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FIXED POINT RESULTS OF GENERALIZED SUZUKI-GERAGHTY CONTRACTIONS ON f-ORBITALLY COMPLETE B-METRIC SPACES

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In this paper, we introduce the concept of a generalized α-Suzuki-Geraghty type contraction mapping and obtain fixed point results in the framework of an f-orbitally complete b-metric space. The results proved herein improve, generalize, and unify various comparable results in the existing literature. Some examples are presented to validate the effectiveness and applicability of our main result. It is also shown that the presented results are proper extensions of corresponding results in the existing literature. As an application of our obtained result, we establish the existence of a solution of integral equations in the setup of b-metric spaces.

Keywords: b-metric, f-orbitally complete, α-Suzuki-Geraghty type, triangular α-orbital, admissible, fixed point.

MSC2010: 47H10, 47H04, 47H07.

1. Introduction and preliminaries

Fixed point theory deals with the conditions that guarantee the existence of one or more points in a nonempty set which remain invariant under the action of mapping. This theory has attracted considerable attention due to its several applications in areas such as variational and linear inequalities, optimization, and approximation theory. Banach contraction principle has been generalized in many directions. In some generalizations, the contractive nature of the mapping is weakened (see [3], [4], [8], [15], [16]) and in other cases, the structure of ambient space is generalized (see [3], [6], [7], [9] and references therein).

In the sequel, $\mathbb{N}$ and $\mathbb{R}$ will denote the set of natural and the set of real numbers, respectively. We set $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 1.1. [8] *If (X, d) is a compact metric space and $f : X \to X$ satisfies*

$$d(fx, fy) < d(x, y) \tag{1}$$

for all $x, y \in X$, with $x \neq y$. Then f has a unique fixed point in X.

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Theorem 1.2. [16] *Let (X, d) be a compact metric space and f a self mapping on X. If for any $x, y \in X$ with $x \neq y$,*

$$\frac{1}{2}d(x, fx) < d(x, y) \Rightarrow d(fx, fy) < d(x, y), \tag{2}$$

then f has a unique fixed point in X.

Suzuki type fixed point theorems characterize the completeness property of underlying metric spaces [15] but Banach contraction principle does not characterize this property [5].

Definition 1.1. [2] *Let X be a nonempty set, $f : X \to X$ and $x_0 \in X$. Choose a point $x_1 \in X$ such that $x_1 = fx_0$. Continuing this process having chosen $x_1, \ldots, x_k$, we choose $x_{k+1} \in X$ such that $x_{n+1} = fx_n = f^{n+1}x_0$, $n = 0, 1, 2, \ldots$. The sequence $\{x_n\}$ thus obtained is called a sequence of successive approximations or Picard sequence with initial point x_0. The set $\{x_0, fx_0, f^2x_0, f^3x_0, \ldots\}$ is called an orbit of f at the point x_0 and is denoted by $O_f(x_0)$.*

Let $\mathcal{S}$ be the class of all mappings $\beta : \mathbb{R}_0^+ \to [0, \frac{1}{s})$ which satisfy the condition: $\lim_{n\to\infty} t_n = 0$ whenever $\lim_{n\to\infty} \beta(t_n) = \frac{1}{s}$ for some $s \geq 1$. Note that $\mathcal{S} \neq \emptyset$. For instance, a mapping $\beta : \mathbb{R}_0^+ \to [0, \frac{1}{s})$ given by $\beta(t) = \frac{1}{s}e^{-3t}$ if $t > 0$ and $\beta(t) = 0$ if $t = 0$, qualifies for a membership of $\mathcal{S}$.

Theorem 1.3. [9] *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. If there exists $\beta : \mathbb{R}_0^+ \to [0, 1)$ which satisfies the condition $\lim_{n\to\infty} \beta(t_n) = 1$ implies that $\lim_{n\to\infty} t_n = 0$ and for any $x, y \in X$, we have*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y), \tag{3}$$

then f has a unique fixed point $z \in X$. Moreover for any choice of an initial point $x_0 \in X$, sequence of successive approximations converges to z.

Definition 1.2. [10, 12, 14] *Let X be any nonempty set and $\alpha : X \times X \to \mathbb{R}$. A mapping $f : X \to X$ is said to be (i) α-admissible if $\alpha(x, y) \geq 1$ implies that $\alpha(fx, fy) \geq 1$; (ii) triangular α-admissible if it is an α-admissible and $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply that $\alpha(x, y) \geq 1$; (iii) an α-orbital admissible if $\alpha(x, fx) \geq 1$ implies that $\alpha(fx, f^2x) \geq 1$; (iv) triangular α-orbital admissible if f is α-orbital admissible and $\alpha(x, y) \geq 1$ and $\alpha(y, fy) \geq 1$ imply that $\alpha(x, fy) \geq 1$.*

Lemma 1.1. [12] *Let f be a self triangular α-admissible mapping on a nonempty set X. If there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$, then $\alpha(f^m x_0, f^n x_0) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Let X be any nonempty set and $\alpha : X \times X \to \mathbb{R}$. A mapping $f : X \to X$ is said to have a property (H), if for any $x, y \in X$ with $x \neq y$, there exists $w \in X$ such that $\alpha(x, w) \geq 1$, $\alpha(y, w) \geq 1$ and $\alpha(w, fw) \geq 1$.

Let $f : X \to X$. We set

$$N(x,y) = \max\left\{ d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2} \right\}, \tag{4}$$

$$L(x,y) = \max\left\{ \begin{array}{l} d(x,y), \dfrac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s\,(d(x,y)+d(fx,fy))}, \\ \dfrac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)} \end{array} \right\}, \tag{5}$$

$$M_f(x,y) = \max\left\{ \begin{array}{l} d(x,y), d(x,fx), d(y,fy), d(f^2x,fx), d(f^2x,y), \\ \dfrac{1}{s}d(f^2x,fy), \dfrac{1}{2s}d(f^2x,x), \dfrac{d(x,fy)+d(y,fx)}{2s}, \\ \dfrac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s\,(d(x,y)+d(fx,fy))}, \\ \dfrac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)} \end{array} \right\}. \tag{6}$$

Definition 1.3. [12] *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$. A mapping $f : X \to X$ is called a generalized α-Geraghty contraction if there exists a $\beta : \mathbb{R}_0^+ \to [0,1)$ such that $\lim_{n\to\infty} \beta(t_n) = 1$ implies that $\lim_{n\to\infty} t_n = 0$ and for any $x,y \in X$, we have $\alpha(x,y)d(fx,fy) \le \beta(N(x,y))N(x,y)$.*

Theorem 1.4. [12] *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ and $f : X \to X$. If following conditions hold:*

(1) *f is a generalized α-Geraghty contraction and a triangular α-orbital admissible mapping;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \ge 1$;*
(3) *f is continuous.*

Then f has a fixed point $x_* \in X$ and $\{f^n x_0\}$ converges to x_*.

