Applications of Hajós-type constructions to the Hedetniemi Conjecture

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Abstract

Hedetniemi conjectured in 1966 that if G and H are finite graphs with chromatic number n, then the chromatic number $\chi(G \times H)$ of the direct product of G and H is also n. We mention two well-known results pertaining to this conjecture and offer an improvement of the one, which partially proves the other. The first of these two results is due to Burr, Erdős and Lovász, who showed that when every vertex of a graph G with $\chi(G) = n+1$ is contained in an n-clique then $\chi(G \times H) = n+1$ whenever $\chi(H) = n+1$. The second, by Duffus, Sands and Woodrow, and, obtained independently by Welzl, states that the same is true when G and H are connected graphs each with clique number n. Our main result reads as follows: If G is a graph with $\chi(G) = n+1$ and has the property that the subgraph of G induced by those vertices of G that are not contained in an n-clique is homomorphic to an $(n+1)$-critical graph H, then $\chi(G \times H) = n+1$. This result is an improvement of the result by the first authors. In addition we will show that our main result implies a special case of the result by the second set of authors. Our approach will employ a construction of a graph F, with chromatic number $n+1$, that is homomorphic to G and H.

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1 Preliminaries and Introduction

All graphs considered here are simple, undirected and unlabelled, and have finite non-empty vertex sets. For a graph G we represent the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. Given a vertex $x \in G$, we denote the set of all neighbours of x in G by $N_G(x)$. For an edge $uv \in E(G)$ and a vertex $x \in V(G)$ we represent the graph obtained from G by deleting the edge uv by $G - uv$, and represent the graph obtained from G by deleting the vertex x along with the edges incident to it by $G - x$. For a subset A of $V(G)$, $G[A]$ will denote the subgraph of G induced by A. For a positive integer n, a *proper n-colouring* of G is a mapping $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ that does not assign the same value to any two adjacent vertices. Since such a mapping naturally induces a partition of the vertex set of G into n independent subsets we shall also refer to any partition of $V(G)$ into n such subsets as a proper n-colouring. The *chromatic number* $\chi(G)$ of a graph G is the least integer n required to achieve a proper

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n-colouring. The *clique number* $\omega(G)$ of a graph G is the largest integer k such that G contains a complete subgraph on k vertices. We use the symbol $G \sqcup H$ for the disjoint union of the graphs G and H. The disjoint union of k graphs $H_1, H_2, \ldots, H_k$ will be denoted by $\bigsqcup_{j=1}^{k} H_j$. The *direct product* $G \times H$ of G and H is that graph with vertex set

$$V(G \times H) = V(G) \times V(H)$$

and edge set $E(G \times H)$ satisfying the following:

$$(a_1, b_1)(a_2, b_2) \in E(G \times H) \text{ if and only if } a_1 a_2 \in E(G) \text{ and } b_1 b_2 \in E(H).$$

A *homomorphism* from a graph G to a graph H is an edge-preserving mapping from the vertex set of G into the vertex set of H. When such a mapping exists, we write $G \longrightarrow H$. A graph G is *critical* if, for all $e \in E(G)$, we have that $\chi(G - e) < \chi(G)$. We say G is *n-critical* if G is critical and $\chi(G) = n$. This definition implies that $\chi(G - x) < \chi(G)$ for all vertices $x \in V(G)$ of positive degree.

Hedetniemi's Conjecture 1. [7] *For all positive integers n and all graphs G and H, if $\chi(G) = \chi(H) = n$, then $\chi(G \times H) = n$.*

Our interest in Hedetniemi's conjecture has been to prove that it holds for those graphs G with $\omega(G) + 1 = \chi(G)$. To do this would imply, as shown by Duffus and Sauer in [3], that the chromatic number of the direct product of two n-chromatic graphs is at least $\frac{n}{2}$. We have not been successful in settling this special case. The following two theorems have a bearing on the above conjecture.

