INTERLACING OF ZEROS OF QUASI-ORTHOGONAL MEIXNER POLYNOMIALS

K. DRIVER* AND A. JOOSTE†

Abstract. We consider the interlacing of zeros of polynomials within the sequences of quasi-orthogonal order one Meixner polynomials $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ characterised by $-\beta, c \in (0,1)$. The interlacing of zeros of quasi-orthogonal Meixner polynomials $M_n(x;\beta,c)$ with the zeros of their nearest orthogonal counterparts $M_l(x;\beta+k,c)$, $l,n \in \mathbb{N}$, $k \in \{1,2\}$, is also discussed.

Key words. Discrete orthogonal polynomials, quasi-orthogonal polynomials, Meixner polynomials, interlacing of zeros.

AMS subject classifications. 33C45, 42C05

1. Introduction. Meixner polynomials (cf. [16]) were introduced in 1934 by Meixner [18] and are defined by

$$M_n(x;\beta,c) = (\beta)_n \, {}_2F_1\Big(-n,-x;\beta;1-\frac{1}{c}\Big), n \in \mathbb{N}, \tag{1.1}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function and $(\ )_n$ denotes the Pochhammer symbol [16, eq. (1.3.1)]. For $\beta > 0$ and $c \in (0,1)$, the polynomials $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ satisfy the discrete orthogonality relation [16, eq. (9.10.2)]

$$\sum_{k=0}^{\infty} \frac{c^k(\beta)_k}{k!} M_m(k;\beta,c)M_n(k;\beta,c) = \frac{(\beta)_n n!}{c^n(1-c)^{\beta}}\,\delta_{mn} \tag{1.2}$$

and the zeros of $M_n(x;\beta,c)$ are real, distinct and lie in $(0,\infty)$. Further, each interval $[k,k+1]$, $k \in \{0,1,2,\dots\}$, contains at most one zero of $M_n(x;\beta,c)$ (cf. [23, p. 1539]). Meixner polynomials satisfy the three-term recurrence relation [16, eq. (9.10.3)]

$$\frac{c}{c-1} M_n(x;\beta,c) = \Big(x - B(n,\beta,c)\Big) M_{n-1}(x;\beta,c) - \frac{(n-1)(\beta+n-2)}{c-1} M_{n-2}(x;\beta,c), \tag{1.3}$$

$n = 2,3,\dots$, where

$$B(n,\beta,c) = \frac{\beta c + (c+1)(n-1)}{1-c},$$

$M_0(x;\beta,c) \equiv 1$, $M_1(x;\beta,c) = \frac{c-1}{c}x + \beta$, and are connected with classical Jacobi polynomials $P_n^{\alpha,\beta}$ via

$$M_n(x;\beta,c) = n! P_n^{(\beta-1,-n-\beta-x)}\Big(\frac{2-c}{c}\Big),$$

[16, p. 236]. Jacobi and Meixner sequences are each located on the ${}_2F_1$ plane of the Askey scheme of hypergeometic orthogonal polynomials [16, p. 183]. Meixner polynomials are associated with stochastic processes [21] and are used to analyse discrete stochastic processes in the context of spectral analysis in the Laplace domain (cf. [10]).

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It is known, [13], that as the parameter β decreases below zero, the zeros of the Meixner polynomial $M_n(x;\beta,c)$ depart from the interval of orthogonality $(0,\infty)$ through the origin. In particular, for each fixed β and c, $-\beta, c \in (0,1)$ one zero of $M_n(x;\beta,c)$ is simple and negative, the remaining $(n-1)$ zeros are simple and positive and the Meixner sequence $M_n(x;\beta,c)\}_{n=0}^{\infty}$ is quasi-orthogonal of order 1. The concept of quasi-orthogonality of a sequence was introduced by Riesz [20] in connection with the moment problem. We recall the definition of quasi-orthogonality of a sequence with respect to a discrete weight (cf. [5]). A polynomial q_n of exact degree $n \geq r,\ \ n \in \{0,1,\ldots,N\}$, where N may be infinite, is discrete quasi-orthogonal of order r with ρ_i the weight at each point $x_i, i \in \{0,1,\ldots,N-1\}$ if

$$\sum_{i=0}^{N-1}(x_i)^j q_n(x_i)\rho_i \begin{cases} = 0, \text{ for } j \in \{0,1,\ldots,n-r-1\}, \\ \neq 0, \text{ for } j = n-r. \end{cases}$$

For the Meixner polynomial $M_n(x;\beta,c)$ we have (cf. [13])

$$\sum_{k=0}^{\infty} k^j M_n(k;\beta,c)\frac{c^k(\beta+1)_k}{k!} = 0, j \in \{0,1,\ldots,n-2\}.$$

Quasi-orthogonality has been investigated by many authors, including Fejér [9], Shohat [22], Chihara [4], Dickinson [6], Draux [7], Maroni [17] and Joulak [15]. The quasi-orthogonality of Jacobi, Gegenbauer and Laguerre sequences is discussed in [2], and the quasi-orthogonality of Meixner sequences in [13] and of Meixner-Pollaczek, Hahn, Dual-Hahn and Continuous Dual-Hahn sequences in [11]. Recently, in [3], interlacing properties of zeros of quasi-orthogonal polynomials were used to prove results on Gaussian-type quadrature.

