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Numerical stabilization with boundary controls for hyperbolic systems of balance laws

by

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Submitted in partial fulfilment of the requirements for the degree of

Magister Scientiae

in the Department of Mathematics and Applied Mathematics

in the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

July 2016

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

Signature:

Name: Gediyon Yemane Weldegiyorgis

Date:

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This dissertation is dedicated to my family and friends.

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Acknowledgment

Firstly, I would like to express my sincere gratitude to my supervisor **Professor Mapundi Banda** for the continuous support of my MSc study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this dissertation. Without his guidance and persistent help this dissertation would not have been possible.

Secondly, I would like to thank the African Institute for Mathematical Sciences (AIMS), South Africa for financial support granted as a matching master's bursary. I also thank the Department of Mathematics and Applied Mathematics at University of Pretoria for financial and material support.

Finally, I must express my very profound gratitude to my family and to my friends for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this dissertation. This accomplishment would not have been possible without them. THANK YOU God!

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Abstract

In this dissertation, boundary stabilization of a linear hyperbolic system of balance laws is considered. Of particular interest is the numerical boundary stabilization of such systems. An analytical stability analysis of the system will be presented as a preamble. A discussion of the application of the analysis on specific examples: telegrapher equations, isentropic Euler equations, Saint-Venant equations and Saint-Venant-Exner equations is also presented. The first order explicit upwind scheme is applied for the spatial discretization. For the temporal discretization a splitting technique is applied. A discrete $\mathbb{L}^2-$Lyapunov function is employed to investigate conditions for the stability of the system. A numerical analysis is undertaken and convergence of the solution to its equilibrium is proved. Further a numerical implementation is presented. The numerical computations also demonstrate the stability of the numerical scheme with parameters chosen to satisfy the stability requirements.

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Chapter 1

Introduction

We consider hyperbolic systems of conservation laws with source terms (also called hyperbolic systems of balance laws) in one-space dimension. If the system involves k hyperbolic balance laws, then it can be written in the form

$$\partial_t U + \partial_x F(U) + G(U) = 0, \tag{1.1}$$

where $U(x,t)$, $F(U)$ and $G(U)$ represent vectors of k physical quantities, vector of flux functions associated with physical quantities and source terms that balance the system of conservation laws, respectively. The balance laws are complemented by initial condition and for finite spatial domain boundary conditions need to be defined. The notations ∂_t and ∂_x denote partial differentiation with respect to t and x, respectively. In the case where there are no source terms involved (i.e. $G(U) \equiv 0, \forall U$), a hyperbolic system of conservation laws in one-space dimension can be described as a set of k conservation laws in the following form

$$\partial_t U + \partial_x F(U) = 0. \tag{1.2}$$

Hyperbolic systems of balance laws are useful in describing the transport of a set of physical quantities that have a mathematical physics and an engineering interest. Therefore, these kinds of systems have a large number of applications in the physical modeling of different phenomena. Some of the relevant examples include the telegrapher equation that describes the propagation of an electric signal along an electric line [31], the isentropic gas dynamics that

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describes the flow of gas through a medium [35], the Aw-Rascle and Greenberg equation that describes the dynamics of road traffic flow [45], the Saint-Venant equation that describes the dynamics of shallow water flow along an open channel [8], the Saint-Venant Exner equation that describes the dynamics of shallow water flow with sediment transportation along an open channel [22] and the Euler equations for gas dynamics that describes the behavior of gas flow in pipelines [13]. In addition, hyperbolic systems of balance laws are used to model networked flow in which case the flow through the edge of the network is governed by the balance law. The flow through the node (or vertex) of the network, outgoing or ingoing to an edge, is coupled by algebraic conditions motivated by physical considerations.

The mathematical analysis of balance laws has been an active field of research for more than fifty years [18]. For more details of the mathematical properties of balance laws, the reader is referred to Introduction part in [18]. In particular, in the recent past the boundary control problem of hyperbolic systems of balance laws has been a very active area of research [6, 17, 16, 46, 22, 21, 25, 27, 34, 39, 42]. In a nutshell, this is the problem of finding boundary control action that will be used at the boundary points (or at the nodes of the network in networked flow) to stabilize the solution of hyperbolic systems of balance laws to a preferred equilibrium state.

In such cases, researchers have focused on well-posedness (i.e. existence, uniqueness and stability) of a hyperbolic system of balance laws [15, 28]. Besides the existences and uniqueness, a stability analysis for boundary control of a hyperbolic system of balance laws has been studied for different categories such as for a linear hyperbolic system of conservation laws in [34], a linear hyperbolic system of balance laws in [42], non-linear hyperbolic systems of conservation laws in [39] and networks of such in [20]. More details of the Lyapunov stability analysis of a linear hyperbolic system of balance laws are discussed in Chapter 3.

For the stability analysis of a boundary control of a hyperbolic system of balance laws, there exists a candidate Lyapunov function which is in a positive quadratic form. According to the Lyapunov stability analysis theorem, the time derivative of the Lyapunov function needs to be in negative quadratic form to show the stability of the system. This technique was introduced for a linear hyperbolic system of conservation laws in [17, 25], a linear hyperbolic system of

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balance laws in [9] and later for networks of hyperbolic 2×2 systems of balance laws in [8]. This approach is briefly analyzed for a linear hyperbolic system of balance laws in [22]. Using the discussion of the linear case, this approach is further extended to a non-linear hyperbolic system of conservation laws in [16] and remarks have been given to extend it further [16].

For practical purposes numerical aspects of a hyperbolic system of balance laws have also seen tremendous development [40, 41]. For instance, in the above mentioned literature and in general, numerical results for the stability analysis of a hyperbolic systems of balance laws have been given to support the theoretical stability analysis. This stability analysis has been applied to some important examples such as shallow water flow along an open channel [17, 8, 22, 19] and gas dynamics [4]. However, there is limited literature on the boundary stability analysis of hyperbolic systems of balance laws. Recently, researchers have investigated conditions for the numerical stability of a discretized linear hyperbolic system of conservation laws by considering a single flow domain [3] and networked flow domain[24]. Beside these, numerical boundary stabilization for a linear hyperbolic system of balance laws with constant coefficients was considered in [26]. In these studies, a discrete Lyapunov function was introduced and the decay rate of the time derivative was shown.

The purpose of this work is to consider a numerical boundary stabilization of a linear hyperbolic system of balance laws with a varying flux function. The analysis of a numerical discretization of stabilization problems with boundary controls for a linear hyperbolic system of balance laws will be undertaken. In particular, an investigation of conditions for the decay of discrete solutions of linear hyperbolic systems of balance laws will be presented. For details of numerical analysis the reader is referred to Chapter 5. The numerical stability analysis technique used in this work is similar to that used in [3]. A discrete Lyapunov function will be applied in the investigation of conditions for decay rates depending on numerical schemes that have been employed. Numerical results for this study will be presented in Section 5.2.

In summary the dissertation is organized as follows: Chapter 2 gives a brief introduction to hyperbolic systems of balance laws, explains some relevant topics for the discussion of this dissertation and provides applications into physical modeling including telegrapher's equations, isentropic Euler equations, Saint-Venant equations and Saint-Venant-Exner equations.

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In Chapter 3, the analytical stability analysis of the systems in general and particular form are presented. Beside this, the analytical stability analysis for examples discussed in Chapter 2 is presented. In particular, details in the investigation of conditions for the stability of such systems are shown.

