Journal of Applied Analysis and Computation
Volume 7, Number 3, August 2017, 1112–1126
Website:http://jaac-online.com/
DOI:10.11948/2017069

NEW TYPE OF FIXED POINT RESULT OF F-CONTRACTION WITH APPLICATIONS

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Abstract The purpose of this paper is to prove theorem which generalize the corresponding results of Rhoades [B. E. Rhoades, Two New Fixed Point Theorems, Gen. Math. Notes, 2015, 27(2), 123–132]. This paper is to introduce the notion of dynamic process for generalized $F-$contraction mappings and to obtain coincidence and common fixed point results for such process. It is worth mentioning that our results do not rely on the commonly used range inclusion condition. We provide some examples to support our results. As an application of our results, we obtain the existence and uniqueness of solutions of dynamic programming and integral equations. Our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords Coincidence point, generalized dynamic process, F-contraction, integral equations, dynamic programming.

MSC(2010) 47H10, 47H04, 60H25, 54H25.

1. Introduction

Let (X, d) be a metric space. Let $CB(X)$ $(CL(X))$ be the family of all nonempty closed and bounded (nonempty closed) subsets of X. For $A, B \in CL(X)$, define a set

$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\}.$$

The Hausdorff metric H on $CL(X)$ induced by metric d is given as:

$$H(A, B) = \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \emptyset, \\ \infty, & \text{if } E_{A,B} = \emptyset. \end{cases}$$

Let $f : X \to X$ and $T : X \to CL(X)$. A hybrid pair $\{f, T\}$ is said to satisfy range inclusion condition if $f(X) \subseteq T(X)$.

A point x in X is called a fixed point of T if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$. Furthermore, a point x in X is called a coincidence point of f and T if $fx \in Tx$. The set of all such points is denoted by $C(f, T)$. If for

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some point x in X, we have $x = fx \in Tx$, then a point x is called a common fixed point of f and T. We denote set of all common fixed points of f and T by $F(f,T)$. A mapping $T : X \to CL(X)$ is said to be continuous at $p \in X$ if for any sequence $\{x_n\}$ in X with $\lim\limits_{n\to\infty} d(x_n, p) = 0$, we have $\lim\limits_{n\to\infty} H(Tx_n, Tp) = 0$.

Let x_0 be an arbitrary but fixed element in X , then we define a set $D(f,T,x_0) = \Big\{(fx_n)_{n\in\mathbb{N}\cup\{0\}} : fx_n \in Tx_{n-1} \text{ for all } n \in \mathbb{N}\Big\}$ is called a generalized dynamic process of f and T starting at x_0. Note that $D(f,T,x_0)$ reduces to dynamic process of T starting at x_0 if $f = I_X$ (an identity map on X) [17]. The generalized dynamic process $D(f,T,x_0)$ will simply be written as (fx_n). The sequence $\{x_n\}$ for which (fx_n) is a generalized dynamic process is called f iterative sequence of T starting at x_0.

Note that, if hybrid pair $\{f,T\}$ is satisfy $f(X) \subseteq T(X)$, then for any $x_0 \in X$, construction of f iterative sequence of T starting at x_0 is immediate and hence $D(f,T,x_0)$ is nonemtpy.

There are many situations where $D(f,T,x_0)$ is nonempty even the range inclusion condition does not hold. Following are the examples of such cases:

Example 1.1. Let $X = [0,\infty)$. Define $f : X \to X$ and $T : X \to CL(X)$ by $f(x) = 2x$, $Tx = [1+x,\infty)$, respectively. Note that, one can construct several f iterative sequences of T starting at some point $x_0 \in X$.

$$x_n = \frac{3}{2}(1 + x_{n-1})$$

is an f iterative sequence of T starting at 0.

Example 1.2. Let $X = [0,\infty)$. Define $f : X \to X$ and $T : X \to CL(X)$ by $f(x) = x^2$, $Tx = [2+x,\infty)$, respectively. The sequence $\{x_n\}$, where

$$x_n = \sqrt{x_{n-1} + 2}.$$

is an f iterative sequence of T starting at a point 0.

