Global finite-time synchronization of different dimensional chaotic systems

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Abstract

This paper addresses the problem of global finite-time synchronization of two different dimensional chaotic systems. Firstly, the definition of global finite-time synchronization of different dimensional chaotic systems are introduced. Based on the finite-time stability methods, the controller is designed such that the chaotic systems are globally synchronized in a finite time. Then, some uncertain parameters are adopted in the chaotic systems, new control law and dynamical parameter estimation are proposed to guarantee that the global finite-time synchronization can be obtained. By considering a dynamical parameter designed in the controller, the adaptive updated controller is also designed to achieve the desired results. At last, the results of two different dimensional chaotic systems are also extended to two different dimensional networked chaotic systems. Finally, three numerical examples are given to verify the validity of the proposed methods.

Key Words: finite-time synchronization, chaotic systems, different orders, uncertain parameters.

I. Introduction

In the past few decades, the problem of synchronization for chaotic systems has attracted much attention in [1], [2], [3], [4], [5], [6]. Since the authors have studied the problem of synchronization in 1990 [1], synchronization between two chaotic systems has been applied into many applications, such as secure communication, neural network, biology and information processing [7], [8], [9], [10], [11]. Up to now, there exist some different types of synchronization, for instance, complete synchronization [12], cluster synchronization [13], [14], projective synchronization [15], phase synchronization [16], anti-synchronization [17], generalized synchronization [18].

It should be noticed that the synchronization problem is almost studied between two identical or similar systems. To the best knowledge of the authors, there exist some natural behaviors modeling with different orders in our life. For instance, the synchronization of heart and lung has been studied in [19]. Thus, it is interesting to investigate the synchronization of two systems with different orders. Up to present, some interesting results have been obtained. For example, the synchronization between two different dimensional systems is considered based on the secure communication [20], [21]. By using reduced order of the drive system, the synchronization between two different dimensional systems is achieved in [22], [23]. The generalized synchronization of two different dimensional systems are studied in [18], [24], [25], [26]. In [18], a Lyapunov approach is introduced to attain the desired results.

In practice, the parameters of the chaotic systems sometimes can not be obtained exactly for the different kinds of external disturbance and noises. Thus, the synchronization problem with unknown parameters is becoming essential for the practical applications. Some relevant results have been investigated in [22], [23], [27], [28], [29]. In [23], an adaptive control scheme is proposed to attain the full and reduced order synchronization with uncertain parameters. In [29], based on the finite-time stability methods [30], faster synchronization of two chaotic systems is achieved in a finite time. In addition, the finite-time techniques have advantages to enhance the robustness and to overcome the disturbance [31]. Some more results about finite-time synchronization can be found in [29], [32], [33], [34]. It should be noted that the systems considered are either of some specific forms or analyzed without considering the unknown parameters. Then, by taking unknown model uncertainties and external disturbances into account, global finite-time synchronization of two chaotic systems with different orders is investigated in [35]. Moreover, a more general derivative of the Lyapunov function is derived to design finite-time observer for a class of nonlinear systems in [37], [38]. Based on the aforementioned reasons, it is important to study the finite-time synchronization of different dimensional chaotic systems with unknown parameters.

In this paper, global finite-time synchronization of two different dimensional chaotic systems is studied. Based on the finite-time stability, the definition of global finite-time synchronization is given first. Then, the synchronization schemes are derived with and without considering the unknown parameters, respectively. At last, the adaptive updated law is taken into consideration, the controller is designed to achieve the desired results. The main contributions of this paper include: i) The synchronization schemes of two different dimensional systems are proposed to achieve global finite-time synchronization with and without considering the unknown parameters, respectively. ii) The adaptive updated law of parameters is also proposed to guarantee the stability. iii) finite-time synchronization of two networked chaotic systems with different order is first considered in this paper.

This paper is organized as follows. In Section II, the problem statement and some definitions and lemmas are introduced. In Section III, global finite-time synchronization of two different dimensional systems is derived. Then, some unknown parameters are taken into account, new controller is designed based on the estimations of unknown parameters. In Section IV, two examples are provided to verify the validity of the our proposed design methods. At last, Section V concludes the paper.

II. Preliminaries

Let $\mathbb{R}^n$ denote n-dimension real space and $\mathbb{R}^+$ denote 1-dimension positive real space. For any $x \in \mathbb{R}^n$, let $\|x\| = (x^T x)^{1/2}$. Superscript "T" denotes the transpose of a vector or matrix. For a matrix $A \in \mathbb{R}^{n\times n}$ ($\in \mathbb{R}^{n\times m}$), A^{-1} denotes the inverse matrix of A (the right inverse matrix of A).

Consider the following drive system:

$$\dot{x}(t) = F(x(t)), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with $F(0) = 0$. The controlled response system is given by

$$\dot{y}(t) = G(y(t)) + u(t), \tag{2}$$

where $y(t) \in \mathbb{R}^m$ denotes the system state, $u(t) \in \mathbb{R}^m$ denotes the control input, $G : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function.

Remark 1: When $m = n$, (1) and (2) are chaotic systems with same order. Lots of relevant works about synchronization and finite-time synchronization have been studied in [29], [32], [33], [34]. When $m \neq n$, the generalized synchronization of (1) and (2) with different order is studied in [18], [24], [27]. There are few results about the finite-time synchronization of chaotic systems with different order. Therefore, it is essential to give some theoretical results.

Remark 2: When $F(x(t)) = Ax + f(x)$, $G(y(t)) = By + g(y)$, the finite-time synchronization of (1) and (2) is investigated in [27]. By taking into unknown system parameters account, then, $F(x(t)) = Ax + f(x) + f_1(x)\Phi$, $G(y(t)) = By + g(y) + g_1(y)\Psi$, where Φ, and Ψ are the vector of unknown parameters. When $A = 0$ and $B = 0$, finite-time synchronization of (1) and (2) is considered in [28] with unknown parameters. In summary, systems (1) and (2) have an more general form of the existing results. In addition, it should be noticed that lots of well-known chaotic systems could be written as the form of (1) and (2), such as, Lorenz system, Chen system, Rössler system, Duffing system, hyperchaotic Lozenz system and hyperchaotic Rössler system.

In this paper, our aim is to design the controller $u(t)$ to guarantee that different dimensional systems (1) and (2) are synchronized in a finite time. Now, we give the following definition of finite-time synchronization of systems (1) and (2).

Definition 1: [28] Consider systems (1) and (2). If there exist two continuously differentiable functions $\phi : \mathbb{R}^n \to \mathbb{R}^r$ and $\varphi : \mathbb{R}^m \to \mathbb{R}^r$, an open neighborhood $\mathcal{U} \subset \mathbb{R}^r$ of the origin such that $e_0 = \phi(x_0) - \varphi(y_0) \in \mathcal{U}$ and a function $\mathcal{T} : \mathcal{U} \setminus \{0\} \to (0, +\infty)$ and

$$\begin{aligned} &\lim_{t \to \mathcal{T}(e_0)} \|e(t)\| = \lim_{t \to \mathcal{T}(e_0)} \|\phi(x) - \varphi(y)\| \to 0, \\ &\|e(t)\| \equiv 0, t > \mathcal{T}(e_0), \end{aligned} \tag{3}$$

where $e(t) \in \mathbb{R}^r$ denotes the synchronization error of different dimensional systems (1) and (2). Then, systems (1) and (2) are said to be finite-time synchronization. If the open set $\mathcal{U}$ can be selected as $\mathbb{R}^r$, systems (1) and (2) are said to be global finite-time synchronization.

