better. An interesting question at this point is whether we can be able to generalise the $SL(3,R)$ arguments which gave rise to a W algebra to higher spin theories (perhaps of general λ).

6 Higher spin black holes partition functions

Now that we have constructed black holes in $SL(3,R)$ gravity, we confront the more challenging task of repeating the same process in the three-dimensional Vasiliev theory containing an infinite tower of higher spins $s \geq 2$. Vasiliev theories are said [32] to be most likely to contribute meaningfully in the business of relating higher spin gravity to string theory. In our case, what is most useful in the Vasiliev framework is that its higher spin sector can be cast as two copies of Chern-Simons theory with connections valued in the infinite Lie algebra $hs[\lambda]$, a one parameter, λ, family of higher spin algebras.

6.1 $hs[\lambda]$ higher spin gravity

The $hs[\lambda]$ theory contains two complex scalars with masses given by $m^2 = \lambda^2 - 1$ and an infinite tower of higher spin fields [35]. For $\lambda = 3 = N$ the Vasiliev theory described by the $hs[\lambda]$ theory can be truncated to two Chern-Simons when the complex scalar fields are set to zero. The $hs[\lambda]$ algebra is described by generators V^s_m labelled by two integers: a spin $s \geq 2$ and a mode index $|m| < s$ with the following commutation relations

$$\left[V^s_m, V^t_n\right] = \sum\nolimits_{u=2,4,6,\cdots}^{s+t-|s-t|-1} g^{st}_u\,(m,n;\lambda)\,V^{s+t-u}_{m+n} \tag{6.1}$$

where g are structure constants given in Appendix B. The parameter λ is the label of inequivalent algebras; g's simplify to polynomials in λ^2 when evaluated for integers (s,m) since they feature via hypergeometric functions. From a mathematical perspective [39, 30], $hs[\lambda]$ can be defined as

$$\mathcal{B}[\mu] = \frac{\mathcal{U}\left((SL(2,R)\right)}{\langle C_2 - \mu\mathbf{1}\rangle} \tag{6.2}$$

where $\mathcal{U} = \{\mathbf{1} + a_i L_i + a_{ij} L_i L_j + \cdots, a \in \mathbf{R}, (i,j) \in \mathbf{Z}^+\}$ is the universal enveloping algebra, where products of generators are elements of the algebra. The triangle-like brackets in the denominator are referred to as the ideal, all polynomials and generators multiplied by $C_2 - \mu\mathbf{1}$. C_2 is the $SL(2,R)$ Casimir, the element at the center of the algebra, i.e. commuting with all elements of the algebra. In $SL(2,R)$ the Casimir has the property

$$C_2 < -\frac{1}{4} \tag{6.3}$$

By convention we set (the mass of the complex scalars) [39, 30]

$$\mu = \frac{1}{4}(\lambda^2 - 1) \tag{6.4}$$

for an $sl(2,R)$ algebra with commutation relations (3.11) C_2 is fixed as

$$C_2 = (L_0)^2 - \frac{1}{2}\left(L_1 L_{-1} + L_{-1} L_1\right) = \frac{1}{4}\left(\lambda^2 - 1\right) \tag{6.5}$$

The $hs[\lambda]$ generators are constructed from the $sl(2,R)$ generators as

$$V^s_m = (-1)^{m+s-1}\frac{(m+s-1)!}{(2s-2)!}\left[L_{-1}, ...\left[L_{-1}, \left[L_{-1}, L_1^{s-1}\right]\right]\right] \tag{6.6}$$

where the commutation relations are done $s-1-m$ times. The $hs[\lambda]$ algebra is generated upon modding out by the ideal formed by (6.2) and dropping the identity element, defined to be V^1_0

The $hs[\lambda]$ algebra contain an $SL(2,R)$ sub-algebra for $s=2$, whilst the remaining generators transform under the adjoint in the following way

$$\left[V_m^2, V_n^t\right] = \left(m(t-1)-n\right) V_{m+n}^t \tag{6.7}$$

Unlike the $SL(N,R)$ algebras, the $hs[\lambda]$ algebra only has a single $SL(2,R)$ sub-algebra; any commutator of two generators each with spin $s>2$ produces a generator with spin $t>s$, at $\lambda = N$ an ideal forms consisting of all generators V_m^s with spin with $s>N$ may decouple and $Tr(V_m^s V_n^r)=0$, thus

$$SL(N,R) \approx \frac{hs[N]}{\langle V_m^s \rangle}, \qquad s>N \tag{6.8}$$

In other words, $SL(N,R)$ is recovered by modding out by this ideal. This means that $SL(N,R)$ gravity theories can be thought of as limiting cases of $hs[\lambda]$ gravity. The radial dependency which is suppressed in moving from A to a is now given by

$$b = e^{\rho V_0^2} \tag{6.9}$$

When $\lambda = \frac{1}{2}$, it is said in [32] that $hs[\lambda]$ is isomorphic to $hs(1,1)$ whose commutator can be written as the antisymmetric part of the Moyal product. The commutation relations (6.1) can be written as a star commutator

$$\left[V_m^s, V_n^t\right] = V_m^s \star V_n^t - V_n^t \star V_m^s \tag{6.10}$$

with the lone star product

$$V_m^s \star V_n^t \equiv \frac{1}{2}\sum\nolimits_{u=1,2,3,\cdots}^{s+t-|s-t|-1} g_u^{st}\left(m,n;\lambda\right) V_{m+n}^{s+t-u} \tag{6.11}$$

We also need to define a bi-linear trace which picks out the identity element, V_0^1, of any lone star product

$$Tr(V_m^s, V_n^t) \propto g_{s+t-1}^{st}(m,n;\lambda)\delta^{st}\delta_{m,-n} \tag{6.12}$$

with the structure constant explicitly given by

$$g_{2s-1}^{ss}(m,-m;\lambda) = (-1)^m \frac{2^{3-2s}\Gamma(s+m)\Gamma(s-m)}{(2s-1)!!(2s-3)!!}\prod\nolimits_{\sigma=1}^{s-1}(\lambda^2-\sigma^2) \tag{6.13}$$

It is worth noting that the trace naturally factors out the ideal when $\lambda = N$ - this will come in handy when computing black hole holonomies which easily compares to the $SL(N,R)$ case. In [33], the trace is normalized as

$$Tr(V_m^s V_{-m}^s) = \frac{12}{\lambda^2-1} g_{2s-1}^{ss}(m,-m;\lambda) \tag{6.14}$$

for $s=2$, the $SL(2,R)$ sub-algebra, the traces can be shown to be

$$Tr(V_1^2 V_{-1}^2) = -4, \qquad Tr(V_0^2 V_0^2) = 2 \tag{6.15}$$

One useful identity of the lone star product is the following result for products of the highest weight $SL(2,R)$ generator

$$\left(V_1^2\right)^{s-1} = V_{s-1}^s \tag{6.16}$$

The analogous conditions for asymptotic AdS in (4.5) to (4.6) of $hs[\lambda]$ has been shown in [7] to be the infinite-dimensional W-algebra, W_∞. To show this we consider an $hs[\lambda]$ valued connection in highest-weight gauge

$$a_z = V_1^2 - \frac{2\pi}{k}\bar{\mathcal{L}}(z)V_{-1}^2 + \sum\nolimits_{s=3}^{\infty} J^{(s)}(z)V_{1-s}^s \tag{6.17}$$

the $J^{(s)}(z)$ are spin-s currents. The W_∞ is a non-linear algebra containing $hs[\lambda]$ not as a proper sub-algebra, but in the infinite central charge limit where the non-linear terms drop out. Hence $hs[\lambda]$ is known as the *wedge sub-algebra* of W_∞. The $hs[\lambda]$ generators can be identified as *wedge modes* in the following way

$$J_m^{(s)} = V_m^s, \qquad |m| < s$$

provided we expand the currents in modes

$$J_m^{(s)} = \sum\nolimits_m \frac{J_m^{(s)}}{z^{m+s}} \tag{6.18}$$

In what follows we work with the flat connection a as given by (4.20) in constructing higher spin black hole in $hs[\lambda]$.

6.2 Constructing $hs[\lambda]$ black holes

We follow the prescription in section 4.2. of constructing higher spin black holes, but now with modifications to $hs[\lambda]$. In the $hs[\lambda]$ theory, the BTZ black hole in particular gauge, known as the 'wormhole gauge' is described, following [32, 11] by

$$a_z = a_z^{BTZ} + (\text{higher spin charges}) \tag{6.19}$$

$$a_{\bar{z}} \sim \mu_s[(a_z)^{s-1} - trace]$$

where μ_s is the spin-s chemical potential. Though the above connections are in the wormhole gauge we know because of $SL(3,R)$ that somewhere on the gauge orbit of the solution is a solution with a manifest black hole metric. And we can see that at large ρ ($\rho \to \infty$) for the chemical potential term the leading part of $(a_z)^{s-1}$ carries the $e^{(s-1)\rho}$ dependence. This is so in the $hs[\lambda]$ case because of (6.6)

$$\left(V_1^2\right)^{s-1} = V_{s-1}^s$$

Just to recap, we had said in section (4.2) that to make the black hole solution smooth we need to fix all charges in terms of the potential (μ_s, τ). This is achieved by considering Wilson loops, since Chern-Simons is a topological theory, which are gauge invariant holonomies around given cycles of a manifold; which in this case are the BTZ holonomy conditions given by the contractible cycle of Euclidean time as in (4.36)

$$\omega = 2\pi(\tau a_z + \bar{\tau} a_{\bar{z}}) \tag{6.20}$$

with the following condition on its eigenvalues, that they equal those of BTZ

$$Tr(\omega^n) = Tr(\omega^n_{BTZ}), \qquad n = 2, 3, \cdots, rk(\omega) \tag{6.21}$$

The smooth black holes solution with spin-3 chemical potential is given by

$$a_z = V_1^2 + \frac{1}{4\tau^2} V_{-1}^2 \tag{6.22}$$

$$a_{\bar{z}} = 0$$

This would mean that the holonomy is given by

$$\omega_{BTZ} = 2\pi\tau \left(V_1^2 + \frac{1}{4\tau^2} V_{-1}^2 \right) \tag{6.23}$$

Odd-n traces vanish because of the in-workings of the Moyal product. The first three even-n traces are

$$\begin{aligned} Tr(\omega_{BTZ}^2) &= -8\pi^2 \\ Tr(\omega^4) &= \frac{8\pi^4}{5}\left(3\lambda^2-7\right) \\ Tr(\omega_{BTZ}^6) &= -\frac{8\pi^6}{7}(3\lambda^4-18\lambda^2+31) \end{aligned} \tag{6.24}$$

We consider the following ansatz for the higher spin black hole in the presence of a spin-3 field, following (6.19)

$$\begin{aligned} a_z &= V_1^2 - \frac{2\pi\mathcal{L}}{k}V_{-1}^2 - N(\lambda)\frac{\pi\mathcal{W}}{2k}V_{-2}^3 + J \\ a_{\bar{z}} &= -\mu N(\lambda)\left(a_z \star a_z - \frac{2\pi\mathcal{L}}{3k}(\lambda^2-1)\right) \end{aligned} \tag{6.25}$$

with

$$J = J^{(4)}V_{-3}^4 + J^{(5)}V_{-4}^5 + \cdots \tag{6.26}$$

where $N(\lambda)$ is a normalization factor simplifying the comparison with the $SL(3,R)$ case, given by

$$N(\lambda) = \sqrt{\frac{20}{(\lambda^2-4)}} \tag{6.27}$$

It is noted in [11] that truncating all spins $s > 3$ produces a solution with the same generator normalizations and bi-linear traces as the spin-3 black hole (4.6) of the $SL(3,R)$ theory. This black hole is thought of as a saddle point contribution to the following partition function

$$Z(\tau,\alpha) = Tr\left[e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right] \tag{6.28}$$

where the potential is defined as

$$\alpha = \bar{\tau}\mu \tag{6.29}$$

where τ as mentioned in section 3 is the modular parameter of the boundary torus, defined through the following identifications

$$z \cong z + 2\pi\tau, \qquad \bar{z} = \bar{z} + 2\pi\bar{\tau}$$

These charges should, of-course, satisfy the integrability conditions

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{\partial\mathcal{W}}{\partial\tau} \tag{6.30}$$