The concept of a b-metric space was introduced by Czerwik in [6]. Since then, several papers have been published on the fixed point theory of various classes of single valued and multivalued operators in b-metric spaces (see [6, 7, 11]).

Consistent with [1, 3, 7, 13], the following definitions and results will be needed in the sequel.

Definition 1.4. [6, 7] *Let X be a nonempty set and $s \ge 1$ a given real number. A function $d : X \times X \to \mathbb{R}_0^+$ is said to be a b-metric if for any $x,y,z \in X$, the following conditions hold:*

(bm1) *$d(x,y) = 0$ if and only if $x = y$;*
(bm2) *$d(x,y) = d(y,x)$;*
(bm3) *$d(x,y) \le s\,(d(x,z) + d(z,y))$*

The pair (X,d) is called a b-metric space.

Example 1.1. [13] *Let (X,d) be a metric space, and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.*

Note that a b-metric $d : X \times X \to \mathbb{R}_0^+$ is not necessarily continuous in each variable. Also, if b-metric d is continuous in one variable, then it is continuous in the other variable (see [1]).

Definition 1.5. [3] *Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called (a) Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for each $n, m > n_\epsilon$, we have $d(x_n, x_m) < \epsilon$; (b) a convergent sequence if and only if there exists $x \in X$ such that for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$ we have $d(x_n, x) < \epsilon$. The b-metric space X is complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some point $x \in X$.*

Latif et al. [11] proved the following Suzuki type theorem in the context of a complete b-metric space.

Theorem 1.5. [11] *Let (X, d) be a complete b-metric space and f a triangular α-admissible self mapping on X. If there exists a $\beta \in \mathcal{S}$ such that for any $x, y \in X$,*

$$\frac{1}{2}d(x, fx) \leq sd(x, y) \Rightarrow s\alpha(x, y)d(fx, fy) \leq \beta(L(x, y))L(x, y).$$

Then f has a fixed point provided that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for $n \in \mathbb{N}_0$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$.*

Definition 1.6. [4] *A b-metric space X is said to be f-orbitally complete if every Cauchy sequence in $O_f(x_0)$ converges in X, where f is a self mapping on X and $x_0 \in X$.*

Note that every complete b-metric space is f-orbitally complete. But the converse does not hold in general.

Example 1.2. *Let $X = \mathbb{R}$. Define $d(x, y) = |e^x - e^y|^2$. Then (X, d) is a b-metric space with parameter $s \geq 2$. It is known that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for all $a, b \geq 0$ and $p > 1$. Let $x, y, z \in X$. Then*

$$|e^x - e^y|^2 \leq (|e^x - e^z| + |e^z - e^y|)^2 \leq 2^{2-1}\left(|e^x - e^z|^2 + |e^z - e^y|^2\right)$$

implies that $d(x, y) \leq 2(d(x, z) + d(z, y))$. The space (X, d) is not a complete b-metric space. Indeed, for the sequence $\{x_n\}$ defined by $x_n = -n$, there exists $N \in \mathbb{N}$ such that $N < n < m$ implies that $d(x_m, x_n) = |e^{x_m} - e^{x_n}|^2 = (e^{-m} - e^{-n})^2 < e^{-2N} \to 0$ as $N \to \infty$. Hence the sequence $\{-n\}$ is a b-Cauchy sequence. However, $\{-n\}$ is not convergent. If $x_n \to x$ for some $x \in X$, then $d(x_n, x) = |e^{-n} - e^x|^2 \to 0$ gives that $e^x = 0$, a contradiction. Let $f : X \to X$ be the mapping defined by $fx = x_0$ for some $x_0 \in X$. Then (X, d) is f-orbitally complete b-metric space.

Definition 1.7. *Let (X,d) be a b-metric space with parameter $s \geq 1$ and $\alpha : X \times X \to \mathbb{R}$. A mapping $f : X \to X$ is called a generalized α-Suzuki-Geraghty contraction if there exists a $\beta \in \mathcal{S}$ such that for any $x, y \in X$,*

$$\frac{1}{2}d(x, fx) \leq sd(x,y) \Rightarrow s\alpha(x,y)d(fx, fy) \leq \beta(M_f(x,y))M_f(x,y). \tag{7}$$

2. Main Result

In the sequel, we assume that a b-metric d is continuous in one variable.

Theorem 2.1. *Let (X,d) be a b-metric space with parameter $s \geq 1$, $\alpha : X \times X \to \mathbb{R}$ and $f : X \to X$. Assume that X is f-orbitally complete and the following conditions hold:*

(a) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(b) *f is a generalized α-Suzuki-Geraghty contraction and a triangular α-orbital admissible;*
(c) *either f is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.*

Then f has a fixed point z in X and $\{f^n x_0\}$ converges to z. Moreover, f has a unique fixed point if condition (a) is replaced with the property (H).