We recall the definition of interlacing of zeros within a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$. Let $\{x_{i,n}\}_{i=1}^{n}$ denote the zeros of p_n in increasing order, $n \in \mathbb{N}$. The zeros of p_n and p_{n-1} are interlacing if

$$x_{1,n} < x_{1,n-1} < x_{2,n} < \cdots < x_{n-1,n-1} < x_{n,n}.$$

Since we shall discuss interlacing of zeros of polynomials p_n and q_n of the same degree, the zeros of p_n and q_n, are said to be interlacing if either

$$x_{1,n} < y_{1,n} < x_{2,n} < y_{1,n} < \cdots < x_{n,n} < y_{n,n}$$

or

$$y_{1,n} < x_{1,n} < y_{2,n} < x_{2,n} < \cdots < y_{n,n} < x_{n,n},$$

where $\{y_{i,n}\}_{i=1}^{n}$ denote the zeros of q_n. Further, the zeros of p_n and $q_m, m \leq n-2$, are interlacing (often called Stieltjes interlacing) if there exist m open intervals with endpoints at successive zeros of p_n, each of which contains exactly one zero of q_m.

Interlacing results for the zeros of quasi-orthogonal and orthogonal polynomials from different classical sequences within the same family were proved in [2] and extended in the case of the family of Laguerre quasi-orthogonal and orthogonal polynomials in [8].

In Section 2 we discuss interlacing of zeros of polynomials of different degree within each quasi-orthogonal Meixner sequence $\{M_n(x;\beta,c)\}_{n=0}^{\infty}$ where β and c are fixed, $-\beta, c \in (0,1)$. In Section 3, we consider the interlacing of zeros of quasi-orthogonal order 1 Meixner polynomials $M_n(x;\beta,c)$, with those of (orthogonal) Meixner polynomials $M_n(x;\beta+1,c), -\beta, c \in (0,1)$. In Section 4 we discuss interlacing of zeros of $M_n(x;\beta,c), -\beta, c \in (0,1)$, and $M_l(x;\beta+2,c), l \leq n, l \in \mathbb{N}$.

The identities we use to prove our results are derived from the contiguous relations for ${}_2F_1$ hypergeometric polynomials [19, p. 71]. A useful algorithm in this regard is available as a computer package [24].

2. Zeros of $M_n(x;\beta,c)$ and $M_l(x;\beta,c), l,n \in \mathbb{N}, l < n, -\beta, c \in (0,1)$. We will use the following result which is an application of [2, Theorem 3 (ii)(b)].

THEOREM 2.1. *Let $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ be the sequence of Meixner polynomials and suppose $-\beta, c \in (0,1)$. For each $n \in \mathbb{N}$, let $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ be the zeros of $M_n(x;\beta,c)$ and $y_{1,n} < y_{2,n} < \cdots < y_{n,n}$ the zeros of $M_n(x;\beta+1,c)$. Then*

(i)

$$x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}; \tag{2.1}$$

(ii) the zeros of $M_n(x;\beta,c)$ and $M_n(x;\beta+1,c)$ are interlacing:

$$x_{1,n} < 0 < y_{1,n} < x_{2,n} < y_{2,n} < \cdots < x_{n,n} < y_{n,n}, \tag{2.2}$$

and the zeros of $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta+1,c)$ are interlacing:

$$x_{1,n} < 0 < y_{1,n-1} < x_{2,n} < y_{2,n-1} < \cdots < y_{n-1,n-1} < x_{n,n}. \tag{2.3}$$

Proof.

(i) Equation (2.1) was proved in [13, Theorem 6].

(ii) We have from (1.1) and [19, p. 71,(2)], that

$$M_n(x;\beta,c) = M_n(x;\beta+1,c) - nM_{n-1}(x;\beta+1,c).$$

Since the leading coefficients of $M_n(x;\beta+1,c)$ and $M_{n-1}(x;\beta+1,c)$ have opposite signs, it follows from [2, Theorem 3 (ii)(b)] that the zeros of $M_n(x;\beta,c)$ and $M_n(x;\beta+1,c)$ are interlacing and the zeros of $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta+1,c)$ interlace, which proves (2.2) and (2.3). □

Remarks:

(1) Note that, since $\{M_n(x;\beta+1,c)\}_{n=1}^{\infty}$ is an orthogonal sequence for each fixed β and c, $-\beta, c \in (0,1)$, (2.2) and (2.3) can be combined to give:

$$x_{1,n} < y_{1,n} < y_{1,n-1} < x_{2,n} < y_{2,n} < y_{2,n-1} < \cdots < y_{n-1,n-1} < x_{n,n} < y_{n,n}.$$

(2) For $-\beta, c \in (0,1)$ and $n \in \mathbb{N}$, the quasi-orthogonal (order 1) Meixner polynomials $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta,c)$ are co-prime. This follows from the three-term recurrence relation (1.3) which shows that if M_n and M_{n-1} have a common zero, this is also a zero of M_{n-2}. Iterating the argument we find that $M_0(x;\beta,c) \equiv 1$ has a zero and we have a contradiction.

The value of the smallest positive zero $x_{2,n}$ of $M_n(x;\beta,c)$ relative to that of $-\beta$, $0 < -\beta < 1$, plays an important role in our results. Numerical evidence (see Table 2.1) suggests that whenever $-\beta, c \in (0,1)$, $x_{2,n}$ is greater than 1 for each $n \in \mathbb{N}$. In addition, we note that the point $x = 1$ cannot be a zero of $M_n(x;\beta,c)$ if $-\beta, c \in (0,1)$, since $M_n(1;\beta,c) =$

TABLE 2.1
The values of $x_{2,n}$ for different values of n, β and c.

$-\beta$	c	$x_{2,5}$	$x_{2,35}$
0.99	0.01	$1+5.05*10^{-10}$	$1+3.61*10^{-69}$
0.99	0.5	1.002	$1+1.08*10^{-11}$
0.99	0.99	3.987	1.18
0.5	0.1	$1+4.54*10^{-4}$	$1+9.78*10^{-33}$
0.5	0.5	1.150	$1+2.32*10^{-9}$
0.5	0.9	4.254	1.11
0.01	0.01	$1+1.91*10^{-7}$	$1+1.12*10^{-65}$
0.01	0.5	1.346	$1+1.56*10^{-8}$
0.01	0.99	73.314	10.37

$(\beta)_n \frac{nc+\beta c-n}{\beta c}$, which is zero if and only if $c = \frac{n}{n+\beta}$ which is not possible if $-\beta, c \in (0,1)$. We make the following conjecture:

Conjecture I
Let $-\beta, c \in (0,1)$. For each $n \in \mathbb{N}$, if $x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}$ denote the zeros of $M_n(x;\beta,c)$, then $x_{2,n} > 1$.