Example 1.3. Let $X = \mathbb{R}$. Define $f : X \to X$ and $T : X \to CL(X)$ by $f(x) = \frac{x-1}{2}$, and

$$Tx = \begin{cases} [\frac{1}{4}, \frac{x}{2}], & \text{when } x > 0, \\ \{0\}, & \text{otherwise,} \end{cases}$$

respectively. Define a sequence $\{x_n\}$ by $x_n = x_{n-1} + 1$. If $x_0 = 1$, then

$$\begin{aligned} f(x_1) &= \frac{1}{2} \in Tx_0 = [\frac{1}{4}, \frac{1}{2}], \\ f(x_2) &= 1 \in Tx_1 = [\frac{1}{4}, 1], \\ f(x_3) &= \frac{3}{2} \in Tx_2 = [\frac{1}{4}, \frac{3}{2}] \quad \text{and so on.} \end{aligned}$$

Here

$$D(f,T,1) = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots\}$$

is a generalized dynamic process of f and T starting at $x_0 = 1$.

Berinde [11] introduced the following concept of a weak contraction mapping.

Definition 1.1 ([11]). Let (X, d) be a metric space. A self mapping f on X is called a weak contraction if there exist constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$d(fx, fy) \leq \theta d(x, y) + Ld(y, fx)$$

holds for every x, y in X.

For more discussion on weak contraction mappings, we refer to [13, 15] and references therein.

Berinde and Berinde [12] extended the notion of weak contraction mappings as follows:

Definition 1.2 ([12, 14]). A mapping $T : X \to CL(X)$ is called a multivalued weak contraction if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) \tag{1.1}$$

holds for every x, y in X.

Following definition of a generalized multivalued $(\theta, L)-$strict almost contraction mapping is due to Berinde and Păcurar [14] .

Definition 1.3 ([14]). A mapping $T : X \to CL(X)$ is called generalized multivalued $(\theta, L)-$strict almost contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + L \min\{d(y, Tx), d(x, Ty), d(x, Tx), d(y, Ty)\} \tag{1.2}$$

holds for every x, y in X.

We have following fixed point theorem in [14].

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ a generalized multivalued $(\theta, L)-$strict almost contraction mapping. Then $F(T) \neq \emptyset$. Moreover, for any $p \in F(T)$, T is continuous at p.*

Kamran [16] extended the notion of a multivalued weak contraction mapping to a hybrid pair $\{f, T\}$ of single valued mapping f and multivalued mapping T.

Definition 1.4. Let (X, d) be a metric space and f a self map on X. A multivalued mapping $T : X \to CL(X)$ is called generalized multivalued $(f, \theta, L)-$weak contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx) \tag{1.3}$$

holds for every x, y in X.

Abbas [1] extended the above definition as follows.

Definition 1.5 ([1]). Let (X, d) be a metric space and f a self map on X. A multivalued mapping $T : X \to CL(X)$ is called generalized multivalued $(f, \theta, L)-$almost contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta M(x, y) + LN(x, y) \tag{1.4}$$

holds for every x, y in X, where

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\},$$
$$N(x, y) = \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$

Let $\mathcal{F}$ be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy the following conditions:

C1 F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;

C2 For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;

C3 There exist $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Wardowski [26] introduced the following concept of $F-$contraction mappings.

Definition 1.6 ([26]). Let (X, d) be a metric space. A self map f on X is said to be an $F-$contraction on X if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)) \tag{1.5}$$

for all $x, y \in X$, where $F \in \mathcal{F}$.

Remark 1.1 ([26]). Every $F-$contraction mapping is continuous.

Abbas et al.([3]) extended the concept of $F-$ contraction mapping and obtained common fixed point results. They employed their results to obtain fixed points of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Recently, Minak [18] proved some fixed point results for Ciric type generalized $F-$contractions on complete metric spaces.

Sgroi and Vetro [25] proved the following result to obtain fixed point of multi-valued mappings as a generalization of Nadler's Theorem [19].

Theorem 1.2 ([25]). *Let (X, d) be a complete metric space and $T : X \to CL(X)$ a multivalued mapping. Assume that there exists an $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx))$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point.

Acar et al. [4] proved the following result.

Theorem 1.3 ([4]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ (Compact subsets of X). Assume that there exist an $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that for any $x, y \in X$, we have*

$$H(Tx, Ty) > 0 \Longrightarrow \tau + F(H(Tx, Ty)) \leq F(M(x, y)),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Then T has a fixed point if T or F is continuous,

Recently, Altun et al. [5] proved the following result.

Theorem 1.4 ([5]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$. Assume that there exist an $F \in \digamma$ and $\tau, \lambda \in \mathbb{R}_+$ such that for any $x, y \in X$, we have*

$$H(Tx,Ty) > 0 \text{ implies that } \tau + F(H(Tx,Ty)) \leq F(d(x,y) + \lambda d(y,Tx)).$$

Then the mapping T is multivalued weakly Picard operator.