Proof. Let x_0 be a given point in X such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$, $n \in \mathbb{N}_0$. Since f is α-orbital admissible, we obtain that $\alpha(fx_0, f^2x_0) \geq 1$, that is, $\alpha(x_1, x_2) \geq 1$. Continuing this way, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then x_{n_0} becomes a fixed point of f and the proof is finished. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Now we show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Clearly, $\frac{1}{2}d(x_n, fx_n) \leq sd(x_n, x_{n+1})$. As $\alpha(x_n, x_{n+1}) \geq 1$ and $s \geq 1$, we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(fx_n, fx_{n+1}) \\ &\leq s\alpha(x_n, x_{n+1})d(fx_n, fx_{n+1}) \leq \beta(M_f(x_n, x_{n+1}))M_f(x_n, x_{n+1}), \end{aligned}$$

where

$$\begin{aligned} M_f(x_n, x_{n+1}) &= \max \left\{ \begin{array}{l} d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), d(f^2x_n, fx_n), \\ d(f^2x_n, x_{n+1}), \frac{1}{s}d(f^2x_n, fx_{n+1}), \frac{1}{2s}d(f^2x_n, x_n), \\ \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s}, \\ \frac{d(x_n, fx_n)d(x_n, fx_{n+1}) + d(x_{n+1}, fx_{n+1})d(x_{n+1}, fx_n)}{1 + s\left(d(x_n, x_{n+1}) + d(fx_n, fx_{n+1})\right)}, \\ \frac{d(x_n, fx_n)d(x_n, fx_{n+1}) + d(x_{n+1}, fx_{n+1})d(x_{n+1}, fx_n)}{1 + d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)} \end{array} \right\} \\ &\leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}. \end{aligned}$$

Thus, we have

$d(x_{n+1}, x_{n+2}) \leq \beta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$.

If $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$, then we obtain

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_{n+1}, x_{n+2}))d(x_{n+1}, x_{n+2}) < \frac{d(x_{n+1}, x_{n+2})}{s} \leq d(x_{n+1}, x_{n+2})$$

a contradiction. Hence

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < d(x_n, x_{n+1}). \tag{8}$$

Thus $\{d(x_n, x_{n+1})\}$ is (strictly) decreasing sequence of positive real numbers. Consequently, there exists $\ell \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \ell$. We claim that $\ell = 0$. If not, then by (8) we have

$$\frac{d(x_{n+1}, x_{n+2})}{sd(x_n, x_{n+1})} \leq \frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) < \frac{1}{s}$$

which implies that $\lim\limits_{n\to\infty} \beta(d(x_n, x_{n+1})) = \frac{1}{s}$ and hence $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

Now we show that $\{x_n\}$ is a b-Cauchy sequence. Assume on the contrary that $\{x_n\}$ is not a b-Cauchy sequence. Then we may choose an $\epsilon > 0$ such that for all $k \geq 1$, there exists $n_k > m_k > k$ with

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{10}$$

Without any loss of generality, we assume that n_k is the smallest index satisfying (10). Then

$$d(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{11}$$

Note that

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{m_k+1}) + sd(x_{m_k+1}, x_{n_k}).$$

On taking upper limit as $k \to \infty$, we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}). \tag{12}$$

Now we claim that either

$$\frac{1}{2}d(x_{m_k}, fx_{m_k}) \leq sd(x_{m_k}, x_{n_k-1}) \tag{13}$$

$$\text{or} \qquad \frac{1}{2}d(x_{m_k+1}, fx_{m_k+1)}) \leq sd(x_{m_k+1}, x_{n_k-1}) \tag{14}$$

hold for all $k \in \mathbb{N}_0$. If not, there exists a $k_0 \in \mathbb{N}_0$ such that

$$\frac{1}{2}d(x_{m_{k_0}}, fx_{m_{k_0}}) > sd(x_{m_{k_0}}, x_{n_{k_0}-1}) \text{ and } \frac{1}{2}d(x_{m_{k_0}+1}, fx_{m_{k_0}+1}) > sd(x_{m_{k_0}+1}, x_{n_{k_0}-1}). \tag{15}$$

Now by (8) and (15), we have

$$\begin{aligned} d(x_{m_{k_0}}, x_{m_{k_0}+1}) &\leq s(d(x_{m_{k_0}}, x_{n_{k_0}-1}) + d(x_{m_{k_0}+1}, x_{n_{k_0}-1})) \\ &< \frac{1}{2}d(x_{m_{k_0}}, fx_{m_{k_0}}) + \frac{1}{2}d(x_{m_{k_0}+1}, fx_{m_{k_0}+1}) \\ &= \frac{1}{2}d(x_{m_{k_0}}, x_{m_{k_0}+1}) + \frac{1}{2}d(x_{m_{k_0}+1}, x_{m_{k_0}+2}) \\ &< \frac{1}{2}d(x_{m_{k_0}}, x_{m_{k_0}+1}) + \frac{1}{2}d(x_{m_{k_0}}, x_{m_{k_0}+1}) = d(x_{m_{k_0}}, x_{m_{k_0}+1}) \end{aligned}$$

a contradiction and hence the claim follows. Suppose that (13) holds. By Lemma 1.1, it follows that $\alpha(x_{m_k}, x_{n_k-1}) \geq 1$. Now by (7) and (12), we have

$$\begin{aligned} &\epsilon \leq s \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup_{k\to\infty} s\alpha(x_{m_k}, x_{n_k-1})d(fx_{m_k}, fx_{n_k-1}) \\ &\leq \limsup_{k\to\infty}(\beta(M_f(x_{m_k}, x_{n_k-1}))M_f(x_{m_k}, x_{n_k-1})). \end{aligned} \tag{16}$$

Note that

$$\begin{aligned} &\limsup_{k\to\infty} M_f(x_{m_k}, x_{n_k-1}) \\ &= \limsup_{k\to\infty} \max \left\{ \begin{array}{l} d(x_{m_k}, x_{n_k-1}), d(x_{m_k}, fx_{m_k}), d(x_{n_k-1}, fx_{n_k-1}), d(f^2x_{m_k}, fx_{m_k}), \\ d(f^2x_{m_k}, x_{n_k-1}), \dfrac{1}{s}d(f^2x_{m_k}, fx_{n_k-1}), \dfrac{1}{2s}d(f^2x_{m_k}, x_{m_k}), \\ \dfrac{d(x_{m_k}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{m_k})}{2s}, \\ \dfrac{d(x_{m_k}, fx_{m_k})d(x_{m_k}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{n_k-1})d(x_{n_k-1}, fx_{m_k})}{1 + s\left[d(x_{m_k}, x_{n_k-1}) + d(fx_{m_k}, fx_{n_k-1})\right]}, \\ \dfrac{d(x_{m_k}, fx_{m_k})d(x_{m_k}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{n_k-1})d(x_{n_k-1}, fx_{m_k})}{1 + d(x_{m_k}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{m_k})} \end{array} \right\} \\ &= \limsup_{k\to\infty} \max \left\{ \begin{array}{l} d(x_{m_k}, x_{n_k-1}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k+2}, x_{m_k+1}), \\ d(x_{m_k+2}, x_{n_k-1}), \dfrac{1}{s}d(x_{m_k+2}, x_{n_k}), \dfrac{1}{2s}d(x_{m_k+2}, x_{m_k}), \\ \dfrac{d(x_{m_k}, x_{n_k}) + d(x_{n_k-1}, x_{m_k+1})}{2s}, \\ \dfrac{d(x_{m_k}, x_{m_k+1})d(x_{m_k}, x_{n_k}) + d(x_{n_k-1}, x_{n_k})d(x_{n_k-1}, x_{m_k+1})}{1 + s\left[d(x_{m_k}, x_{n_k-1}) + d(x_{m_k+1}, x_{n_k})\right]}, \\ \dfrac{d(x_{m_k}, x_{m_k+1})d(x_{m_k}, x_{n_k}) + d(x_{n_k-1}, x_{n_k})d(x_{n_k-1}, x_{m_k+1})}{1 + d(x_{m_k}, x_{n_k}) + d(x_{n_k-1}, x_{m_k+1})} \end{array} \right\} \\ &\leq \max\{\epsilon, s\epsilon, s\epsilon, 0, \epsilon\} = s\epsilon. \end{aligned}$$