In the next theorem we analyse the interlacing of zeros of $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta,c)$ and will use the mixed recurrence relation

$$\frac{c}{c-1}M_n(x;\beta,c) = \frac{\beta+n-1}{c-1}M_{n-1}(x;\beta,c) + (x+\beta)M_{n-1}(x;\beta+1,c), \tag{2.4}$$

obtained from [19, p. 71,(4)].

THEOREM 2.2. *Fix n, β and c where $n \in \mathbb{N}$ and $-\beta, c \in (0,1)$. Let $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ be the sequence of Meixner polynomials and let $x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}$ denote the zeros of $M_n(x;\beta,c)$. Then $x_{2,n-1} > -\beta$ is a necessary and sufficient condition for the zeros of $xM_{n-1}(x;\beta,c)$ and $M_n(x;\beta,c)$ to interlace.*

Proof. For each $n \in \mathbb{N}$, let $0 < y_{1,n} < y_{2,n} < \cdots < y_{n,n}$ be the zeros of $M_n(x;\beta+1,c)$. Evaluating (2.4) at successive zeros $x_{i,n-1}$ and $x_{i+1,n-1}, i \in \{1,2,\ldots,n-2\}$, of $M_{n-1}(x;\beta,c)$, we have

$$\frac{M_n(x_{i,n-1};\beta,c)M_n(x_{i+1,n-1};\beta,c)}{M_{n-1}(x_{i,n-1};\beta+1,c)M_{n-1}(x_{i+1,n-1};\beta+1,c)} = \left(\frac{c-1}{c}\right)^2(x_{i,n-1}+\beta)(x_{i+1,n-1}+\beta). \tag{2.5}$$

Note that, by (2.2) and (2.4), neither $M_{n-1}(x;\beta+1,c)$ nor $M_n(x;\beta,c)$ is equal to zero when $x = x_{i,n-1}, i \in \{1,2,\ldots,n-1\}$.

From (2.2), we know that $M_{n-1}(x;\beta+1,c)$ has a different sign at successive zeros of $M_{n-1}(x;\beta,c)$, therefore the left-hand side of (2.5) is negative for each $i \in \{1,2,\ldots,n-2\}$.

(1) Suppose $x_{2,n-1} > -\beta$. Then, since $x_{1,n-1}+\beta < 0$ and $x_{2,n-1}+\beta > 0$, the product $(x_{1,n-1}+\beta)(x_{2,n-1}+\beta) < 0$, while $(x_{i,n-1}+\beta)(x_{i+1,n-1}+\beta) > 0$ for each $i \in \{2,3,\ldots,n-2\}$. It follows from (2.5) that $M_n(x;\beta,c)$ has a different sign at $x_{i,n-1}$ and $x_{i+1,n-1}$ for each $i \in \{2,3,\ldots,n-2\}$ and the same sign at $x_{1,n-1}$ and $x_{2,n-1}$. Therefore at least $(n-3)$ positive, distinct zeros of $M_n(x;\beta,c)$ lie between $x_{2,n-1}$ and $x_{n-1,n-1}$. In addition, $M_n(x;\beta,c)$ has exactly one negative zero and the largest zero $x_{n,n}$ of $M_n(x;\beta,c)$ is greater than $y_{n-1,n-1}$ by (2.3), whereas (2.2)

shows $y_{n-1,n-1} > x_{n-1,n-1}$. We have therefore accounted for $(n-3)$ positive zeros of $M_n(x;\beta,c)$ between $x_{2,n-1}$ and $x_{n-1,n-1}$, one negative zero and one zero $> x_{n-1,n-1}$ and we must still have a configuration of zeros that ensures $M_n(x;\beta,c)$ has the same sign at $x_{1,n-1}$ and $x_{2,n-1}$. The only possibility that accommodates all constraints is

$$x_{1,n-1} < x_{1,n} < 0 < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n-1} < x_{n,n},$$

which shows that the zeros of $xM_{n-1}(x;\beta,c)$ interlace with the zeros of $M_n(x;\beta,c)$.

(2) Suppose now that the zeros of $xM_{n-1}(x;\beta,c)$ and $M_n(x;\beta,c)$ are interlacing. Since each polynomial $M_{n-1}(x;\beta,c)$ and $M_n(x;\beta,c)$ has exactly one negative zero, the only possible configuration is

$$x_{1,n-1} < x_{1,n} < 0 < x_{2,n} < x_{2,n-1} < x_{3,n} < \cdots < x_{n-1,n-1} < x_{n,n}. \tag{2.6}$$

It follows from (2.6) that $M_n(x_{1,n-1};\beta,c)M_n(x_{2,n-1};\beta,c) > 0$. Also, from (2.2), with n replaced by $n-1$, the zeros of $M_{n-1}(x;\beta,c)$ and $M_{n-1}(x;\beta+1,c)$ are interlacing and therefore $M_{n-1}(x_{i,n-1};\beta+1,c)M_{n-1}(x_{i+1,n-1};\beta+1,c) < 0$ for each $i \in \{1,2,\ldots,n-1\}$. Since $x_{1,n-1}+\beta < 0$, it follows from (2.5) that $x_{2,n-1} > -\beta$. □

Remarks:

(1) If $-\beta$ is a (positive) zero of $M_n(x;\beta,c)$ for any positive integer $n \geq 2$, then, from (2.4) it follows that $-\beta$ is a zero of $M_n(x;\beta,c)$ for every positive integer $n \geq 2$. However, letting $n=2$ we have

$$M_2(x;\beta,c) = \frac{(c-1)^2x^2 + (c-1)(2\beta c + c + 1)x + \beta(\beta+1)c^2}{c^2}$$

and $M_2(-\beta;\beta,c) = \frac{\beta(\beta+1)}{c^2} \neq 0$, since $-1 < \beta < 0$. Therefore $x_{i,n} \neq -\beta, i \in \{2,3,\ldots,n\}$, for any $n \in \mathbb{N}_{\geq 2}$;

(2) $x_{3,n} > 1$. Since the zeros of orthogonal Meixner polynomials $M_n(x;\beta+1,c)$ are separated by the numbers $0,1,2,\ldots$, we have $y_{2,n} \geq 1$, while from (2.2), $x_{3,n} > y_{2,n}$, so that $x_{3,n} > 1$ for each $n \in \mathbb{N}$.