The following lemmas are useful for our main results.

Lemma 1: [36] For any real numbers α_i, $i = 1, 2, \cdots, n$ and $d \in (0, 1)$, the following inequality holds:

$$(|\alpha_1| + |\alpha_2| \cdots + |\alpha_n|)^d \leq |\alpha_1|^d + |\alpha_2|^d \cdots + |\alpha_n|^d. \tag{4}$$

Lemma 2: [37] Assume that there is a Lyapunov function $V(e(t))$ defined on a neighborhood $\mathcal{U} \subset \mathbb{R}^r$ of the origin, and it satisfies the following differential inequality

$$\dot{V}(e(t)) \leq -kV^{\eta}(e(t)) - pV(e(t)), \ \forall t \geq t_0, \tag{5}$$

where $k > 0$, $0 < \eta < 1$, and $p > 0$ are three constants. Then, for any given t_0, the following inequality

$$V^{1-\eta}(t) \leq V^{1-\eta}(t_0)e^{-(1-\eta)p(t-t_0)} - \frac{k}{p}[1 - e^{-(1-\eta)p(t-t_0)}], \ t_0 \leq t \leq T',$$

holds and $V(t) \equiv 0$, $\forall t > T'$, T' denotes the settling time, given by $T' = t_0 + \frac{\ln(1+(p/k)V(e_0)^{1-\eta})}{(1-\eta)p}$.

Remark 3: When $p = 0$, $\forall t \geq t_0$, $\dot{V}(e(t)) \leq -kV^{\eta}(e(t))$, the global finite-time stability has been studied in [30] with the settling time $T'' = t_0 + \frac{V^{1-\eta}(t_0)}{k(1-\eta)}$. It is easy to verify $\lim_{p \to 0} T' = T''$. Thus, inequality (5) can be applied with a wider range of forms.

For instance, finite-time observers have been studied for triangular nonlinear systems in [37], [38]. In [39], global finite-time observers are designed for a class of Lipschtiz nonlinear systems.

Remark 4: In [27], Lemma 2 is used to prove the finite-time stability with $p = 0$. By comparing the settling time T'' in [27] and T' in this paper, we have that $T' - T'' = \frac{\ln(1+(p/k)V(t_0)^{1-\eta})}{p(1-\eta)} - \frac{V^{1-\eta}(t_0)}{k(1-\eta)} = \frac{1}{1-\eta}(\frac{\ln(1+(p/k)V(t_0)^{1-\eta})}{p} - \frac{V^{1-\eta}(t_0)}{k}) \leq 0$ and $T' - T'' = 0$ if and only if $V(t_0) = 0$. Thus, by using (5), a faster convergence performance will be achieved. In this paper, our aim is to construct an appropriate controller to satisfy inequality (5).

III. Main results

In this section, our main results include three parts: Global finite-time synchronization of different dimensional chaotic systems is studied in subsection A. Then, some uncertain parameters are considered in the chaotic systems (1) and (2), sufficient conditions are also derived to guarantee global finite-time synchronization in subsection B. To the end, the adaptive updated laws of parameters designed in controllers are adopted, and some new sufficient conditions are given to achieve the desired results. In subsection C, finite-time synchronization of two chaotic systems is extended to two networked chaotic systems, the corresponding controller is designed to guarantee the stability.

A. Global finite-time synchronization of chaotic systems

Consider chaotic systems (1) and (2), define $e(t) = \phi(x) - \varphi(y)$, the error dynamics are given by

$$\dot{e} = J_\phi(x)\dot{x} - J_\varphi(y)\dot{y} = J_\phi(x)F(x) - J_\varphi(y)(G(y) + u(t)), \tag{6}$$

where $J_\phi(x)$ and $J_\varphi(y)$ denote the Jacobin matrices of functions $\phi(x)$ and $\varphi(y)$, respectively. Particularly,

$$J_\phi(x) = \begin{pmatrix} \frac{\partial\phi_1(x)}{\partial x_1} & \frac{\partial\phi_1(x)}{\partial x_2} & \cdots & \frac{\partial\phi_1(x)}{\partial x_n} \\ \frac{\partial\phi_2(x)}{\partial x_1} & \frac{\partial\phi_2(x)}{\partial x_2} & \cdots & \frac{\partial\phi_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_r(x)}{\partial x_1} & \frac{\partial\phi_r(x)}{\partial x_2} & \cdots & \frac{\partial\phi_r(x)}{\partial x_n} \end{pmatrix},$$

$$J_\varphi(y) = \begin{pmatrix} \frac{\partial\varphi_1(y)}{\partial y_1} & \frac{\partial\varphi_1(y)}{\partial y_2} & \cdots & \frac{\partial\varphi_1(y)}{\partial y_m} \\ \frac{\partial\varphi_2(y)}{\partial y_1} & \frac{\partial\varphi_2(y)}{\partial y_2} & \cdots & \frac{\partial\varphi_2(y)}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi_r(y)}{\partial y_1} & \frac{\partial\varphi_r(y)}{\partial y_2} & \cdots & \frac{\partial\varphi_r(y)}{\partial y_m} \end{pmatrix}.$$

Assumption 1: $r \leq \min\{m, n\}$ and matrix J_φ is row full-rank.

In order to achieve global finite-time synchronization of systems (1) and (2), the following controller is proposed

$$u(t) = -G(y) + J_\varphi^{-1}(y)(J_\phi(x)F(x) + k \cdot \eta(\alpha, e(t)) + pe(t)), \tag{7}$$

where $\eta(\alpha, e(t)) = [sign(e_1(t))|e_1(t)|^\alpha, \cdots, sign(e_r(t))|e_r(t)|^\alpha]^T$, $\alpha \in (0, 1]$ are constants, $k > 0$, $p > 0$, and $e(t) = (e_1(t), e_2(t), \cdots, e_r(t))^T$ and $sign(\cdot)$ denotes standard the sign function.

Remark 5: In [27], the parameter $\alpha = m_1/m_2$ is a rational number, where m_1 and m_2 are two positive odd integers and $m_2 > m_1$. This is a very strong restriction of parameter selection. In this paper, α can be selected in $(0, 1]$ arbitrarily.

Substituting (7) into (6), we have

$$\dot{e}(t) = -k \cdot \eta(\alpha, e(t)) - pe(t). \tag{8}$$

Now we give our main results.

Theorem 1: Consider chaotic systems (1) and (2) under Assumption 1, if the controller $u(t)$ is designed in the form of (7), then the error dynamics (6) are global finite-time stable at the origin. We can also say that systems (1) and (2) are global finite-time synchronized with the settling time, given by $T' = t_0 + \frac{\ln(1+(p/k)V(e_0)^{(1-\alpha)/2})}{(1-\alpha)p}$.