In Chapter 4, the numerical methods for linear hyperbolic systems of balance laws are presented and the numerical discretization of the above mentioned examples are discussed.

Chapter 5 is the main part of this dissertation. The numerical stability of the linear system in general and particular form is analyzed. In particular, conditions for the numerical stability analysis are investigated. Furthermore, the numerical stability analysis and simulations are performed on the above mentioned examples. The main purpose of the computation is to test the validity of the numerical stability analysis and to compare the implications of the analytical results.

Finally, the main conclusions are presented and possible topics for further work are also mentioned.

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Chapter 2

Hyperbolic Systems of Balance Laws

The focus of this chapter is to introduce a hyperbolic systems of balance laws in one space dimension. We restrict ourselves to linear (or linearized) hyperbolic systems of balance laws, which have wide application to some physical problems such as shallow water flow with and without transportation of sediment along a channel and current flow along an electrical transmission line. In particular, we discuss the general, quasilinear, linearly coupled and decoupled representations of hyperbolic systems of balance laws. Furthermore, we deal with the general initial and boundary conditions for the hyperbolic systems of balance laws. Finally, shallow water flow along a channel with and without transportation of sediment, current flow along an electrical transmission line and isentropic gas dynamics are presented and they are among relevant examples of hyperbolic systems of balance laws, which will be discussed in this dissertation.

2.1 Balance Laws in One-Space Dimension

Consider a system of k first-order partial differential equations in one-space dimension of the form

$$\partial_t u_i + \partial_x f_i(u_1, \ldots, u_k) + g_i(u_1, \ldots, u_k) = 0, \quad i = 1, \ldots, k, \tag{2.1}$$

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where the unknowns u_i depend on space, x, and time, t. Equations in (2.1) can be expressed in the form

$$\partial_t U + \partial_x F(U) + G(U) = 0, \quad x \in \mathbb{R} \quad \text{and} \quad t \geq 0, \tag{2.2}$$

where $U := [u_1, \ldots, u_k]^T : \mathbb{R} \times [0, +\infty) \to \Omega$, with $\Omega \subset \mathbb{R}^k$ being a non-empty, open and connected set, is a vector of k physical quantities, $F(U) := [f_1(U), \ldots, f_k(U)]^T : \Omega \to \mathbb{R}^k$ is a vector of k flux functions which depend on the components of U only and $G(U) := [g_1(U), \ldots, g_k(U)]^T : \Omega \to \mathbb{R}^k$ is a vector of source terms also depending on the components of U only. The system (2.2) is called a system of balance laws. If the source terms are neglected (or $G \equiv 0$), the system is written in the form

$$\partial_t U + \partial_x F(U) = 0. \tag{2.3}$$

The system (2.3) is called a system of conservation laws.

2.2 Hyperbolicity and Strict Hyperbolicity

In order to study hyperbolicity for the system (2.2), we assume a smooth solution vector U with smooth flux function $F(U)$. By applying the chain rule, the system (2.2) can be expressed in a quasilinear form

$$\partial_t U + A(U)\partial_x U + G(U) = 0, \quad x \in \mathbb{R} \quad \text{and} \quad t \geq 0, \tag{2.4}$$

with a Jacobian matrix

$$A(U) := \nabla_U F(U) = \begin{bmatrix} \partial_{u_1} f_1(U) & \ldots & \partial_{u_k} f_1(U) \\ \vdots & \ldots & \vdots \\ \partial_{u_1} f_k(U) & \ldots & \partial_{u_k} f_k(U) \end{bmatrix}.$$

Definition 1 (Hyperbolicity, [18])**.** *The system* (2.4) *is called hyperbolic if for every* $U \in \Omega$*, the* $k \times k$ *Jacobian matrix* $A(U)$ *has real eigenvalues* $\lambda_1(U) \leq \cdots \leq \lambda_k(U)$ *and* k *linearly independent right eigenvectors* $r_1(U), \ldots, r_k(U)$.

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Let us recall that the right eigenvectors of $A(U)$ satisfy

$$A(U)r_i(U) = \lambda_i(U)r_i(U), \quad i = 1, \dots, k.$$

Similarly, left eigenvectors, $l_1(U), \dots, l_k(U)$ associated with the eigenvalues, $\lambda_1(U), \dots, \lambda_k(U)$ satisfy

$$l_i(U)A(U) = \lambda_i(U)l_i(U), \quad i = 1, \dots, k. \tag{2.5}$$

Moreover, if eigenvalues are distinct, then left and right eigenvectors are orthogonal and can be normalized [38]. In this case, the normalized right and left eigenvectors associated with distinct eigenvalues satisfy

$$l_i(U)r_j(U) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Definition 2 (Strict Hyperbolicity, [18])**.** *The system* (2.4) *is called strictly hyperbolic if for every* $U \in \Omega$*, the* $k \times k$ *Jacobian matrix* $A(U)$ *has real and distinct eigenvalues* $\lambda_1(U) < \dots < \lambda_k(U)$*.*

2.3 Characteristic Curves and Variables

In the system (2.4), each equation can contain the relation of a temporal derivative of a physical quantity with spatial derivatives of one or more physical quantities. Therefore, the system (2.4) is known as a strongly coupled one [5]. However, the system (2.4) can be expressed in characteristic form [7]. This can be done by applying a coordinate transformation. Let us define a coordinate curve by $V := \phi(U)$ under a transformation $\phi : \Omega \to \Gamma \subset \mathbb{R}^k$ such that the pre-image is defined by $U := \phi^{-1}(V)$ and

$$H(U)A(U) = D(U)H(U), \tag{2.6}$$

with $H(U) := \nabla_U \phi(U)$ and $D(U) := \text{diag}\{\lambda_i(U) : i = 1, \dots, k\}$. Therefore, pre-multiplying both sides of the system (2.4) by $H(U)$ and using (2.6), we transform the system (2.4) into the following form

$$\partial_t V + \mathcal{J}(V)\partial_x V + M(V) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{2.7}$$

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with

$$\mathcal{J}(V) := D(\phi^{-1}(V)) \quad \text{and} \quad M(V) := H(\phi^{-1}(V))G(\phi^{-1}(V)).$$

Remark 1. *In condition* (2.6)*, the matrix of left eigenvectors of the Jacobian matrix,* $L_E(U)$*, such that* $L_E(U)A(U) = D(U)L_E(U)$*, can be taken to complete the transformation of the system* (2.4) *into the system* (2.7)*, which is commonly referred to as decoupled or weakly coupled [5], since each equation does not involve spatial derivatives of other characteristic variables.*

Remark 2. *In linear systems case, if the Jacobian matrix is diagonalizable, has distinct eigenvalues, then there exist coordinate curves that satisfy* (2.6) *[7]. Otherwise (i.e. in the nonlinear case), we can only obtain coordinate curves that satisfy the condition in* (2.6) *when* $k = 2$*. For further detail, the reader is referred to literature [18, 32].*

2.4 Steady State and Linearization of Hyperbolic Systems of Balance Laws

For the reason that we are interested in linear or linearized hyperbolic systems of balance laws, a review of some concepts about a linear hyperbolic system of balance laws is useful. Using the discussion in [12], the system (2.7) is said to be linear with constant coefficients if all diagonal entries of $\mathcal{J}(V)$ and components of $M(V)$ are constant. If all diagonal entries of $\mathcal{J}(V)$ and components of $M(V)$ depend on the independent variables x and t, then the system (2.7) is said to be linear with variable coefficients. The system (2.7) is still linear if components of $M(V)$ depend linearly on the characteristic variables.