The structure of this ansatz is such that a_z is the asymptotically AdS_3 connection in the highest-weight gauge. The $a_{\bar{z}}$ is a traceless source term (containing the spin-3 chemical potential) that, as noted in [11], deforms the ultra violet asymptotics by

$$a_z = \mu N(\lambda)V_2^3 + (\text{subleading}) \tag{6.31}$$

Perhaps it is worth noting the divergences which occur in the $hs[\lambda]$ construction of black holes. First, there are an infinite number of holonomy equations (6.21). This is because we are fixing smoothness at the event horizon of the black hole metric and an infinite tower of higher spin fields. The number of non-zero higher spin charges $J^{(s)}$ of the infinite set of holonomy equations would always occur in these solutions, demanding that they are always turned on. This can be seen from a $W_\infty[\lambda]$ perspective as noted in [32, 11] by noting the higher spin charges appearing in the $\mathcal{W}\mathcal{W}$ OPE

$$\mathcal{W}(z)\mathcal{W}(0) \sim \cdots + \frac{\mu J_4(0)}{z^2} + \cdots \tag{6.32}$$

Another way is by considering operator expectation values

$$\langle J^{(s)} \rangle_\alpha = \frac{Tr\left[J^{(s)} e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right]}{Tr\left[e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right]} \tag{6.33}$$

A pertubative expansion in α of (6.33) is given as [11]

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \alpha^2 \mathcal{L}_2 + \cdots \\ \mathcal{W} &= \alpha \mathcal{W}_1 + \alpha^3 \mathcal{W}_3 + \cdots \\ J^{(s)} &= \alpha^{s-2} J^{(s)}_{s-2} + \alpha^s J^{(s)}_s + \cdots \end{aligned} \tag{6.34}$$

6.3 Black hole partition function

The holonomy matrix for the W_∞ black hole ansatz (6.25) is

$$\omega = 2\pi \left[\tau a_z + \alpha N(\lambda)\left(a_z \star a_z - \frac{2\pi\mathcal{L}}{3k}\left(\lambda^2-1\right)\right)\right] \tag{6.35}$$

The holonomy constraints were given to be (6.21)

$$Tr\left(\omega^n\right) = Tr\left(\omega^n_{BTZ}\right) \tag{6.36}$$

Upon using the pertubative expansions (6.34), the holonomy equations up to $n = 4$ are given to be the following in [11]

$n = 2:$

$$0 = \alpha^2 J^4(144k^2(\lambda^2-9)) - 1792\pi^2\alpha^2\mathcal{L}^2 - 504\pi\alpha\tau k\mathcal{W} - 168\pi\tau^2 k\mathcal{L} - 21k^2$$

$n = 3:$

$$\begin{aligned} 0 = {} & \alpha J_4(36k^2(\lambda^2-4)(\lambda^2-9)[80\pi\alpha^2\mathcal{L}(\lambda^2-16) + 9k\tau^2(\lambda^2-4)]) \\ & - 40\alpha^3\pi^2[45\mathcal{W}^2 k(5\lambda^4 - 65\lambda^2 + 264) + 256\mathcal{L}^3\pi(\lambda^2-4)(\lambda^2-16)] \\ & - 4320\alpha^2\mathcal{W}\mathcal{L}\pi^2\tau k(\lambda^2-4)(4\lambda^2-29) \\ & - 4032\alpha\mathcal{L}^2\pi^2\tau^2 k(\lambda^2-4)^2 \\ & - 189\mathcal{W}\pi\tau^3 k^2(\lambda^2-4)^2 \end{aligned}$$

$n = 4:$

$$\begin{aligned} 0 = {} & \alpha^4 J_4^2(57600k^4(\lambda^2-4)(\lambda^2-9)[35\lambda^6 - 1330\lambda^4 + 21707\lambda^2 - 134748]) \\ & - J_4(624k^2(\lambda^2-4)(\lambda^2-9)[12800\alpha^4\mathcal{L}^2\pi^2(7\lambda^4 - 199\lambda^2 + 1788) \\ & + 8400\alpha^3\mathcal{W}\pi\tau k(5\lambda^4 - 95\lambda^2 + 636) \\ & + 23760\alpha^2\mathcal{W}\pi\tau^2 k(\lambda^2-4)(\lambda^2-11) + 99\tau^4 k^2(\lambda^2-4)^2]) \\ & + 665600\alpha^4\mathcal{L}\pi^3[75\mathcal{W}^2 k(\lambda^2-9)(5\lambda^4 - 95\lambda^2 + 636) \\ & + 352\mathcal{L}^3\pi(\lambda^2-4)(\lambda^4 - 17\lambda^2 + 100)] \\ & + 131788800\alpha^3\mathcal{W}\mathcal{L}^3\pi^3\tau k(\lambda^2-4)[3\lambda^4 - 51\lambda^2 + 244] \\ & + 137280\alpha^2\pi^2\tau^2 k(\lambda^2-4)[64\mathcal{L}^3\pi(\lambda^2-4)(11\lambda^2-71) \\ & + 45\mathcal{W}^2 k(5\lambda^4 - 65\lambda^2 + 264)] \\ & + 823680\alpha\mathcal{W}\mathcal{L}\pi^2\tau^3 k^2(\lambda^2-4)^2[23\lambda^2 - 123] \\ & - 9009k^2(3\lambda^2-7)(\lambda^2-4)(k + 8\pi\mathcal{L}\tau^2)(k - 8\pi\mathcal{L}\tau^2) \end{aligned} \tag{6.37}$$

These holonomy results which show the J_4 dependence have been reproduced and shown in Appendix C. The exact match can only be seen, however, after simplification. The charges through $\mathcal{O}(\alpha^8)$ which are recovered from (6.34) are given by

$$\begin{aligned}
\mathcal{L} &= -\frac{k}{8\pi\tau^2} + \frac{5k}{6\pi\tau^6}\alpha^2 - \frac{50k}{3\pi\tau^{10}}\frac{\lambda^2-7}{\lambda^2-4}\alpha^4 + \frac{2600k}{27\pi\tau^{14}}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\alpha^6 \\
&\quad - \frac{68000k}{81\pi\tau^{18}}\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\alpha^8 + \cdots \\
\mathcal{W} &= -\frac{k}{3\pi\tau^5}\alpha + \frac{200k}{27\pi\tau^9}\frac{\lambda^2-7}{\lambda^2-4}\alpha^3 - \frac{400k}{9\pi\tau^{13}}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\alpha^5 \\
&\quad + \frac{32000k}{81\pi\tau^{17}}\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\alpha^8 + \cdots \\
J^{(4)} &= \frac{35}{9\tau^8}\frac{1}{\lambda^2-4}\alpha^2 - \frac{700}{9\tau^{12}}\frac{2\lambda^2-21}{(\lambda^2-4)^2}\alpha^4 + \frac{2800}{9\tau^{16}}\frac{20\lambda^4-480\lambda^2+3189}{(\lambda^2-4)^3}\alpha^6 + \cdots
\end{aligned} \tag{6.38}$$

These charges satisfy the integrability condition and integrating either $\mathcal{W}$ or $\mathcal{L}$ we get the holomorphic part of the black hole partition function

$$\begin{aligned}
\ln Z(\tau,\alpha) = \frac{\pi i k}{2\tau}\Bigg[1 &- \frac{4}{3}\frac{\alpha^2}{\tau^4} + \frac{400}{27}\frac{\lambda^2-7}{\lambda^2-4}\frac{\alpha^4}{\tau^8} - \frac{1600}{27}\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\frac{\alpha^6}{\tau^{12}} \\
&+ \frac{32000}{81}\frac{20\lambda^6-600\lambda^4+6387\lambda^2-23357}{(\lambda^2-4)^3}\frac{\alpha^8}{\tau^{16}} + \cdots
\end{aligned} \tag{6.39}$$

This is the partition function of a black hole with a higher spin-3 field. The entropy including the spin-3 field and charge is given by

$$S = \ln Z(\tau,\alpha) - 4\pi^2(\tau\mathcal{L} + \alpha\mathcal{W} - \bar{\tau}\bar{\mathcal{L}} - \bar{\alpha}\bar{\mathcal{W}}) \tag{6.40}$$

At $\mathcal{O}(\alpha^0)$ the entropy is the Bekenstein-Hawking entropy

$$S = \frac{A}{4G}$$

there is no understood geometric interpretation as yet for large α.

6.4 Matching to CFT

For specific values of $\lambda = 0, 1$ the partition function has a dual CFT theory that admits a free field description. For these values, the partition function is given by

$$\begin{aligned}
\lambda = 1 : \ln Z(\tau,\alpha) &= \frac{\pi i k}{2\tau} - \frac{2\pi i k}{3}\frac{\alpha^2}{\tau^5} + \frac{400\pi i k}{27}\frac{\alpha^4}{\tau^9} - \frac{8800\pi i k}{9}\frac{\alpha^6}{\tau^{13}} + \frac{10400000\pi i k}{81}\frac{\alpha^8}{\tau^{17}} + \cdots \\
\lambda = 0 : \ln Z(\tau,\alpha) &= \frac{\pi i k}{2\tau} - \frac{2\pi i k}{3}\frac{\alpha^2}{\tau^5} + \frac{350\pi i k}{27}\frac{\alpha^4}{\tau^9} - \frac{18850\pi i k}{27}\frac{\alpha^6}{\tau^{13}} + \frac{5839250\pi i k}{81}\frac{\alpha^8}{\tau^{17}} + \cdots
\end{aligned} \tag{6.41}$$

focusing only on the holomorphic part, but the structure also holds for the non-holomorphic part of the partition function. Perhaps now its a good time to say something about W-algebras. The $\mathcal{W}_N$ algebra has currents with spin $s = 2, 3, \cdots, N$ such that $\mathcal{W}_2$ is the Virasoro algebra. The $\mathcal{W}_\infty$ algebra has currents of spins $s = 2, 3, \cdots, \infty$ while the $\mathcal{W}_{1+\infty}$ has currents of spin $\mathcal{W}_\infty$ plus a spin $s = 1$ current as suggested by the notation. There are more as discussed in [40].