Theorem 1. (Burr, Erdős and Lovász [1]) *For all positive integers n and all graphs G and H with $\chi(G) = \chi(H) = n + 1$, if every vertex of G lies in an n-clique, then $\chi(G \times H) = n + 1$.*

Theorem 2. (Duffus, Sands and Woodrow [2], Welzl [9]) *For all positive integers n and all connected graphs G and H with $\chi(G) = \chi(H) = n + 1$, if $\omega(G) = \omega(H) = n$, then $\chi(G \times H) = n + 1$.*

From these results two questions immediately arise. The first, inspired by Theorem 1, is how many of the vertices of G are required to be in an n-clique. The second, arising from Theorem 2, is whether we can obtain the same result if $\omega(H) \leq n - 1$. Both of these questions we will partially answer, but we require that some restrictions be placed on G and H. Our answer to both is as follows: If those vertices of G not belonging to an n-clique induce a subgraph of G that has a homomorphism to an $(n + 1)$-critical subgraph of H, then the clique number of H may be relaxed to below n, and the number of vertices of G not in an n-clique need not to be prescribed.

We intend to show that Theorem 1 is in fact a corollary of our Theorem 3, and lastly, we will show that our Theorem 4, our main result, is not only an improvement of Theorem 1 but it also proves the special case of Theorem 2 when G and H are $(n + 1)$-critical. Our proof will involve the construction of a graph F that is homomorphic to G and H, and has the same chromatic number.

For the interested reader, an alternative proof of Theorem 1, involving uniquely n-colourable graphs, can be found in [2]. E. El-Zahar and N. Sauer have offered alternative proofs of Theorem 1 and Theorem 2 in [4].

2 Main results

We introduce in this section a construction that is reminiscent of the Hajós Construction [5], followed by another construction. Both are chromatic number preserving constructions.

Let G and H be graphs, $e = uv$ an edge of H, and X a subset of $V(G)$. For each $x \in X$, let H_x be a copy of $H - e$ in which each vertex w in $H - e$ has been renamed w_x. Thus u_x and v_x are the vertices in H_x that correspond to the vertices u and v in H, respectively. For each $x \in X$, replace x with H_x, then, for each pair H_x and H_y with $x \neq y$, make u_x adjacent to v_y or v_x adjacent to u_y, but not both, if and only if x and y are adjacent in G. Lastly, for each H_x make u_x or v_x adjacent to a vertex y in $V(G) \setminus X$, but not both, if and only if x and y are adjacent in G. Any resulting graph we refer to as a *Hajós-type construct (Ht-construct) of G and H obtained through X and e.*

Lemma 1. *Let G be a graph with $\chi(G) = n \geq 2$, and H be an n-critical graph. If $X \subseteq V(G)$ then every Ht-construct of G and H obtained through X and any edge e of H, has chromatic number n.*

Proof. Let G, H, and X be as described above, and let $e = uv$ be an edge of H, and F be an Ht-construct of G and H obtained through X and e.

First we show that $\chi(F) \geq n$. Suppose, to the contrary, that $\chi(F) < n$, and let f be a proper $(n-1)$-colouring of F. Since H is a n-critical graph it follows that in every proper $(n-1)$-colouring of $H - e$ the vertices u and v are assigned the same colour. Thus, for all $x \in X$, it follows that $f(u_x) = f(v_x)$. Now let g be a colouring of G determined by f as follows: For each $x \in V(G)$,

$$g(x) = \begin{cases} f(u_x) & \text{if } x \in X, \text{ and} \\ f(x) & \text{if } x \in V(G) \setminus X. \end{cases}$$

Since $\chi(G) = n$, it follows that $g(x) = g(y)$ for some adjacent vertices x and y in G. Furthermore, at least one of these vertices belongs to X. Suppose first that both x and y belong to X. Then it follows that $f(u_x) = f(v_x) = f(u_y) = f(v_y)$. But, by the construction of F, either $u_x v_y \in E(F)$ or $v_x u_y \in E(F)$. This, of course, implies that f is not a proper $(n-1)$-colouring of F. Clearly this is a contradiction. Now suppose that x belongs to X and y belongs to $V(G) \setminus X$. Then $f(u_x) = f(v_x) = f(y)$, but, by the construction of F, either $u_x y \in E(F)$ or $v_x y \in E(F)$. This, clearly, implies that f is not a proper $(n-1)$-colouring of F, which is a contradiction. Therefore it follows that $\chi(F) \geq n$.