In case we have a nonlinear system, we deal with linearization about a steady state. We introduce below the definition of a steady state.

Definition 3 (Steady state [7])**.** *A steady state (or equilibrium) is a time-invariant solution* $U(x,t) = U^*(x)$ *for all* $t \in [0, +\infty)$ *of the system* (2.2)*, that is* U^* *satisfies*

$$\partial_x F(U^*(x)) + G(U^*(x)) = 0, \quad x \in \mathbb{R}. \tag{2.8}$$

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We use now Taylor's approximation in order to linearize the vector valued function $F(U)$ and $G(U)$, which are assumed to be at least twice continuously differentiable, about the equilibrium point $U^*(x)$. We obtain

$$F(U) = F(U^*(x)) + \nabla_U F(U)|_{U=U^*(x)}(U - U^*(x)) + R_F, \tag{2.9a}$$

and

$$G(U) = G(U^*(x)) + \nabla_U G(U)|_{U=U^*(x)}(U - U^*(x)) + R_G, \tag{2.9b}$$

with remainder in Lagrange form

$$\begin{aligned} R_F &:= \frac{1}{2!}\nabla_U^2 F(U)|_{U=U_c(x)}(U - U^*(x))^2, \quad \text{and} \\ R_G &:= \frac{1}{2!}\nabla_U^2 G(U)|_{U=U_c(x)}(U - U^*(x))^2, \end{aligned}$$

for some $U_c(x)$ component-wise between $U^*(x)$ and U. Thus, substituting the linearized form (2.9) of $F(U)$ and $G(U)$, by neglecting the remainder R_F and R_G, into the system (2.2) and using (2.8), we obtain

$$\begin{aligned} &\partial_t(U - U^*(x)) + \left(\nabla_U F(U)|_{U=U^*(x)}\right)\partial_x(U - U^*(x)) + \partial_x F(U^*(x)) + G(U^*(x)) \\ &+ \left[\partial_x\left(\nabla_U F(U)|_{U=U^*(x)}\right) + \nabla_U G(U)|_{U=U^*(x)}\right](U - U^*(x)) = 0, \quad x \in \mathbb{R} \quad t \geq 0. \end{aligned} \tag{2.10}$$

By the discussion in [7], one sees the system (2.10) is the linearization of the system (2.2) about the steady state, which can be rewritten as

$$\partial_t Z + B(x)\partial_x Z + C(x)Z = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{2.11}$$

with

$$\begin{aligned} &Z := U - U^*(x), \quad B(x) := \nabla_U F(U)|_{U=U^*(x)} \quad \text{and} \\ &C(x) := \partial_x\left(\nabla_U F(U)|_{U=U^*(x)}\right) + \nabla_U G(U)|_{U=U^*(x)}. \end{aligned} \tag{2.12}$$

If the Jacobian matrix $B(x)$ is diagonalizable, then there are real eigenvalues $\lambda_i(x) := \lambda_i(U^*(x)), i = 1, \dots, k$.

In order to decouple the linearized system (2.11), let us define a coordinate curve by $W := \psi(Z)$ under a transformation $\psi : \Omega \to \Gamma \subset \mathbb{R}^k$ such that the pre-image is defined by

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$Z = \psi^{-1}(W)$ and by definition of coordinate transformation for each steady state we have $W^*(x) = \psi(Z^*(x)) \equiv 0$. In addition, the transformation satisfies

$$N(Z)B(x) = \Lambda(x)N(Z), \tag{2.13}$$

where

$$N(Z) := \nabla_Z \psi(Z) \quad \text{and} \quad \Lambda(x) := \text{diag}\{\lambda_i(x) : i = 1, \ldots, k\}. \tag{2.14}$$

Thus, pre-multiplying both sides of the system (2.11) by $N(Z)$ and using (2.13), we transform the system (2.11) into the following form

$$\partial_t W + \Lambda(x)\partial_x W + \Pi(x)W = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{2.15}$$

where

$$\Pi(x)W := N(\psi^{-1}(W))C(x)\psi^{-1}(W). \tag{2.16}$$

Remark 3. *We observe now the nonlinear system* (2.2) *is linearized about a steady state solution, $U^*(x)$, and is expressed as a decoupled system* (2.15)*. Furthermore, at steady state of the system* (2.2)*, the decoupled system* (2.15) *has a steady state solution $W^*(x) \equiv 0$.*

Remark 4. *If $G(U^*(x)) = 0$, then the system* (2.2) *has a constant steady state [7], which is both time and space invariant, and the linearized decoupled system* (2.15) *can be written as*

$$\partial_t W + \Lambda \partial_x W + \Pi W = 0, \quad x \in \mathbb{R} \quad t \geq 0, \tag{2.17}$$

where Λ and Π are constant matrices.

In the case the system (2.2) can be transformed into the decoupled system (2.7), the system (2.7) can be expressed using the total time derivative [7, 37] of the following form

$$\frac{dv_i}{dt} := \partial_t v_i + \frac{dx}{dt}\partial_x v_i, \quad i = 1, \ldots, k, \tag{2.18}$$

with

$$\frac{dx}{dt} = \lambda_i(V) + \frac{M_i(V)}{\partial_x v_i}, \quad i = 1, \ldots, k, \tag{2.19}$$

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where $\lambda_i(V) := \lambda_i(\phi^{-1}(U))$, $M_i(V)$ are the components of $M(V)$ and v_i are also the components of V or characteristic variables, which are constant along the corresponding characteristic curves defined by the differential equations (2.19).

Due to strict hyperbolicity of the decoupled system (2.7), the non-zero eigenvalues, which are also called characteristic speeds, can be separated into positive and negative. Therefore, depending on the sign of the characteristic speeds, the characteristic curves are illustrated in Figure 2.1 below.

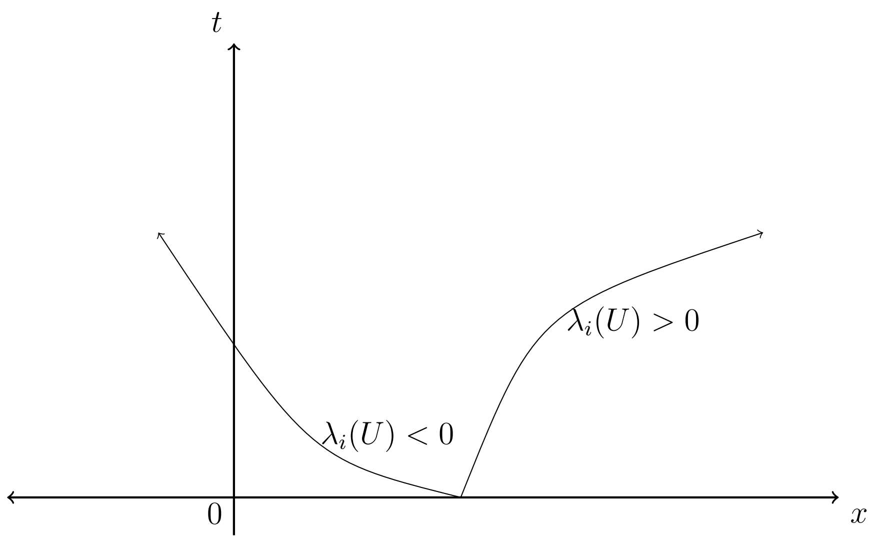

Figure 2.1: Characteristic curves.