Thus

$$\epsilon \leq \limsup_{k\to\infty} (\beta(M_f(x_{m_k}, x_{n_k-1}))M_f(x_{m_k}, x_{n_k-1})) \leq s\epsilon \limsup_{k\to\infty} \beta(M_f(x_{m_k}, x_{n_k-1})).$$

As $\beta \in \mathcal{S}$, we obtain $\limsup_{k\to\infty} \beta(M_f(x_{m_k}, x_{n_k-1})) = \frac{1}{s}$. Consequently, we have $\lim_{k\to\infty} M_f(x_{m_k}, x_{n_k-1}) = 0$ and $\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = 0$. This further gives that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0$, a contradiction to (10). We now assume that (14) holds.

Then by (10), we obtain that $\epsilon \leq d(x_{m_k}, x_{n_k}) \leq sd(x_{m_k}, x_{m_k+2}) + sd(x_{m_k+2}, x_{n_k})$ which implies that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m_k+2}, x_{n_k}). \tag{17}$$

It follows from (9) and (11) that

$$\limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k-1}) \leq s\epsilon. \tag{18}$$

Also, from Lemma 1.1, we have $\alpha(x_{m_k+1}, x_{n_k-1}) \geq 1$. Now

$$\begin{aligned}\epsilon \leq s \limsup_{k\to\infty} d(x_{m_k+2}, x_{n_k}) &\leq \limsup_{k\to\infty} s\alpha(x_{m_k+1}, x_{n_k-1}) d(fx_{m_k+1}, fx_{n_k-1}) \\ &\leq \limsup_{k\to\infty} (\beta(M_f(x_{m_k+1}, x_{n_k-1}))) M_f(x_{m_k+1}, x_{n_k-1}).\end{aligned} \tag{19}$$

where

$$\begin{aligned}&\limsup_{k\to\infty} M_f(x_{m_k+1}, x_{n_k-1}) \\ &\leq \limsup_{k\to\infty} \max \left\{ \begin{array}{l} d(x_{m_k+1}, x_{n_k-1}), d(x_{m_k+1}, fx_{m_k+1}), d(x_{n_k-1}, fx_{n_k-1}), \\ d(f^2x_{m_k+1}, fx_{m_k+1}), d(f^2x_{m_k+1}, x_{n_k-1}), \frac{1}{s} d(f^2x_{m_k+1}, fx_{n_k-1}), \\ \frac{1}{2s} d(f^2x_{m_k+1}, x_{m_k+1}), \frac{d(x_{m_k+1}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{m_k+1})}{2s}, \\ \frac{d(x_{m_k+1}, fx_{m_k+1}) d(x_{m_k+1}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{n_k-1}) d(x_{n_k-1}, fx_{m_k+1})}{1 + s\left[d(x_{m_k+1}, x_{n_k-1}) + d(fx_{m_k+1}, fx_{n_k-1})\right]}, \\ \frac{d(x_{m_k+1}, fx_{m_k+1}) d(x_{m_k+1}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{n_k-1}) d(x_{n_k-1}, fx_{m_k+1})}{1 + d(x_{m_k+1}, fx_{n_k-1}) + d(x_{n_k-1}, fx_{m_k+1})} \end{array} \right\} \\ &= \max\left\{s\epsilon, s\epsilon, s\epsilon, \left(\tfrac{s+1}{2}\right)\epsilon\right\} = s\epsilon.\end{aligned}$$

Thus

$$\epsilon \leq \limsup_{k\to\infty} (\beta(M_f(x_{m_k+1}, x_{n_k-1}) M_f(x_{m_k+1}, x_{n_k-1})) \leq \epsilon s \limsup_{k\to\infty} \beta(M_f(x_{m_k+1}, x_{n_k-1}))$$

gives $\limsup_{k\to\infty} \beta(M_f(x_{m_k+1}, x_{n_k-1})) = \dfrac{1}{s}$ as $\beta \in \mathcal{S}$ which further implies that $\lim_{k\to\infty} M_f(x_{m_k+1}, x_{n_k-1}) = 0$. Consequently, by (6) we have $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k-1}) = 0$ which further gives $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0$, a contradiction to (10). Thus $\{x_n\}$ is a b-Cauchy sequence in the orbit $O_f(x_0)$. Since X is an f-orbitally complete metric space, $\{x_n\}$ b-converges to a point $z \in X$. Now we prove that either

$$\frac{1}{2} d(x_n, fx_n) \leq sd(x_n, z) \tag{20}$$

$$\text{or} \qquad \frac{1}{2} d(x_{n+1}, fx_{n+1}) \leq sd(x_{n+1}, z) \tag{21}$$

hold for all $n \in \mathbb{N}_0$. If not, there exists $n_0 \in \mathbb{N}_0$ such that

$$\frac{1}{2} d(x_{n_0}, fx_{n_0}) > sd(x_{n_0}, z) \text{ and } \frac{1}{2} d(x_{n_0+1}, fx_{n_0+1}) > sd(x_{n_0+1}, z). \tag{22}$$