(3) Assuming that $x_{2,n-1} > -\beta$, Theorem 2.2 proves that the zeros of $M_{n-1}(x;\beta,c)$, together with the point $x=0$, interlace with the zeros of $M_n(x;\beta,c)$. However, (2.6) shows that two zeros of $M_n(x;\beta,c)$ lie between the smallest two zeros of $M_{n-1}(x;\beta,c)$, and full interlacing between the zeros of $M_{n-1}(x;\beta,c)$ and $M_n(x;\beta,c)$ breaks down for each $n \in \mathbb{N}$. This breakdown of full interlacing is similar to results proved for quasi-orthogonal Laguerre [8], Jacobi and ultraspherical polynomials. Note that the positive zeros of M_{n-1} and the positive zeros of M_n are interlacing for any $n \in \mathbb{N}$ by Theorem 2.2, provided $x_{2,n-1} > -\beta$.

COROLLARY 2.3. *Let $-\beta, c \in (0,1)$ and $n \in \mathbb{N}$, and suppose $x_{2,n-1} > -\beta$. Then the negative zero of $M_n(x;\beta,c)$ increases with n. Moreover, $x_{1,1} = \frac{\beta c}{1-c}$ is a lower bound for the negative zero of $M_n(x;\beta,c)$ for each $n \in \mathbb{N}$.*

Our next result investigates interlacing between the zeros of $M_{n-2}(x;\beta,c)$ and $M_n(x;\beta,c)$. Equation (2.6) shows that $x_{1,n-2} < x_{1,n-1} < x_{1,n} < 0$, i.e., the smallest zero of the polynomial $M_{n-2}(x;\beta,c)$ does not lie in any interval with endpoints at successive zeros of $M_n(x;\beta,c)$. Therefore, Stieltjes interlacing between the zeros of $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta,c)$ cannot hold for any $n \in \mathbb{N}$ and fixed β and c, with $-\beta, c \in (0,1)$, for which $x_{2,n-1} > -\beta$.

However, again subject to the assumption $x_{2,n-1} > -\beta$, interlacing holds between the positive zeros of $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta,c)$ in the following way:

THEOREM 2.4. *Fix n, β and c where $n \in \mathbb{N}$, $-\beta, c \in (0,1)$. Let $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ be the sequence of Meixner polynomials. Assume $x_{2,n-1} > -\beta$ and that $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta,c)$ are co-prime. For each $n \in \mathbb{N}_{\geqq 3}$, the positive zeros of $M_{n-2}(x;\beta,c)$, together with the point $B(n,\beta,c) = \frac{\beta c+(c+1)(n-1)}{1-c}$, interlace with the positive zeros of $M_n(x;\beta,c)$.*

Proof. Let $x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}$ be the zeros of $M_n(x;\beta,c)$. We note from (1.3) that $x_{i,n} \neq B$, $i \in \{1,2,\ldots,n\}$, since we are assuming that $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta,c)$ are co-prime.

Evaluating (1.3) at successive positive zeros $x_{i,n}$ and $x_{i+1,n}$, $i \in \{2,3,\ldots,n-1\}$, of $M_n(x;\beta,c)$, we have, for $i \in \{2,3,\ldots,n-1\}$,

$$\frac{M_{n-1}(x_{i,n};\beta,c)M_{n-1}(x_{i+1,n};\beta,c)}{M_{n-2}(x_{i,n};\beta,c)M_{n-2}(x_{i+1,n};\beta,c)} = \frac{(n-1)^2(\beta+n-2)^2}{(c-1)^2(x_{i,n}-B)(x_{i+1,n}-B)}. \tag{2.7}$$

From Theorem 2.2, the positive zeros of $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta,c)$ are interlacing. Therefore $M_{n-1}(x_{i,n};\beta,c)M_{n-1}(x_{i+1,n};\beta,c) < 0$ for each $i \in \{2,3,\ldots,n-1\}$. Further, the right-hand side of (2.7) is positive for each $i \in \{2,3,\ldots,n-1\}$ except possibly for one value, say $i = j$, where $B \in (x_{j,n}, x_{j+1,n})$, $j \in \{2,3,\ldots,n-1\}$. It follows from (2.7) that $M_{n-2}(x_{i,n};\beta,c)M_{n-2}(x_{i+1,n};\beta,c) < 0$ for each $i \in \{2,3,\ldots,n-1\}$, except if $B \in (x_{j,n}, x_{j+1,n})$ for some $j \in \{2,3,\ldots,n-1\}$. Therefore in at least $(n-3)$ intervals of the form $(x_{i,n}, x_{i+1,n})$, $i \in \{2,3,\ldots,n-1\}$, $M_{n-2}(x;\beta,c)$ changes sign. However, $M_{n-2}(x;\beta,c)$ only has $(n-3)$ positive zeros so we can deduce that the positive zeros of $M_{n-2}(x;\beta,c)$, together with the point B, (which must lie between two positive zeros of $M_n(x;\beta,c)$) interlace with the positive zeros of $M_n(x;\beta,c)$. ∎

3. Zeros of $M_n(x;\beta,c)$ and $M_l(x;\beta+1,c), l \leq n, -\beta, c \in (0,1)$. We have proved in Theorem 2.1 that, provided $x_{2,n-1} > -\beta$, the zeros of the (quasi-orthogonal order 1) Meixner polynomials $M_n(x;\beta,c)$ interlace with the zeros of the (orthogonal) Meixner polynomials $M_n(x;\beta+1,c)$ and with the zeros of $M_{n-1}(x;\beta+1,c)$.