For the definition of multivalued weakly Picard operator and the related results, we refer to [12].

Now, we give the following definition.

Definition 1.7. Let f be a self map on metric space X and $T : X \to CL(X)$ a multivalued mapping, then T is called generalized multivalued $(f,L)-$almost $F-$contraction mapping if there exist $F \in \digamma$ and $\tau \in \mathbb{R}_+$ and $L \geq 0$ such that

$$2\tau + F(H(Tx,Ty)) \leq F(M(x,y) + LN(x,y)) \tag{1.6}$$

for every x, y in X, with $Tx \neq Ty$ and

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty) + d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}). \end{aligned}$$

Remark 1.2. Take $F(x) = \ln x$ in the Definition (1.7). Then (1.6) becomes

$$2\tau + \ln(H(Tx,Ty)) \leq \ln(M(x,y) + LN(x,y),$$

that is,

$$\begin{aligned} H(Tx,Ty)) &\leq e^{-2\tau} M(x,y) + e^{-2\tau} LN(x,y) \\ &= \theta_1 M(x,y) + L_1 N(x,y), \end{aligned}$$

where $\theta_1 = e^{-2\tau} \in (0,1)$ and $L_1 = e^{-2\tau} L \geq 0$. Thus we obtain the generalized multivalued $(f, \theta_1, L_1)-$almost contraction mapping [1].

Remark 1.3. Take $\alpha = \beta = \gamma = \dfrac{1}{4}$, $\delta = \dfrac{1}{8} = L$. Note that $\alpha + \beta + \gamma + 2\delta = 1$. Then a contraction condition in Theorem 1.2 becomes

$$\begin{aligned} 2\tau + F(H(Tx,Ty)) &\leq F\left(\frac{1}{4}\left(d(x,y) + (d(x,Tx) + d(y,Ty)) + \frac{d(x,Ty) + d(y,Tx)}{2}\right)\right) \\ &\leq F\left(\frac{1}{4}(4M(x,y))\right) = F((M(x,y) + 0N(x,y))) \end{aligned}$$

for all $x, y \in X$, with $Tx \neq Ty$. Thus, for $L = 0$ and $f = I_X$ in

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty) + d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,fy), d(fx,Tx), d(fy,Ty)\}, \end{aligned}$$

a contraction condition in Theorem 1.3 is an $(f,0)-$almost $F-$contraction, a special case of generalized multivalued $(f,L)-$almost $F-$contraction (for $L = 0$ and $\tau = 2\tau_1$).

Definition 1.8. Let $f : X \to X$ and x_0 an arbitrary point in X. A multivalued mapping $T : X \to CL(X)$ is called a generalized multivalued $F-$contraction with respect to a dynamic process $D(f, T, x_0)$ if there exist $F \in \digamma$ and $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing such that

$$\begin{aligned}\forall_{n\in\mathbb{N}}\ d(fx_n, fx_{n+1}) > 0 \Longrightarrow &\tau\left(M(x_{n-1}, x_n)\right) + F(d(fx_n, fx_{n+1})) \\ &\leq F(M(x_{n-1}, x_n)N(x_{n-1}, x_n)),\end{aligned}$$

where

$$\begin{aligned}&M(x_{n-1}, x_n)\\ &=\max\left\{d(fx_{n-1},fx_n),d(fx_{n-1},Tx_{n-1}),d(fx_n,Tx_n),\frac{d(fx_{n-1},Tx_n)+d(fx_n,Tx_{n-1})}{2}\right\},\\ &N(x_{n-1}, x_n)\\ &=\frac{\max\{d(fx_{n-1},fx_n),d(fx_{n-1},Tx_{n-1})+d(fx_n,Tx_n),d(fx_{n-1},Tx_n)+d(fx_n,Tx_{n-1})\}}{d(fx_{n-1},Tx_{n-1})+d(fx_n,Tx_n)+1}\end{aligned}$$

and $\liminf_{s\to t^+} \tau(s) > 0$ for all $t \geq 0$.

Remark 1.4. Take $F(x) = \ln x$ in the Definition 1.18, we obtain

$$\tau\left(M(x_{n-1}, x_n)\right) + \ln(d(fx_n, fx_{n+1})) \leq \ln(M(x_{n-1}, x_n) + LN(x_{n-1}, x_n),$$

that is,

$$\begin{aligned}d(fx_n, fx_{n+1}) &\leq e^{-\tau(M(x_{n-1},x_n))}M(x_{n-1}, x_n) + e^{-\tau(M(x_{n-1},x_n))}LN(x, y)\\ &= \theta_2 M(x_{n-1}, x_n) + L_2 N(x_{n-1}, x_n),\end{aligned}$$

where $\theta_2 = e^{-\tau(M(x_{n-1},x_n))} \in (0,1)$ and $L_2 = e^{-\tau(M(x_{n-1},x_n))}L \geq 0$. Thus we obtain the generalized multivalued $(f, L)-$almost $F-$contraction with respect to a dynamic process.