6.4.1 Free bosons ($\lambda = 1$)

For $\lambda = 1$ the higher spin theory compares to a CFT of D free complex bosons with $\mathcal{W}_\infty[1]$ global symmetry [30, 40] with central charge $c = 2D$. The interactions between free complex bosons, the OPE is given by [22]

$$\partial\bar{\phi}^i(z_1)\partial\phi_j(z_2) \sim -\frac{\delta^i_j}{(z_1 - z_2)^2} \tag{6.42}$$

where $(i, j) = 1, 2, \cdots, D$. The stress tensor and the spin-3 current for free complex bosons are given by [30]

$$\begin{aligned} T &= -\partial\bar{\phi}^i\partial\phi_i \\ \mathcal{W} &= ia\left(\partial^2\bar{\phi}^i\partial\phi_i - \partial\bar{\phi}^i\partial^2\phi_i\right) \end{aligned} \tag{6.43}$$

where a is a normalization factor and $\mathcal{W}$ is a Virasoro primary field to match with the spin-3 current in our bulk theory. We compute the leading part of the $\mathcal{W}\mathcal{W}$ OPE to fix the normalization factor, a

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{4a^2D}{z^6} + \cdots \tag{6.44}$$

We know from (4.10) that our standard normalization is

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{5k}{\pi^2 z^6} \tag{6.45}$$

$\Rightarrow$

$$a = \sqrt{\frac{5k}{4\pi^2 d}} \tag{6.46}$$

we can write D in terms of k or vice-versa using $D = 3k$ and the Brown-Henneaux central charge, $c = \frac{3l}{2G}$. We then have for a

$$a = \sqrt{\frac{5}{12\pi^2}} \tag{6.47}$$

The mode expansions of our fields ϕ are given by

$$\partial\phi(w) = -\sum\nolimits_m \beta_m e^{imw} \qquad \partial\bar{\phi}(w) = -\sum\nolimits_m \bar{\beta}_m e^{imw} \tag{6.48}$$

obeying the following commutation relations

$$[\bar{\beta}_m, \beta_n] = m\delta_{m,-n} = [\beta_m, \bar{\beta}_n] \tag{6.49}$$

The normal ordered stress tensor is given by [22]

$$T = -\sum\left(\bar{\beta}_{-m}\beta_m + \beta_{-m}\bar{\beta}_m\right) + \frac{k}{4} + \text{nonconstant} \tag{6.50}$$

What is important for our purposes, however, is the zero mode of this stress tensor since it is the quantity appearing in our partition function. The non-constant terms are, therefore, not relevant. Following [11] we drop the ground state term, $\frac{k}{4}$ is also dropped since it does not contribute in the high temperature expansion that is studied in [11]. The quantity $\mathcal{L}$ is given in terms of the constant part of the stress tensor as

$$\mathcal{L} = -\frac{1}{2\pi}T \tag{6.51}$$

$\rightarrow$

$$\mathcal{L} = \frac{1}{2\pi}\sum\nolimits_{m=1}^{\infty}\left(\bar{\beta}_{-m}\beta_m + \beta_{-m}\bar{\beta}_m\right) \tag{6.52}$$

and the modes of the spin-3 current after normal ordering is

$$\mathcal{W} = 2a\sum_{m=1}^{\infty} m^2 \left(\bar{\beta}_{-m}\beta_m - \beta_{-m}\bar{\beta}_m\right) \tag{6.53}$$

Defining a state of the form

$$|n_m, \bar{n}_m\rangle = (\beta_{-m})^{n_m}(\bar{\beta}_{-m})^{\bar{n}_m}|0\rangle \tag{6.54}$$

obeying, as in [11]

$$\begin{aligned} \mathcal{L} &= \frac{1}{2\pi} m(\bar{n}_m + n_m)|n_m, \bar{n}_m\rangle \\ \mathcal{W}|n_m, \bar{n}_m\rangle &= 2am^2(\bar{n}_m - n_m)|n_m, \bar{n}_m\rangle \end{aligned} \tag{6.55}$$

we can, therefore, easily compute the partition function

$$Z(\tau, \alpha) = Tr\left[e^{4\pi^2 i(\tau\mathcal{L} + \alpha\mathcal{W})}\right] \tag{6.56}$$

which can be given by

$$\ln Z(\tau, \alpha) = -D\sum_{m=1}^{\infty}\left[\ln\left(1 - e^{2\pi i\tau m - 8\pi^2 i a\alpha m^2}\right) + \ln\left(1 - e^{2\pi i\tau m + 8\pi^2 i a\alpha m^2}\right)\right] \tag{6.57}$$

For high temperature considerations, $\tau \to 0$, the partition function using $D = 3k$ and $x = -2\pi i\tau m$ is

$$\ln Z(\tau, \alpha) = -\frac{3ik}{2\pi\tau}\int_0^{\infty} dx\left[\ln\left(1 - e^{-x + \frac{2ia\alpha}{\tau^2}x^2}\right) + \ln\left(1 - e^{-x - \frac{2ia\alpha}{\tau^2}x^2}\right)\right] \tag{6.58}$$

We use the identity [13]

$$\frac{\partial}{\partial\alpha}\ln\left[\left(1 - \varepsilon e^{-x + \frac{2ia\alpha}{\tau^2}x^2}\right) + \ln\left(1 - \varepsilon e^{-x - \frac{2ia\alpha}{\tau^2}x^2}\right)\right] = 4a\sum_{p=1}^{\infty} e^{-px}x^2\sin(2ap\alpha x^2)\varepsilon^p \tag{6.59}$$

Setting ε to unity we get

$$\begin{aligned} \sum_{p=1}^{\infty} e^{-px}x^2\sin\left(2ap\alpha x^2\right) &= \sum_{p=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n e^{-px}x^2\frac{(2ap\alpha x^2)^{2n+1}}{(2n+1)!} \\ &= \sum_{p=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n e^{-px}x^{4n+4}\frac{(2ap\alpha)^{2n+1}}{(2n+1)!} \end{aligned} \tag{6.60}$$

We need the following result [13]

$$\int_0^{\infty} e^{-px}x^{4n+4}dx = p^{-4n-5}\Gamma\left(4n+5\right) \tag{6.61}$$

using the Hurwitz function, for the p summation, given by

$$\sum_{p=1}^{\infty}\frac{1}{p^{2n+4}} = \zeta(2n+4) \tag{6.62}$$

at positive integers the Hurwitz function can be written as

$$\zeta(2n) = (-1)^{n-1}2^{2n-1}\pi^{2n}\frac{B_{2n}}{(2n)!} \tag{6.63}$$

where B_n are Bernoulli numbers. Integrating in α [13] recovers

$$\ln Z(\tau,\alpha) = \frac{\pi i k}{2\tau} - \frac{2\pi i k}{3}\frac{\alpha^2}{\tau^5} + \frac{400\pi i k}{27}\frac{\alpha^4}{\tau^9} - \frac{8800\pi i k}{9}\frac{\alpha^6}{\tau^{13}} + \frac{10400000\pi i k}{81}\frac{\alpha^8}{\tau^{17}} + \cdots \tag{6.64}$$

showing agreement with (6.41), the bulk results. In [13], a closed formula for the generic term of the pertubative expansion of the CFT partition function for $\lambda = 1$ with τ set to 1 is derived to be

$$\ln Z(1,\alpha) = \frac{6\pi i}{\sqrt{\pi}}\sum\nolimits_0^\infty \frac{B_{2n+2}\Gamma\left(2n+\frac{1}{2}\right)}{(2n+2)!}\left(\frac{320}{3}\right)^n \alpha^{2n} \tag{6.65}$$

where B_{2n} are Bernoulli numbers given by the series [46]

$$N - 1 = \sum\nolimits_{n=1}^{N-1} B_{2n}\binom{2n}{2} \tag{6.66}$$

For $\lambda = 1$ as can be easily checked, (6.65) gives (6.41). It is also noted in [13] that for small α expansion, the partition function (6.64) can be written in terms of the Hurwitz function, ζ as

$$\ln Z(1,\alpha) = \pi i\left[-\frac{3}{40\alpha^2} + \frac{3}{4}\sqrt{\frac{3}{5}}\frac{1}{\alpha} - 3\sqrt{2}\sqrt{\sqrt{\frac{3}{5}}}\zeta\left(-\frac{1}{2}, \frac{1}{8\alpha}\sqrt{\frac{3}{5}}\right)\frac{1}{\sqrt{\alpha}}\right] \tag{6.67}$$

with

$$\zeta(a,s) = \sum\nolimits_{n=0}^\infty \frac{1}{(a+n)^s} \tag{6.68}$$

The motivation of reproducing these closed forms is with the intention of using them to check modular properties of partition functions which depend on the modular parameter, which has been set to 1 in (6.67).

6.4.2 Free fermions ($\lambda = 0$)

For $\lambda = 0$ the higher spin theory we dealing with compares to a CFT of D free complex fermions with $\mathcal{W}_{1+\infty}$ symmetry and central charge $c = D$ [46]. We have already mentioned the relatedness of $\mathcal{W}_{1+\infty}$ with $\mathcal{W}_\infty$[1]. To proceed as in 6.41 we use a chemical potential to require that the spin-1 current is set to zero in the partition function. The fermions have the following OPE

$$\bar{\psi}^i(z_1)\psi_j(z_2) \sim \frac{\delta^i_j}{z_1 - z_2} \tag{6.69}$$

and the stress tensor

$$T = -\frac{1}{2}\bar{\psi}^i\partial\psi_i - \frac{1}{2}\psi_i\partial\bar{\psi}^i \tag{6.70}$$

In [46] the spin-3 current, up to a normalization constant b is given by

$$\mathcal{W} = ib\left(\partial^2\bar{\psi}^i\psi_i - 4\partial\bar{\psi}^i\partial\psi_i + \bar{\psi}^i\partial^2\psi_i\right) \tag{6.71}$$

The $\mathcal{TW}$ OPE contains a term which scales like J/z^4, J being the spin-1 current. We, therefore set this spin-1 current J to zero forcing $\mathcal{W}$ to be primary so that we can compare with results in our bulk theory. The leading part of the $\mathcal{WW}$ OPE is

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{24Db^2}{z^6} + \cdots \tag{6.72}$$

Our standard normalization from (4.10) is given by

$$\mathcal{W}(z)\mathcal{W}(0) \sim -\frac{5k}{\pi^2 z^6} \tag{6.73}$$

using $c = 6k = D \Rightarrow$

$$b = \sqrt{\frac{5}{144\pi^2}} \tag{6.74}$$

Proceeding as in 6.41, using the fermionic mode expansions, stress tensor, the spin-3 current as given in [11] and the spin-1 charge operator given by

$$Q \sim \int \bar{\psi}\psi \tag{6.75}$$

Defined to be [11]

$$Q|n_m, \bar{n}_m\rangle = (\bar{n}_m - n_m)\,|n_m, \bar{n}_m\rangle \tag{6.76}$$

For high temperature considerations we impose $Q = 0$ as an exact condition on states. We do this by incorporating a chemical potential for Q in our partition function

$$Z(\tau, \alpha, \gamma) = Tr\left[e^{4\pi^2 i[\tau\mathcal{L}+\alpha\mathcal{W}]+i\gamma Q}\right] \tag{6.77}$$

$\Rightarrow$

$$\ln Z(\tau, \alpha, \gamma) = D\sum\nolimits_{m=1}^{\infty}\left[\ln\left(1 + e^{\left(2\pi i\tau m - 24\pi^2 ib\alpha m^2 + i\gamma\right)}\right) + \ln\left(1 + e^{\left(2\pi i\tau m + 24\pi^2 ib\alpha m^2 - i\gamma\right)}\right)\right] \tag{6.78}$$

using $D = 6k$ and $x = -2\pi i\tau m$ and converting the sum to an integral we get

$$\ln Z(\tau, \alpha, \gamma) = \frac{3ik}{\pi\tau}\int_0^{\infty}\left[\ln\left(1 + e^{-x + \frac{6ib\alpha}{\tau^2}x^2 + i\gamma}\right) + \ln\left(1 + e^{-x - \frac{6ib\alpha}{\tau^2} - i\gamma}\right)\right] \tag{6.79}$$

In [11] pertubative approaches are employed in solving (6.79), producing

$$\ln Z(\tau, \alpha) = \frac{\pi ik}{2\tau} - \frac{2\pi ik}{3}\frac{\alpha^2}{\tau^5} + \frac{350\pi ik}{27}\frac{\alpha^4}{\tau^9} - \frac{18850\pi ik}{27}\frac{\alpha^6}{\tau^{13}} + \frac{5839250\pi ik}{81}\frac{\alpha^8}{\tau^{17}} + \cdots. \tag{6.80}$$

matching with the bulk results (6.41)