Now we give an n-colouring of F. Let c be a proper n-colouring of G, then, for each $x \in X$, allow c_x to be a proper $(n-1)$-colouring of H_x satisfying $c_x(u_x) = c_x(v_x) = c(x)$. Next, let c' be an n-colouring of F defined, for each $z \in V(F)$, in the following way:

$$c'(z) = \begin{cases} c_x(z) & \text{if } z \in V(H_x), \text{ and} \\ c(z) & \text{if } z \in V(G) \setminus X. \end{cases}$$

Then c' is a proper n-colouring of F. □

The following result, which is given without proof, will be used in the proof of Theorem 3.

Lemma 2. *Let G and H be graphs with $\chi(G) = \chi(H) = n$, then $\chi(G \times H) = n$ if and only if there exists a graph F with $\chi(F) = n$ satisfying $F \longrightarrow G$ and $F \longrightarrow H$.*

Theorem 3. *Let G and H be graphs with $\chi(H) \geq \chi(G) = n + 1 \geq 3$ and $\omega(G) = n$. Let A be the set of vertices of G that are not contained in an n-clique. If there exists a proper $(n+1)$-colouring $V_1, \ldots, V_{n+1}$ of G satisfying $A \subseteq V_n \cup V_{n+1}$, then $\chi(G \times H) = n + 1$.*

Proof. Let G, H, A, and $V_1, \ldots, V_{n+1}$ be as described above. We will construct a graph F with $\chi(F) = n + 1$, satisfying $F \longrightarrow G$ and $F \longrightarrow H$. This will imply, by Lemma 2, that $\chi(G \times H) = n + 1$.

Let H' be any $(n + 1)$-critical subgraph of H. Such a subgraph exists and is connected. Begin by fixing an edge $e = uv$ of H'. Then, for each $x \in V_1 \cup V_2 \cup \ldots \cup V_{n-1}$, let H'_x be a copy of $H' - e$ as described earlier. Next, let F be the Ht-construct of G and H' obtained through $V_1 \cup V_2 \cup \ldots \cup V_{n-1}$ and e by introducing edges as follows.

For a pair H'_x and H'_y with $xy \in E(G)$, it follows, without loss of generality, that $x \in V_i$ and $y \in V_j$ for some integers i and j satisfying $1 \leq i < j \leq n - 1$. We introduce the edge $u_x v_y$ if j is even, and the edge $v_x u_y$ if j is odd. Lastly, for each H'_x and vertex $y \in V_n \cup V_{n+1}$ satisfying $xy \in E(G)$, make y adjacent to u_x if n is even and $y \in V_n$ or if n is odd and $y \in V_{n+1}$, otherwise make y adjacent to v_x. By Lemma 1 we have that $\chi(F) = n + 1$.

Let $U = \{u_x \in V(F) \mid x \in V_1 \cup \ldots \cup V_{n-1}\}$, and $V = \{v_x \in V(F) \mid x \in V_1 \cup \ldots \cup V_{n-1}\}$. If $n+1$ is odd then, for every vertex $x \in V_n$, we have that $N_F(x) \subseteq V_{n+1} \cup U$, and, for every vertex $x \in V_{n+1}$, we have that $N_F(x) \subseteq V_n \cup V$. If $n + 1$ is even then, for every vertex $x \in V_n$, we have $N_F(x) \subseteq V_{n+1} \cup V$, and, for every vertex $x \in V_{n+1}$, we have $N_F(x) \subseteq V_n \cup U$. In addition, it follows by the construction of F, that U and V are independent subsets of $V(F)$.

Now we prove that $F \longrightarrow H'$. Let $\phi : V(F) \longrightarrow V(H')$ be the mapping defined as follows: For each vertex w_x in H_x, $\phi(w_x) = w$. Thus the image of the restriction of ϕ to U is u, and the image of the restriction of ϕ to V is v. If $n + 1$ is even, the image of the restriction of ϕ to V_n is u, and the image of the restriction of ϕ to V_{n+1} is v. If $n + 1$ is odd, the image of the restriction of ϕ to V_n is v, and the image of the restriction of ϕ to V_{n+1} is u. Then ϕ is a homomorphism from F to H'. Since H' is a subgraph of H it follows that $F \longrightarrow H$.