Now by (8) and (22), we have

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) &\leq s\left(d(x_{n_0}, z) + d(x_{n_0+1}, z)\right) < \frac{1}{2}d(x_{n_0}, fx_{n_0}) + \frac{1}{2}d(x_{n_0+1}, fx_{n_0+1}) \\ &= \frac{1}{2}d(x_{n_0}, x_{n_0+1}) + \frac{1}{2}d(x_{n_0+1}, x_{n_0+2}) < \frac{1}{2}d(x_{n_0}, x_{n_0+1}) + \frac{1}{2}d(x_{n_0}, x_{n_0+1}) \\ &= d(x_{n_0}, x_{n_0+1}) \end{aligned}$$

a contradiction. Hence either (20) or (21) hold for all $n \in \mathbb{N}_0$. Next, we prove that $z = fz$. If not, then $d(z, fz) > 0$. Note that

$$\begin{aligned} &\lim_{n\to\infty} M_f(x_n, z) \\ &= \lim_{n\to\infty} \max\left\{ \begin{array}{l} d(x_n, z), d(x_n, fx_n), d(z, fz), d(f^2x_n, fx_n), d(f^2x_n, z), \\ \frac{1}{s}d(f^2x_n, fz), \frac{1}{2s}d(f^2x_n, x_n), \frac{d(x_n, fz) + d(z, fx_n)}{2s}, \\ \frac{d(x_n, fx_n)d(x_n, fz) + d(z, fz)d(z, fx_n)}{1 + s\left[d(x_n, z) + d(fx_n, fz)\right]}, \\ \frac{d(x_n, fx_n)d(x_n, fz) + d(z, fz)d(z, fx_n)}{1 + d(x_n, fz) + d(z, fx_n)} \end{array} \right\} \\ &= d(z, fz). \end{aligned}$$

If (20) holds for all $n \in \mathbb{N}_0$, then we have,

$$\begin{aligned} d(z, fz) &\leq s \lim_{n\to\infty} \left(d(z, x_{n+1}) + d(x_{n+1}, fz)\right) = s \lim_{n\to\infty} \left(d(fx_n, fz) + d(x_{n+1}, z)\right) \\ &\leq s \lim_{n\to\infty} \left(s\alpha(x_n, z)d(fx_n, fz) + d(x_{n+1}, z)\right) \\ &\leq s \lim_{n\to\infty} \left(\beta(M_f(x_n, z))M_f(x_n, z) + d(x_{n+1}, z)\right) \\ &\leq s \lim_{n\to\infty} \beta(M_f(x_n, z)) \lim_{n\to\infty} M_f(x_n, z) + s \lim_{n\to\infty} d(x_{n+1}, z) \\ &\leq s \lim_{n\to\infty} \beta(M_f(x_n, z))d(z, fz). \end{aligned}$$

This further implies that $\frac{1}{s} \leq \lim\limits_{n\to\infty} \beta(M_f(x_n, z))$. Consequently, we have

$$\lim_{n\to\infty} \beta(M_f(x_n, z)) = \frac{1}{s}$$

and hence $\lim\limits_{n\to\infty} M_f(x_n, z) = 0$. That is $d(z, fz) = 0$, a contradiction. Similarly, if (21) holds for all $n \in \mathbb{N}_0$, then we have $d(z, fz) = 0$, a contradiction. Hence z is a fixed point of f. To prove the uniqueness: Let z and z^* be two fixed points of f such that $z \neq z^*$. Then by condition (H), there exists $w \in X$ such that $\alpha(z, w) \geq 1$, $\alpha(z^*, w) \geq 1$ and $\alpha(w, fw) \geq 1$. Since f is a triangular α-orbital admissible, we obtain that $\alpha(z, f^n w) \geq 1$ and $\alpha(z^*, f^n w) \geq 1$ for all $n \in \mathbb{N}_0$. As $\frac{1}{2}d(z, fz) = 0 \leq sd(z, f^n w)$, by (7) we have

$$d(z, f^{n+1}w) \leq s\alpha(z, f^n w)d(z, f^{n+1}w) \leq \beta(M_f(z, f^n w))M_f(z, f^n w) \tag{23}$$

for all $n \in \mathbb{N}_0$, where

$$M_f(z, f^n w) = \max \left\{ \begin{array}{l} d(z, f^n w), d(z, fz), d(f^n w, f^{n+1} w), d(f^2 z, fz), d(f^2 z, f^n w), \\ \frac{1}{s} d(f^2 z, f^{n+1} w), \frac{1}{2s} d(f^2 z, z), \frac{d(z, f^{n+1} w) + d(f^n w, fz)}{2s}, \\ \frac{d(z, fz) d(z, f^{n+1} w) + d(f^n w, f^{n+1} w) d(f^n w, fz)}{1 + s\left[d(z, f^n w) + d(fz, f^{n+1} w)\right]}, \\ \frac{d(z, fz) d(z, f^{n+1} w) + d(f^n w, f^{n+1} w) d(f^n w, fz)}{1 + d(z, f^{n+1} w) + d(f^n w, fz)} \end{array} \right\}.$$

Note that the sequence $\{f^n w\}$ is a Picard sequence that converges to a fixed point z^{**} of f. On taking limit as $n \to \infty$ on both sides of the above equality, we get

$$\begin{aligned} \lim_{n\to\infty} M_f(z, f^n w) &= \lim_{n\to\infty} \max \left\{ \begin{array}{l} d(z, f^n w), d(f^2 z, f^n w), \frac{1}{s} d(f^2 z, f^{n+1} w), \\ \frac{d(z, f^{n+1} w) + d(f^n w, fz)}{2s} \end{array} \right\} \\ &= d(z, z^{**}). \end{aligned}$$

If $z \neq z^{**}$, then from (23) and (**??**) we have

$$\frac{1}{s} = \lim_{n\to\infty} \frac{d(z, f^{n+1} w)}{s M_f(z, f^n w)} \leq \lim_{n\to\infty} \frac{d(z, f^{n+1} w)}{M_f(z, f^n w)} \leq \lim_{n\to\infty} \beta(M_f(z, f^n w)) < \frac{1}{s},$$

that is, $\lim\limits_{n\to\infty} \beta(M_f(z, f^n w)) = \frac{1}{s}$. Consequently $\lim\limits_{n\to\infty} M_f(z, f^n w) = d(z, z^{**}) = 0$, a contradiction. Therefore, $z = z^{**}$. Similarly $z^* = z^{**}$. Thus we have $z = z^*$, a contradiction. Hence the result follows. □

Remark 2.1. As complete b-metric space is f-orbitally complete b-metric space and triangular α-admissible mapping is triangular α-orbital admissible mapping (but the converse does not hold in general [12, Example 7]), replacing $M_f(x, y)$ with $L(x, y)$ in Theorem 2.1, we obtain Theorem 1.5 as a special case of Theorem 2.1. We also observe that Theorem 2.1 extends Theorem 1.4 which in turn generalize Banach contraction principle and Theorems 1.1, 1.2, and 1.3.