6.4.3 Other values of λ, $(\lambda = \infty)$

The partition function (6.39) for $\lambda \to \infty$, the limit where all higher spin charges have been switched on takes the form

$$\ln Z(\tau, \alpha) = \frac{\pi ik}{2\tau}\left(1 - \frac{4}{3}\varsigma^2 + \frac{400}{27}\varsigma^4 - \frac{8000}{27}\varsigma^6 + \frac{640000}{81}\varsigma^8 + \cdots\right) \tag{6.81}$$

with

$$\varsigma = \frac{\alpha}{\tau^2} \tag{6.82}$$

It turns out that (6.81) can be written in the following form

$$\ln Z(\tau, \alpha) = \frac{12\pi ik}{\tau}\sum\nolimits_0^{\infty}(-1)^p\left(\frac{20}{3}\right)^p\frac{\Gamma(4p)}{\Gamma(p)\Gamma(3p+4)}\varsigma^{2p} \tag{6.83}$$

where the $p = 0$ term is evaluated with $\frac{\Gamma(4p)}{\Gamma(p)} \to \frac{1}{4}$. Writing out the Γ functions in full, the pochammer symbols are evident and the infinite series can be written as a closed expression using hypergeometric function as

$$\ln Z(\tau, \alpha) = \frac{3\pi i\tau}{160\alpha}\frac{1}{\varsigma}\left[{}_3F_2\left(\begin{matrix}-\frac{3}{4}, & -\frac{1}{2}, & -\frac{1}{4} \\ \frac{1}{3}, & \frac{2}{3} & \end{matrix}\middle| -\frac{5120}{81}\varsigma^2\right) - 1\right] \tag{6.84}$$

It is worth noting that it is only for $\lambda \to \infty$ where we can write the partition function as a hypergeometric function. This is due to the holonomy constraints which are algebraic at $\lambda = N$ [13].

6.5 Higher Spin corrections to CFT thermodynamics

Following [47], we consider the partition function from the CFT side

$$Z(\hat{\tau}, \alpha) = Tr\left(\hat{q}^{L_0 - \frac{c}{24}} y W_0\right) \tag{6.85}$$

where $\hat{q} = \exp 2\pi i \hat{\tau}$, $y = \exp 2\pi i \alpha$ and the shift $L_0 \to L_0 - \frac{c}{24}$. The spin-3 current is given by the zero mode of the spin-3 generator of $\mathcal{W}_\infty[\lambda]$

$$\mathcal{W} = \frac{1}{2\pi} W_0.$$

The modular parameter $\hat{\tau}$ is related to the temperature of the black hole β and angular momentum Ω_H by

$$\hat{\tau} = \frac{i}{2\pi\beta}(1 + \Omega_H) \tag{6.86}$$

such that $\hat{\tau} \to 0$ $(q \to 0)$ is the high temperature limit, since the $\mathcal{S}$ transformation is

$$\tau = -\frac{1}{\hat{\tau}}, \quad q = \exp 2\pi i \tau$$

Expanding (6.85) in α yields, up to order $\mathcal{O}(\alpha^6)$

$$\begin{aligned} Z(\hat{\tau}, \alpha) \approx\, & Tr\left(\hat{q}^{L_0 - \frac{c}{24}}\right) + \frac{(2\pi i \alpha)^2}{2!} Tr\left((W_0)^4 \hat{q}^{L_0 - \frac{c}{24}}\right) + \frac{(2\pi i \alpha)^4}{4!} Tr\left((W_0)^4 \hat{q}^{L_0 - \frac{c}{24}}\right) \\ & + \frac{(2\pi i \alpha)^6}{6!} Tr\left((W_0)^6 \hat{q}^{L_0 - \frac{c}{24}}\right) \end{aligned} \tag{6.87}$$

Using the method of obtaining a partition function from a dual conformal field theory, by taking the $\mathcal{S}$ transformation in the high temperature regime. We find for the vacuum contribution $(L_0 = 0)$ of the first term of (6.87)

$$Tr\left(\hat{q}^{L_0 - \frac{c}{24}}\right) = \sum_{s,r} S_{sr} Tr_r\left(q^{L_0 - \frac{c}{24}}\right) \sim \left(\sum_s S_{s0}\right) q^{-\frac{c}{24}} + \cdots \tag{6.88}$$

S_{sr} is the modular S-matrix; the terms which are exponentially suppressed at high temperature are indicated by the dots.
Taking the logarithm of (6.88) gives the leading behaviour

$$\ln Tr\left(\hat{q}^{L_0 - \frac{c}{24}}\right) = -\frac{\pi i k}{2}\tau + \cdots \tag{6.89}$$

which matches exactly with the first term, the α independent term in (6.39). In the following section we will follow this method in investigating the leading behaviour of our partition function. In [47] a zero mode of a dimension-1 current is considered, with the modular transformation of its exponential described by a Jacobi form [47]

$$\phi_r\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \sum_s \exp\left\{2\pi i k \frac{cz^2}{c\tau + d}\right\} M_{rs} \phi_s(\tau, z) \tag{6.90}$$

Where M_{rs} is the representation of the modular group. We have assumed that the CFT at our disposal is rational with a $U(1)$ current J_n, we define the trace in the representation r

$$\phi_r(\tau, z) = Tr_r\left(y^{J_0} q^{L_0 - \frac{c}{24}}\right). \tag{6.91}$$

And the level k given by the commutator

$$[J_m, J_n] = 2km\delta_{m,-n} \tag{6.92}$$

In [47] the modular properties of the partition function order by order in insertions of higher spin zero modes were computed for the partition function (6.85). For the full partition function we might expect something like (6.90) which is relevant for CFT's with dimension 1 currents. To understand this better, in the following section we try to look in detail at how the higher spin partition functions (6.41) can be related to a new class of generalised Jacobi forms introduced in Kaneko-Zagier [48], which might eventually lead to formulas similar to (6.90).

7 $hs[\lambda]$ modular properties

In this section we investigate the modular properties of the partition function (6.39). We follow section 5.2., testing the $\mathcal{S} : \tau \to -\frac{1}{\tau}$ modular transformation. The challenge we face, however, is that we do not have (at the time of writing, maybe in future) a compact form of the partition function (6.39); which would make it neat to check modular invariance or specifically, the modular properties. We check modular properties for the values for which we have closed forms. Following which we check the modular properties of the first few terms of (6.39). We also make use of the transformation of the modular and generalized-Jacobi forms as given by Kaneko and Zagier [48].

7.1 Modular forms

A *modular form*, of weight k, on a subgroup Γ of finite index of Γ_1 is a holomorphic function f on $\mathcal{H}$ such that

$$Z\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k Z(\tau) \quad \forall \tau \in \mathcal{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{7.1}$$

where $\mathcal{H} = \{\tau \in \mathbf{C} | \Im(\tau) > 0\}$ is the complex upper half-plane. We also set $q = e^{2\pi i \tau}$ and $Y = 4\pi\Im(\tau)$. For (7.1) to be considered a modular form we have to add the condition that f must grow polynomially in $\frac{1}{Y}$ as $y \to 0$.
The space of all holomorphic modular forms of weight k on Γ is denoted by $M_k(\Gamma)$. The graded ring[6] $\oplus_k M_k(\Gamma)$ (see Appendix C), is denoted by $M_*(\Gamma)$. Following [48] we also define an *almost holomorphic modular form* $F(\tau)$ of weight k, $F(\tau)$ which satisfies the same transformation properties and growth conditions as Z but have the form

$$F(\tau) = \sum_{m=0}^{M} Z_m(\tau) Y^{-m} \tag{7.2}$$

where $0 \leq M \leq \frac{k}{2}$ is some integer. The space of all *almost holomorphic modular forms* of weight k is denoted by $\hat{M}_k(\Gamma)$, whilst $Z_0(\tau)$ which is a constant term with respect to $\frac{1}{Y}$ of f is known as a *quasi-modular form* of weight k. The space of all *quasi-modular forms* is denoted by $\tilde{M}_k(\Gamma)$. It is noted in [48] that the spaces $\hat{M}_\star(\Gamma) = \oplus \hat{M}_k(\Gamma)$ and $\tilde{M}_\star(\Gamma) = \oplus \tilde{M}_h(\Gamma)$ are graded rings and the map $\hat{M}_\star(\Gamma) \to \tilde{M}_\star(\Gamma)$ is a homomorphism [48]. For example, the following function is a modular form of weight k [48]

$$G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n \qquad (k = 2, 4, 6..., B_k = k\text{th Bernoulli number}) \tag{7.3}$$

whereas $G_2(\tau)$ transforms as

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c(c\tau+d)}{4\pi i} \qquad \forall \tau \in \mathcal{H} \tag{7.4}$$

Meaning that $G_2(\tau)$ is quasi-modular. We now turn our attention to Jacobi forms and quasi-Jacobi forms as proposed in [48]

[6] A graded ring is a ring (a set equipped with addition and multiplication) M with a set of subgroups $\oplus_k M_k$, $k \geq 0$ such that:

1. $\oplus_k M_k$ forms an Abelian group
2. $x \cdot y \in M_{k+l} \forall x \in M_k, \forall y \in M_l$

7.1.1 Jacobi forms

A Jacobi form [50] on Γ is a holomorphic function ϕ of two variables $\tau \in \mathcal{H}$ and $z \in C$ satisfying, among other properties, the modular transformation property with respect to

$$(\tau, z) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$
$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^h \exp\left(\frac{2\pi imcz^2}{c\tau + d}\right) \phi(\tau, z) \tag{7.5}$$

where h is the weight and m the index of the Jacobi form ϕ. Related to the concept of Jacobi forms is a concept of quasi-Jacobi forms [48], which are forms that exhibit Jacobi forms properties in a modified form

$$Z(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k \sum_{i \leq s, j \leq t} S_{i,j} Z(\tau, z) \left(\frac{cz}{c\tau + d}\right)^i \left(\frac{c}{c\tau + d}\right)^j \tag{7.6}$$

Following [48], a generalized classical Jacobi theta function, the analogue of our partition function (6.39) is given by

$$\Theta(X, q, \zeta) = \prod_{n>0}(1 - q^n) \prod_{n(odd)>0} (1 - e^{n^2X/8} q^{n/2}\zeta)(1 - e^{-n^2X/8} q^{n/2}\zeta^{-1}) \tag{7.7}$$

Where we define

$$w = e^X \qquad \zeta = e^Z \tag{7.8}$$

and denote by Γ_2 the group of matrices in (7.1) $\in \Gamma_1$ with b even. We also introduce $\theta(\tau)$

$$\theta_4(\tau) = \sum_r (-1)^r q^{r^2/2} \tag{7.9}$$

a modular form of weight $\frac{1}{2}$ on Γ_2. The function $\Theta(X, \tau, Z)$ has an expansion of the form

$$\Theta(X, \tau, Z) = \theta_4(\tau) \sum_{j,l \geq 0} H_{j,l}(\tau) \frac{X^j}{j!} \frac{Z^l}{l!} \tag{7.10}$$

where $H_{0,0}(\tau) = 1$ and each $H_{j,l}(\tau)$ is a quasi-modular form of weight $3j + l$ on Γ_2. If we set

$$H(w, q, \zeta) = q^{-1/24} \prod_{n(odd)>0} (1 - w^{n^2/8} q^{n/2}\zeta)(1 - w^{-n^2/8} q^{n/2}\zeta^{-1}) \tag{7.11}$$

$H_0(w, q)$ is the coefficient of ζ^0 in $H(w, q, \zeta)$ as a Laurent series in ζ. The function $H(w, q, \zeta)$ has the expansion [48]

$$H(w, q, \zeta) = \sum_{r \in \mathbf{Z}} (-1)^r H_0(w, w^r q) w^{r^3/6} q^{r^2/6} \zeta^r. \tag{7.12}$$

H_0 is given by

$$H_0(\alpha, \tau) = \sum_{n=0}^{\infty} \frac{A_n(\tau)}{\eta(\tau)} \alpha^{2n} \tag{7.13}$$

with A_n a quasi-modular form of weight $k = 6n$ [48].