Example 1.4. Consider Example 1.3. Let any two arbitrary points $x = 0$ and $y = 2$, we have

$$\begin{aligned}M(0, 2) &= \max\{d(f0, f2), d(f0, T0), d(f2, T2), \frac{d(f0, T2) + d(f2, T0)}{2}\}\\ &= \max\left\{d(-\frac{1}{2}, \frac{1}{2}), d(-\frac{1}{2}, 0), d(\frac{1}{2}, [\frac{1}{4}, 1]), \frac{d(-\frac{1}{2}, [\frac{1}{4}, 1) + d(\frac{1}{2}, 0)}{2}\right\}\\ &= \max\{1, \frac{1}{2}, 0, \frac{5}{8}\} = 1,\end{aligned}$$

and

$$\begin{aligned}N(0, 2) &= \min\{d(f0, T0), d(f2, T2), d(f0, T2), d(f2, T0)\}\\ &= \min\left\{\frac{1}{2}, 0, \frac{3}{4}, \frac{1}{2}\right\} = 0.\end{aligned}$$

Take $F(x) = \ln x$ and $\tau > 0$ and $L \geq 0$, we get

$$\begin{aligned}&2\tau + F(H(T0, T2)) \nleq F(M(0, 2) + LN(0, 2)),\\ &2\tau + \ln\frac{1}{4} \nleq \ln(1).\end{aligned}$$

Hence T is not generalized multivalued $(f, L)-$almost $F-$contraction. On the other hand, the contractive condition is satisfied for every point in the set $D(f, T, 1)$. For example, take $\frac{1}{2}$, and 1 in the set $D(f, T, 1)$, we obtain

$$\begin{aligned} M(\frac{1}{2}, 1) &= \max\{d(f\frac{1}{2}, f1), d(f\frac{1}{2}, T\frac{1}{2}), d(f1, T1), \frac{d(f\frac{1}{2}, T1) + d(f1, T\frac{1}{2})}{2}\} \\ &= \max\left\{d(-\frac{1}{4}, 0), d(-\frac{1}{4}, \frac{1}{4},), d(0, [\frac{1}{4}, \frac{1}{2}]), \frac{d(-\frac{1}{4}, [\frac{1}{4}, \frac{1}{2}]) + d(0, \frac{1}{4})}{2}\right\} \\ &= \max\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}\} = \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} N(\frac{1}{2}, 1) &= \frac{\max\{d(f\frac{1}{2}, f1), d(f\frac{1}{2}, T\frac{1}{2}) + d(f1, T1), d(f\frac{1}{2}, T1) + d(f1, T\frac{1}{2})\}}{d(f\frac{1}{2}, T\frac{1}{2}) + d(f1, T1) + 1} \\ &= \frac{\max\left\{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right\}}{\frac{3}{4} + 1} = \frac{7}{3}. \end{aligned}$$

We have

$$d(f(x_2), f(x_3)) = d(1, \frac{3}{2}) = \frac{1}{2} > 0.$$

Take $F(x) = \ln x$ and $\tau(t) = \begin{cases} -\ln(t + \frac{1}{2}), & \text{for } t \in (0, 1)\,, \\ \ln 3, & \text{for } t \in [1, \infty)\,, \end{cases}$ we obtain

$$\begin{aligned} &\tau\left(M(\frac{1}{2}, 1)\right) + F(\frac{1}{2}) \leq F(M(\frac{1}{2}, 1)N(\frac{1}{2}, 1)), \\ &\tau\left(\frac{1}{2}\right) + F(\frac{1}{2}) \leq F(\frac{1}{2}.\frac{7}{3}), \\ &-\ln 1 + \ln\frac{1}{2} \leq \ln\frac{7}{6}. \end{aligned}$$

Hence T is a generalized multivalued $F-$contraction with respect to a generalized dynamic process $D(f, T, 1)$.