The following corollary is the generalization of Theorem 1.4 in the context of b-metric spaces.

Corollary 2.1. *Let (X, d) be a b-metric space with parameter $s \geq 1$, $\beta \in \mathcal{S}$, and $\alpha : X \times X \to \mathbb{R}$. Suppose that a mapping $f : X \to X$ is such that for any $x, y \in X$,*

$$\frac{1}{2} d(x, fx) \leq s d(x, y) \Rightarrow s\alpha(x, y) d(fx, fy) \leq \beta(N(x, y)) N(x, y).$$

If X is f-orbitally complete and the following conditions hold:

(a) *f is a triangular α-orbital admissible mapping;*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*

(c) *either f is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for $n \in \mathbb{N}_0$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.*

Then f has a fixed point z in X and $\{f^n x_0\}$ converges to z. Moreover f has a unique fixed point if condition (b) is replaced with condition (H).

If we set $\beta(t) = \frac{1}{2s}$, $M_f(x, y) = N(x, y)$ and $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.1, then we have the following result.

Corollary 2.2. *Let (X, d) be a b-metric space with parameter $s \geq 1$ and $f : X \to X$. If for any $x, y \in X$, $\frac{1}{2}d(x, fx) \leq sd(x, y)$ implies $sd(fx, fy) \leq \frac{1}{2s}N(x, y)$. Then f has a unique fixed point z in X and $\{f^n x_0\}$ converges to z for any choice of x_0 in X provided that X is f-orbitally complete.*

Corollary 2.3. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and $f : X \to X$. If for any $x, y \in X$, $\frac{1}{2}d(x, fx) \leq sd(x, y)$ implies $sd(fx, fy) \leq \frac{1}{2s}d(x, y)$. Then f has a unique fixed point z in X and $\{f^n x_0\}$ converges to z for any choice of x_0 in X.*

Proof. Take $N(x, y) = d(x, y)$ for all $x, y \in X$ in Corollary 2.2. □

Example 2.1. *Let $X = \{x_1, x_2, x_3, x_4\}$. Define $d : X \times X \to \mathbb{R}_0^+$ by*

$$d(x_1, x_2) = d(x_1, x_3) = \tfrac{1}{4}, \quad d(x_1, x_4) = d(x_2, x_3) = d(x_3, x_4) = \tfrac{1}{2}, \quad d(x_2, x_4) = 1,$$
$$d(x, x) = 0 \text{ and } d(x, y) = d(y, x) \text{ for all } x, y \in X.$$

As $1 = d(x_2, x_4) \nleq d(x_2, x_1) + d(x_1, x_4) = \frac{3}{4}$, d is not a metric on X. Clearly, (X, d) is a complete b-metric space with parameter $s \geq \frac{4}{3} > 1$. Define a mapping $f : X \to X$ by

$$fx = \begin{cases} x_1, & \text{if } x = x_1, x_2, x_3, \\ x_2, & \text{if } x = x_4. \end{cases}$$

Let $\beta : \mathbb{R}_0^+ \to [0, \frac{1}{s})$ be defined by $\beta(t) = \frac{1}{s+t}$. Clearly for any $x \in X$, any sequence in $O_f(x)$ converges to $x_1 \in X$. Hence (X, d) is f-orbitally complete b-metric space. Let

$$B = \left\{ \begin{array}{l} (x_1, x_1), (x_1, x_2), (x_2, x_1), (x_1, x_3), (x_2, x_2), \\ (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_3, x_3), (x_2, x_4) \end{array} \right\}.$$

Define $\alpha : X \times X \to \mathbb{R}_0^+$ by $\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in B \\ 0 & \text{otherwise} \end{cases}$. If $x \in \{x_1, x_2, x_3\}$, then $\alpha(x, fx) = \alpha(x, x_1) = 1$ implies that $\alpha(fx, f^2x) = \alpha(x_1, x_1) = 1$. If $x = x_4$, then $\alpha(x, fx) = 0$. Hence for any $x \in X$, $\alpha(x, fx) \geq 1$ implies that $\alpha(fx, f^2x) \geq 1$. If $x, y \in \{x_1, x_2, x_3\}$, then $\alpha(x, y) = 1$ and $\alpha(y, fy) = \alpha(y, x_1) = 1$ imply that $\alpha(x, fy) = \alpha(x, x_1) = 1$.If $x = x_2$ and $y = x_4$, then $\alpha(x, y) = 1$ and $\alpha(y, fy) = \alpha(y, x_2) = 0$. Hence for any $x, y \in X$, $\alpha(x, y) \geq 1$

and $\alpha(y, fy) \geq 1$ *imply that* $\alpha(x, fy) \geq 1$. *Therefore* f *is a triangular* α*-orbital admissible mapping. Note that, for any* $x, y \in X$, *the inequalities*

$$\frac{1}{2}d(x, fx) \leq \frac{4}{3}d(x, y) and \alpha(x, y) \geq 1,$$

give

$$(x, y) \in \left\{ \begin{array}{l} (x_1, x_1), (x_1, x_2), (x_2, x_1), (x_1, x_3), \\ (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_2, x_4) \end{array} \right\}$$

As b- metric d is symmetric, we focus only on the set

$$\{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_2, x_3), (x_2, x_4)\}$$

If $x, y \in \{x_1, x_2, x_3\}$, *then we have*

$$\frac{4}{3}\alpha(x, y)d(fx, fy) = 0 \leq \beta\left(M_f(x, y)\right) M_f(x, y).$$