We now turn to computing modular properties of the partition function at $\lambda = 0, 1$ against the backdrop of these Jacobi (quasi) forms.

7.2 Partition function modular properties

In this section we compute the modular properties ($\mathcal{S}$ transformation) of the partition function at values of λ for which a CFT match has been computed in 6.4.

7.2.1 $\lambda = 0$

We check the standard free-fermion calculation. Starting with

$$Z = \left[\prod_{m=1}^{\infty}\left(1+e^{2\pi i\tau m-24\pi^2 ib\alpha m^2+i\gamma}\right)\left(1+e^{-2\pi i\tau m+24\pi^2 ib\alpha m^2-i\gamma}\right)\right]^{6k} \tag{7.14}$$

Take $\alpha = 0$, this also means that $\gamma = 0$

$$Z = \left[\prod_{m=1}^{\infty}(1+q^m)(1+q^m)\right]^{6k} = \left[\prod_{m=1}^{\infty}(1+q^m)^2\right]^{6k} \tag{7.15}$$

The expression in the r.h.s is given by [36]

$$\prod_{m=0}^{\infty}(1+q^m) = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} \tag{7.16}$$

$$\begin{aligned} Z &= \left[\prod_{m=1}^{\infty}(1+q^m)\right]^{12k} \\ &= \left[\frac{1}{2}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}\right]^{12k} = \frac{1}{2^{12k}}\left(\frac{\theta_2(\tau)}{\eta(\tau)}\right) \end{aligned} \tag{7.17}$$

We now check if this partition function (7.17) gives the correct high temperature behaviour, $\tau \to 0$ ($\hat{\tau} = -1/\tau \to i\infty$)[7]

$$\begin{aligned} Z(-1/\hat{\tau}) &= \frac{1}{2^{12k}}\left(\frac{\theta_2(-1/\hat{\tau})}{\eta(-1/\hat{\tau})}\right)^{6k} \\ &= \frac{1}{2^{12k}}\left(\frac{\sqrt{-i\hat{\tau}}\,\theta_4(\hat{\tau})}{\sqrt{-i\hat{\tau}}\,\eta(\hat{\tau})}\right) \sim \left(\frac{\theta_4(\hat{\tau})}{\eta(\hat{\tau})}\right)^{6k} \end{aligned} \tag{7.18}$$

Using the following relation of $\theta_4(\hat{\tau})$ (proof in Appendix C)

$$\theta_4(\hat{\tau}) = \frac{\eta(\hat{\tau}/2)^2}{\eta(\hat{\tau})} \tag{7.19}$$

Yielding for Z

$$\begin{aligned} Z &\sim \left(\frac{\eta(\hat{\tau}/2)}{\eta(\hat{\tau})}\right)^{12k} \\ &= \left(\frac{\hat{q^{1/48}}}{\hat{q^{1/24}}}\right)^{12k} = (\hat{q}^{-1/48})^{12k} = e^{-\frac{\pi i\hat{\tau}k}{2}} = e^{\frac{\pi ik}{2\tau}} \end{aligned} \tag{7.20}$$

[7]To get to low temperature behaviour we need to account for the vacuum contribution of $q^{-k/4}$ which has been suppressed above

Showing that $Z(\tau) \to e^{\frac{\pi i k}{2\tau}}$ as $\tau \to 0$, which agrees with the first term of (6.41). Now as a second check, we wish to relate (6.78)

$$\ln Z(\tau,\alpha,\gamma) = D\sum\nolimits_{m=1}^{\infty}\left[\ln\left(1+e^{\left(2\pi i\tau m-24\pi^2 ib\alpha m^2+i\gamma\right)}\right)+\ln\left(1+e^{\left(2\pi i\tau m+24\pi^2 ib\alpha m^2-i\gamma\right)}\right)\right]$$

to $H(w,q,\zeta)$ (7.10)

$$H(w,q,\zeta) = \sum_{r\in\mathbf{Z}}(-1)^r H_0(w,w^r q)w^{r^3/6}q^{r^2/6}\zeta^r$$

We set

$$q = e^{2\pi i\tau}, \quad w = e^{8\cdot 24\pi i^2 b\alpha}, \quad \zeta = -e^{-i\gamma} \tag{7.21}$$

This means that we have

$$\begin{aligned}
\ln Z(\alpha,\tau,\gamma) &= D\sum\nolimits_{m=1}^{\infty}\left[\ln\left(1+e^{\left(2\pi i\tau m-24\pi^2 ib\alpha m^2+i\gamma\right)}\right)+\ln\left(1+e^{\left(2\pi i\tau m+24\pi^2 ib\alpha m^2-i\gamma\right)}\right)\right]\\
&= D\sum_{m=1}^{\infty}\left[\ln(1-q^m w^{m^2/8}\zeta)+\ln(1-q^m w^{-m^2/8}\zeta^{-1})\right]\\
&= D\sum_{m=1}^{\infty}\left[\ln\left((1-q^m w^{m^2/8}\zeta)(1-q^m w^{-m^2/8}\zeta^{-1})\right)\right]\\
&= D\left[\ln\prod_{m=1}^{\infty}\left((1-q^m w^{m^2/8}\zeta)(1-q^m w^{-m^2/8}\zeta^{-1})\right)\right]\\
&= D\ln\prod_{m_{odd}>0}^{\infty}\left((1-q^m w^{m^2/8}\zeta)(1-q^m w^{-m^2/8}\zeta^{-1})\right)\\
&\prod_{m_{even}>0}^{\infty}\left((1-q^m w^{m^2/8}\zeta)(1-q^{m/2} w^{-m^2/8}\zeta^{-1})\right)
\end{aligned} \tag{7.22}$$

$$\begin{aligned}
\ln Z(\alpha,\tau,\gamma) = D\ln\prod_{m_{odd}>0}^{\infty}&\left((1-q^m w^{m^2/8}\zeta)(1-q^m w^{-m^2/8}\zeta^{-1})\right)\\
\prod_{m_{odd}>0}^{\infty}&\left((1-q^{m+1} w^{(m+1)^2/4}\zeta)(1-q^{m+1} w^{-(m+1)^2/4}\zeta^{-1})\right)
\end{aligned} \tag{7.23}$$

But we can write [48]

$$\begin{aligned}
\prod_{m(odd)=1}(1-w^{-(m+1)^2/8}q^{m+1}\zeta^{-1}) &= \prod_{m=3}(1-w^{-(m-1)^2/8}q^{m-1}\zeta^{-1})\\
&= \prod_{m(odd)=1}\frac{(1-w^{-(m-1)^2/8}q^{m+1}\zeta^{-1})}{(1-w^0q^0\zeta^{-1})}
\end{aligned} \tag{7.24}$$

$$\begin{aligned}
\ln Z(\alpha,\tau,\gamma) = D\ln &\prod_{m_{odd}>0}^{\infty}\left((1-w^{m^2/8}q^m\zeta)(1-q^m w^{-m^2/8}\zeta^{-1})\right)\\
&\prod_{m_{odd}>0}^{\infty}\left((1-w^{m^2/8}(w^{m/4}q^m)(w^{1/8}q\zeta)\right)\left(1-w^{-m^2/8}(w^{m/4}q^m)(w^{-1/8}q^{-1}\zeta^{-1})\right)\\
&\times\left(\frac{1}{1-\zeta^{-1}}\right)
\end{aligned} \tag{7.25}$$

Therefore Z is given by

$$\begin{aligned} Z(\alpha,\tau,\gamma) &= \left[H(w,q^2,\zeta)\cdot H(w,w^{1/2}q^2,w^{1/8}q\zeta)\cdot(q^{1/2}(w^{1/2}q^2)^{1/24})\times\left(\frac{1}{1-\zeta^{-1}}\right)\right]^D \\ Z(\alpha,\tau,\gamma) &= \left[\frac{q^{1/6}w^{1/48}}{(1-\zeta^{-1})}H(w,q^2,\zeta)\cdot H(w,w^{1/2}q^2,w^{1/8}q\zeta)\right]^D \end{aligned} \tag{7.26}$$

What we have done in the above equation, is to relate the $\lambda = 0$ partition function to the Kaneko-Zagier generalized-Jacobi forms, which implies that a better understanding of the modular properties of these forms will help us also understand the physics of higher spin black holes at least at $\lambda = 0$. H is given by [48]

$$\begin{aligned} H(\alpha,\tau,\gamma) &= \frac{\theta(\tau)}{\eta(\tau)}\sum_{m,n\geq 0} H_{m,n}(\tau)\frac{\alpha^m}{m!}\frac{\gamma^n}{n!} \\ &= \frac{\theta(\tau)}{\eta(\tau)}\left(H_{0,0}(\tau)+H_{1,0}(\tau)\alpha+H_{0,1}(\tau)\gamma+\cdots\right) \end{aligned} \tag{7.27}$$

γ (ζ) is only known in a high temperature expansion, which would at least mean that the (7.26) is non-singular. Since, as a first step, we are interested in the $\alpha = 0$ term, i.e. $w = 1$ and $\zeta = -1$, we therefore reconsider the fermionic partition function with these values. For $D = 6k$ we have (7.15)

$$Z = \left[\prod_{m=1}^{\infty}(1+q^m)(1+q^m)\right]^{6k} \tag{7.28}$$

We introduce ζ to bring it to the form of the H functions of [48], later we let $\zeta \to -1$.

$$Z = \left[\prod_{m(odd)=1}^{\infty}(1-q^m\zeta)(1-q^m\zeta^{-1})\prod_{m(even)=2}^{\infty}(1-q^m\zeta)(1-q^m\zeta^{-1})\right]^{6k} \tag{7.29}$$

We use $\theta_1(z|\tau)$ and $\theta_4(z|\tau)$ of [36]

$$\begin{aligned} \theta_1(z|\tau) &= -i\zeta^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-\zeta q^{n+1})(1-\zeta^{-1}q^n) \\ &= -i\zeta^{1/2}q^{1/8}q^{-1/24}\eta(\tau)\prod_{n=0}^{\infty}(1-\zeta q^{n+1})(1-\zeta q^n) \\ \theta_4(z|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbf{Z}}^{\infty}(1-\zeta q^r)(1-\zeta^{-1}q^r) \\ &= q^{-1/24}\eta(\tau)\prod_{n(odd)=1}^{\infty}(1-q^{n/2}\zeta)(1-q^{n/2}\zeta^{-1}) \end{aligned} \tag{7.30}$$

Substituting the θ's we get for Z

$$Z = \left[\frac{q^{-1/6}}{(1-\zeta^{-1})}\frac{1}{\eta(2\tau)^2}\frac{\theta_4(z|2\tau)\theta_1(z|2\tau)}{\eta(2\tau)^2}\right]^{6k} \tag{7.31}$$

In agreement with (7.26). To see this we take $w = 1$ in (7.26)

$$\begin{aligned} Z &= \left[\frac{q^{1/6}w^{1/48}}{(1-\zeta^{-1})}H(w,q^2,\zeta)H(w,w^{1/2}q^2,w^{1/8}q\zeta)\right]^{6k} \\ &= \left[\frac{q^{1/6}}{(1-\zeta^{-1})}H(1,q^2,\zeta)H(1,q^2,q\zeta)\right]^{6k} \end{aligned} \tag{7.32}$$