Finally we prove that $F \longrightarrow G$. Since $\chi(H'_x) = n$ there exists a homomorphism γ from H'_x to K_n, and γ is such that $\gamma(u_x) = \gamma(v_x)$. For each $x \in V_1 \cup \ldots \cup V_{n-1}$, we have that $x \notin A$, therefore it belongs to a copy of K_n. So we can let γ_x be a homomorphism that maps H'_x into a complete subgraph K_n of G containing x. In addition let γ_x satisfy $\gamma_x(u_x) = \gamma_x(v_x) = x$. Finally, let $\phi : V(F) \longrightarrow V(G)$ be the mapping defined as follows: for each $x \in V_1 \cup \ldots \cup V_{n-1}$, the restriction of ϕ to H'_x is γ_x, and, for each $x \in V_n \cup V_{n+1}$, $\phi(x) = x$. Then ϕ is a homomorphism from F to G.

This completes our proof. □

The following corollary shows that the result obtained by Burr, Erdős and Lovász can be derived quite easily from Theorem 3.

Corollary 1. [1] *Let G and H be graphs with $\chi(G) = \chi(H) = n + 1$, $\omega(G) = n$. Let A, the set of vertices of G that are not contained in any n-clique of G, be empty. Then $\chi(G \times H) = n + 1$.*

Proof. Let G, H and A be as described above, then assume that $A = \emptyset$. In addition, let H' be an $(n + 1)$-critical subgraph of H. Then it follows that, for each $(n + 1)$-colouring $V_1, \ldots, V_{n+1}$ of $V(G)$ we have that $A \subseteq V_n \cup V_{n+1}$. By Theorem 3, we obtain that $\chi(G \times H') = n + 1$. Since $G \times H'$ is a subgraph of $G \times H$ it follows that $\chi(G \times H) \geq \chi(G \times H') = n + 1$. From this we obtain that $\chi(G \times H) = n + 1$. □

Next we develop the second of the two constructions mentioned earlier.

For an $(n + 1)$-critical graph H, $n \geq 2$, and a path $P = (x_1, x_2, \ldots, x_{k+1})$ in H with $k \geq 2$, let H_i, for each $1 \leq i \leq k$, be a copy of the graph $H - x_i x_{i+1}$ in which each vertex w has been renamed w^i. Next, starting with $\bigsqcup_{j=1}^{k} H_j$, for each integer $1 \leq j \leq k - 1$, identify the vertex

x^j_{j+1} of H_j with the vertex x^{j+1}_{j+1} of H_{j+1} to form a new vertex x'_{j+1}. This new graph we shall refer to as the *P-string of H with respect to P.*

If G is an $(n+1)$-chromatic graph, and $e = ab$ is an edge of G, then the *Hajos-Pe-string construct*, also written as $HPes$-construct, of G and H, is obtained by "replacing" the edge e of G with the P-string of H as follows: First, identify the vertex a of $G - ab$ with the vertex x^1_1 of the P-string of H. We rename this newly formed vertex a. Next, we make the vertex b of $G - ab$ adjacent to the vertex x^k_{k+1} of the P-string of H. This replacement of the edge $e = ab$ with a P-string in such a manner that b is made adjacent to x^k_{k+1} shall also be referred to as *hanging the P-string on b* or *attaching the P-string on a.*

For a set $\mathcal{P}$ of paths in H of length at least 2, and a set E of edges of G, consider the graph obtained by replacing, as described above, each and every edge in E with a P-string of H for some $P \in \mathcal{P}$. Let these replacements be done in such a way that for all $P \in \mathcal{P}$ there exists an edge $e \in E$ such that the edge e was replaced with the P-string of H. We call this graph a *Hajos-$\mathcal{P}E$-string-construct* or $H\mathcal{P}Es$-construct of G and H.