THEOREM 3.1. *Fix n, β and c where $n \in \mathbb{N}$ and $-\beta, c \in (0,1)$. Let $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ be the sequence of Meixner polynomials. Assume $x_{2,n} > -\beta$ and that $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta+1,c)$ are co-prime. For each $n \in \mathbb{N}_{\geqq 3}$, the zeros of $M_{n-2}(x;\beta+1,c)$, together with the point $B_1 = \frac{\beta c+n-1}{1-c}$ interlace with the $(n-1)$ positive zeros of $M_n(x;\beta,c)$.*

Proof. Let $x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}$ be the zeros of $M_n(x;\beta,c)$. We have from [12, p. 54],

$$\left(\frac{c}{c-1}\right) M_n(x;\beta,c) = (x-B_1)\, M_{n-1}(x;\beta,c) + (n-1)(x+\beta)M_{n-2}(x;\beta+1,c), \tag{3.1}$$

where $B_1 = \frac{\beta c+n-1}{1-c}$. Also, $x_{i,n} \neq B_1$, $i \in \{1,2,\ldots,n\}$, since we are assuming that $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta+1,c)$ are co-prime.

Evaluating (3.1) at successive positive zeros $x_{i,n}$ and $x_{i+1,n}$, $i \in \{2,3,\ldots,n-1\}$, of $M_n(x;\beta,c)$, we obtain

$$\frac{M_{n-1}(x_{i,n};\beta,c)M_{n-1}(x_{i+1,n};\beta,c)}{M_{n-2}(x_{i,n};\beta+1,c)M_{n-2}(x_{i+1,n};\beta+1,c)} = \frac{(n-1)^2(x_{i,n}+\beta)(x_{i+1,n}+\beta)}{(x_{i,n}-B_1)(x_{i+1,n}-B_1)}. \tag{3.2}$$

Since $x_{2,n} > -\beta$, $x_{i,n}+\beta > 0$ for each $i \geq 2$, the right-hand side of (3.2) is positive provided $B_1 \notin (x_{i,n}, x_{i+1,n})$.

Suppose $B_1 \notin (x_{i,n}, x_{i+1,n})$ for any $i \in \{2, 3, \ldots, n-1\}$. Since by (2.6), if $x_{2,n} > -\beta$, then $x_{2,n-1} > -\beta$ and it follows from Theorem 2.2 that the positive zeros of $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta,c)$ interlace.

Therefore $M_{n-1}(x_{i,n};\beta,c)M_{n-1}(x_{i+1,n};\beta,c) < 0$ for $i \in \{2,3,\ldots,n-1\}$, and from (3.2), $M_{n-2}(x_{i,n};\beta+1,c)M_{n-2}(x_{i+1,n};\beta+1,c) < 0$ for each $i \in \{2,3,\ldots,n-1\}$. It follows from (3.1) that if $B_1 \notin (x_{i,n}, x_{i+1,n})$ for any $i \in \{2,3,\ldots,n-1\}$, the $(n-2)$ positive zeros of $M_{n-2}(x;\beta+1,c)$ interlace with the $(n-1)$ positive zeros of $M_n(x;\beta,c)$. Further, by our assumption, the point B_1 lies outside the interval with endpoints at the smallest positive zero $x_{2,n}$ of $M_n(x;\beta,c)$ and its largest zero $x_{n,n}$, so interlacing holds between the $(n-2)$ positive zeros of $M_{n-2}(x;\beta+1,c)$, together with the point B_1, and the $(n-1)$ positive zeros of $M_n(x;\beta,c)$.

Suppose now $B_1 \in (x_{i,n}, x_{i+1,n})$ for some $i \in \{2,3,\ldots,n-1\}$. Then in this single interval containing B_1, there will be no sign change of $M_{n-1}(x;\beta,c)$, but its sign will change in each of the remaining $(n-3)$ intervals with endpoints at successive zeros of $M_n(x;\beta,c)$. However, evaluating (3.1) at $x_{1,n}$ and $x_{2,n}$, we obtain

$$\frac{M_{n-1}(x_{1,n};\beta,c)M_{n-1}(x_{2,n};\beta,c)}{M_{n-2}(x_{1,n};\beta+1,c)M_{n-2}(x_{2,n};\beta+1,c)} = \frac{(n-1)^2(x_{1,n}+\beta)(x_{2,n}+\beta)}{(x_{1,n}-B_1)(x_{2,n}-B_1)}. \tag{3.3}$$

Now $B_1 \in (x_{i,n}, x_{i+1,n})$ for some $i \in \{2,3,\ldots,n-1\}$, which implies $B_1 \notin (x_{1,n}, x_{2,n})$. Furthermore $x_{1,n} + \beta < 0$ and therefore the right-hand side of (3.3) is negative, since $x_{2,n} > -\beta$. From (2.2), $M_{n-1}(x_{1,n};\beta,c)M_{n-1}(x_{2,n};\beta,c) > 0$. We deduce from (3.3) that $M_{n-2}(x;\beta+1,c)$ differs in sign at $x_{1,n}$ and $x_{2,n}$ and therefore has an odd number (which must be equal to 1) of zeros $< x_{2,n}$.