Using (7.12)

$$H(1,q,\zeta)=\frac{\theta_4(z|\tau)}{\eta(\tau)} \tag{7.33}$$

Which means

$$Z=\left[\frac{q^{1/6}}{(1-\zeta^{-1})}\frac{1}{\eta(\tau)^2}\theta_4(z|2\tau)\theta_4(z+\tau|2\tau)\right]^{6k} \tag{7.34}$$

where $\zeta q=e^{2\pi i(z+\tau)}$. In Appendix C, (C.3) gives the following relation

$$\theta_4(z+\frac{\tau}{2}|2\tau)=i\zeta^{-1/2}q^{-1/8}\theta_1(z|\tau) \tag{7.35}$$

Setting $\tau'=2\tau$

$$\theta_4(z+\frac{\tau'}{2}|\tau')=i\zeta^{-1/2}(q')^{-1/2}\theta_1(z|\tau')=i\zeta^{-1/2}q^{-1/4}\theta_1(z|2\tau) \tag{7.36}$$

Which gives

$$Z=\left[i\frac{q^{-1/12}\zeta^{-1/2}}{\eta(\tau)^2}\frac{1}{(1-\zeta^{-1})}\theta_4(z|2\tau)\theta_1(z|2\tau)\right]^{6k} \tag{7.37}$$

Now we check the modular properties, $\mathcal{S}$ transformation of (7.37). We have for θ_4 and θ_1

$$\begin{aligned}\theta_4\left(\frac{z}{\tau},e^{-\frac{i\pi}{\tau}}\right)&=\frac{\sqrt{\tau}}{\sqrt{i}}e^{\frac{\pi i z^2}{\tau}}\theta_2\left(z,e^{\pi i\tau}\right)\\ \theta_1\left(\frac{z}{\tau},e^{\frac{-\pi i}{\tau}}\right)&=-i\left(e^{-\frac{\pi i}{\tau}}\right)\sqrt{-\pi i}e^{\frac{\pi i z^2}{\tau}+\frac{\pi i}{4\tau}}\theta_1(z,e^{\pi i\tau})\end{aligned} \tag{7.38}$$

Starting with $Z(\hat{z},\hat{\tau})$, with $\hat{z}=\frac{1}{2}$ since we want, at the end, to take $\hat{\zeta}\to-1$. For high temperature considerations, $\hat{\tau}\to 0$

$$\theta_4(\hat{z}|2\hat{\tau})=\theta_4\left(\hat{z},e^{2\pi i\hat{\tau}}\right)=\theta_4\left(\hat{z},e^{-\frac{\pi i}{\tau'}}\right) \tag{7.39}$$

Defining $\hat{z}=\frac{z'}{\tau'}$ we get

$$\begin{aligned}\theta_4(\hat{z}|2\hat{\tau})&=\theta_4\left(\frac{z'}{\tau'},e^{-\frac{\pi i}{\tau'}}\right)\\ \theta_4\left(\hat{z}|2\hat{\tau}\right)&=\theta_4\left(\frac{z'}{\tau'},e^{-\frac{\pi i}{\tau'}}\right)=\sqrt{\frac{\tau'}{i}}e^{\frac{\pi i z'^2}{\tau'}}\theta_2\left(z',e^{\pi i\tau'}\right)\\ &=\sqrt{-i\tau/2}e^{\frac{\pi i\tau}{8}}\theta_2(\hat{z}\tau',e^{\pi i\tau'})=\sqrt{-\frac{i\tau}{2}}e^{\frac{\pi i\tau}{8}}e^{\pi i\hat{z}^2\frac{\tau}{2}}\theta_2\left(\hat{z}\frac{\tau}{2},e^{\pi i\frac{\tau}{2}}\right)\\ &=\sqrt{-\frac{i\tau}{2}}e^{\frac{\pi i\tau}{8}}\theta_2\left(\frac{\tau}{4},e^{\frac{\pi i\tau}{2}}\right)\end{aligned} \tag{7.40}$$

where in the second line we used $\hat{z}=\frac{z'}{\tau'}$ and substituted $\hat{z}=1/2$ in the last line. The product expansion

$$\theta_2(z|\tau)=\zeta^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1+\zeta q^{n+1})(1+\zeta^{-1}q^n) \tag{7.41}$$

As we take $\tau\to i\infty$, $q\to 0$, meaning $\theta_2\to\zeta^{1/2}q^{1/8}(1+\zeta^{-1}q^0)$. This means

$$\begin{aligned}\theta_4(\hat{z}|2\hat{\tau})&\to\sqrt{-\frac{i\tau}{2}}e^{\frac{\pi i\tau}{8}}\theta_2\left(\frac{\tau}{4},e^{\frac{\pi i\tau}{2}}\right)\\ &=\sqrt{-\frac{i\tau}{2}}e^{\frac{\pi i\tau}{8}}\left(e^{2\pi i\frac{\tau}{4}}\right)^{1/2}\left(e^{\frac{2\pi i}{8}}\right)^{1/2}\left(1+e^{-\frac{2\pi i\tau}{4}}\right)\\ &=\sqrt{-\frac{i\tau}{2}}\cdot\mathbf{1}\end{aligned} \tag{7.42}$$

where in the second line we drop 1 in the last term of the product due to the largeness of the exponential term when $\tau \to i\infty$. Meaning that for high temperature behaviour we have $\theta_4(\hat{z}|2\hat{\tau}) \to \sqrt{-\frac{i\tau}{2}}$.
Similarly, we look at the high temperature behaviour of $\theta_1(\hat{z}|2\hat{\tau})$

$$\begin{aligned}\theta_1(\hat{z}|2\hat{\tau}) &= \theta_1\left(\hat{z}, e^{-\frac{\pi i}{\tau'}}\right) = \theta_1\left(\frac{\hat{z}'}{\tau'}, e^{-\frac{\pi i}{\tau'}}\right) \\ &= -i\sqrt[4]{e^{-\frac{\pi i}{\tau'}}}\sqrt{-i\tau'}e^{\frac{\pi i(z')^2}{\tau'}+\frac{\pi i}{4\tau'}}\theta_1\left(z', e^{\pi i\tau'}\right) \\ &= -ie^{-\frac{\pi i}{4\tau'}}\sqrt{-i\tau'}e^{\pi i\hat{z}^2\tau'+\frac{\pi i}{4\tau'}}\theta_1\left(z', e^{\pi i\tau'}\right) \\ &= -ie^{-\frac{\pi i}{4\tau'}}\sqrt{-i\tau'}e^{\pi i\hat{z}^2\tau'+\frac{\pi i}{4\tau'}}\theta_1\left(\hat{z}\tau', e^{\pi i\tau'}\right) \\ &= -ie^{-\frac{2\pi i}{\tau}}\sqrt{-\frac{i\tau}{2}}e^{\pi i\frac{\tau}{8}+\frac{2\pi i}{\tau}}\theta_1\left(\frac{\tau}{4}, e^{\frac{\pi i\tau}{2}}\right)\end{aligned} \tag{7.43}$$

Now we need

$$\begin{aligned}\theta_1(z, e^{\pi i\tau}) &= -i\zeta^{1/2}\left(e^{2\pi i\tau}\right)^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-\zeta q^{n+1})(1-\zeta^{-1}q^n) \\ &\to -i\zeta^{1/2}e^{\frac{\pi i\tau}{4}}\left(1-\zeta^{-1}\right)\end{aligned} \tag{7.44}$$

Meaning for θ_1

$$\theta_1(\hat{z}, 2\hat{\tau}) \to \left(-i\sqrt{-\frac{i\tau}{2}}e^{\frac{\pi i\tau}{8}}\right)\left(-i\left(e^{\frac{2\pi i\tau}{4}}\right)^{1/2}\left(e^{\frac{\pi i\tau}{8}}\right)\left(1-e^{-\frac{2\pi i\tau}{4}}\right)\right) = \sqrt{-\frac{i\tau}{2}} \tag{7.45}$$

Looking at the full partition function (7.37)

$$\begin{aligned}Z &= \left[i\frac{q^{-1/12}\zeta^{-1/2}}{\eta(\tau)^2}\frac{1}{(1-\zeta^{-1})}\theta_4(z|2\tau)\theta_1(z|2\tau)\right]^{6k} \\ &\to \left[\frac{1}{2}\frac{1}{\eta(\tau/2)^2}\right]^{6k}\end{aligned} \tag{7.46}$$

where in moving to line 2 we took the high temperature behaviour $\left(\hat{q} \to 1, \hat{\zeta} \to -1\right)$. Showing that the high temperature behaviour is due to $\eta(\tau)$, the Dedekind's function, giving for Z

$$Z \to \frac{1}{2^{6k}}\left(\frac{1}{e^{\frac{\pi ik\tau}{2}}}\right) \sim e^{-\frac{\pi ik\tau}{2}} \tag{7.47}$$

in agreement[8] with (7.18) up to an exponential factor of 2 in the pre-factor. Under $\mathcal{S}$ transformation we have

$$Z = e^{\frac{\pi i}{2\hat{\tau}}} \tag{7.48}$$

Which corresponds to the high temperature behaviour of (6.41) for leading order in α as (7.20), thus giving credibility to the idea of writing these higher spin partition functions in terms of these Kaneko-Zagier generalized Jacobi forms.

[8] we have used $\tau \to i\infty$ ($q \to 0$) to write $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ as $\eta(\tau) \to q^{1/24}$.

7.2.2 $\lambda = 1$

We start with the partition function (6.56)

$$Z(\tau, \alpha) = Tr\left[e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right] \tag{7.49}$$

We have D complex bosons, with stress tensor given by

$$T = -\sum_{m=1}^{\infty}\left(\bar{\beta}_{-m}\beta_m + \beta_{-m}\bar{\beta}_m\right) + \frac{k}{4} \tag{7.50}$$

Unlike in section 6.4.1. and 6.4.2., we keep the ground state term, $\frac{k}{4}$, for purposes of investigating the low temperature behaviour. The spin-3 primary is defined as

$$\mathcal{W} = -2a\sum_{m=1}^{\infty} m^2\left(\bar{\beta}_{-m}\beta_m + \beta_{-m}\bar{\beta}_m\right) \tag{7.51}$$

Using (6.53), on a state of the form (6.42) we have

$$\begin{aligned}\mathcal{L}|n_m, \bar{n}_m\rangle &= \frac{1}{2\pi}m + (\bar{n}_m + n_m)|n_m, \bar{n}_m\rangle - \frac{k}{8\pi}\\ W|n_m, \bar{n}_m\rangle &= 2am^2(\bar{n}_m - n_m)|n_m, \bar{n}_m\rangle\end{aligned} \tag{7.52}$$

Looking at the first few contributions to Z we write

$$\ln Tr\left[e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right] = e^{-\frac{\pi ik}{2}\tau}[1+\sum_{j=1}^{D}\sum_{m=1}^{\infty}(e^{2\pi i\tau m-8\pi^2 ia\alpha m^2}+e^{2\pi i\tau m+8\pi^2 ia\alpha m^2}+e^{4\pi i\tau m}+e^{6\pi i\tau m}+\cdots)] \tag{7.53}$$

Summing over D scalars as well as m, meaning

$$Tr\left[e^{4\pi^2 i(\tau\mathcal{L}+\alpha\mathcal{W})}\right] = e^{-\frac{\pi k}{2}\tau}\prod_{i=1}^{D}\prod_{m=1}^{\infty}\frac{1}{(1-e^{2\pi i\tau m-8\pi^2 ia\alpha m^2})(1-e^{2\pi i\tau m+8\pi^2 ia\alpha m^2})} \tag{7.54}$$

The D scalars are independent, meaning we can write

$$Z(\tau, \alpha) = e^{-\frac{\pi ik}{2}\tau}\frac{1}{[(1-e^{2\pi i\tau m-8\pi^2 ia\alpha m^2})(1-e^{2\pi i\tau m+8\pi^2 ia\alpha m^2})]^D} \tag{7.55}$$