2. Main Result

Throughout this section, we assume that the mapping F is right continuous. In the sequel, we will consider only the dynamic processes (fx_n) satisfying the following condition:

$$(D) \text{ For any } n \text{ in } \mathbb{N},\ d(fx_n, fx_{n+1}) > 0 \Rightarrow d(fx_{n-1}, fx_n) > 0.$$

If dynamic processes (fx_n) does not satisfy property (D), then there exists $n_0 \in \mathbb{N}$ such that $d(fx_{n_0}, fx_{n_0+1}) > 0$ and $d(fx_{n_0-1}, fx_{n_0}) = 0$ which implies that $fx_{n_0-1} = fx_{n_0} \in Tx_{n_0-1}$, that is, the set of coincidence point of hybrid pair (f, T) is nonempty. Under suitable conditions on hybrid pair (f, T), one obtaines the existence of common fixed point of (f, T).

Theorem 2.1. *Let x_0 be an arbitrary point in X and $T : X \to CL(X)$ a generalized multivalued F- contraction with respect to dynamic process $D(f, T, x_0)$. Then $C(f, T) \neq \phi$ provided that $f(X)$ is complete and F is continuous or T is closed multivalued mapping. Moreover $F(f, T) \neq \emptyset$ if one of the following conditions hold:*
(a) for some $x \in C(f, T)$, f is T- weakly commuting at x, $f^2x = fx$.
(b) $f(C(f, T))$ is a singleton subset of $C(f, T)$.

Proof. Let x_0 be a given point in X. Since T is generalized multivalued F- contraction with respect to dynamic process $D(f, T, x_0)$, so we have

$$\tau(M(x_{n-1}, x_n)) + F(d(fx_n, fx_{n+1})) \leq F(M(x_{n-1}, x_n)N(x_{n-1}, x_n)).$$

Now

$$\begin{aligned} M(x_{n-1}, x_n) =& \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \\ & \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\} \\ \leq& \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ & \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}. \end{aligned}$$

and

$$\begin{aligned} & N(x_{n-1}, x_n) \\ =& \frac{\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n), d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})\}}{d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n) + 1} \\ \leq& \frac{\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)\}}{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) + 1} \\ =& \frac{\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1}) + 0\}}{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) + 1} \end{aligned}$$

defining $d_n = d(fx_n, fx_{n+1})$, so we have

$$\begin{aligned} N(x_{n-1}, x_n) \leq& \frac{\max\{d_{n-1}, d_{n-1} + d_n, d(fx_{n-1}, fx_{n+1}) + 0\}}{d_{n-1} + d_n + 1} \\ =& \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1} = \beta_{n-1}. \end{aligned}$$

and

$$M(x_{n-1}, x_n) \leq \max\{d_{n-1}, d_{n-1}, d_n, \frac{d(fx_{n-1}, fx_{n+1}) + 0}{2}\} = \max\{d_{n-1}, d_n\}.$$

Thus

$$F(d_n) \leq F(\beta_{n-1} \max\{d_{n-1}, d_n\}) - \tau(\max\{d_{n-1}, d_n\}).$$

Since $0 < \beta_{n-1} < 1$, and $d_n \neq 0$, it satisfies

$$F(d_n) \leq F(\max\{d_{n-1}, d_n\}) - \tau(\max\{d_{n-1}, d_n\}) \tag{2.1}$$

for all $n \in \mathbb{N}$. As F is strictly increasing, so we have

$$d_n < \max\{d_{n-1}, d_n\}.$$

If

$$\max\{d_{n-1}, d_n\} = d_n$$

for some n, then,

$$d_n < d_n$$

gives a contradiction and hence we have

$$d_n < d_{n-1}. \tag{2.2}$$

Consequently,

$$\tau(d_{n-1}) + F(d_n) \le F(d_{n-1}), \tag{2.3}$$

for all $n \in \mathbb{N}$. By given assumption on τ, there exists $b > 0$ and $n \in \mathbb{N}$ such that $\tau(d_n) > b$ for all $n > n_0$. Thus, we obtain that

$$\begin{aligned} F(d_n) &\le F(d_{n-1}) - \tau(d_{n-1}) \\ &\le F(d_{n-2}) - \tau(d_{n-2}) - \tau(d_{n-1}) \\ &\;\;\vdots \\ &\le F(d_0) - \tau(d_0) - \cdots - \tau(d_{n-1}) \\ &= F(d_0) - (\tau(d_0) + \cdots + \tau(d_{n_0-1})) \\ &\quad - (\tau(d_{n_0}) + \cdots + \tau(d_{n-1})) \\ &\le F(d_0) - (n - n_0)\, b. \end{aligned}$$