Similarly, the decoupled linearized system (2.15) can be written in terms of the total time derivative in the form

$$\frac{dw_i}{dt} := \partial_t w_i + \frac{dx}{dt}\partial_x w_i, \quad i = 1, \ldots, k, \tag{2.20}$$

with

$$\frac{dx}{dt} = \lambda_i(W(x,t)) + \frac{\Pi_i(x)W(x,t)}{\partial_x w_i}, \quad i = 1, \ldots, k, \tag{2.21}$$

where $\Pi_i(x)$ are the rows of the matrix $\Pi(x)$ and w_i are the components of W or characteristic variables, which are constant along the corresponding characteristic curves defined by the differential equations (2.21).

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From both equations (2.18) and (2.20), one obtains $\frac{dv_i}{dt} = 0$ and $\frac{dw_i}{dt} = 0$ for each $i = 1, \ldots, k$, respectively. Therefore, these characteristic variables are called Riemann invariants [7].

2.5 Initial and Boundary Conditions

In order to formulate a Cauchy problem with the system (2.2), which has a unique solution, we need to specify initial and boundary conditions. Therefore, the spatial domain can be restricted to a finite interval $[0, l]$.

The initial condition for each physical quantity can be specified in the following form

$$U(x, 0) = U_0(x), \quad x \in [0, l], \tag{2.22}$$

where $U_0 : \mathbb{R} \to \Omega$ is a vector function depending on x. Since we are interested in the linear system (2.15), the transformation given by (2.12) and (2.13) can be used to express the initial condition (2.22) in terms of the characteristic variables

$$W(x, 0) = \psi(Z(x, 0)) = W_0(x), \quad x \in [0, l], \tag{2.23}$$

with

$$Z(x, 0) = U(x, 0) - U^*(x) = U_0(x) - U^*(x), \tag{2.24}$$

where $W_0 : [0, l] \to \Omega$ is a vector function depending on x.

The boundary conditions for the system (2.2) at the boundary points of $[0, l]$ are required to include information about the incoming and outgoing signals [7]. It is described by the following form

$$\Upsilon(U(0, t), U(l, t)) = 0, \quad t \geq 0, \tag{2.25}$$

with $\Upsilon : \Omega \times \Omega \to \mathbb{R}^k$. Then, the incoming wave should be determined by the outgoing wave at the boundary points.

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In the interest of expressing boundary conditions for the linear system (2.15) in terms of characteristic variables, the system can be assumed to be strictly hyperbolic. Therefore, there are non-zero distinct eigenvalues or characteristic speeds. Without loss of generality, the characteristic speeds can be arranged in the following order

$$\lambda_k(x) < \cdots < \lambda_{m+1}(x) < 0 < \lambda_m(x) < \cdots < \lambda_1(x), \quad \forall x \in [0, l]. \tag{2.26}$$

Thus, the diagonal matrix $\Lambda(x)$ in the system (2.15) can be expressed in the following form

$$\Lambda(x) := \text{diag}\{\Lambda^+(x), -\Lambda^-(x)\}, \quad \forall x \in [0, l],$$

with $\Lambda^+(x) := \text{diag}\{\lambda_1(x), \ldots, \lambda_m(x)\}$ and $\Lambda^- := \text{diag}\{|\lambda_{m+1}(x)|, \ldots, |\lambda_k(x)|\}$.

Since the characteristic variables represent travelling waves along rightward and leftward directions corresponding to m positive and $k-m$ negative characteristic speeds, which is illustrated in Figure 2.2 below, the number of boundary conditions specified in terms of the characteristic variables at $x = 0$ and $x = l$ should be equal to the number of positive and negative characteristic speeds, respectively [5, 7]. Therefore, the characteristic variables can be categorized in the following form

$$W := \left[W^+, W^-\right]^T, \tag{2.27}$$

with

$$W^+ := [w_1, \ldots, w_m]^T \quad \text{and} \quad W^- := [w_{m+1}, \ldots, w_k]^T. \tag{2.28}$$

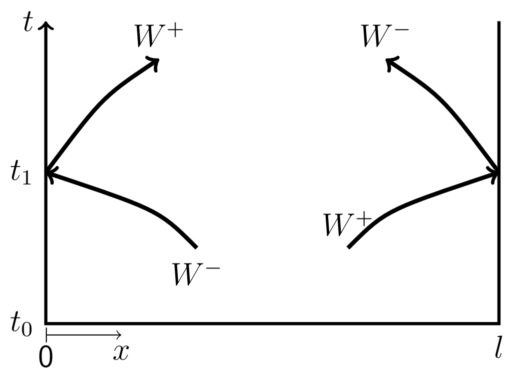

Figure 2.2: The moving waves to the leftward and rightward directions.

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Then, using the transformation given by (2.12) and (2.13), the incoming and outgoing information can be expressed in terms of the characteristic variables in the following form [7]

$$W_{\text{in}}(t) := \left[W^+(0,t), W^-(l,t)\right]^T \quad \text{and} \quad W_{\text{out}}(t) := \left[W^+(l,t), W^-(0,t)\right]^T, \quad t \geq 0. \tag{2.29}$$

The linear boundary condition for the system (2.15) is defined by

$$W_{\text{in}}(t) = KW_{\text{out}}(t), \tag{2.30}$$

with

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

where K_{11}, K_{12}, K_{21} and K_{22} are real matrices of dimensions $m \times m$, $m \times (k-m)$, $(k-m) \times m$ and $(k-m) \times (k-m)$, respectively and the coefficients are determined from the boundary conditions (2.25) using the transformation given by (2.12) and (2.13) [5, 7, 22].

The Cauchy problem in terms of the characteristic variables is formulated as follows

$$\partial_t W + \Lambda(x)\partial_x W + \Pi(x)W = 0, \quad x \in [0,l], \quad t \geq 0, \tag{2.31a}$$

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \quad t \geq 0, \tag{2.31b}$$

$$W(x,0) = W_0(x), \quad x \in [0,l]. \tag{2.31c}$$

The Cauchy problem under the special case of the linear system with constant coefficients is formulated as follows:

$$\partial_t W + \Lambda\partial_x W + \Pi W = 0, \quad x \in [0,l], \quad t \geq 0, \tag{2.32a}$$

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \quad t \geq 0, \tag{2.32b}$$

$$W(x,0) = W_0(x), \quad x \in [0,l]. \tag{2.32c}$$

Remark 5. *The condition for the initial condition* (2.23) *compatible with the boundary conditions* (2.30) *may be required in the well-posedness of the Cauchy problems* (2.31) *and* (2.32)*, which will be discussed in the next section.*

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2.6 Well-posedness of The Cauchy Problem for Linear Hyperbolic Systems of Balance Laws

Consider the Cauchy problem (2.31) associated with the linear hyperbolic systems of balance laws (2.15), the boundary conditions (2.30) and the initial condition (2.23). In case, the initial condition, $W_0 \in \mathbb{H}^1((0,l);\mathbb{R}^k)$, satisfies the compatibility condition

$$\begin{bmatrix} W_0^+(0) \\ W_0^-(l) \end{bmatrix} = K \begin{bmatrix} W_0^+(l) \\ W_0^-(0) \end{bmatrix}, \tag{2.33}$$

then, we have the following theorem.