Taking the natural logarithm one obtains (6.57) with a pre-factor of $e^{-\frac{\pi k\tau}{2}}$

$$\ln Z(\tau, \alpha) = D\sum\nolimits_{m=1}^{\infty}\left[\ln\left(1-e^{2\pi i\tau m-8\pi^2 ia\alpha m^2}\right)+\ln\left(1-e^{2\pi i\tau m+8\pi^2 ia\alpha m^2}\right)\right] \tag{7.56}$$

Relating (7.33) with H (7.10) we set

$$q = e^{2\pi i\tau}, \quad w = e^{64\pi^2 ia\alpha}, \quad \zeta = 1 \tag{7.57}$$

Then we get

$$
\begin{aligned}
Z(\alpha,\tau) &= (e^{-\frac{\pi i k\tau}{2}})\left[\prod_{m=1}^{\infty}(1-q^m w^{m^2/8})(1-q^m w^{-m^2/8}\right]^{-D} \\
&= (e^{-\frac{\pi i k\tau}{2}})[\prod_{m(odd)=1}^{\infty}(1-q^m w^{m^2/8})(1-q^m w^{-m^2/8}) \\
&\prod_{m(even)=2}^{\infty}(1-q^m w^{m^2/8})(1-q^m w^{-m^2/8})]^{-D} \\
&= (e^{-\frac{\pi i k\tau}{2}})[\prod_{m(odd)=1}^{\infty}(1-q^m w^{m^2/8})(1-q^m w^{-m^2/8}) \\
&\prod_{m(odd)=1}^{\infty}(1-q^{m+1} w^{(m+1)^2/8})(1-q^{m+1} w^{-(m+1)^2/8})]^{-D}
\end{aligned} \tag{7.58}
$$

Taking the logarithm gives (6.57), corrected by the vacuum contribution $\frac{-\pi i k\tau}{2}$. Applying (7.17) to (7.58)

$$
\begin{aligned}
\prod_{m(odd)=1}(1-w^{-(m+1)^2/8}q^{m+1}\zeta^{-1}) &= \prod_{m=3}(1-w^{-(m-1)^2/8}q^{m-1}\zeta^{-1}) \\
&= \prod_{m(odd)=1}\frac{(1-w^{-(m-1)^2/8}q^{m+1}\zeta^{-1})}{(1-w^0q^0\zeta^{-1})}
\end{aligned} \tag{7.59}
$$

makes Z singular since in the $\lambda=0$ case we dealing with W_∞ algebra, with no $U(1)$ current, i.e. $\zeta=1$. This behaviour is not completely surprising because the two complex scalars in the Vasiliev theory which are proportional to $1-\lambda^2$ become massless at $\lambda=1$ and perhaps the partition function must be corrected, by including these scalars in the bulk partition function together with their dual operators in the CFT partition function in order to be able to express the partition function in terms of modular forms.

7.2.3 $\lambda=\infty$

We now turn to the case of $\lambda=\infty$ given by (6.84)

$$
\begin{aligned}
\ln Z(\tau,\alpha) &= \frac{3\pi i\tau}{160\alpha}\frac{\tau^2}{\alpha}\left[{}_3F_2\left(\begin{matrix}-\frac{3}{4}, & -\frac{1}{2}, & -\frac{1}{4}\\ \frac{1}{3}, & \frac{2}{3} & \end{matrix}\middle| -\frac{5120}{81}\varsigma^2\right)-1\right] \\
\ln Z\left(\frac{a\tau+b}{c\tau+d},\frac{\alpha}{(c\tau+d)^3}\right) &= \frac{3\pi i\tau^3}{160\alpha^2}\left[{}_3F_2\left(\begin{matrix}-\frac{3}{4}, & -\frac{1}{2}, & -\frac{1}{4}\\ \frac{1}{3}, & \frac{2}{3} & \end{matrix}\middle| -\frac{5120}{81}\frac{\alpha^2}{\tau^4}(\tau^2)\right)-1\right]
\end{aligned} \tag{7.60}
$$

We use the 3F2 hypergeometric transformations to write the hypergeometric part of the r.h.s of (7.37) as [53]

$$
\begin{aligned}
&\left[{}_3F_2\left(\begin{matrix}-\frac{3}{4}, & -\frac{1}{2}, & -\frac{1}{4}\\ \frac{1}{3}, & \frac{2}{3} & \end{matrix}\middle| -\frac{5120}{81}\frac{\alpha^2}{\tau^4}(\tau^2)\right)\right] = (1-\tau^2)^{3/4} \\
&\sum_{n=0}^{\infty}\frac{(-\frac{3}{4})_n}{n!}\left[{}_3F_2\left(\begin{matrix}-n, & -\frac{1}{2}, & -\frac{1}{4}\\ \frac{1}{3}, & \frac{2}{3} & \end{matrix}\middle| -\frac{5120}{81}\frac{\alpha^2}{\tau^4}\right)\right]\left(\frac{\tau^2}{\tau^2-1}\right)^n
\end{aligned} \tag{7.61}
$$

We see that for $\lambda=\infty$ we get an infinite sum. We can think of this minimal model as a limit of minimal models with an ever-increasing number of representations so we would expect to have to sum over an infinite number of characters. It would be interesting, perhaps in the future, to see whether the partition function at $\lambda=\infty$ can be written in terms of the Kaneko-Zagier [48] generalized Jacobi forms.

8 Conclusion and future work

In this project we have shown that the partition functions of black holes in the presence of a spin-3 chemical potential can be written in terms of the Kaneko-Zagier generalised Jacobi forms for $\lambda = 0$. This is a strong indication that generalised Jacobi forms have a central role to play in the study of higher spin black holes. We also observe for $\lambda = 1$ that the partition function might need to be corrected for it to be non-singular and therefore be written as a generalized Jacobi-like form. This will be undertaken in future work. It would be worth exploring also in future work if there might exist higher-order analogues of the triple product identity and thus have functions such as $H(w', w, q, \zeta)$ which can be useful for, say spin-4 chemical potential. This can, in the future, be checked for spin-4 black holes. The case for bringing $\lambda = \infty$ into a known modular form proves to be more involved for the scope of this project. However, it can be observed that (7.61) just needs a transformation that would give the hypergeometric on r.h.s., appearing in the sum, as the original variables (specifically getting n to produce the same results as $-3/4$).

The ultimate goal of this project, hopefully to be accomplished in future work, is to check the modular properties of the higher spin black hole partition function (6.39) for any λ, thus extend the focus from leading order in α to higher orders in α. The expectation is that this partition function should be modular invariant, since it can be reproduced from 2d CFT calculations. To check this, however, we might need to let λ transform in a certain way for $\mathcal{S}$, $\lambda \to \lambda/(\lambda - 1)$, and along the $\mathcal{T}$ transformation $\lambda \to 1 - \lambda$. Obviously these would need to be checked since in our case we are dealing with a higher spin black hole. What is also important as shown by [58] might be to consider conical defects in order for the full partition function to be modular invariant.

A Some proofs

Consider the Fefferman-Graham metric (3.48)

$$ds^2 = d\rho^2 + 8\pi Gl(\mathcal{L}dw^2 + \bar{\mathcal{L}}d\bar{w}^2) + (l^2 e^{\frac{2\rho}{l}} + (8\pi G)^2 \mathcal{L}\bar{\mathcal{L}} e^{\frac{-2\rho}{l}} dwd\bar{w}) \tag{A.1}$$

We argue that there exists a diffeormophism such that

$$(w, \bar{w}, \rho) \mapsto \left(w', \bar{w}', \rho'\right)$$

without loss of generality, we set $\bar{L} = 0$ and use the following infinitesimal diffeomorphisms

$$\rho' = \rho + \frac{1}{2}\partial_w \varepsilon(w) \bar{w}' = \bar{w} + \frac{1}{2} e^{-\rho} \partial_w^2 \varepsilon(w) \tag{A.2}$$

the claim is that the metric remains the same under these diffeomorphisms with only $\mathcal{L}(w)$ changing the form as

$$\mathcal{L}'(w) = \mathcal{L}(w) + \delta_\varepsilon \mathcal{L}(w) \tag{A.3}$$

with

$$\delta_\varepsilon \mathcal{L}(w) = \varepsilon(w)\partial_w \mathcal{L}(w) + 2\partial_w \varepsilon(w)\mathcal{L}(w) - \frac{l}{8G}\partial_w^3 \varepsilon(w) \tag{A.4}$$

This result is a transformation law of the energy stress tensor of a CFT [22]. We can therefore read off the central charge from the above result using the normalized TT OPE

$$T(z)T(w) = \frac{c/12}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + ...$$

We fix c as

$$\begin{aligned} \frac{c}{12} &= -\frac{l}{8G} \\ c &= -\frac{3l}{2G} \end{aligned} \tag{A.5}$$

B $hs[\lambda]$ structure constants

The $hs[\lambda]$ algebra structure constants are given by

$$g_u^{st}(m, n; \lambda) = \frac{q^{u-2}}{2(u-1)!} \phi_u^{st}(\lambda) N_u^{st}(m, n) \tag{B.1}$$

with

$$N_u^{st}(m, n) = \sum_{k=0}^{u-1} (-1)^k \frac{u-1}{k} [s-1+m]_{u-1-k} [s-1-m]_k [t-1-m]_k [t-1-n]_{u-1-k} \tag{B.2}$$

and

$$\phi_u^{st}(\lambda) = {}_4F_3 \left[\begin{matrix} \frac{1}{2}+\lambda, & \frac{1}{2}-\lambda, & \frac{2-u}{2}, & \frac{1-u}{2} \\ \frac{3}{2}-s, & \frac{3}{2}-t, & \frac{1}{2}+s+t-u & \end{matrix} \middle| 1 \right] \tag{B.3}$$

also making use of the Pocchammer symbol

$$[a]_n = a(a-1)...(a-n+1) \tag{B.4}$$

q is a normalization that is usually set to $q = 1/4$. The following identities are also useful

$$\begin{aligned}
\phi_u^{st}\left(\frac{1}{2}\right) &= \phi_2^{st}(\lambda) = 1 \\
N_u^{st}(m,n) &= (-1)^{u+1} N_u^{ts}(n,m) \\
N^{st}(0,0) &= 0 \\
N_u^{st}(n,-n) &= N_u^{ts}(n,-n)
\end{aligned} \tag{B.5}$$

The first three identities give testimony to the isomorphism, $hs\left[\frac{1}{2}\right] \cong hs(1,1)$, meaning that the lone star product can be used to define the $hs[\lambda]$ algebra [11]. The cyclic trace is defined to be zero for all generators except the identity element. We use the conventional normalization

$$Tr\left(V_m^s V_{-m}^s\right) = \frac{24}{\lambda^2 - 1} g_{2s-1}^{ss}(m,-m,\lambda) \tag{B.6}$$

C Dedekind and Jacobi theta function

Jacobi's triple product identity is given by [36]

$$\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}/t) = \sum_{n\in\mathbf{Z}} q^{n^2/2} t^n \qquad \forall |q|<1, \quad t \neq 0 \tag{C.1}$$