On taking limit as $n \to \infty$, we have $\lim\limits_{n\to\infty} F(d_n) = -\infty$. By (C1), $\lim\limits_{n\to\infty} d_n = 0$. By (C3), there exists an $r \in (0,1)$ such that

$$\lim_{n\to\infty} \{d_n^r\}\, F(d_n) = -\infty.$$

Hence it follows that

$$\begin{aligned} &\{d_n^r\}\, F(d_n) - \{d_n^r\}\, F(d_0) \\ &\le d_n^r \left[F(d_0 - (n - n_0)\, b)\right] - d_n^r F(d_0) \\ &= -(n - n_0)\, b\, [d_n^r] \le 0. \end{aligned}$$

On taking limit as n tends to ∞, we obtain that $\lim_{n\to\infty} n\{d_n^r\} = 0$, that is, $\lim_{n\to\infty} n^{1/r} d_n = 0$. This implies that $\sum_{n=1}^{\infty} d_n$ is convergent and hence the sequence $\{fx_n\}$ is a Cauchy sequence in $f(X)$. There is $p \in f(X)$ such that $\lim_{n\to\infty} fx_n = p$. Suppose that u^* is in X such that $fu^* = p$. Now we claim that $fu^* \in Tu^*$. If not, then $d(fu^*, Tu^*) > 0$ as Tu^* is closed. Since F is strictly increasing, we deduce from Definition 1.18 that for all $n \in \mathbb{N}$. Therefore

$$d(fx_n, Tu^*) \le H(Tx_n, Tu^*) < M(x_n, u^*)N(x_n, u^*).$$

Since from condition (C1), we have

$$\tau(M(x_n, u^*)) + F(d(fx_n, Tu^*)) \le F(M(x_n, u^*)N(x_n, u^*))$$

for all $n \in \mathbb{N}$. Next suppose that F is continuous. Since

$$\lim_{n\to\infty} d(fx_n, Tu^*) = d(fu^*, Tu^*),$$

we deduce that

$$\lim_{n\to\infty} M(x_n, u^*) = d(fu^*, Tu^*).$$

Moreover

$$\lim_{n\to\infty} N(x_n, u^*) = \frac{2d(fu^*, Tu^*)}{2d(fu^*, Tu^*) + 1} < 1,$$

so, by continuity of F, we have

$$\tau(d(fu^*, Tu^*)) + F(d(fu^*, Tu^*)) \leq F(d(fu^*, Tu^*)),$$

which provides a contradiction. We conclude that $d(fu^*, Tu^*) = 0$, and thus $fu^* \in Tu^*$.

Now let (a) holds, that is for $x \in C(f,T)$, f is $T-$weakly commuting at x. So we get $f^2x \in Tfx$. By the given hypothesis $fx = f^2x$ and hence $fx = f^2x \in Tfx$. Consequently $fx \in F(f,T)$. (b) Since $f(C(f,T)) = \{x\}$ (say) and $x \in C(f,T)$, this implies that $x = fx \in Tx$. Thus $F(f,T) \neq \emptyset$. $\square$

Example 2.1. Let $X = [1, \infty)$ be the usual metric space. Define $f : X \to X$, $\tau : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$and $T : X \to CL(X)$ by $fx = x^2$ and $Tx = [x+2, \infty)$ for all $x \in X$ and $\tau(t) = \begin{cases} -\ln t, & \text{for } t \in (0,1)\,, \\ \ln 3, & \text{for } t \in [1,\infty)\,, \end{cases}$ and $F(t) = \ln(t)$ for all $t > 0$. Note that $f(X)$ is complete. It is easy to check that for all $x, y \in X$ with $Tx \neq Ty$ (equivalently with $x \neq y$), one has

$$\tau(M(x,y)) + F(H(Tx,Ty)) \leq F(M(x,y)N(x,y)).$$

So we can apply Theorem 2.1.

(1) Application to solution of system of functional equations in dynamic programming:

Decision space and a state space are two basic components of dynamic programming problem. State space is a set of states including initial states, action states and transitional states. So a state space is set of parameters representing different states. A decision space is the set of possible actions that can be taken to solve the problem. These general settings allow us to formulate many problems in mathematical optimization and computer programming. In particular the problem of dynamic programming related to multistage process reduces to the problem of solving functional equations

$$p(x) = \sup_{y\in D}\{g(x,y) + G_1(x, y, p(\xi(x,y)))\}, \text{ for } x \in W, \tag{2.4}$$

$$q(x) = \sup_{y\in D}\{g'(x,y) + G_2(x, y, q(\xi(x,y)))\}, \text{ for } x \in W, \tag{2.5}$$