Proof: Consider the following positive definite function

$$V_\alpha(t) = e^T(t)e(t). \tag{9}$$

Then, the derivative of $V_\alpha(t)$ along (8) can be calculated as

$$\dot{V}_\alpha(t) = 2e^T(t)\dot{e}(t) = -2ke^T(t)\eta(\alpha, e(t)) - 2pe^T(t)e(t). \tag{10}$$

When $\alpha = 1$, the error system can be rewritten as

$$\dot{e}(t) = -ke(t) - pe(t). \tag{11}$$

The derivative of $V_\alpha(t)$ along the system (11), we have

$$\frac{dV_\alpha(t)}{dt}|_{(11)} = -2(k + p)e^T(t)e(t) < 0,$$

which implies the system (11) is globally asymptotically stable. Moreover, for the continuity of α, there exists $\varepsilon > 0$ such that $\frac{dV_\alpha(t)}{dt}|_{(8)} < 0$ for $\alpha \in (1 - \varepsilon, 1)$. More details can be found in [32], [37]. Next, will prove the global finite-time stability of system (8). From (10), by Lemma 1, it follows that

$$\begin{aligned}
&\dot{V}_\alpha(t) = -2ke^T(t) \cdot sign(e)|e(t)|^\alpha - 2pe^T(t)e(t) \\
&= -2k \textstyle\sum_{i=1}^{r} |e_i(t)|^{\alpha+1} - 2pe^T(t)e(t) \\
&\leq -2k(e^T(t)e(t))^{\frac{\alpha+1}{2}} - 2pe^T(t)e(t) \\
&\leq -2kV_\alpha(t)^{\frac{\alpha+1}{2}} - 2pV_\alpha(t).
\end{aligned} \tag{12}$$

By Lemma 2, the error dynamics (8) are globally finite-time stable with $T' = t_0 + \frac{\ln(1+(p/k)V(e_0)^{(1-\alpha)/2})}{(1-\alpha)p}$. The proof is completed here.

B. Global finite-time synchronization of chaotic systems with unknown parameters

For chaotic systems (1) and (2), if there exist some unknown parameters in them, then, they can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= F_1(x(t)) + f(x)\Phi, \\
\dot{y}(t) &= G_1(y(t)) + g(y)\Psi + u(t),
\end{aligned} \tag{13}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $F_1 : \mathbb{R}^n \to \mathbb{R}^n$, $G_1 : \mathbb{R}^m \to \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $g : \mathbb{R}^m \to \mathbb{R}^{m\times q}$ are continuous functions, $\Phi \in \mathbb{R}^p$ and $\Psi \in \mathbb{R}^q$ denote the vector of unknown parameters. The controller (7) can not be used again for the existence of unknown parameters. Thus, the following adaptive law is proposed to estimate the unknown parameters Φ and Ψ:

$$\begin{aligned}&\dot{\hat{\Phi}} = (J_\phi(x)\cdot f(x))^T e(t) - k\cdot sign(\tilde{\Phi})|\tilde{\Phi}|^\eta - p\tilde{\Phi},\\&\dot{\hat{\Psi}} = -(J_\varphi(y)\cdot g(y))^T e(t) - k\cdot sign(\tilde{\Psi})|\tilde{\Psi}|^\eta - p\tilde{\Psi},\end{aligned}\tag{14}$$

where $\hat{\Phi}$, $\hat{\Psi}$ denote the estimations of Φ and Ψ, $\tilde{\Phi} = \hat{\Phi} - \Phi$, $\tilde{\Psi} = \hat{\Psi} - \Psi$ and $\eta \in (0, 1]$, $k > 0$, $p > 0$.

Now, based on (14), the controller $u(t)$ can be designed as

$$u(t) = -G_1(y) - g(y)\hat{\Psi} + J_\varphi^{-1}(y)[J_\phi(x)(F_1(x) + f(x)\hat{\Phi}) + k\cdot sign(e)|e|^\eta + pe(t)].\tag{15}$$

Then, the error dynamics are derived

$$\begin{aligned}&\dot{e}(t) = J_\phi(x)(F_1(x) + f(x)\Phi) - J_\varphi(y)(G_1(y) + g(y)\Psi + u)\\&= -J_\phi(x)f(x)\tilde{\Phi} + J_\varphi(y)g(y)\tilde{\Psi} - k\cdot sign(e)|e|^\eta - pe(t).\end{aligned}\tag{16}$$

Now, we give our main results.

Theorem 2: Consider chaotic systems (13) and (14) under Assumption 1, if the controller $u(t)$ is designed in the form of (15), then the error dynamics (16) are global finite-time stable at the origin. We can also say that systems (13) and (14) are global finite-time synchronized with the settling time, given by $T' = t_0 + \frac{\ln(1+(p/k)V(e_0)^{(1-\eta)/2})}{(1-\eta)p}$.

Proof: Consider the following positive definite function

$$V_\eta(t) = e(t)^T e(t) + \tilde{\Phi}^T\tilde{\Phi} + \tilde{\Psi}^T\tilde{\Psi}.\tag{17}$$

Then, the derivative of $V_\eta(t)$ along (16) can be calculated as

$$\begin{aligned}&\dot{V}_\eta(t) = 2e^T(t)\dot{e}(t) + 2\tilde{\Phi}^T\dot{\tilde{\Phi}} + 2\tilde{\Psi}^T\dot{\tilde{\Psi}} = 2e^T(t)\dot{e}(t) + 2\tilde{\Phi}^T\dot{\hat{\Phi}} + 2\tilde{\Psi}^T\dot{\hat{\Psi}}\\&= -2e^T(t)J_\phi(x)f(x)\tilde{\Phi} + 2e^T(t)J_\varphi(y)g(y)\tilde{\Psi} - 2ke^T(t)\cdot sign(e)|e|^\eta - 2pe^T(t)e(t)\\&+2\tilde{\Phi}^T(J_\phi(x)\cdot f(x))^T e(t) - 2k\tilde{\Phi}^T\cdot sign(\tilde{\Phi})|\tilde{\Phi}|^\eta - 2p\tilde{\Phi}^T\tilde{\Phi}\\&-2\tilde{\Psi}^T(J_\varphi(y)\cdot g(y))^T e(t) - 2k\tilde{\Psi}^T\cdot sign(\tilde{\Psi})|\tilde{\Psi}|^\eta - 2p\tilde{\Psi}^T\tilde{\Psi}\\&= -2ke^T(t)\cdot sign(e)|e|^\eta - 2k\tilde{\Phi}^T\cdot sign(\tilde{\Phi})|\tilde{\Phi}|^\eta - 2k\tilde{\Psi}^T\cdot sign(\tilde{\Psi})|\tilde{\Psi}|^\eta\\&-2pe^T(t)e(t) - 2p\tilde{\Phi}^T\tilde{\Phi} - 2p\tilde{\Psi}^T\tilde{\Psi}.\end{aligned}\tag{18}$$

Similar proof to the Theorem 1, it is easy to obtain that the error system (16) is asymptotically stable. Moreover, we also have

$$\dot{V}_\eta(t) \le -2kV_\eta(t)^{\frac{\eta+1}{2}} - 2pV_\eta(t).$$

Then, the error system (16) is also globally finite-time stable with the settling time $T' = t_0 + \frac{\ln(1+(p/k)V(e_0)^{(1-\eta)/2})}{(1-\eta)p}$. The proof is completed.

For system (13), we can also design an adaptive controller as follows:

$$\begin{aligned}&\dot{\hat{\Phi}} = (J_\phi(x)\cdot f(x))^T e(t) - k\cdot sign(\tilde{\Phi})|\tilde{\Phi}|^\eta - \gamma(t)\tilde{\Phi},\\&\dot{\hat{\Psi}} = -(J_\varphi(y)\cdot g(y))^T e(t) - k\cdot sign(\tilde{\Psi})|\tilde{\Psi}|^\eta - \gamma(t)\tilde{\Psi},\\&u(t) = -G_1(y) - g(y)\hat{\Psi} + J_\varphi^{-1}(y)[J_\phi(x)(F_1(x) + f(x)\hat{\Phi}) + k\cdot sign(e)|e|^\eta + \gamma(t)e(t)],\\&\dot{\gamma}(t) = -e(t)^T e(t) - \tilde{\Phi}^T\tilde{\Phi} - \tilde{\Psi}^T\tilde{\Psi} - k\cdot sign(\gamma(t))|\gamma(t)|^\eta - p\gamma(t),\end{aligned}\tag{19}$$

where η, p, k are the same as (15).