If $x = x_2$, $y = x_4$, *then we obtain that*

$$\begin{aligned} \frac{4}{3}\alpha(x, y)d(fx, fy) &= \frac{4}{3}\alpha(x_2, x_4)d(fx_2, fx_4) = \frac{4}{3}d(x_1, x_2) = \frac{1}{3} \\ &\leq \frac{3}{7} = \beta\left(d(x_2, x_4)\right) d(x_2, x_4) \leq \beta\left(M_f(x_2, x_4)\right) M_f(x_2, x_4). \end{aligned}$$

Therefore, for any $x, y \in X$, $\frac{1}{2}d(x, fx) \leq \frac{4}{3}d(x, y)$ *implies that*

$$s\alpha(x, y)d(fx, fy) \leq \beta(M_f(x, y))M_f(x, y).$$

Thus all the conditions of Theorem 2.1 are satisfied. Moreover, $z = x_1$ *is a fixed point of* f. *On the other hand,*

$$\alpha(x_1, x_2) = 1, \ \alpha(x_2, x_4) = 1 \text{ and } \alpha(x_1, x_4) = 0.$$

Thus f *is not a triangular* α*-admissible. Hence Theorem 1.5 is not applicable in this case.*

Example 2.2. *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$. *Define* $d : X \times X \to \mathbb{R}_0^+$ *by*

$$\begin{array}{l} d(x_1, x_2) = d(x_1, x_3) = 8, \quad d(x_1, x_4) = d(x_1, x_5) = 6, \quad d(x_2, x_4) = 12, \\ d(x_2, x_3) = d(x_2, x_5) = d(x_3, x_4) = d(x_3, x_5) = 2, \quad d(x_4, x_5) = 4, \\ d(x, x) = 0 \text{ and } d(x, y) = d(y, x) \text{ for all } x, y \in X. \end{array}$$

Since $12 = d(x_2, x_4) \nleq d(x_2, x_5) + d(x_5, x_4) = 6$,*and* $12 = d(x_2, x_4) \nleq d(x_2, x_3) + d(x_3, x_4) = 4$,*hence* d *is not a metric on* X. *Clearly,* (X, d) *is a complete b-metric space with* $s \geq 3 > 1$. *Define the mapping* $f : X \to X$ *by*

$$fx = \begin{cases} x_1, & \text{if } x = x_1, x_2, x_3, \\ x_2, & \text{if } x = x_4, \\ x_3, & \text{if } x = x_5. \end{cases}$$

Let $\beta : \mathbb{R}_0^+ \to [0, \frac{1}{s})$ *be defined by* $\beta(t) = \frac{5}{6}$. *Clearly for any* $x \in X$, *any sequence in* $O_f(x)$ *converges to* $x_1 \in X$. *Hence* (X, d) *is* f*-orbitally complete b-metric space. Let*

$$C=\left\{\begin{array}{l}(x_1,x_1),(x_1,x_2),(x_2,x_1),(x_1,x_3),(x_2,x_2),(x_2,x_3),(x_3,x_1),(x_3,x_2),\\(x_3,x_3),(x_4,x_4),(x_4,x_5),(x_5,x_4),(x_5,x_5)\end{array}\right\}.$$

Define $\alpha : X\times X\to\mathbb{R}_0^+$ *by* $\alpha(x,y)=1$ *if* $(x,y)\in C$ *and* $\alpha(x,y)=0$ *otherwise. If* $x,y\in\{x_1,x_2,x_3\}$*, then* $\alpha(x,y)=1$ *implies that* $\alpha(fx,fy)=\alpha(x_1,x_1)=1$*. If* $x,y\in\{x_4,x_5\}$*, then* $\alpha(x,y)=1$ *implies that* $\alpha(fx,fy)=1$*. Thus for any* $x,y\in X$*,* $\alpha(x,y)\geq 1$ *implies that* $\alpha(fx,fy)\geq 1$*. If* $x,y,z\in\{x_1,x_2,x_3\}$*, then* $\alpha(x,y)=1$ *and* $\alpha(y,z)=1$ *imply that* $\alpha(x,z)=1$*. If* $x,y,z\in\{x_4,x_5\}$*, then* $\alpha(x,y)=1$ *and* $\alpha(y,z)=1$ *imply that* $\alpha(x,z)=1$*. Hence, for all* $x,y,z\in X$ *we have* $\alpha(x,y)\geq 1$ *and* $\alpha(y,z)\geq 1$ *imply that* $\alpha(x,z)\geq 1$*. Also, if* $x\in\{x_1,x_2,x_3\}$*, then* $\alpha(x,fx)=1$ *implies that* $\alpha(fx,f^2x)=1$ *and if* $x\in\{x_4,x_5\}$*, then we have* $\alpha(x,fx)=0$*. Thus, for any* $x\in X$*,* $\alpha(x,fx)\geq 1$ *implies that* $\alpha(fx,f^2x)\geq 1$*. For all* $x,y\in\{x_1,x_2,x_3\}$*,* $\alpha(x,y)=1$ *and* $\alpha(y,fy)=1$ *imply that* $\alpha(x,fy)=1$*. For* $x,y\in\{x_4,x_5\}$*, we have* $\alpha(x,y)=1$ *and* $\alpha(y,fy)=0$*. Hence, for any* $x,y\in X$*,* $\alpha(x,y)\geq 1$ *and* $\alpha(y,fy)\geq 1$ *imply that* $\alpha(x,fy)\geq 1$*. Therefore* f *is a triangular* α*-orbital admissible mapping. Note that, for any* $x,y\in X$*, the following inequalities* $\frac{1}{2}d(x,fx)\leq 3d(x,y)$ *and* $\alpha(x,y)\geq 1$*, give*

$$(x,y)\in\{(x_1,x_1),(x_1,x_2),(x_2,x_1),(x_1,x_3),(x_2,x_3),(x_3,x_1),(x_3,x_2),(x_4,x_5),(x_5,x_4)\}.$$