We deduce that the $(n-2)$ simple, positive zeros of $M_{n-2}(x;\beta+1,c)$, together with the point B_1, interlace with the $(n-1)$ positive zeros of $M_n(x;\beta,c)$ if $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta+1,c)$ are co-prime and $x_{2,n} > -\beta$. □

4. Zeros of $M_n(x;\beta,c)$ and $M_l(x;\beta+2,c), l \le n, -\beta, c \in (0,1)$. In 1989, Richard Askey [1] proved that the zeros of the orthogonal Jacobi polynomials $P_n^{\alpha,\beta}$ and $P_n^{\alpha+1,\beta}, \alpha,\beta > -1$, are interlacing. He conjectured, based on graphical evidence, that the zeros of $P_n^{\alpha,\beta}$ and $P_n^{\alpha+2,\beta}, \alpha,\beta > -1$, are interlacing, in other words, interlacing of the zeros of Jacobi polynomials is retained if one parameter is increased by two. In this section we investigate an extension of the Askey conjecture to zeros of quasi-orthogonal Meixner polynomials of order one by considering interlacing between the zeros of $M_n(x;\beta,c)$ and those of $M_n(x;\beta+2,c)$. We also investigate interlacing of the zeros of $M_n(x;\beta,c)$ with the zeros of $M_{n-1}(x;\beta+2,c)$. The recurrence relations used in this section follow from contiguous relations for ${}_2F_1$ hypergeometric functions [19, p. 71] and for the convenience of the reader we list them here:

$$\begin{aligned}\frac{n+\beta}{1-c}M_n(x;\beta,c) &= \left(x + \frac{\beta(c-2)}{c-1} + n + 1\right) M_n(x;\beta+1,c) \\ &\quad - (x+\beta+1)M_n(x;\beta+2,c)\end{aligned} \tag{4.1}$$

$$\frac{n+\beta}{n(c-1)}M_n(x;\beta,c) = \frac{n+\beta-nc}{n(c-1)}M_n(x;\beta+1,c) + (x+\beta+1)M_{n-1}(x;\beta+2,c) \tag{4.2}$$

$$\begin{aligned}\frac{n+\beta}{n(1-c)}M_n(x;\beta,c) &= \frac{(n+\beta)(c-1)-c}{n(1-c)}M_n(x;\beta+2,c) \\ &\quad + (x-A_n)\, M_{n-1}(x;\beta+2,c)\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{((n-1)(1-c)+\beta)c}{(c-1)(\beta+n-1)}M_n(x;\beta,c) &= (x-B_2)\, M_{n-1}(x;\beta,c) \\ &\quad + \frac{(n-1)(1-c)(x+\beta)_2}{\beta+n-1}M_{n-2}(x;\beta+2,c),\end{aligned} \tag{4.4}$$

where $B_2 = \frac{\beta c}{1-c} + n - 1$.

THEOREM 4.1. *Let $-\beta, c \in (0,1)$ be fixed and suppose $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ is the sequence of Meixner polynomials. Let $x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}$ be the zeros of $M_n(x;\beta,c)$, $0 < z_{1,n} < z_{2,n} < \cdots < z_{n,n}$ the zeros of $M_n(x;\beta+2,c)$ and $A_n = \frac{-\beta(c-2)}{c-1} - n - 1$. Assume that $M_n(x;\beta,c)$ and $M_n(x;\beta+2,c)$ are co-prime. Then*

(i) If n, β and c are such that $A_n \notin (z_{i,n}, z_{i+1,n})$ for any $i \in \{1,2,\ldots,n-1\}$, the zeros of $M_n(x;\beta,c)$ interlace with the zeros of $M_n(x;\beta+2,c)$;

(ii) If n, β and c are such that $A_n \in (z_{j,n}, z_{j+1,n})$ for some $j \in \{1,2,\ldots,n-1\}$, then the zeros of $(x-A_n)M_n(x;\beta,c)$ interlace with the zeros of $xM_n(x;\beta+2,c)$.

Proof. Evaluate (4.1) at successive (positive) zeros $z_{i,n}$ and $z_{i+1,n}$ of $M_n(x;\beta+2,c)$, to obtain, for $i \in \{1,2,\ldots,n-1\}$,

$$\frac{M_n(z_{i,n};\beta,c)M_n(z_{i+1,n};\beta,c)}{M_n(z_{i,n};\beta+1,c)M_n(z_{i+1,n};\beta+1,c)} = \left(\frac{1-c}{n+\beta}\right)^2 (z_{i,n}-A_n)(z_{i+1,n}-A_n). \tag{4.5}$$

It was proved in [14, Corollary 2.2] that the zeros of $M_n(x;\beta+1,c)$ and $M_n(x;\beta+2,c)$ are interlacing and therefore the left-hand side of (4.5) is negative for each $i \in \{1,2,\ldots,n-1\}$.

(i) Suppose $A_n \notin (z_{i,n}, z_{i+1,n})$ for any $i \in \{1,2,\ldots,n-1\}$. Then from (4.5), $M_n(x;\beta,c)$ has a different sign at each successive pair of zeros $z_{i,n}$ and $z_{i+1,n}, i \in \{1,2,\ldots,n-1\}$ of $M_n(x;\beta+2,c)$. This accounts for $(n-1)$ positive zeros of $M_n(x;\beta,c)$. Also, the smallest zero of $M_n(x;\beta,c)$ is negative, while each $z_{i,n}, i \in \{1,2,\ldots,n-1\}$ is positive, so the only possible configuration of the zeros is

$$x_{1,n} < z_{1,n} < x_{2,n} < z_{2,n} < \cdots < x_{n,n} < z_{n,n},$$

and the result follows.