Based on Theorem 2, we have the following Corollary.

Corollary 1: Consider chaotic systems (13) and (14) under Assumption 1, if the controller $u(t)$ is designed in the form of (19), then the error dynamics (16) are global finite-time stable at the origin. We can also say that systems (13) and (14) are global finite-time synchronized.

Proof: Consider the following positive definite function

$$V_\eta(t) = e(t)^T e(t) + \gamma(t)^2 + \tilde{\Phi}^T \tilde{\Phi} + \tilde{\Psi}^T \tilde{\Psi}. \tag{20}$$

Then, the derivative of $V_\eta(t)$ along (16) can be calculated as

$$\begin{aligned}
\dot{V}_\eta(t) &= 2e^T(t)\dot{e}(t) + 2\gamma(t)\dot{\gamma(t)} + 2\tilde{\Phi}^T\dot{\tilde{\Phi}} + 2\tilde{\Psi}^T\dot{\tilde{\Psi}} \\
&= -2ke^T(t)\cdot sign(e)|e|^\eta - 2k\tilde{\Phi}^T\cdot sign(\tilde{\Phi})|\tilde{\Phi}|^\eta - 2k\tilde{\Psi}^T\cdot sign(\tilde{\Psi})|\tilde{\Psi}|^\eta - 2\gamma(t)e^T(t)e(t) \\
&\quad -2\gamma(t)\tilde{\Phi}^T\tilde{\Phi} - 2\gamma(t)\tilde{\Psi}^T\tilde{\Psi} + 2\gamma(t)e(t)^Te(t) + 2\gamma(t)\tilde{\Phi}^T\tilde{\Phi} + 2\gamma(t)\tilde{\Psi}^T\tilde{\Psi} + 2k\gamma(t)\cdot sign(\gamma(t))|\gamma(t)|^\eta - 2p\gamma(t)^2 \\
&= -2ke^T(t)\cdot sign(e)|e|^\eta - 2k\tilde{\Phi}^T\cdot sign(\tilde{\Phi})|\tilde{\Phi}|^\eta - 2k\tilde{\Psi}^T\cdot sign(\tilde{\Psi})|\tilde{\Psi}|^\eta \\
&\quad -2k\gamma(t)\cdot sign(\gamma(t))|\gamma(t)|^\eta - 2p\gamma(t)^2 \\
&\leq -2kV_\eta(t)^{\frac{\eta+1}{2}} - 2pV_\eta(t).
\end{aligned} \tag{21}$$

The rest proof is similar to Theorem 2, it is omitted here. The proof is completed.

Remark 6: If $m = n$, let $\phi(x(t)) = x(t)$, $\varphi(y(t)) = y(t)$, the error dynamics are $e(t) = x(t) - y(t)$, then, the it can be transformed as complete synchronization problem. Obviously, it is a special case of our proposed design methods.

Remark 7: In [27], finite-time synchronization of two chaotic systems with different order is studied, and the relation between the design parameter, the initial state of system and the convergent time are also discussed. Moreover, the authors investigated the finite-time synchronization with the error dynamics $e(t) = y(t) - \phi(x(t))$ without considering the unknown parameters. Then, by considering the unknown parameters, finite-time synchronization with the error dynamics $e(t) = \phi(x(t)) - \varphi(y(t))$ is attained in [28]. It should be noticed that the parameters designed in controller must be the proportion of two odd integers in both [27] and [28]. In this paper, the restriction on the parameter is removed. Moreover, the adaptive updated law is proposed to estimate the parameter, which is designed in the controller.

C. Global finite-time synchronization of networked different dimensional chaotic systems

In this subsection, finite-time synchronization of two different dimensional chaotic systems is extended to finite-time synchronization of networked chaotic systems. Consider the following dynamical networks consisting of N identical nodes:

$$\dot{x}_i(t) = f_i(x_i(t)) + \textstyle\sum_{j=1}^N a_{ij}x_j(t), i = 1, 2\cdots, N, \tag{22}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t),]^T \in \mathbb{R}^n$ dentes the state vector of ith node, The abbreviation $x(t) = [x_1(t)^T, \cdots, x_N(t)^T]^T$, $f(x) := [f_1(x_1), \cdots, f_N(x_N)]^T$, where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ denotes the weight configuration matrix. If there is a connection between nodes i and j ($i \neq j$), then $a_{ij} = a_{ji} > 0$; otherwise, $a_{ij} = a_{ji} = 0$. The diagonal elements of matrices A are defined as

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}. \tag{23}$$

For simplicity, the drive system (22) can be written in following form:

$$\dot{x}(t) = f(x(t)) + Ax(t) = F(x). \tag{24}$$

The response system can be given as follows:

$$\dot{y}(t) = g(y(t)) + Ay(t) + u(t) = G(y) + u(t), \tag{25}$$

where $y(t) = [y_1(t)^T, \cdots, y_N(t)^T]^T$, $y_i(t) = [y_{i1}(t), y_{i2}(t), \cdots, y_{im}(t)]^T \in \mathbb{R}^m$ denotes the response state vector of $x_i(t)$ of the ith node. $u(t) = [u_1(t)^T, \cdots, u_N(t)^T]^T$, $u_i(t) \in \mathbb{R}^m$ denotes the control input of the ith node. $g(\cdot) := [g_1(\cdot), \cdots, g_N(\cdot)]^T$, where $g_i : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function.

In view of (24)- (25), the synchronization error system can be obtained

$$\dot{e}(t) = J_\phi(x)\dot{x} - J_\varphi(y)\dot{y} = J_\phi(x)F(x) - J_\varphi(y)(G(y) + u(t)), \tag{26}$$

where $e(t) = [e_1(t)^T, e_2(t)^T, \cdots, e_N(t)^T]^T$, $e_i(t) = \phi(x_i(t)) - \varphi(y_i(t))$. $J_\phi(x) = \mathrm{diag}\{J_\phi(x_1), J_\phi(x_2), \cdots, J_\phi(x_N)\}$ and $J_\varphi(y) = \mathrm{diag}\{J_\varphi(y_1), J_\varphi(y_2), \cdots, J_\varphi(y_N)\}$, where $J_\phi(x_i)$ and $J_\varphi(y_i)$ denote the Jacobin matrices of functions $\phi(x_i)$ and $\varphi(y_i)$, respectively. Particularly,

$$J_\phi(x_i) = \begin{pmatrix} \frac{\partial\phi_1(x_i)}{\partial x_{i1}} & \frac{\partial\phi_1(x_i)}{\partial x_{i2}} & \cdots & \frac{\partial\phi_1(x_i)}{\partial x_{in}} \\ \frac{\partial\phi_2(x_i)}{\partial x_{i1}} & \frac{\partial\phi_2(x_i)}{\partial x_{i2}} & \cdots & \frac{\partial\phi_2(x_i)}{\partial x_{in}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_r(x_i)}{\partial x_{i1}} & \frac{\partial\phi_r(x_i)}{\partial x_{i2}} & \cdots & \frac{\partial\phi_r(x_i)}{\partial x_{in}} \end{pmatrix},$$

$$J_\varphi(y_i) = \begin{pmatrix} \frac{\partial\varphi_1(y_i)}{\partial y_{i1}} & \frac{\partial\varphi_1(y_i)}{\partial y_{i2}} & \cdots & \frac{\partial\varphi_1(y_i)}{\partial y_{im}} \\ \frac{\partial\varphi_2(y_i)}{\partial y_{i1}} & \frac{\partial\varphi_2(y_i)}{\partial y_{i2}} & \cdots & \frac{\partial\varphi_2(y_i)}{\partial y_{im}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi_r(y_i)}{\partial y_{i1}} & \frac{\partial\varphi_r(y_i)}{\partial y_{i2}} & \cdots & \frac{\partial\varphi_r(y_i)}{\partial y_{im}} \end{pmatrix}.$$