Theorem 1 ([7]). *For every $W_0 \in \mathbb{H}^1((0,l);\mathbb{R}^k)$ satisfying the compatibility condition* (2.33), *there exists a unique solution*

$$W \in C^1(\mathbb{L}^2((0,l);\mathbb{R}^k);[0,+\infty)) \cap C^0(\mathbb{H}^1((0,l);\mathbb{R}^k);[0,+\infty))$$

of the Cauchy problem (2.31)*. Moreover, there exists $C_0 > 0$ such that, for every $W_0 \in \mathbb{H}^1((0,l);\mathbb{R}^k)$ satisfying the compatibility condition* (2.33)*, this unique solution W satisfies*

$$\|W(\cdot,t)\|_{\mathbb{H}^1((0,l);\mathbb{R}^k)} + \|W(\cdot,t)\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)} \leq C_0 e^{C_0 t}\|W_0\|_{\mathbb{H}^1((0,l);\mathbb{R}^k)}, \quad \forall t \in [0,+\infty),$$
$$\|W(\cdot,t)\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)} \leq C_0 e^{C_0 t}\|W_0\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)}, \quad \forall t \in [0,+\infty).$$

In case the compatibility condition is not satisfied, we deal with the weak solutions of the Cauchy problem (2.31), which is defined as follows:

Definition 4 (Weak solution [7]). *Let $W_0 \in \mathbb{L}^2((0,l);\mathbb{R}^k)$. An $\mathbb{L}^2$-solution $W : (0,l) \times (0,+\infty) \to \mathbb{R}^k$ of the Cauchy problem* (2.31) *is a map $W \in C^0(\mathbb{L}^2((0,l);\mathbb{R}^k),[0,+\infty))$ such that, for every $T > 0$ and every $\varphi^T = [\varphi^+, \varphi^-] \in C^1([0,l] \times [0,T];\mathbb{R}^k)$ satisfying*

$$\begin{bmatrix} \varphi^+(l,t) \\ \varphi^-(0,t) \end{bmatrix} = \begin{bmatrix} \Lambda^+(l)^{-1}K_{11}^T\Lambda^+(0) & \Lambda^+(l)^{-1}K_{21}^T\Lambda^-(l) \\ \Lambda^-(0)^{-1}K_{12}^T\Lambda^+(0) & \Lambda^-(0)^{-1}K_{22}^T\Lambda^-(l) \end{bmatrix} \begin{bmatrix} \varphi^+(0,t) \\ \varphi^-(l,t) \end{bmatrix}, \quad \forall t \in [0,T],$$

we have

$$\int_0^T \int_0^l \left(\partial_t\varphi^T(x,t) + \partial_x\varphi^T(x,t)\Lambda(x) + \varphi^T(x,t)\left(\partial_x\Lambda(x) - \Pi(x)\right)\right) W dx dt = \int_0^l \varphi^T(x,T)W(x,T)dx - \int_0^l \varphi^T(x,0)W_0(x)dx.$$

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Theorem 2 ([7]). *For every $W_0 \in \mathbb{L}^2((0,l);\mathbb{R}^k)$, the Cauchy problem* (2.31) *has a unique $\mathbb{L}^2$-solution. Moreover, there exists $C_1 > 0$ such that, for every $W_0 \in \mathbb{L}^2((0,l);\mathbb{R}^k)$, the solution W to the Cauchy problem* (2.31) *satisfies*

$$\|W(\cdot,t)\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)} \leq C_1 e^{C_1 t}\|W_0\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)}, \quad \forall t \in [0,+\infty).$$

Consider the Cauchy problem associated with the linear system with constant coefficients (2.17), the boundary conditions (2.30) and the initial condition (2.23). Then, we have the following definition of a weak solution:

Definition 5 (Weak solution [22]). *Let $W_0 \in \mathbb{L}^2((0,l);\mathbb{R}^k)$. A map $W : (0,l)\times[0,+\infty) \to \mathbb{R}^k$ is a solution of a Cauchy problem* (2.32) *if $W \in C^0(\mathbb{L}^2((0,l);\mathbb{R}^k),[0,+\infty))$ is such that, for every $\varphi^T = [\varphi^+,\varphi^-] \in C^1([0,l]\times[0,+\infty);\mathbb{R}^k)$ with compact support and satisfying*

$$\begin{bmatrix}\varphi^+(l,t)\\ \varphi^-(0,t)\end{bmatrix} = \begin{bmatrix}(\Lambda^+)^{-1}K_{11}^T\Lambda^+ & (\Lambda^+)^{-1}K_{21}^T\Lambda^- \\ (\Lambda^-)^{-1}K_{12}^T\Lambda^+ & (\Lambda^-)^{-1}K_{22}^T\Lambda^-\end{bmatrix}\begin{bmatrix}\varphi^+(0,t)\\ \varphi^-(L,t)\end{bmatrix},$$

we have

$$\int_0^{+\infty}\int_0^l \left[\partial_t\varphi^T(x,t) + \partial_x\varphi^T(x,t)\Lambda - \varphi^T(x,t)\Pi\right] W dxdt + \int_0^l \varphi^T(x,0)W_0(x)dx = 0.$$

With this definition, we have the following proposition.

Proposition 1 ([22]). *For every $W_0 \in \mathbb{L}^2((0,l);\mathbb{R}^k)$, the Cauchy problem* (2.32) *has a unique solution. Moreover, for every $T > 0$, there exists $C(T) > 0$ such that, for every $W_0 \in \mathbb{L}^2((0,l);\mathbb{R}^k)$, the solution to the Cauchy problem* (2.32) *satisfies*

$$\|W(\cdot,t)\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)} \leq C(T)\|W_0\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)}, \quad \forall t \in [0,T].$$

Finally, consider a Cauchy problem associated with a linear system of conservation laws

$$\begin{aligned}&\partial_t W + \Lambda(x)\partial_x W = 0, \quad x \in [0,l], \quad t \geq 0,\\ &\begin{bmatrix}W^+(0,t)\\ W^-(l,t)\end{bmatrix} = K\begin{bmatrix}W^+(l,t)\\ W^-(0,t)\end{bmatrix}, \quad t \geq 0, \qquad (2.34a)\\ &W(x,0) = W_0(x), \quad x \in [0,l],\end{aligned}$$

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and a Cauchy problem associated with the source term

$$\begin{aligned} &\partial_t W + \Pi(x)W = 0, \quad x \in [0,l], \quad t \geq 0, \\ &\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \quad t \geq 0, \\ &W(x,0) = W_0(x), \quad x \in [0,l], \end{aligned} \tag{2.34b}$$

separately. Then, the well-posedness of the Cauchy problem (2.31) makes the system (2.34) well-posed [14].

2.7 Examples of Hyperbolic Systems of Balance Laws

In this section, we deal with hyperbolicity, characteristic decomposition and linearization of some examples of hyperbolic systems of balance laws.

2.7.1 The Telegrapher Equations

Consider a piece of wire that connects a power supply and a resistive load, which can be used in modeling the propagation of current and voltage along this wire (or transmission line). This physical phenomena is modeled by the following system of balance laws [31]

$$L\partial_t I(x,t) + \partial_x V(x,t) + RI(x,t) = 0, \tag{2.35a}$$

$$C\partial_t V(x,t) + \partial_x I(x,t) + GV(x,t) = 0, \quad x \in [0,l], \quad t \geq 0, \tag{2.35b}$$

where I and V represent the current density and the voltage along the transmission line at any time, respectively. The parameters L, C, R and G (all per unit length) represent the line self-inductance, the line capacitance, the resistance of the two conductors and the admittance of the dielectric material separating the conductors, respectively (see Figure 2.3 below).