(ii) Suppose $A_n \in (z_{j,n}, z_{j+1,n})$ for some $j \in \{1,2,\ldots,n-1\}$. Then, from (4.5), $M_n(x;\beta,c)$ has no sign change in $(z_{j,n}, z_{j+1,n})$ and therefore an even number of zeros in $(z_{j,n}, z_{j+1,n})$ and it will have at least $(n-2)$ sign changes in the intervals $(z_{i,n}, z_{i+1,n}), i \neq j, i \in \{1,2,\ldots,n-1\}$. Also, the smallest zero of $M_n(x;\beta,c)$ is $x_{1,n} < 0$, so we have accounted for at least $(n-2)$ positive zeros of $M_n(x;\beta,c)$ and one negative zero, i.e., $(n-1)$ zeros of $M_n(x;\beta,c)$.

Since $M_n(x;\beta,c)$ has precisely one negative zero, there are two possibilities for the location of the remaining zero: it can lie between 0 and the smallest (positive) zero $z_{1,n}$ of $M_n(x;\beta+2,c)$ or it can be larger than the largest zero $z_{n,n}$ of $M_n(x;\beta+2,c)$. From [14, Corollary 2.2] we know that $x_{n,n} < z_{n,n}$, which rules out the second possibility and we deduce that

$$\begin{aligned} &x_{1,n} < 0 < x_{2,n} < z_{1,n} < x_{3,n} < \ldots \\ &< x_{j+1,n} < z_{j,n} < A_n < z_{j+1,n} < x_{j+2,n} < z_{j+2,n} < \cdots < x_{n,n} < z_{n,n}, \end{aligned}$$

which shows that the zeros of $(x-A_n)M_n(x;\beta,c)$ interlace with the zeros of $xM_n(x;\beta+2,c)$. □

THEOREM 4.2. *Let $-\beta, c \in (0,1)$ be fixed and $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ be the sequence of Meixner polynomials. Let $x_{1,n} < 0 < x_{2,n} < \cdots < x_{n,n}$ be the zeros of $M_n(x;\beta,c)$ and $A_n = \frac{-\beta(c-2)}{c-1} - n - 1$. For each $n \in \mathbb{N}$,*

(i) the zeros of $(x+\beta+1)M_{n-1}(x;\beta+2,c)$ and $M_n(x;\beta,c)$ are interlacing;

(ii) if n, β and c are such that $A_n < x_{1,n}$, the zeros of $M_{n-1}(x;\beta+2,c)$ interlace with the zeros of $M_n(x;\beta,c)$.

Proof. Let $0 < z_{1,n} < z_{2,n} < \cdots < z_{n,n}$ be the zeros of $M_n(x;\beta+2,c)$.

(i) Evaluating (4.2) at successive zeros $x_{i,n}$ and $x_{i+1,n}, i \in \{1, 2, \ldots, n-1\}$, of $M_n(x;\beta,c)$, we obtain

$$\left(\frac{n+\beta-nc}{n(c-1)}\right)^2 M_n(x_{i,n};\beta+1,c)M_n(x_{i+1,n};\beta+1,c) \\ = (x_{i,n}+\beta+1)(x_{i+1,n}+\beta+1)M_{n-1}(x_{i,n};\beta+2,c)M_{n-1}(x_{i+1,n};\beta+2,c). \tag{4.6}$$

We know from (2.2) that the zeros of $M_n(x;\beta,c)$ and $M_n(x;\beta+1,c)$ interlace, therefore $M_n(x;\beta+1,c)$ will differ in sign at each $x_{i,n}, i \in \{1,2,\ldots,n\}$ and the left-hand side of (4.6) is negative for each $i \in \{1,2,\ldots,n-1\}$. The term $(x_{i,n}+\beta+1)(x_{i+1,n}+\beta+1)$ on the right-hand side is positive unless $-\beta-1 \in (x_{i,n}, x_{i+1,n})$. Since $-\beta-1$ is negative, the only interval in which this could occur is $(x_{1,n}, x_{2,n})$ and actually in the sub-interval $(x_{1,n}, 0)$. Therefore, the right-hand side of (4.6) is negative for each $i \in \{2,3,\ldots,n-1\}$ and $M_{n-1}(x;\beta+2,c)$ changes sign in each interval $(x_{i,n}, x_{i+1,n}), i \in \{2,3,\ldots n-1\}$.

There are therefore two cases to consider: (a) $x_{1,n} < -\beta-1$ and (b) $x_{1,n} > -\beta-1$.

Suppose $x_{1,n} < -\beta-1$. Then the right-hand side of (4.6) is negative for each $i \in \{2,3,\ldots,n-1\}$ and at least $(n-2)$ (positive) zeros of $M_{n-1}(x;\beta,c)$ are accounted for. Also, there are an even number of zeros of $M_{n-1}(x;\beta+2,c)$ in $(x_{1,n}, x_{2,n})$, so the only possibility is that this interval contains no zeros and

$$x_{1,n} < -\beta-1 < x_{2,n} < z_{1,n-1} < x_{3,n} < z_{2,n-1} < \cdots < x_{n,n} < z_{n-1,n-1}. \tag{4.7}$$

If, on the other hand, $x_{1,n} > -\beta-1$, then $M_{n-1}(x;\beta+2,c)$ has $(n-1)$ sign changes between the zeros $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ of $M_n(x;\beta,c)$. We deduce that

$$-\beta-1 < x_{1,n} < z_{1,n-1} < x_{2,n} < z_{2,n-1} < \cdots < z_{n-1,n-1} < x_{n,n}. \tag{4.8}$$

(ii) Suppose $A_n < x_{1,n}$. Let $0 < z_{1,n} < z_{2,n} < \cdots < z_{n,n}$ be the zeros of $M_n(x;\beta+2,c)$. Then $A_n < x_{1,n} < 0 < z_{1,n}$. Evaluating (4.3) at successive zeros $x_{i,n}$ and $x_{i+1,n}, i \in \{1,2,\ldots,n-1\}$ of $M_n(x;\beta,c)$, we have

$$\frac{M_n(x_{i,n};\beta+2,c)M_n(x_{i+1,n};\beta+2,c)}{M_{n-1}(x_{i,n};\beta+2,c)M_{n-1}(x_{i+1,n};\beta+2,c)} = \frac{n^2(1-c)^2(x_{i,n}-A_n)(x_{i+1,n}-A_n)}{((n+\beta)(c-1)-c)^2}. \tag{4.9}$$