Lemma 3. *Let G and H be connected graphs such that $\chi(G) = \chi(H) = n + 1 \geq 3$ and H is critical. If $\mathcal{P}$ is a set paths of H, each of length at least 2, and $E \subseteq E(G)$ is a set of edges of G, then every $H\mathcal{P}Es$-construct of G and H has chromatic number $n + 1$.*

Proof. Let G, H, $\mathcal{P}$, and E be as described above, and let F be an $H\mathcal{P}Es$-construct of G and H. Assume that $\chi(F) < n+1$, and let $P = (x_1, x_2, \ldots, x_{k+1})$ be a path in $\mathcal{P}$. Then there exists an edge $e = ab \in E$ that was replaced by hanging the P-string of H, without loss of generality, on b. Let $C : V(F) \longrightarrow \{1, \ldots, n\}$ be a proper n-colouring of F. The subgraph of F induced by the vertices of the P-string of H hung on b is composed of graphs that are each isomorphic to $H - x_i x_{i+1}$ for some integer $1 \leq i \leq k$. Since H is $(n+1)$-critical it follows, for each $1 \leq i \leq k$, that every proper n-colouring of $H - x_i x_{i+1}$ assigns the same colour to the vertices x_i and x_{i+1}. Therefore C satisfies

$$C(a) = C(x'_2) = C(x'_3) = \cdots = C(x'_k) = C(x^k_{k+1}).$$

Since x^k_{k+1} is adjacent to b in F it follows that $C(x^k_{k+1}) \neq C(b)$, and hence $C(a) \neq C(b)$. We can therefore conclude, for all edges $ab \in E$, that $C(a) \neq C(b)$. This implies that the restriction of C to $V(G)$ is a proper n-colouring of G, which, of course, is a contradiction since $\chi(G) = n+1$. Therefore it follows that $\chi(F) \geq n + 1$.

Now we show that $\chi(F) \leq n + 1$. Let C be a proper $(n+1)$-colouring of G. Associate each edge $e = ab \in E$ with the path $P = (x_1, x_2, \ldots, x_{k+1})$ in $\mathcal{P}$ that is related to the P-string that replaced e. For each edge $ab \in E$ and its related path P, let b be the vertex which the P-string of H was hung on, then assign a proper n-colouring D^e_P to the P-string of H that satisfies

$$C(a) = D^e_P(x^1_1) = D^e_P(x'_2) = D^e_P(x'_3) = \cdots = D^e_P(x'_k) = D^e_P(x^k_{k+1}).$$

Such an n-colouring exists since all P-strings discussed here are n-colourable. These n-colourings D^e_P are not restricted to using only the colours in $\{1, 2, \ldots n\}$ but can use any n colours from the set $\{1, 2, \ldots, n, n+1\}$ as long as the above equality has been met. Now, for all vertices $u \in V(F)$, let $D : V(F) \longrightarrow \{1, \ldots, n+1\}$ be defined as follows

$$D(u) = \begin{cases} C(u) & \text{if } u \in V(G), \text{ and} \\ D^e_p(u) & \text{if } u \text{ belongs to the } P\text{-string of } H \text{ that replaces } e. \end{cases}$$

Then D is a proper $(n+1)$-colouring of F.

From the two above arguments we can conclude that $\chi(F) = n + 1$. □

Theorem 4. *Let G and H be connected graphs such that $\chi(G) = \chi(H) = n+1 \geq 3$, $\omega(G) = n$, and H is critical. Let A be the set of vertices of G that are not contained in any n-clique of G. If $G[A] \longrightarrow H$ then $\chi(G \times H) = n+1$.*

Proof. Let G, H, and A be as described above, and assume that $G[A] \longrightarrow H$. We will prove that $\chi(G \times H) = n+1$ by constructing a graph F, with chromatic number $n+1$, that is homomorphic to G and H.

Note that if $A = \emptyset$ then we can arrive at $\chi(G \times H) = n+1$ by the use of Corollary 1, therefore we can assume that $|A| \geq 1$. We remark that the graph $G[A]$ may or may not be connected. Let $f : A \longrightarrow V(H)$ be a homomorphism from $G[A]$ to H, and let B be the subset of A that contains those vertices of G that are adjacent to at least one vertex in $V(G) \setminus A$. Note that each component of $G[A]$ contains at least one vertex that belongs to B. For some integer $k \geq 1$, we can partition B into k non-empty sets $B_1, \ldots, B_k$ satisfying the following: For all $b, b' \in B$, we have that

$$f(b) = f(b') \text{ if and only if } b \text{ and } b' \text{ belong to the same set } B_i$$

From this it follows that there exist k vertices $u, v_2 \ldots, v_k \in V(H)$ such that $u \in f(B_1)$ and $v_i \in f(B_i)$ for all $2 \leq i \leq k$. This means that $f(B) = \{u, v_2, \ldots, v_k\}$. For all $2 \leq i \leq k$, let P_i be a shortest (in length) uv_i-path in H. For each $2 \leq i \leq k$, such a path P_i exists since H is connected.