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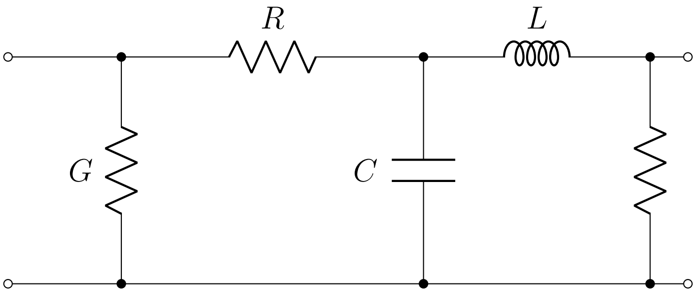

Figure 2.3: Example of a transmission line

In order to define a Cauchy problem associated with the telegrapher equations (2.35), the following initial and boundary conditions are specified

$$\begin{bmatrix} I(x,0) \\ V(x,0) \end{bmatrix} = \begin{bmatrix} I_0(x) \\ V_0(x) \end{bmatrix}, \quad x \in [0,l], \tag{2.36}$$

and

$$V(0,t) = -R_0 I(0,t) \quad \text{and} \quad V(l,t) = R_l I(l,t), \quad t \geq 0, \tag{2.37}$$

where R_0 and R_l represent the internal resistance of the power and the load, respectively.

Assuming non-zero value of the constants L, C, R and G, the system (2.35) can be expressed in a linear system with constant coefficients as given by

$$\partial_t Y + A\partial_x Y + BY = 0, \tag{2.38}$$

with

$$Y := \begin{bmatrix} I \\ V \end{bmatrix}, \quad A := \begin{bmatrix} 0 & L^{-1} \\ C^{-1} & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} L^{-1}R & 0 \\ 0 & C^{-1}G \end{bmatrix}.$$

Remark 6 ([31])**.** *One obtains an equation*

$$a\partial_t^2 V - \partial_x^2 V + b\partial_t V + cV = 0, \tag{2.39}$$

with

$$a := LC, \quad b := RC + LG \quad \text{and} \quad c := RG,$$

by eliminating I from the system (2.35)*. This can be done by differentiating equations* (2.35a) *and* (2.35b) *with respect to x and t, respectively and substituting one into the other. If $b = c = 0$, then equation* (2.39) *has a wave equation form.*

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With steady state solution $I^*(x)$, $V^*(x)$ such that

$$\partial_x V^*(x) + RI^*(x) = 0 \quad \text{and} \quad \partial_x I^*(x) + GV^*(x) = 0, \tag{2.40}$$

the system (2.38) can be written as a linear system about a steady state as follows:

$$\partial_t Z + A\partial_x Z + BZ = 0, \tag{2.41}$$

with

$$Z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} I - I^*(x) \\ V - V^*(x) \end{bmatrix}.$$

The system (2.41) has two distinct eigenvalues, which have opposite signs

$$\lambda_2 := -\frac{1}{\sqrt{LC}} < 0 < \frac{1}{\sqrt{LC}} =: \lambda_1.$$

Thus, the system (2.41) is strictly hyperbolic, which can be transformed into characteristic form. Therefore, each characteristic variable is defined using left eigenvectors of A corresponding to eigenvalues or characteristic speeds

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \sqrt{\frac{L}{C}} & 1 \\ -\sqrt{\frac{L}{C}} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 + z_1\sqrt{\frac{L}{C}} \\ z_2 - z_1\sqrt{\frac{L}{C}} \end{bmatrix}.$$

and the inverse transformation gives

$$z_1 = \frac{1}{2}(w_1 - w_2)\sqrt{\frac{C}{L}} \quad \text{and} \quad z_2 = \frac{1}{2}(w_1 + w_2).$$

Hence, by using the characteristic variables the system (2.41) is expressed as a decoupled linear system with constant coefficients

$$\partial_t W + \Lambda\partial_x W + \Pi W = 0, \quad x \in [0, l], \quad t \geq 0, \tag{2.42}$$

with

$$W := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad \Pi := \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{bmatrix},$$

where

$$\gamma_1 := \frac{1}{2}\left(\frac{G}{C} + \frac{R}{L}\right) \quad \text{and} \quad \gamma_2 := \frac{1}{2}\left(\frac{G}{C} - \frac{R}{L}\right).$$

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The initial and boundary conditions can be expressed in terms of the characteristic variables by using the characteristic variables and inverse transformation into the initial condition (2.36) and the boundary conditions (2.37). i.e.

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} := \begin{bmatrix} (V_0(x) - V^*(x)) + (I_0(x) - I^*(x))\sqrt{\frac{L}{C}} \\ (V_0(x) - V^*(x)) - (I_0(x) - I^*(x))\sqrt{\frac{L}{C}} \end{bmatrix}, \tag{2.43}$$

and

$$\begin{bmatrix} w_1(0,t) \\ w_2(l,t) \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix}, \tag{2.44}$$

with

$$k_{12} := \frac{R_0\sqrt{C} - \sqrt{L}}{R_0\sqrt{C} + \sqrt{L}} \quad \text{and} \quad k_{21} := \frac{R_l\sqrt{C} - \sqrt{L}}{R_l\sqrt{C} + \sqrt{L}}.$$

2.7.2 The Isentropic Euler Equations

Consider an inviscid ideal gas flowing in a rigid cylindrical pipe with l unit cross-section (see Figure 2.4) with no heat transfer through the pipe (called isentropic flow). This model is governed by isentropic Euler equations, which are derived from Euler equations for gas dynamics by neglecting the enthalpy [7, 35]. The simplified isentropic Euler equations in the form of a system of balance laws are given by

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho V) = 0, \\ &\partial_t V + \partial_x\left(\frac{V^2}{2}\right) + \frac{1}{\rho}\partial_x P(\rho) + CV|V| = 0, \end{aligned} \qquad x \in [0,l], \quad t \geq 0, \tag{2.45}$$

where ρ, V and $P(\rho)$ (with $P'(\rho) > 0$) denote the density, velocity and pressure of gas and $C|V|$ corresponds to a friction term.

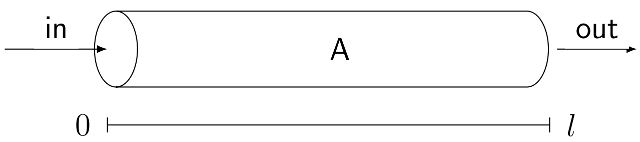

Figure 2.4: An example of a pipe.

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Initial and boundary conditions for the system (2.45) can be specified in the following form

$$\begin{bmatrix} \rho(x,0) \\ V(x,0) \end{bmatrix} = \begin{bmatrix} \rho_0(x) \\ V_0(x) \end{bmatrix}, \quad x \in [0,l], \tag{2.46}$$

and

$$V(0,t) = -k_0\rho(0,t) \quad \text{and} \quad V(l,t) = k_l\rho(l,t), \quad t \geq 0, \tag{2.47}$$

where k_0 and k_l are arbitrary constants.

In quasilinear form, the system (2.45) can be written as follows

$$\partial_t Y + A(Y)\partial_x Y + G(Y) = 0, \tag{2.48}$$

with

$$Y := \begin{bmatrix} \rho \\ V \end{bmatrix}, \quad A(Y) := \begin{bmatrix} V & \rho \\ P'(\rho)/\rho & V \end{bmatrix} \quad \text{and} \quad G(Y) := \begin{bmatrix} 0 \\ CV|V| \end{bmatrix},$$

where $\sqrt{P'(\rho)}$ is the sound velocity.