From Theorem 4.1 (i) we know that the zeros of $M_n(x;\beta,c)$ interlace with the zeros of $M_n(x;\beta+2,c)$ when $A_n \notin (z_{j,n}, z_{j+1,n}), j \in \{1,2,\ldots,n-1\}$, and therefore the right-hand side of (4.9) is negative if $A_n < x_{1,n}$. $M_{n-1}(x;\beta+2,c)$ will have a different sign at $x_{i,n}$ and $x_{i+1,n}$ for each $i \in \{1,2,\ldots,n-1\}$, which proves

$$x_{1,n} < z_{1,n-1} < x_{2,n} < z_{2,n-1} < \cdots < z_{n-1,n-1} < x_{n,n}. \qquad \square$$

Remarks:

(1) We observe from (4.2) that $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta+2,c)$ cannot have a common zero since any common zero would also be a zero of $M_n(x;\beta+1,c)$ and we know from (2.2) that $M_n(x;\beta,c)$ and $M_n(x;\beta+1,c)$ are co-prime.

(2) For β and c fixed with $-\beta, c \in (0,1)$, we have

$$A_n = -\beta - \frac{\beta}{1-c} - (n+1),$$

which tends to $-\infty$ as $n \to \infty$. We also deduce from Theorem 4.1 (i) and Theorem 4.2 (ii) that if $A_n < x_{1,n} < z_{1,n}$, the zeros of $M_n(x;\beta,c)$, $M_n(x;\beta+2,c)$ and $M_{n-1}(x;\beta+2,c)$ interlace as follows:

$$x_{1,n} < z_{1,n} < z_{1,n-1} < x_{2,n} < z_{2,n} < \cdots < z_{n-1,n} < z_{n-1,n-1} < x_{n,n} < z_{n,n}.$$

(3) The two inequalities (4.7) and (4.8) show that for a fixed $n \in \mathbb{N}$ the zeros of $M_n(x;\beta,c)$ and $M_{n-1}(x;\beta+2,c)$ are interlacing if and only if $x_{1,n} > -\beta - 1$.

THEOREM 4.3. *Let $n \in \mathbb{N}$ and fix β and c, $-\beta, c \in (0,1)$. Let $\{M_n(x;\beta,c)\}_{n=1}^{\infty}$ be the sequence of Meixner polynomials and assume $x_{2,n} > -\beta$ and that $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta+2,c)$ are co-prime. The zeros of $(x-B_2)M_{n-2}(x;\beta+2,c)$ interlace with the positive zeros of $M_n(x;\beta,c)$ where $B_2 = \frac{\beta c}{1-c} + n - 1$.*

Proof. Let $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ be the zeros of $M_n(x;\beta,c)$ and $z_{1,n-2} < z_{2,n-2} < \cdots < z_{n-2,n-2}$ the zeros of $M_{n-2}(x;\beta+2,c)$. From (4.4) we obtain $B_2 = \frac{\beta c}{1-c} + n - 1 = x_{1,1} + n - 1$, and we note that $x_{i,n} \neq B_2$, $i \in \{1,2,\ldots,n\}$, since we are assuming that $M_n(x;\beta,c)$ and $M_{n-2}(x;\beta+2,c)$ are co-prime. Evaluating (4.4) at successive positive zeros $x_{i,n}$ and $x_{i+1,n}$, $i \in \{2,3,\ldots,n-1\}$, of $M_n(x;\beta,c)$, we obtain

(4.10)
$$\frac{M_{n-1}(x_{i,n};\beta,c)M_{n-1}(x_{i+1,n};\beta,c)}{M_{n-2}(x_{i,n};\beta+2,c)M_{n-2}(x_{i+1,n};\beta+2,c)} = \frac{(n-1)^2(1-c)^2(x_{i,n}+\beta)_2(x_{i+1,n}+\beta)_2}{(\beta+n-1)^2(x_{i,n}-B_2)(x_{i+1,n}-B_2)}.$$

The numerator on the left-hand side is negative for each $i \in \{2,3,\ldots,n-1\}$. Since we are assuming $x_{2,n} > -\beta$, the numerator on the right-hand side of (4.10) is positive for each $i \in \{2,3,\ldots,n-1\}$. Therefore, $M_{n-2}(x_{i,n};\beta+2,c)M_{n-2}(x_{i+1,n};\beta+2,c) < 0$ for each $i \in \{2,3,\ldots,n-1\}$, unless $B_2 \in (x_{j,n}, x_{j+1,n})$ for some $j \in \{2,3,\ldots,n-1\}$.

If $B_2 \notin (x_{i,n}, x_{i+1,n})$, for any $i \in \{2,3,\ldots,n-1\}$, then $M_{n-2}(x;\beta+2,c)$ has $(n-2)$ zeros between $x_{2,n}$ and $x_{n,n}$ (all its zeros are positive) and interlacing holds between the $(n-2)$ simple, positive zeros of $M_{n-2}(x;\beta+2,c)$, together with the point B_2, and the $(n-1)$ positive zeros of $M_n(x;\beta,c)$.

If $B_2 \in (x_{j,n}, x_{j+1,n})$ for some $j \in \{2,3,\ldots,n-1\}$, then in the interval containing B_2, there will be no sign change of $M_{n-1}(x;\beta,c)$, but its sign will change in each of the remaining $(n-3)$ intervals with endpoints at successive zeros of $M_n(x;\beta,c)$. The remaining zero will be either $< x_{2,n}$ or $> x_{n,n}$, which proves our result. ∎

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