In order to achieve global finite-time synchronization of systems (24) and (25), the following controller is proposed

$$u(t) = -G(y) + J_\varphi^{-1}(y)(J_\phi(x)F(x) + k \cdot \eta(\alpha, e(t)) + pe(t)), \tag{27}$$

where $\eta(\alpha, e(t)) = [\eta(\alpha, e_1(t))^T, \cdots, \eta(\alpha, e_N(t))^T]^T$, $\eta(\alpha, e_i(t)) = [sign(e_{i1}(t))|e_{i1}(t)|^\alpha, \cdots, sign(e_{ir}(t))|e_{ir}(t)|^\alpha]^T$, $\alpha \in (0, 1]$ are constants, and $sign(\cdot)$ denotes standard the sign function, $k > 0$, $p > 0$.

Substituting (27) into (26), we have

$$\dot{e}(t) = -k \cdot \eta(\alpha, e(t)) - pe(t). \tag{28}$$

The main result is stated as follows.

Theorem 3: Consider chaotic systems (24) and (25) under Assumption 1, if the controller $u(t)$ is designed in the form of (27), then the error dynamics (28) are global finite-time stable at the origin. We can also say that systems (24) and (25) are global finite-time synchronized with the settling time, given by $T' = t_0 + \frac{\ln(1+(p/k)V(e_0)^{(1-\alpha)/2})}{(1-\alpha)p}$.

The rest of proof is similar to Theorem 1 and omitted here.

IV. Numerical simulations

In this section, we give two examples to verify the effectiveness of our proposed methods.

Example 1: Consider the following hyperchaotic Rössler system (20) and the Duffing system (30):

$$\begin{cases} \dot{x}_1(t) = -x_2(t) - x_3(t), \\ \dot{x}_2(t) = x_1(t) + ax_2(t) + x_4(t), \\ \dot{x}_3(t) = x_1(t)x_3(t) + b, \\ \dot{x}_4(t) = -cx_3(t) + dx_4(t), \end{cases} \tag{29}$$

and

$$\begin{cases} \dot{y}_1(t) = y_2(t) + u_1(t), \\ \dot{y}_2(t) = y_1(t) - y_1^3(t) - h_1 y_2(t) + h_2 \cos(t) + u_2(t), \end{cases} \tag{30}$$

where the system parameters are selected as $a = 0.25$, $b = 3$, $c = 0.5$, $d = 0.05$, $h_1 = 1$ and $h_2 = 0.8$ such that systems (29) and (30) are chaotic. Select (29) as the drive system and (30) as the controlled system ($n > m$). Assume that $\phi(x) = [x_1 + x_2 + x_3, x_4]^T$ and $\varphi(y) = [y_1, y_2]^T$. Then, we have $J_\phi(x) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $J_\varphi(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

From (6), the error dynamics can be obtained as

$$\begin{cases} \dot{e}_1(t) = -y_2(t) + x_1(t) - 0.75x_2(t) - x_3(t) + x_4(t) + x_1(t)x_3(t) + 3 - u_1(t), \\ \dot{e}_2(t) = -y_1(t) + y_1^3(t) + h_1 y_2(t) - h_2 \cos(t) - 0.5x_3(t) + 0.05x_4(t) - u_2(t). \end{cases} \tag{31}$$

By (7), the controller is given by

$$\begin{cases} u_1(t) = -y_2(t) + x_1(t) - 0.75x_2(t) - x_3(t) + x_4(t) + x_1(t)x_3(t) + 3 + k \cdot sign(e_1(t))|e_1(t)|^\alpha + pe_1(t), \\ u_2(t) = -y_1(t) + y_1^3(t) + h_1 y_2(t) - h_2 \cos(t) - 0.5x_3(t) + 0.05x_4(t) + k \cdot sign(e_2(t))|e_2(t)|^\alpha + pe_2(t). \end{cases} \tag{32}$$

In the simulations, the initial values of systems (29) and (30) are $(x_1(0), x_2(0), x_3(0), x_4(0)) = (1, 1, 1, 1)$, $(y_1(0), y_2(0)) = (2.5, 1.5)$. In order to show better convergence performance of our controller, a comparison is made with the controller designed in [27]. Based on the controller design (9) in [27], the following controller is given by

$$\begin{cases} u_1(t) = -y_2(t) + x_1(t) - 0.75x_2(t) - x_3(t) + x_4(t) + x_1(t)x_3(t) + 3 + e_1(t)^\beta + e_1(t), \\ u_2(t) = -y_1(t) + y_1^3(t) + h_1 y_2(t) - h_2 \cos(t) - 0.5x_3(t) + 0.05x_4(t) + e_2(t)^\beta + e_2(t), \end{cases} \tag{33}$$

where parameter $\beta > 0$ is a rational number. The trajectories of errors $e_i(t)$ by using controllers (32) and (33) are shown in Figure 1 with different parameters. When $\beta = 1/3$, by using the controller design (33) in [27], the trajectories of errors $e_1(t)$ and $e_2(t)$ are shown in Figure 1(a), where the settling time can be estimated by $T' = \frac{1}{1-\beta}||e(t_0)||^{1-\beta} = 1.1906$s. By comparing with this result, we select $k = 1$, $\alpha = 1/3$, $p = 1$ in our controller design (32), then, the trajectories of errors can be obtained shown in Figure 1(b) with the settling time $T' = \frac{\ln(1+(p/k)||e(t_0)||^{1-\alpha})}{(1-\alpha)p} = 0.8764$s. By fixing k and α, select $p = 4$, then a faster convergence performance is shown in Figure 1(c) with the settling time $T' = \frac{\ln(1+(p/k)||e(t_0)||^{1-\alpha})}{(1-\alpha)p} = 0.5359$s. By fixing p and α, select $k = 3$, we can also obtain a faster convergence performance shown in Figure 1(d) with the settling time $T' = \frac{\ln(1+(p/k)||e(t_0)||^{1-\alpha})}{(1-\alpha)p} = 0.3521$s. From the above comparisons between (33) and (32), it is easy to find that controller (32) has more advantages and shows a better convergence performance.