The system (2.45) about a steady state solution $Y^*(x) := [\rho^*(x), V^*(x)]^T$ such that

$$\begin{aligned} &\partial_x\left(\rho^*(x)V^*(x)\right) = 0, \\ &\partial_x\left(\frac{V^{*2}(x)}{2}\right) + \frac{1}{\rho^*(x)}\partial_x P(\rho^*(x)) + CV^*(x)|V^*|(x) = 0, \end{aligned} \tag{2.49}$$

can be linearized in the following form

$$\partial_t Z + A(x)\partial_x Z + B(x)Z = 0, \tag{2.50}$$

with

$$Z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} \rho - \rho^*(x) \\ V - V^*(x) \end{bmatrix}, \quad A(x) := A(Y^*(x)) \quad \text{and}$$

$$B(x) := \begin{bmatrix} \partial_x V^*(x) & \partial_x \rho^*(x) \\ \left(\frac{P''(\rho^*(x))}{\rho^*(x)} - \frac{P'(\rho^*(x))}{\rho^{*2}(x)}\right)\partial_x\rho^*(x) & \partial_x V^*(x) + 2C|V^*(x)| \end{bmatrix}.$$

The linearized system (2.50) under subsonic conditions (i.e. $V^2 < P'(\rho)$) has two distinct eigenvalues, which are opposite in sign

$$\lambda_2(Y^*(x)) := V^*(x) - \sqrt{P'(\rho^*(x))} < 0 < V^*(x) + \sqrt{P'(\rho^*(x))} =: \lambda_1(Y^*(x)).$$

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In order to decouple the linearized system (2.50), the matrix of left eigenvectors of $A(x)$ can be used and then the characteristic variables are

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \frac{\sqrt{P'(\rho^*(x))}}{\rho^*(x)} & 1 \\ -\frac{\sqrt{P'(\rho^*(x))}}{\rho^*(x)} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 + z_1 \frac{\sqrt{P'(\rho^*(x))}}{\rho^*(x)} \\ z_2 - z_1 \frac{\sqrt{P'(\rho^*(x))}}{\rho^*(x)} \end{bmatrix},$$

and the inverse transformation gives

$$z_1 = \frac{\rho^*(x)}{2\sqrt{P'(\rho^*(x))}} (w_1 - w_2) \quad \text{and} \quad z_2 = \frac{1}{2}(w_1 + w_2).$$

The linearized system (2.50) can be written in characteristic variables, which is a decoupled linear system with variable coefficients, as follows

$$\partial_t W + \Lambda(x)\partial_x W + \Pi(x)W = 0, \quad x \in [0, l], \quad t \geq 0, \tag{2.51}$$

with

$$W := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \Lambda(x) := \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} \quad \text{and} \quad \Pi(x) := \begin{bmatrix} \gamma_{11}(x) & \gamma_{12}(x) \\ \gamma_{21}(x) & \gamma_{22}(x) \end{bmatrix},$$

where the characteristic speeds $\lambda_1(x) := \lambda_1(Y^*(x))$ and $\lambda_2(x) := \lambda_2(Y^*(x))$ and the coefficients

$$\gamma_{11}(x) := \frac{CV^*(x)|V^*(x)|}{V^{*2}(x) - P'(\rho^*(x))} \left(\frac{\rho^*(x)P''(\rho^*)}{2\sqrt{P'(\rho^*(x))}} - \frac{\rho^*(x)P''(\rho^*)}{4P'(\rho^*(x))} - V^*(x) + \frac{1}{2} \right) + C|V^*(x)|,$$

$$\gamma_{12}(x) := \frac{CV^*(x)|V^*(x)|}{V^{*2}(x) - P'(\rho^*(x))} \left(\sqrt{P'(\rho^*(x))} + \frac{\rho^*(x)P''(\rho^*)}{4P'(\rho^*(x))} - \frac{\rho^*(x)P''(\rho^*)}{2\sqrt{P'(\rho^*(x))}} - \frac{1}{2} \right) + C|V^*(x)|,$$

$$\gamma_{21}(x) := -\frac{CV^*(x)|V^*(x)|}{V^{*2}(x) - P'(\rho^*(x))} \left(\frac{\rho^*(x)P''(\rho^*)}{4P'(\rho^*(x))} - \sqrt{P'(\rho^*(x))} + \frac{\rho^*(x)P''(\rho^*)}{2\sqrt{P'(\rho^*(x))}} - \frac{1}{2} \right) + C|V^*(x)| \quad \text{and}$$

$$\gamma_{22}(x) := -\frac{CV^*(x)|V^*(x)|}{V^{*2}(x) - P'(\rho^*(x))} \left(\frac{1}{2} - \frac{\rho^*(x)P''(\rho^*)}{4P'(\rho^*(x))} - \frac{\rho^*(x)P''(\rho^*)}{2\sqrt{P'(\rho^*(x))}} - V^*(x) \right) + C|V^*(x)|.$$

The initial condition (2.46) and the boundary conditions (2.47) in the new coordinates can be expressed in the following form

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} := \begin{bmatrix} (V_0(x) - V^*(x)) + (\rho_0(x) - \rho^*(x)) \frac{\sqrt{P'(\rho^*(x))}}{\rho^*(x)} \\ (V_0(x) - V^*(x)) - (\rho_0(x) - \rho^*(x)) \frac{\sqrt{P'(\rho^*(x))}}{\rho^*(x)} \end{bmatrix}, \quad x \in [0, l], \tag{2.52}$$

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and

$$\begin{bmatrix} w_1(0,t) \\ w_2(l,t) \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix}, \quad t \geq 0, \tag{2.53}$$

with

$$k_{12} := \frac{k_0\rho^*(0) - \sqrt{P'(\rho^*(0))}}{k_0\rho^*(0) + \sqrt{P'(\rho^*(0))}} \quad \text{and} \quad k_{21} := \frac{k_l\rho^*(l) - \sqrt{P'(\rho^*(l))}}{k_l\rho^*(l) + \sqrt{P'(\rho^*(l))}}.$$

2.7.3 The Saint-Venant Equations with Source Terms

Consider water flow along a prismatic channel with rectangular cross-section, a width of l unit and constant bottom slope (see Figure 2.5). In this case, the Saint-Venant Equations [8] is given by

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t V + \partial_x\left(\frac{1}{2}V^2 + gH\right) + (C_f\frac{V^2}{H} - gS_b) = 0, \end{aligned} \qquad x \in [0,l], \quad t \geq 0, \tag{2.54}$$

where the physical quantities H and V represent the depth and velocity of the water, respectively and the source terms g, C_f and S_b represent a gravitational constant, a friction parameter and the constant bottom slope of the channel, respectively.

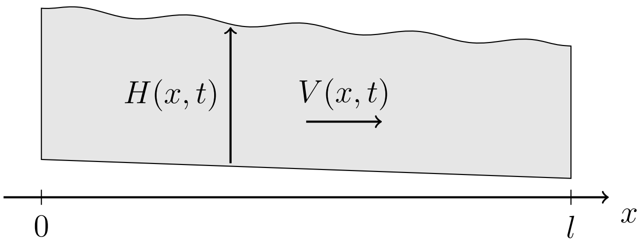

Figure 2.5: Lateral view of a prismatic channel with rectangular cross-section, width of l unit and constant bottom slope.