where U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$ and

$$\begin{aligned} \xi : W \times D &\longrightarrow W, \\ g, g' : W \times D &\longrightarrow \mathbb{R}, \\ G_1, G_2 : W \times D \times \mathbb{R} &\longrightarrow \mathbb{R}, \end{aligned}$$

for more details on dynamic programming we refer to [7–10, 23]. Suppose that W and D are the state and decision spaces respectively. We aim to give the existence and uniqueness of common and bounded solution of functional equations given in (2.4) and (2.5). Let $B(W)$ denotes the set of all bounded real valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x\in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric d defined as

$$d(h,k) = \sup_{x\in W} |hx - kx|. \tag{2.6}$$

Suppose that the following conditions hold:
$(C1)$: G_1, G_2, g, and g' are bounded.
$(C2)$: For $x \in W$, $h \in B(W)$ and $b > 0$, define

$$Kh(x) = \sup_{y\in D}\{g(x,y) + G_1(x,y,h(\xi(x,y)))\}, \tag{2.7}$$
$$Jh(x) = \sup_{y\in D}\{g'(x,y) + G_2(x,y,h(\xi(x,y)))\}. \tag{2.8}$$

Moreover assume that $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ and $L \geq 0$ such that for every $(x,y) \in W \times D$, $h,k \in B(W)$ and $t \in W$ implies

$$|G_1(x,y,h(t)) - G_1(x,y,k(t))| \leq e^{-\tau(t)}[M(h(t),k(t))N(h(t),k(t))], \tag{2.9}$$

where

$$M((h(t),k(t)) =$$
$$\max\{d(Jh(t),Jk(t)), d(Jk(t),Kk(t)), d(Jh(t),Kh(t)), \frac{d(Jh(t),Kk(t))+d(Jk(t),Kh(t))}{2}\},$$
$$N((h(t),k(t)) =$$
$$\frac{\max\{d(Jh(t),Jk(t)), d(Jk(t),Kk(t))+d(Jh(t),Kh(t)), d(Jh(t),Kk(t))+d(Jk(t),Kh(t))\}}{d(Jk(t),Kk(t)) + d(Jh(t),Kh(t)) + 1}.$$

$(C3)$: For any $h \in B(W)$, there exists $k \in B(W)$ such that for $x \in W$

$$Kh(x) = Jk(x).$$

$(C4)$: There exists $h \in B(W)$ such that

$$Kh(x) = Jh(x) \text{ implies that } JKh(x) = KJh(x).$$

Theorem 2.2. *Assume that the conditions $(C1)$–$(C4)$ are satisfied. If $J(B(W))$ is a closed convex subspace of $B(W)$, then the functional equations (2.4) and (2.5) have a unique, common and bounded solution.*

Proof. Note that $(B(W), d)$ is a complete metric space. By $(C1)$, J, K are self-maps of $B(W)$. The condition $(C3)$ implies that $K(B(W)) \subseteq J(B(W))$. It follows from $(C4)$ that J and K commute at their coincidence points. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Choose $x \in W$ and $y_1, y_2 \in D$ such that

$$Kh_j < g(x,y_j) + G_1(x,y_j,h_j(x_j) + \lambda, \tag{2.10}$$

where $x_j = \xi(x,y_j)$, $j = 1,2$. Further from (2.7) and (2.8), we have

$$Kh_1 \geq g(x,y_2) + G_1(x,y_2,h_1(x_2)), \tag{2.11}$$
$$Kh_2 \geq g(x,y_1) + G_1(x,y_1,h_2(x_1)). \tag{2.12}$$

Then (2.10) and (2.12) together with (2.9) imply

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2)) + \lambda \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| + \lambda \\ &\leq e^{-\tau(t)}(M((h(t), k(t))N(h(t), k(t))) + \lambda. \end{aligned} \tag{2.13}$$

Then (2.10) and (2.11) together with (2.9) imply

$$\begin{aligned} Kh_2(x) - Kh_1(x) &\leq G_1(x, y_1, h_2(x_2)) - G_1(x, y_1, h_1(x_1)) \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| \\ &\leq e^{-\tau(t)}(M((h(t), k(t))N(h(t), k(t))). \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we have

$$|Kh_1(x) - Kh_2(x)| \leq e^{-\tau(t)}(M((h(t), k(t))N(h(t), k(t))). \tag{2.15}$$

The inequality (2.15) implies

$$d(Kh_1(x) - Kh_2(x)) \leq e^{-\tau(t)}[(M((h(t), k(t))N(h(t), k(t)))], \tag{2.16}$$

$$\tau(t) + \ln[d(Kh_1(x) - Kh_2(x))] \leq \ln[(M((h(t), k(t))N(h(t), k(t)))]. \tag{2.17}$$