Next, let $\mathcal{P}$ be the set of all these paths, then partition it into $\mathcal{P}_1$ and $\mathcal{P}_2$, where $\mathcal{P}_1$ is the set of those paths in $\mathcal{P}$ of length 1. Finally let $v \in V(H)$ be any vertex adjacent to u. We point out that v may belong to the set $\{v_2, \ldots, v_k\}$, and $\mathcal{P}_1$ or $\mathcal{P}_2$ may be empty, possibly both. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are empty, then $f(x) = u$ for each $x \in B$. Later, each path in $\mathcal{P}_2$ will be used to create P-strings of H provided that $\mathcal{P}_2$ is not empty.

Now, consider the subgraph $G[V(G) \setminus A]$ of G. If $\chi(G[V(G) \setminus A]) = n+1$ then, by Corollary 1, we have that $\chi(G[V(G) \setminus A] \times H) = n+1$ which implies that $\chi(G \times H) = n+1$. Therefore we may assume that $\chi(G[V(G) \setminus A]) = n$. Let $V_1, \ldots, V_n$ be a proper n-colouring of $G[V(G) \setminus A]$.

For each $x \in V(G) \setminus A$, let H_x be a copy of $H - uv$ as described earlier. Next, let F' be the Ht-construct of G and H obtained through $V(G) \setminus A$ and uv by introducing edges as follows: For a pair H_x and H_y with $xy \in E(G[V(G) \setminus A])$, it follows, without loss of generality, that $x \in V_i$ and $y \in V_j$ for some integers i and j satisfying $1 \leq i < j \leq n$. We introduce the edge $u_x v_y$ if j is even, and the edge $v_x u_y$ if j is odd. Lastly, for each H_x and vertex $y \in A$ satisfying $xy \in E(G)$, make y adjacent to u_x if $y \in B \setminus B_1$, and make y adjacent to v_x if $y \in B_1$. By Lemma 1 we have that $\chi(F') = n+1$.

We are now at the final step of our construction. Let E be the set of all edges of F' that connect a vertex of B to a vertex not in A, that is

$$E = \{wy \in E(F') \mid w \in V(F') \setminus A \text{ and } y \in B\}.$$

Thus the set E is not empty since $|A| \geq 1$ and G is connected. Notice that if $wy \in E$ and $y \in B \setminus B_1$ then $w = u_x$ for some $x \in V(G) \setminus A$. For each integer i with $1 \leq i \leq k$ let $E_i = \{wy \in E \mid y \in B_i\}$. Then the sets $E_1, \ldots, E_k$ are mutually disjoint, and $E = E_1 \cup \ldots \cup E_k$. If $\mathcal{P}_2 = \emptyset$ then we let F be the graph F', and we are done. If $\mathcal{P}_2 \neq \emptyset$ then we proceed as follows. For each integer $2 \leq i \leq k$, replace each edge $xy \in E_i$, where $y \in B_i$, with a P_i-string of H hung on y if and only if $P_i \in \mathcal{P}_2$. It should be clear that if, for some $2 \leq j \leq k$, the path P_j belongs to $\mathcal{P}_1$ then none of the edges in E_j are replaced. We call this new graph F. For some $E^* \subset E$, the graph F is an $H\mathcal{P}_2(E^*)s$-construct of F' and H, and thus, by Lemma 3, $\chi(F) = n+1$.

Let $U = \{u_x \in V(F) \mid x \in V(G) \setminus A\}$, and $V = \{v_x \in V(F) \mid x \in V(G) \setminus A\}$. Then the sets U and V are independent sets of vertices of F. This follows as a result of the construction of F.