As b- metric d *is symmetric, we focus only on the set*

$$\{(x_1,x_1),(x_1,x_2),(x_1,x_3),(x_2,x_3),(x_4,x_5)\}.$$

If $x,y\in\{x_1,x_2,x_3\}$*, then we have* $3\alpha(x,y)d(fx,fy)=0\leq\beta\left(M_f(x,y)\right)M_f(x,y)$*.If* $x\in\{x_4,x_5\}$*, then* $3\alpha(x,x)d(fx,fx)=0\leq\beta\left(M_f(x,x)\right)M_f(x,x)$*.If* $x=x_4$*,* $y=x_5$*, then*

$$\begin{aligned}3\alpha(x,y)d(fx,fy)&=3\alpha(x_4,x_5)d(fx_4,fx_5)=3d(x_2,x_3)=6\\&\leq 10=\beta\left(d(x_2,x_4)\right)d(x_2,x_4)\\&\leq\beta\left(M_f(x_4,x_5)\right)M_f(x_4,x_5).\end{aligned}$$

Therefore, for any $x,y\in X$ $\frac{1}{2}d(x,fx)\leq 3d(x,y)$ *implies that* $3\alpha(x,y)d(fx,fy)\leq\beta(M_f(x,y))M_f(x,y)$*. Thus all the conditions of Theorem 2.1 are satisfied. Moreover,* $z=x_1$ *is a fixed point of* f*. On the other hand, if we take* $x=x_4$*,* $y=x_5$*, then* $3\alpha(x,y)d(fx,fy)=3\alpha(x_4,x_5)d(fx_4,fx_5)=3d(x_2,x_3)=6\nleq\frac{14}{3}=\beta\left(L(x_4,x_5)\right)L(x_4,x_5)$*,where*

$$L(x_4,x_5)=\max\left\{\begin{array}{l}d(x_4,x_5),\dfrac{d(x_4,x_2)d(x_4,x_3)+d(x_5,x_3)d(x_5,x_2)}{1+2\left(d(x_4,x_5)+d(x_2,x_3)\right)},\\\dfrac{d(x_4,x_2)d(x_4,x_3)+d(x_5,x_3)d(x_5,x_2)}{1+d(x_4,x_3)+d(x_5,x_2)}\end{array}\right\}=\frac{28}{5}.$$

Hence Theorem 1.5 does not hold in this case.

3. Applications to existence of solutions of integral equations

We consider the following integral equation:

$$u(r) = v(r) + \lambda \int_a^b G(r,z)F(z,u(z))dz, \tag{24}$$

for all $r \in [a,b]$, where $\lambda \in \mathbb{R}$, $G : [a,b] \times [a,b] \to \mathbb{R}_0^+$, $F : [a,b] \times \mathbb{R} \to \mathbb{R}$, $v : [a,b] \to \mathbb{R}$ are known continuous functions. Let $X = C[a,b] = \{\ g : [a,b] \to \mathbb{R}$ is continuous$\}$ and $d : X \times X \to \mathbb{R}_0^+$ be the b- metric defined by

$$d(u,v) = \max_{a \le r \le b} |u(r) - v(r)|^2. \tag{25}$$

Define the mapping $f : X \to X$ by

$$fu(r) = v(r) + \lambda \int_a^b G(r,z)F(z,u(z))dz,\ r \in [a,b]. \tag{26}$$

Note that (X,d) is a complete b-metric space. Let the mapping $\alpha : X \times X \to \mathbb{R}_0^+$ be defined by $\alpha(x,y) = 1$ for all $x,y \in X$.

Theorem 3.1. *If the following conditions hold:*

(1) $|\lambda| \le 1$ *for* $\lambda \in \mathbb{R}$*;*
(2) $\max\limits_{a \le r \le b} \int_a^b G^2(r,z)dz \le \dfrac{1}{b-a}$*;*
(3) *for any* $z_1, z_2 \in \mathbb{R},\ s \ge 1,$ *we have* $|F(z,z_1) - F(z,z_2)|^2 \le \frac{1}{2s^2} |z_1 - z_2|^2$*;*
(4) *for any* $x,y \in X$ *and* $s \ge 1,$ *we have* $\dfrac{1}{2} d(x,fx) \le sd(x,y)$*.*

Then the equation (24) has a unique solution $u^* \in X$.

Proof. By Cauchy-Schwarz inequality and the conditions (1)-(3), we obtain that

$$\begin{aligned}
sd(fu_1, fu_2) &\leq s \max_{a\leq r\leq b} |fu_1(r) - fu_2(r)|^2 \\
&= s|\lambda|^2 \max_{a\leq r\leq b} \left\{ \left| \int_a^b G(r,z)(F(z,u_1(z)) - F(z,u_2(z)))dz \right|^2 \right\} \\
&\leq s|\lambda|^2 \max_{a\leq r\leq b} \left\{ \int_a^b G^2(r,z)dz \int_a^b |F(z,u_1(z)) - F(z,u_2(z))|^2 dz \right\} \\
&= s|\lambda|^2 \left\{ \max_{a\leq r\leq b} \int_a^b G^2(r,z)dz \right\} . \left\{ \int_a^b |F(z,u_1(z)) - F(z,u_2(z))|^2 dz \right\} \\
&\leq s|\lambda|^2 \left\{ \frac{1}{b-a} \right\} . \left\{ \frac{1}{2s^2} \int_a^b |u_1(z) - u_2(z)|^2 dz \right\} \\
&\leq s|\lambda|^2 \left\{ \frac{1}{b-a} \right\} . \frac{1}{2s^2} \int_a^b \max_{a\leq r\leq b} |u_1(r) - u_2(r)|^2 dz \\
&= \frac{|\lambda|^2}{2s} \max_{a\leq r\leq b} |u_1(r) - u_2(r)|^2 = \frac{|\lambda|^2}{2s} d(u_1,u_2) \leq \frac{1}{2s} d(u_1,u_2).
\end{aligned}$$

Hence, all the conditions of Corollary 2.3 are satisfied. Therefore the mapping f has a unique fixed point $u^* \in X$, that is, the integral (24) has a unique solution in $X = C[a,b]$. □

4. Conclusions

In this paper, we presented the concept of a generalized α-Suzuki-Geraghty type contraction mapping and obtained fixed point theorems for such mappings in the framework of f-orbitally complete b-metric space which improve, generalize, and unify various comparable results [8, 9, 11, 12, 16] in the existing literature. Examples are given to prove that the generalization is proper and important one. These main results obtained here can be applied in the existence of a solution of Integral equations.

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