Therefore by Theorem 2.1, the pair (K, J) has a common fixed point h^*, that is, $h^*(x)$ is unique, bounded and common solution of (2.4) and (2.5). □

(2) Application to the system of integral equations:

Now we discuss an application of fixed point theorem we proved in the previous section in solving the system of Volterra type integral equations. Such system is given by the following equations:

$$u(t) = \int_0^t K_1(t, s, u(s))ds + g(t), \tag{2.18}$$

$$w(t) = \int_0^t K_2(t, s, w(s))ds + f(t), \tag{2.19}$$

for $t \in [0, a]$, where $a > 0$. We find the solution of the system (2.18) and (2.19). Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau(t)t}\}$, where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is taken as a function. Let $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t\in[0,a]} \| \, |u(t) - v(t)| \, e^{-\tau(t)t} \|_\tau \tag{2.20}$$

for all $u, v \in C([0, a], \mathbb{R})$. With these setting $C([0, a], \mathbb{R}, \| \cdot \|_\tau)$ becomes Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations. For more details on such applications we refer the reader to [6, 21, 22].

Theorem 2.3. *Assume the following conditions are satisfied:*

(i) $K_1, K_2 : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ *and* $f, g : [0, a] \to \mathbb{R}$ *are continuous;*

(ii) Define

$$Tu(t) = \int_0^t K_1(t, s, u(s))ds + g(t),$$
$$Su(t) = \int_0^t K_2(t, s, u(s))ds + f(t).$$

Suppose there exist $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $L \geq 0$*such that*

$$|K_1(t, s, u) - K_1(t, s, v)| \leq \tau(t)e^{-\tau(t)}[M(u, v)N(u, v)]$$

for all $t, s \in [0, a]$ *and* $u, v \in C([0, a], \mathbb{R})$, *where*

$$M(u, v) = \max\{|Su(t)-Sv(t)|, |Sv(t)-Tv(t)|, |Su(t)-Tu(t)|, \frac{|Su(t)-Tv(t)|+|Sv(t)-Tu(t)|}{2}\},$$

$$N(u, v) = \frac{\max\{|Su(t)-Sv(t)|, |Sv(t)-Tv(t)|+|Su(t)-Tu(t)|, |Su(t)-Tv(t)|+|Sv(t)-Tu(t)|\}}{|Sv(t) - Tv(t)| + |Su(t) - Tu(t)| + 1};$$

(iii) there exists $u \in C([0, a], \mathbb{R})$ *such that* $Tu(t) = Su(t)$ *implies* $TSu(t) = STu(t)$. *Then the system of integral equations given in (2.18) and (2.19) has a solution.*

Proof. By assumption (iii)

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^t |K_1(t, s, u(s) - K_1(t, s, v(s)))|\, ds \\ &\leq \int_0^t \tau(t)e^{-\tau(t)}([M(u, v)N(u, v)]e^{-\tau(t)s})e^{\tau(t)s}ds \\ &\leq \int_0^t \tau(t)e^{-\tau(t)}\|M(u, v)N(u, v)\|_\tau e^{\tau(t)s}ds \\ &\leq \tau(t)e^{-\tau(t)}\|M(u, v)N(u, v)\|_\tau \int_0^t e^{\tau(t)s}ds \\ &\leq \tau(t)e^{-\tau(t)}\|M(u, v)N(u, v)\|_\tau \frac{1}{\tau(t)}e^{\tau(t)t} \\ &\leq e^{-\tau(t)}\|M(u, v)N(u, v)\|_\tau e^{\tau(t)t}. \end{aligned}$$

This implies

$$|Tu(t) - Tv(t)|\, e^{-\tau(t)t} \leq e^{-\tau(t)}\|M(u, v)N(u, v)\|_\tau.$$

That is

$$\|Tu(t) - Tv(t)\|_\tau \leq e^{-\tau(t)}\|M(u, v)N(u, v)\|_\tau,$$

which further implies

$$\tau(t) + \ln \|Tu(t) - Tv(t)\|_\tau \leq \ln \|M(u, v)N(u, v)\|_\tau.$$

So all the conditions of Theorem 2.1 are satisfied. Hence the system of integral equations given in (2.18) and (2.19) has a unique common solution. $\square$

Acknowledgements

The authors sincerely thank the learned referees for a careful reading and thoughtful comments. The present version of the paper owes much to the precise and kind remarks of anonymous referees.

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