Next we describe a homomorphism ϕ from F to H. For each path $P_i = (u = x_1, x_2, \ldots, x_{r+1} = v_i)$ in $\mathcal{P}_2$ let T_i be the P_i-string of H. The mapping $\phi_{P_i} : V(T_i) \longrightarrow V(H)$ defined, for all $z \in V(T_i)$, by

$$\phi_{P_i}(z) = \begin{cases} x_j & \text{if } z = x'_j \text{ for some } 2 \leq j \leq r, \text{ and} \\ w & \text{if } z = w^j \text{ for some } 1 \leq j \leq r \end{cases}$$

is a homomorphism from T_i to H. Let $\phi : V(F) \longrightarrow V(H)$ be the mapping defined as follows: The restriction of ϕ to A is the homomorphism f described in the beginning. The restriction of ϕ to a P_i-string of H is the homomorphism ϕ_{P_i}. For each vertex z in F that belongs to an H_x for some $x \in V(G) \setminus A$, since $z = w_x$ for some $w \in V(H)$, let $\phi(z) = w$.

By this definition, it follows that the image of the restriction of ϕ to U is u, and the image of the restriction of ϕ to V is v. One can quite easily verify that the mapping ϕ is a homomorphism from F to H.

Lastly we give a description of a homomorphism γ from F to G. For each H_x, let γ_x be a homomorphism that maps H_x into an n-clique of G that contains x. Select γ_x to be such that $\gamma_x(u_x) = \gamma_x(v_x) = x$.

For each edge $wy \in E$, where $y \in B \setminus B_1$, that was replaced by some P_i-string, we have that $w = u_x$ for some $x \in V(G) \setminus A$. Let $P_i = (u = x_1, x_2, \ldots, x_{r+1} = v_i)$, and let T_{wy} be that subgraph of F which is the P_i-string that replaced the edge wy. Next, for each integer j with $1 \leq j \leq r$, let γ_{H_j} be a homomorphism from H_j to an n-clique of G containing w. In addition let γ_{H_j} be such that $\gamma_{H_j}(x_j) = \gamma_{H_j}(x_{j+1}) = w$.

Now define the mapping $\gamma_{wy} : V(T_{wy}) \longrightarrow V(G)$ as follows. For all $z \in V(T_{wy})$,

$$\gamma_{wy}(z) = \begin{cases} w & \text{if } z \in \{x'_2, \ldots, x'_r\} \\ \gamma_{H_j}(z) & \text{if } z = x^j \text{ for some } 1 \leq j \leq r. \end{cases}$$

Then γ_{wy} is a homomorphism. Finally, define $\gamma : V(F) \longrightarrow V(G)$ as follows: the restriction of γ to A is the identity mapping, the restriction of γ to a P_i-string of H that replaced an edge $wy \in E$ is the homomorphism γ_{wy}, and, the restriction of γ to any $V(H_x)$ is γ_x. Then γ is a homomorphism from F to G. This completes our proof. □

It is not too difficult to see that Theorem 3 is a special case of Theorem 4.

3 Corollaries

In this section we describe some corollaries to Theorem 4 and link them to known results from [2] and [9]. We then offer one last result which is obtained by further increasing the restrictions on the graphs G and H. All the proofs that follow exploit the existence of a homomorphism from $G[A]$ to H.

Corollary 2. *Let G and H be graphs with $\chi(G) = \chi(H) = n + 1 \geq 3$, $\omega(H) \leq n = \omega(G)$, and H is $(n+1)$-critical. Let A be the set of vertices of G that are not contained in any n-clique of G. If $\chi(G[A]) \leq \omega(H)$ then $\chi(G \times H) = n + 1$.*

Proof. Let G, H, and A be as described. Since $\chi(G[A]) \leq \omega(H)$, there is a homomorphism from $G[A]$ to H, and thus, by Theorem 4, it follows that $\chi(G \times H) = n + 1$. □

The next corollary is a special case of the result in [2] and [9]. Although it bears a striking similarity to the aforementioned result, it requires that G and H be critical. This is to ensure that $\chi(G[A]) \leq n$, which guarantees that $G[A] \longrightarrow H$.