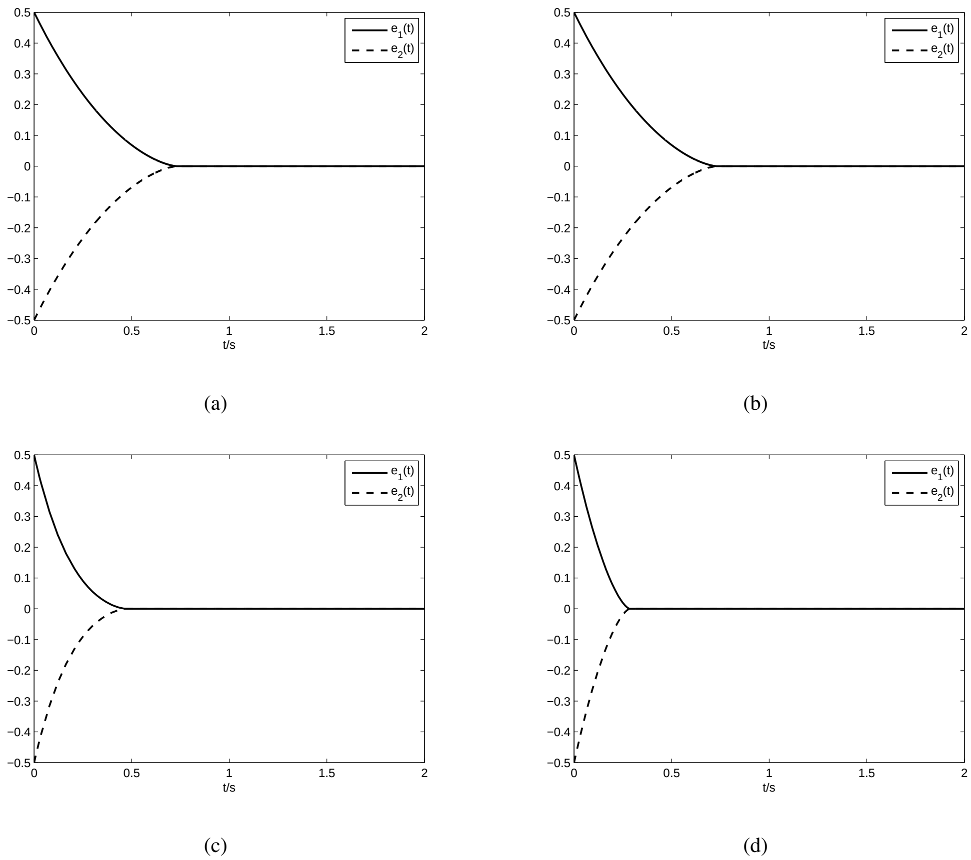


Fig. 1. Trajectories of the errors $e_i(t)$ $(i = 1, 2)$: (a) $\beta = 1/3$; (b) $k = 1$, $\alpha = 1/3$ and $p = 1$; (c) $k = 1$, $\alpha = 1/3$ and $p = 4$; (d) $k = 3$, $\alpha = 1/3$ and $p = 1$.

Example 2: Consider the Lorenz system as the drive system and the Chen hyperchaotic system as the controlled system. The dynamics are given as follows:

$$\begin{cases} \dot{x}_1(t) = a_1(x_2(t) - x_1(t)), \\ \dot{x}_2(t) = -x_2(t) - x_1(t)x_3(t) + a_2x_1(t), \\ \dot{x}_3(t) = x_1(t)x_2(t) - a_3x_3(t), \end{cases} \tag{34}$$

and

$$\begin{cases} \dot{y}_1(t) = b_1(y_2(t) - y_1(t)) + y_4(t) + u_1(t), \\ \dot{y}_2(t) = b_2y_1(t) - y_1(t)y_3(t) + b_3y_2(t) + u_2(t), \\ \dot{y}_3(t) = y_1(t)y_2(t) - b_4y_3(t) + u_3(t), \\ \dot{y}_4(t) = -y_2(t)y_3(t) + b_5y_4(t) + u_4(t), \end{cases} \tag{35}$$

where the parameters are selected as $a_1 = 10$, $a_2 = 28$, $a_3 = 8/3$, $b_1 = 35$, $b_2 = 7$, $b_3 = 12$, $b_4 = 3$ and $b_5 = 0.5$ to guarantee that systems (31) and (32) are chaotic. Assume that $\phi(x) = [x_1x_2, x_1 - x_3]^T$ and $\varphi(y) = [0.5y_1 - 0.5y_2, 0.5y_3 - 0.5y_4]^T$. Then, we have

$$J_\phi(x) = \begin{pmatrix} x_2 & x_1 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

$$J_\varphi(y) = \begin{pmatrix} 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 \end{pmatrix}.$$

Then, from (14) and (16), the updating laws and the error dynamics can be designed as

$$\begin{cases} \dot{\hat{a}}_1(t) = (x_2(t) - x_1(t))(x_2(t)e_1(t) + e_2(t)) - k \cdot sign(\tilde{a}_1)|\tilde{a}_1|^\eta - p\tilde{a}_1, \\ \dot{\hat{a}}_2(t) = x_1(t)^2 e_1(t) - k \cdot sign(\tilde{a}_2)|\tilde{a}_2|^\eta - p\tilde{a}_2, \\ \dot{\hat{a}}_3(t) = -x_3(t)e_2(t) - k \cdot sign(\tilde{a}_3)|\tilde{a}_3|^\eta - p\tilde{a}_3, \end{cases} \tag{36}$$

and

$$\begin{cases} \dot{\hat{b}}_1(t) = 0.5(y_1(t) - y_2(t))e_1(t) - k \cdot sign(\tilde{b}_1)|\tilde{b}_1|^\eta - p\tilde{b}_1, \\ \dot{\hat{b}}_2(t) = 0.5y_1(t)e_1(t) - k \cdot sign(\tilde{b}_2)|\tilde{b}_2|^\eta - p\tilde{b}_2, \\ \dot{\hat{b}}_3(t) = 0.5y_2(t)e_1(t) - k \cdot sign(\tilde{b}_3)|\tilde{b}_3|^\eta - p\tilde{b}_3, \\ \dot{\hat{b}}_4(t) = 0.5y_3(t)e_2(t) - k \cdot sign(\tilde{b}_4)|\tilde{b}_4|^\eta - p\tilde{b}_4, \\ \dot{\hat{b}}_5(t) = 0.5y_4(t)e_2(t) - k \cdot sign(\tilde{b}_5)|\tilde{b}_5|^\eta - p\tilde{b}_5, \end{cases} \tag{37}$$

and

$$\begin{cases} \dot{e}_1(t) = -x_2(t)(x_2(t) - x_1(t))\tilde{a}_1 - x_1(t)^2\tilde{a}_2 + 0.5(y_2(t) - y_1(t))\tilde{b}_1 - 0.5y_1(t)\tilde{b}_2 - 0.5y_2(t)\tilde{b}_3 - k \cdot sign(e_1(t))|e_1(t)|^\eta - pe_1(t), \\ \dot{e}_2(t) = -(x_2(t) - x_1(t))\tilde{a}_1 + x_3(t)\tilde{a}_3 - 0.5y_3(t)\tilde{b}_4 - 0.5y_4(t)\tilde{b}_5 - k \cdot sign(e_2(t))|e_2(t)|^\eta - pe_2(t). \end{cases} \tag{38}$$