Using the above identity we write Jacobi's theta functions as [36]

$$\begin{aligned}
\theta_1(z \mid \tau) &= -iy^{1/2} q^{1/8} \prod_{n=1}^{\infty}(1-q^n) \prod_{n=0}^{\infty}(1-yq^{n+1})(1-y^{-1}q^n) \\
\theta_2(z \mid \tau) &= y^{1/2} q^{1/8} \prod_{n=1}^{\infty}(1-q^n) \prod{}_{n=0}^{\infty}(1+yq^{n+1})(1+y^{-1}q^n) \\
\theta_3(z \mid \tau) &= \prod_{n=1}^{\infty}(1-q^n) \prod_{r\in\mathbf{N}+\frac{1}{2}}^{\infty}(1+yq^r)(1+y^{-1}q^r) \\
\theta_4(z \mid \tau) &= \prod_{n=1}^{\infty}(1-q^n) \prod_{r\in\mathbf{N}+\frac{1}{2}}^{\infty}(1-yq^r)(1-y^{-1}q^r)
\end{aligned} \tag{C.2}$$

where τ is the modular parameter living on the upper half plane as defined in 5.2. and z a complex variable; $q = \exp 2\pi i\tau$ and $y = \exp 2\pi i z$. Theta functions are all related to each other by shifting their arguments as follows

$$\begin{aligned}
\theta_4(z \mid \tau) &= \theta_3(z + \frac{1}{2}|\tau) \\
\theta_1(z \mid \tau) &= -ie^{\pi i z} q^{1/8} \theta_4(z + \frac{1}{2}\tau|\tau) \\
\theta_2(z \mid \tau) &= \theta_1(z + \frac{1}{2}|\tau)
\end{aligned} \tag{C.3}$$

Theta functions play an important role in defining double periodic functions on the complex plane [36]. The theta function at $z = 0$

$$\theta_i(\tau) \equiv \theta_i(0|\tau), \qquad \forall i = 2,3,4 \qquad \theta_1(0|\tau) = 0 \tag{C.4}$$

In terms of sums and products, theta functions at $z = 0$

$$\begin{aligned}
\theta_2(\tau) &= \sum_{n\in\mathbf{Z}} q^{(n+\frac{1}{2})^2/2} = 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \\
\theta_3(\tau) &= \sum_{n\in\mathbf{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2 \\
\theta_4(\tau) &= \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2
\end{aligned} \tag{C.5}$$

Before turning to modular transformations of theta functions, we define the Dedekind function $\eta(\tau)$ [36]

$$\eta(\tau) = q^{\frac{1}{24}}\varphi(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{C.6}$$

where φ is the Euler function. The Dedekind function has the following relationship with theta functions

$$\eta^3(\tau) = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \tag{C.7}$$

Since we want to investigate the modular transformations of theta functions. The $\mathcal{S} : \tau \to -\frac{1}{\tau}$ requires the use of the Poisson re-summation formula

$$\sum_{n\in\mathbf{Z}} \exp(-a\pi n^2 + bn) = \frac{1}{\sqrt{a}}\sum_{k\in\mathbf{Z}} \exp -\frac{\pi}{a}\left(k + \frac{b}{2\pi i}\right)^2 \tag{C.8}$$

Applying the formula to $\theta_3(\tau)$ using

$$a = -i\tau \qquad b = 0$$

we find

$$\theta_3\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\theta_3(\tau) \tag{C.9}$$

Whilst setting

$$a = -i\tau \quad b = -\pi i$$

we find

$$\theta_2\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\theta_4(\tau) \tag{C.10}$$

Applying $\mathcal{S}$ for a second time yields

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\theta_2(\tau) \tag{C.11}$$

Using these transformations and the relationship between the Dedekind function and theta functions we get

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) \tag{C.12}$$

The $\mathcal{T}$ modular transformations are trivial. We therefore give only the $\mathcal{S}$ transformations of theta functions and the Dedekind function

$$\theta_2\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\theta_4(\tau), \quad \theta_3\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\theta_3(\tau), \quad \theta_4\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\theta_2(\tau) \tag{C.13}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) \tag{C.14}$$

In addition to Dedekind's function and Jacobi theta functions, we need to use the concept of Weber modular functions. Weber modular functions are three functions f, f_1, f_2 [52]

$$\begin{aligned} f(\tau) &= q^{-1/48}\prod_{n>0}\left(1+q^{n-\frac{1}{2}}\right) = \frac{\eta^2(\tau)}{\eta(\frac{\tau}{2})\eta(2\tau)} \\ f_1(\tau) &= q^{-1/48}\prod_{n>0}(1-q^{n-\frac{1}{2}}) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)} \\ f_2 &= \sqrt{2}q^{1/24}\prod_{n>0}(1+q^n) = \frac{\sqrt{2}\eta(2\tau)}{\eta(\tau)} \end{aligned} \tag{C.15}$$

$f_1(\tau)$ is also given in terms of the theta functions as

$$f_1(\tau) = \sqrt{\frac{\theta_4(q)}{\eta(\tau)}} \tag{C.16}$$

Therefore, we have for $\theta_4(\tau)$

$$\theta_4(\tau) = \frac{\eta(\tau/2)^2}{\eta(\tau)} \tag{C.17}$$

D Bulk partition function currents

$n = 2:$

$$\left(\frac{8\pi^2}{21k^2}\right)\left(504\pi\alpha k\tau\mathcal{W} + 1792\pi^2\alpha^2\mathcal{L}^2 + 168\pi k\tau^2\mathcal{L} - 144\alpha^2k^2\mathcal{L}^2J_4\right.$$
$$\left. + 1296\alpha^2k^2J_4 + 21k^2\right) = 0 \tag{D.1}$$

$n = 3:$

$$-\left(\frac{32\pi^3}{\left(63\sqrt{5}k^3\left(l-2\right)\left(l+2\right)\right)}\right)\left(9000\pi^2\alpha^3kl^4\mathcal{W}^2 - 117000\pi^2\alpha^3kl^2\mathcal{W}^2 + 475200\pi^2\alpha^3k\mathcal{W}^2\right.$$
$$+ 17280\pi^2\alpha^2kl^4\tau L\mathcal{W} - 194400\pi^2\alpha^2kl^2\tau\mathcal{L}\mathcal{W} + 501120\pi^2\alpha^2k\tau\mathcal{L}\mathcal{W}$$
$$+ 189\pi k^2l^4\tau^3\mathcal{W} - 1512\pi k^2l^2\tau^3\mathcal{W} + 3024\pi k^2\tau^3\mathcal{W} + 10240\pi^3\alpha^3l^4\mathcal{L}^3$$
$$- 204800\pi^3\alpha^3l^2\mathcal{L}^3 + 655360\pi^3\alpha^3\mathcal{L}^3 + 4032\pi^2\alpha kl^4\tau^2\mathcal{L}^2$$
$$- 32256\pi^2\alpha kl^2\tau^2\mathcal{L}^2 + 64512\pi^2\tau^2\mathcal{L}^2 - 2880\pi\alpha^3k^2l^6J_4\mathcal{L}$$
$$+ 83520\pi\alpha^3k^2l^4J_4\mathcal{L} - 702720\pi\alpha^3k^2l^2J_4\mathcal{L} + 1658880\pi\alpha^3k^2J_4\mathcal{L}$$
$$\left. - 324\alpha k^3l^6\tau^2J_4 + 5508\alpha\, k^3l^4\tau^2J_4 - 28512\alpha k^3l^2\tau^2J_4 + 46656\alpha k^3\tau^2J_4\right) = 0 \tag{D.2}$$

$n = 4:$

$$\left(\frac{128\pi^4}{45045k^4(l-2)^2(l+2)^2}\right)$$
$$\left(15600000\pi^3\alpha^4kl^6\mathcal{L}\mathcal{W}^2 - 436800000\pi^3\alpha^4kl^4\mathcal{L}\mathcal{W}^2 + 4651920000\pi^3\alpha^4kl^2\mathcal{L}\mathcal{W}^2\right.$$
$$- 17858880000\pi^3\alpha^4k\mathcal{L}\mathcal{W}^2 + 1930500\pi^2\alpha^2k^2l^6\tau^2\mathcal{W}^2 - 32818500\pi^2\alpha^2k^2l^4\tau^2\mathcal{W}^2$$
$$+ 202316400\pi^2\alpha^2k^2l^2\tau^2\mathcal{W}^2 - 407721600\pi^2\alpha^2k^2\tau^2\mathcal{W}^2 + 24710400\pi^3\alpha^3kl^6\tau\mathcal{L}^2\mathcal{W}$$
$$- 518918400\pi^3\alpha^3kl^4\tau\mathcal{L}^2\mathcal{W} + 3690086400\pi^3\alpha^3kl^2\tau\mathcal{L}^2\mathcal{W} - 8039116800\pi^3\alpha^3k\tau\mathcal{L}^2\mathcal{W}$$
$$+ 1184040\pi^2\alpha k^2l^6\tau^3\mathcal{L}\mathcal{W} - 15804360\pi^2\alpha k^2l^4\tau^3\mathcal{L}\mathcal{W} + 69600960\pi^2\alpha k^2l^2\tau^3\mathcal{L}\mathcal{W}$$
$$- 101312640\pi^2\alpha k^2\tau^3\mathcal{L}\mathcal{W} - 1638000\pi\alpha^3k^3l^8\tau J_4\mathcal{W} + 52416000\pi\alpha^3k^3l^6\tau J_4\mathcal{W}$$
$$- 671907600\pi\alpha^3k^3l^4\tau J_4\mathcal{W} + 3828988800\pi\alpha^3k^3l^2\tau J_4\mathcal{W} - 7500729600\pi\alpha^3k^3\tau J_4\mathcal{W}$$
$$+ 14643200\pi^4\alpha^4l^6\mathcal{L}^4 - 307507200\pi^4\alpha^4l^4\mathcal{L}^4 + 2460057600\pi^4\alpha^4l^2\mathcal{L}^4$$
$$- 5857280000\pi^4\alpha^4\mathcal{L}^4 + 6040320\pi^3\alpha^2kl^6\tau^2\mathcal{L}^3 - 87310080\pi^3\alpha^2kl^4\tau^2\mathcal{L}^3$$
$$+ 408545280\pi^3\alpha^2kl^2\tau^2\mathcal{L}^3 - 623800320\pi^3\alpha^2k\tau^2\mathcal{L}^3 - 3494400\pi^2\alpha^4k^2l^8J_4\mathcal{L}^2$$
$$+ 144768000\pi^2\alpha^4k^2l^6J_4\mathcal{L}^2 - 2309798400\pi^2\alpha^4k^2l^4J_4\mathcal{L}^2 + 15179673600\pi^2\alpha^4k^2l^2J_4\mathcal{L}^2$$
$$- 32132505600\pi^2\alpha^4k^2J_4\mathcal{L}^2 + 108108\pi^2k^2l^6\tau^4\mathcal{L}^2 - 1117116\pi^2k^2l^4\tau^4\mathcal{L}^2$$
$$+ 3747744\pi^2k^2l^2\tau^4\mathcal{L}^2 - 4036032\pi^2k^2\tau^4\mathcal{L}^2 - 926640\pi\alpha^2k^3l^8\tau^2J_4\mathcal{L}$$
$$+ 25945920\pi\alpha^2k^3l^6\tau^2J_4\mathcal{L} - 254826000\pi\alpha^2k^3l^4\tau^2J_4\mathcal{L} + 1030423680\pi\alpha^2k^3l^2\tau^2J_4\mathcal{L}$$
$$- 1467797760\pi\alpha^2k^3\tau^2J_4\mathcal{L} + 126000\alpha^4k^4l^{10}{J_4}^2 - 6426000\alpha^4k^4l^8{J_4}^2$$
$$+ 144925200\alpha^4k^4l^6{J_4}^2 - 1673348400\alpha^4k^4l^4{J_4}^2 + 9119433600\alpha^4k^4l^2{J_4}^2$$
$$- 17463340800\alpha^4k^4{J_4}^2 - 3861k^4l^8\tau^4J_4 + 81081k^4l^6\tau^4J_4$$
$$\left. - 602316k^4l^4\tau^4J_4 + 1915056k^4l^2\tau^4J_4 - 2223936k^4\tau^4J_4\right) = 0 \tag{D.3}$$

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