Corollary 3. *Let G and H be connected $(n+1)$-critical graphs with $\omega(G) = \omega(H) = n$. Then $\chi(G \times H) = n + 1$.*

Proof. Let G and H be as described above and let A be the set of all vertices not contained in an n-clique of G. We may assume A is not empty, since otherwise we would have $\chi(G \times H) = n+1$ by Corollary 1. Since G is critical it follows that $\chi(G[A]) \leq n$, and thus $\chi(G[A]) \longrightarrow H$ since $\omega(H) = n$. Then, by Theorem 4, we have that $\chi(G \times H) = n + 1$. □

For our final result we will require two definitions; the first of these two concepts is introduced in [6]. A graph G with $\chi(G) = n$ is *uniquely n-colourable* if there is exactly one partition of its vertex set $V(G)$ into n independent sets. A graph G is said to be a *homomorphic image* of a graph H if there exists a homomorphism ϕ from H onto G satisfying the following: For every edge $xy \in E(G)$ there exists an edge $uv \in E(H)$ such that $\phi(u) = x$ and $\phi(v) = y$. We note that this concept is essentially the same as the one defined in [8].

Theorem 5. *Let G and H be connected graphs such that $\chi(G) = \chi(H) = n+1$, $\omega(G) = n$, and H is $(n+1)$-critical. In addition, let A, the set of vertices of G that are not contained in an n-clique of G, be such that $\chi(G[A]) = n$. If there exists a uniquely n-colourable subgraph H' of H with $G[A]$ a homomorphic image of H', then $\chi(G \times H) = n + 1$.*

Proof. Let G, H, H' and A be as described above. Let ϕ be a homomorphism from H' to $G[A]$ and let $B \subseteq A$ to be the set of those vertices of G that are adjacent to a vertex in $V(G) \setminus A$. Finally let G' be that graph with vertex set $V(G') = (V(G) \setminus A) \cup V(H')$ and edge set

$$E(G') = E(G[V(G) \setminus A]) \cup E(H') \cup N,$$

where $N = \{xy \mid x \in V(G) \setminus A, y \in V(H'), \phi(y) \in B \text{ and } x\phi(y) \in E(G)\}$. It follows immediately that $G' \longrightarrow G$: This can be seen from the mapping $\gamma : V(G') \longrightarrow V(G)$ whose restriction to $V(G) \setminus A$ is the identity mapping and whose restriction to $V(H')$ is ϕ. From this it follows that $\chi(G') \leq \chi(G) = n + 1$. We claim that $\chi(G') = n + 1$. Suppose this is not so, then $\chi(G') = n$ since $\omega(G') = n$. From this it follows that there exists a proper n-colouring C of G'. In addition, $|C(\phi^{-1}(x))| = 1$ for all $x \in A$, since the assumption that $|C(\phi^{-1}(x))| > 1$ for some $x \in A$ implies that there exists an alternative proper n-colouring of H', one in which the vertices of $\phi^{-1}(x)$ have been assigned at least two different colours, which would imply that H' is not uniquely n-colourable. But then $|C(\phi^{-1}(x))| = 1$ for all $x \in A$, which implies that the mapping $C' : V(G) \longrightarrow \{1, 2, \ldots, n\}$ defined below is a proper n-colouring of G. For all $x \in V(G)$, let

$$C'(x) = \begin{cases} C(x) & \text{if } x \in V(G) \setminus A \text{ and} \\ C(\phi^{-1}(x)) & \text{if } x \in A. \end{cases}$$

This is clearly a contradiction, thus it follows that $\chi(G') = n + 1$. By Theorem 4 it now follows that $\chi(G' \times H) = n + 1$ since the vertices of G' which are not contained in an n-clique induce a subgraph of G' that is homomorphic to H. From $\chi(G \times H) \geq \chi(G' \times H)$ we obtain that $\chi(G \times H) = n + 1$. □

If the hypothesis of Theorem 5 permitted $\chi(G[A])$ to be less than n and H' to be uniquely $\chi(G[A])$-colourable, then we would be faced with the difficulty of proving that the colouring C in the proof of Theorem 5 is such that $|C(\phi^{-1}(x))| = 1$ for all $x \in A$. The reason for this is that the restriction of C to $V(H')$ may use more colours than $\chi(G[A])$ which would not allow us to use the unique $\chi(G[A])$-colourability of H'.

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