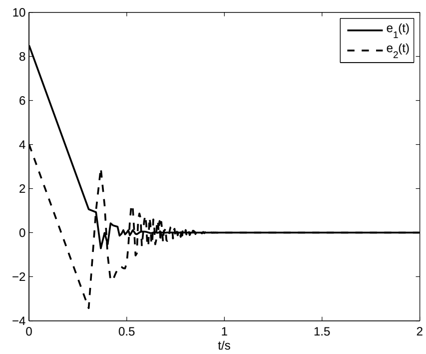
From (15), the controller can be designed as follows:

$$\begin{cases} u_1(t) = -\hat{b}_1(y_2(t) - y_1(t)) - y_4(t) + x_2(t) - x_1(t)x_2(t) - x_1(t)^2x_3(t) + x_2(t)(x_2(t) - x_1(t))\hat{a}_1 \\ +x_1(t)^2\hat{a}_2 + k \cdot sign(e_1(t))|e_1(t)|^\eta + pe_1(t), \\ u_2(t) = -\hat{b}_2y_1(t) + y_1(t)y_3(t) - \hat{b}_3y_2(t) - x_2(t) + x_1(t)x_2(t) + x_1(t)^2x_3(t) - x_2(t)(x_2(t) - x_1(t))\hat{a}_1 \\ -x_1(t)^2\hat{a}_2 - k \cdot sign(e_1(t))|e_1(t)|^\eta - pe_1(t), \\ u_3(t) = -y_1(t)y_2(t) + \hat{b}_4y_3(t) - x_1(t)x_2(t) + (x_2(t) - x_1(t))\hat{a}_1 - x_3(t)\hat{a}_3 + k \cdot sign(e_2(t))|e_2(t)|^\eta + pe_2(t), \\ u_4(t) = -y_2(t)y_3(t) - \hat{b}_5y_4(t) + x_1(t)x_2(t) - (x_2(t) - x_1(t))\hat{a}_1 + x_3(t)\hat{a}_3 - k \cdot sign(e_2(t))|e_2(t)|^\eta - pe_2(t), \end{cases} \tag{39}$$

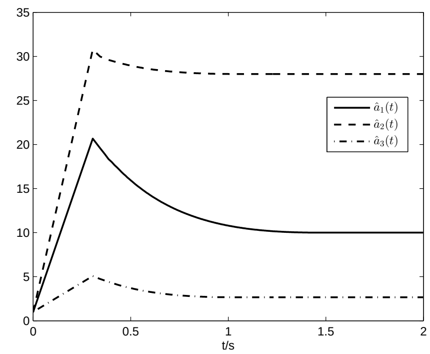
where the parameters are given as $k = 3$, $p = 2$ and $\eta = 0.5$. The initial values of systems (34)-(35) are $(x_1(0), x_2(0), x_3(0)) = (4, 2, 2)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (3, 4, 1, 5)$. The initial values of estimations $\hat{a}_i$ and $\hat{b}_j$ are $(\hat{a}_1(0), \hat{a}_2(0), \hat{a}_3(0)) = (1, 1, 1)$ and $(\hat{b}_1(0), \hat{b}_2(0), \hat{b}_3(0), \hat{b}_4(0), \hat{b}_5(0)) = (2, 2, 2, 2, 2)$. Then, the trajectories of errors $e_i(t)$ are shown in Figure 2(a). The trajectories of estimations $\hat{a}_i$ and $\hat{b}_j$ are shown in Figure 2(b-c).

From (19), the following adaptive controller can be designed

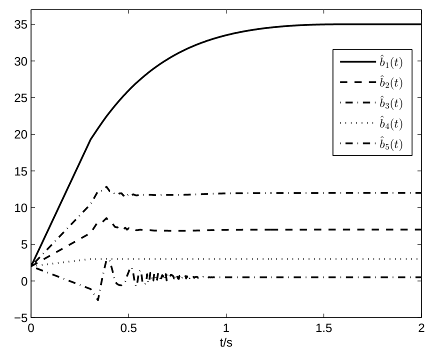
$$\begin{cases} u_1(t) = -\hat{b}_1(y_2(t) - y_1(t)) - y_4(t) + x_2(t) - x_1(t)x_2(t) - x_1(t)^2x_3(t) + x_2(t)(x_2(t) - x_1(t))\hat{a}_1 \\ +x_1(t)^2\hat{a}_2 + k \cdot sign(e_1(t))|e_1(t)|^\eta - \gamma e_1(t), \\ u_2(t) = -\hat{b}_2y_1(t) + y_1(t)y_3(t) - \hat{b}_3y_2(t) - x_2(t) + x_1(t)x_2(t) + x_1(t)^2x_3(t) - x_2(t)(x_2(t) - x_1(t))\hat{a}_1 \\ -x_1(t)^2\hat{a}_2 - k \cdot sign(e_1(t))|e_1(t)|^\eta + \gamma e_1(t), \\ u_3(t) = -y_1(t)y_2(t) + \hat{b}_4y_3(t) - x_1(t)x_2(t) + (x_2(t) - x_1(t))\hat{a}_1 - x_3(t)\hat{a}_3 + k \cdot sign(e_2(t))|e_2(t)|^\eta - \gamma e_2(t), \\ u_4(t) = -y_2(t)y_3(t) - \hat{b}_5y_4(t) + x_1(t)x_2(t) - (x_2(t) - x_1(t))\hat{a}_1 + x_3(t)\hat{a}_3 - k \cdot sign(e_2(t))|e_2(t)|^\eta + \gamma e_2(t), \\ \dot{\gamma}(t) = -(e_1^2(t) + e_2^2(t)) - \sum_{i=1}^3 \tilde{a}_i^2(t) - \sum_{i=1}^5 \tilde{b}_i^2(t) - k \cdot sign(\gamma)|\gamma|^\eta + p\gamma(t), \end{cases} \tag{40}$$



(a)



(b)



(b)

Fig. 2. (a) Trajectories of the error states $e_i(t)(1 \leq i \leq 4)$; (b) Trajectories of estimations $\hat{a}_i$ $(1 \leq i \leq 3)$; (c) Trajectories of estimations $\hat{b}_j$ $(1 \leq j \leq 5)$.

where $k = 2$, $p = 1$, $\eta = 0.5$. The initial value of $\gamma(t)$ is given as $\gamma(0) = 20$, the other initial values are the same. Then, we can obtain that the trajectories of errors $e_i(t)$ are shown in Figure 3(a). The trajectories of estimations $\hat{a}_i$ and $\hat{b}_j$ are shown in Figure 3(b-c). At last, the trajectory of $\gamma(t)$ is shown in Figure 3(d).

Example 3: Consider the following multi-layer complex network with 10 identical nodes:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + Ax(t), \\ \dot{y}(t) &= g(y(t)) + Ay(t) + u(t), \end{aligned} \tag{41}$$

where

$$f_i(x_i(t)) = \begin{pmatrix} -a & a & 0 \\ b & -1 & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}(t)x_{i3}(t) \\ x_{i1}(t)x_{i2}(t) \end{pmatrix},$$

and

$$g_i(y_i(t)) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_{i1}(t) \\ y_{i2}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -y_{i1}^3(t) + 0.8\cos(t) \end{pmatrix},$$

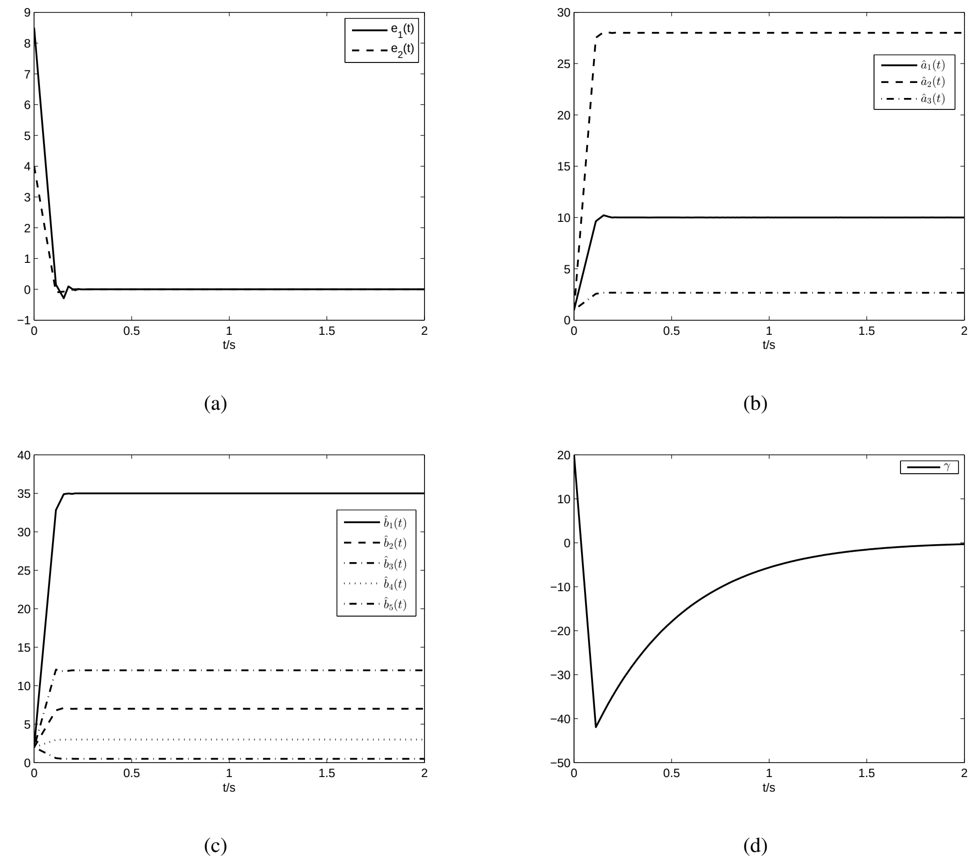


Fig. 3. (a) Trajectories of the error states $e_i(t)(1 \leq i \leq 4)$; (b) Trajectories of estimations $\hat{a}_i$ $(1 \leq i \leq 3)$; (c) Trajectories of estimations $\hat{b}_j$ $(1 \leq j \leq 5)$; (d) Trajectory of the adaptive updated law $\gamma(t)$.

$$A = \begin{pmatrix} -4 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -4 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -5 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -6 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -7 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -6 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & -5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 \end{pmatrix},$$

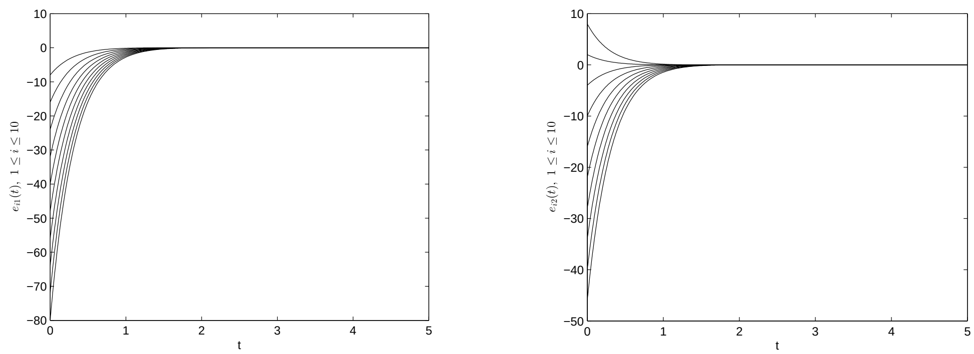
where the parameters are selected as $a = 10$, $b = 30$, $c = 8/3$, $A \in \mathbb{R}^{10\times 10}$ is a symmetrically coupling matrix. Moreover, the initial values are given as: $x(0) = (3 + i, -5 - 2i, 7 + 2i)^T$, $y(0) = (-2 + 7i, -7 + 8i)^T$ $(i = 1, \cdots, 10)$. Assume that $\phi(x_i) = [x_{i1} + x_{i2}, x_{i3}]^T$ and $\varphi(y_i) = [y_{i1}, y_{i2}]^T$. Then, we have $J_\phi(x_i) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $J_\varphi(y_i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. From (26), the error dynamics can be obtained as

$$\begin{cases} \dot{e}_{i1}(t) = 20x_{i1}(t) + 9x_{i2}(t) - x_{i1}(t)x_{i3}(t) - y_{i2}(t) + \sum_{j=1}^{10} a_{ij}(x_{i1}(t) + x_{i2}(t) - y_{i1}(t)) - u_{i1}(t), \\ \dot{e}_{i2}(t) = -8/3x_{i3}(t) + x_{i1}(t)x_{i2}(t) - y_{i1}(t) + y_{i2}(t) + y_{i1}^3(t) - 0.8\cos(t) + \sum_{j=1}^{10} a_{ij}(x_{i3}(t) - y_{i2}(t)) - u_{i2}(t). \end{cases} \tag{42}$$

By (27), the controller is given by

$$\begin{cases} u_{i1}(t) = 20x_{i1}(t) + 9x_{i2}(t) - x_{i1}(t)x_{i3}(t) - y_{i2}(t) + \sum_{j=1}^{10} a_{ij}(x_{i1}(t) + x_{i2}(t) - y_{i1}(t)) + k \cdot sign(e_{i1}(t))|e_{i1}(t)|^{\alpha} + pe_{i1}(t), \\ u_{i2}(t) = -8/3x_{i3}(t) + x_{i1}(t)x_{i2}(t) - y_{i1}(t) + y_{i2}(t) + y_{i1}^3(t) - 0.8\cos(t) + \sum_{j=1}^{10} a_{ij}(x_{i3}(t) - y_{i2}(t)) + k \cdot sign(e_{i2}(t))|e_{i2}v|^{\alpha} + pe_{i2}(t). \end{cases} \tag{43}$$

In the simulation, we select parameters $k = 1,\ p = 3,\ \eta = 3/5$, the simulation results are shown in Figure 4.



(a) (b)

Fig. 4. (a) Trajectories of the errors $e_{i1}(t)(1 \le i \le 10)$; (b) Trajectories of the errors $e_{i2}(t)(1 \le i \le 10)$.

V. Conclusion

This paper has addressed the problem of global finite-time synchronization of two different dimensional chaotic systems. Firstly, the definition of global finite-time synchronization of different dimensional chaotic systems were introduced. Based on the finite-time stability, the controller was proposed such that the chaotic systems were globally synchronized in a finite time. Then, by taking some uncertain parameters into account, a new control law and dynamical parameters estimation were proposed to guarantee that the global finite-time synchronization could be achieved. Moreover, by considering a dynamical parameter adopted in the controller, the adaptive updated controller was also designed to achieve the desired results. Subsequently, the results of two different dimensional chaotic systems were also extended to two different dimensional networked chaotic systems. Finally, three numerical examples were given to verify the efficiency of the proposed methods.

In this paper, the finite-time synchronization of different dimensional chaotic systems is considered based on continuous controller design. To the best of the authors' knowledge, the amount of information needs to be transferred may be limited in some cases and synchronization can be still achieved. For two same dimensional networked chaotic systems, finite-time synchronization has been studied by designing intermittent controllers in our previous research [40]. In our future work, we will achieve finite-time synchronization of two networked chaotic systems with different orders via intermittent feedback control.

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