The initial and boundary conditions for the system (2.54) are specified as follows

$$\begin{bmatrix} H(x,0) \\ V(x,0) \end{bmatrix} = \begin{bmatrix} H_0(x) \\ V_0(x) \end{bmatrix}, \quad x \in [0,l], \tag{2.55}$$

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and

$$V(0,t) = -k_0 H(0,t) \quad \text{and} \quad V(l,t) = k_l H(l,t), \quad t \geq 0, \tag{2.56}$$

where k_0 and k_l are parameters.

The system (2.54) can be written in quasilinear form

$$\partial_t Y + A(Y)\partial_x Y + G(Y) = 0, \tag{2.57}$$

with

$$Y := \begin{bmatrix} H \\ V \end{bmatrix}, \quad A(Y) := \begin{bmatrix} V & H \\ g & V \end{bmatrix} \quad \text{and} \quad G(Y) := \begin{bmatrix} 0 \\ g(C_f \frac{V^2}{H} - S) \end{bmatrix}.$$

The Jacobian matrix $A(Y)$ has the following eigenvalues

$$\lambda_1 := V + \sqrt{gH} \quad \text{and} \quad \lambda_2 := V - \sqrt{gH}.$$

Under the sub-critical flow condition i.e. $V < \sqrt{gH}$, we have $\lambda_2 < 0 < \lambda_1$. Therefore, the system (2.57) is strictly hyperbolic, which can be transformed into characteristic form. Since we are dealing with a linear system, a linearized system about steady state is considered.

Let $Y^*(x) := [H^*(x), V^*(x)]^T$ be a steady state solution for the system (2.57) such that

$$\begin{aligned} &\partial_x(H^* V^*(x)) = 0, \\ &\partial_x \left(\frac{1}{2} V^{*2}(x) + gH^*(x) \right) + (C_f \frac{V^{*2}(x)}{H^*(x)} - gS_b) = 0, \end{aligned} \qquad x \in [0, l]. \tag{2.58}$$

The linearization of the system (2.54) about the steady state is given by

$$\partial_t Z + A(x)\partial_x Z + B(x) Z = 0, \tag{2.59}$$

with

$$Z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} H - H^*(x) \\ V - V^*(x) \end{bmatrix}, \quad A(x) := \begin{bmatrix} V^*(x) & H^*(x) \\ g & V^*(x) \end{bmatrix} \quad \text{and}$$

$$B(x) := \begin{bmatrix} \partial_x V^*(x) & \partial_x H^*(x) \\ -C_f \left(\frac{V^*(x)}{H^*(x)} \right)^2 & \partial_x V^*(x) + 2C_f \left(\frac{V^*(x)}{H^*(x)} \right) \end{bmatrix}.$$

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The characteristic variables for the system (2.59) are defined using left eigenvectors of $A(x)$ in the following form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \sqrt{\frac{g}{H^*(x)}} & 1 \\ -\sqrt{\frac{g}{H^*(x)}} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 + z_1\sqrt{\frac{g}{H^*(x)}} \\ z_2 - z_1\sqrt{\frac{g}{H^*(x)}} \end{bmatrix}.$$

and the inverse transformation gives

$$z_1 = \frac{1}{2}(w_1 - w_2)\sqrt{\frac{H^*(x)}{g}} \quad \text{and} \quad z_2 = \frac{1}{2}(w_1 + w_2).$$

Using the steady state condition (2.58), the characteristic variables and the inverse transformation, the system (2.59) can be written in the following characteristic form

$$\partial_t \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} \partial_x \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \gamma_{11}(x) & \gamma_{12}(x) \\ \gamma_{21}(x) & \gamma_{22}(x) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0, \tag{2.60}$$

with the characteristic speeds $\lambda_1(x) := V^*(x) + \sqrt{gH^*(x)}$ and $\lambda_2(x) := V^*(x) - \sqrt{gH^*(x)}$ and the coefficients

$$\begin{aligned}
\gamma_{11}(x) :=& \frac{C_f V^{*2}(x)}{H^*(x)} \left[\frac{4V^*(x) - 2\sqrt{gH^*(x)} - 1}{4(gH^*(x) - V^{*2}(x))} + \frac{1}{V^*(x)} - \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&+ \frac{gS_b}{gH^*(x) - V^{*2}(x)} \left[\frac{1 + 2\sqrt{gH^*(x)} - 4V^*(x)}{4} \right], \\
\gamma_{12}(x) :=& \frac{C_f V^{*2}(x)}{H^*(x)} \left[\frac{1 - 2\sqrt{gH^*(x)}}{4(gH^*(x) - V^{*2}(x))} + \frac{1}{V^*(x)} + \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&- \frac{gS_b}{gH^*(x) - V^{*2}(x)} \left[\frac{1 - 2\sqrt{gH^*(x)}}{4} \right], \\
\gamma_{21}(x) :=& \frac{C_f V^{*2}(x)}{H^*(x)} \left[\frac{1 + 2\sqrt{gH^*(x)}}{4(gH^*(x) - V^{*2}(x))} + \frac{1}{V^*(x)} - \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&- \frac{gS_b}{gH^*(x) - V^{*2}(x)} \left[\frac{1 + 2\sqrt{gH^*(x)}}{4} \right], \quad \text{and} \\
\gamma_{22}(x) :=& \frac{C_f V^{*2}(x)}{H^*(x)} \left[\frac{1}{V^*(x)} + \frac{1}{2\sqrt{gH^*(x)}} - \frac{1 - 2\sqrt{gH^*(x)} - 4V^*(x)}{4(gH^*(x) - V^{*2}(x))} \right] \\
&+ \frac{gS_b}{gH^*(x) - V^{*2}(x)} \left[\frac{1 - 2\sqrt{gH^*(x)} - 4V^*(x)}{4} \right].
\end{aligned}$$

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The initial condition (2.55) and the boundary conditions (2.56) in characteristic variables can be expressed in the following form

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} := \begin{bmatrix} (V_0(x) - V^*(x)) + (H_0(x) - H^*(x))\sqrt{\frac{g}{H^*(x)}} \\ (V_0(x) - V^*(x)) - (H_0(x) - H^*(x))\sqrt{\frac{g}{H^*(x)}} \end{bmatrix}, \quad x \in [0,l], \tag{2.61}$$

and

$$\begin{bmatrix} w_1(0,t) \\ w_2(l,t) \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix}, \quad t \geq 0, \tag{2.62}$$

with

$$k_{12} := \frac{k_0 - \sqrt{\frac{g}{H^*(0)}}}{k_0 + \sqrt{\frac{g}{H^*(0)}}} \quad \text{and} \quad k_{21} := \frac{k_l - \sqrt{\frac{g}{H^*(l)}}}{k_l + \sqrt{\frac{g}{H^*(l)}}}.$$

2.7.4 The Saint-Venant-Exner Equations with Source Terms

Consider the transport of sediments in a water flow along the prismatic channel with rectangular cross-section and constant bottom slope where the sediment moves predominantly as bed load (see Figure 2.6). In this case, the Saint-Venant-Exner equations [22] is achieved by coupling of the Exner equation to the Saint-Venant equations, which is given by

$$\begin{aligned} &\partial_t H + \partial_x (HV) = 0, \\ &\partial_t V + \partial_x \left(\frac{1}{2}V^2 + g(H+B)\right) + \left(C_f \frac{V^2}{H} - gS_b\right) = 0, \quad x \in [0,l], \quad t \geq 0, \\ &\partial_t B + \partial_x \left(\frac{1}{3}aV^3\right) = 0, \end{aligned} \tag{2.63}$$

where the physical quantities H, V and B represent the depth of the water and the velocity of the water, the bathymetry, respectively and the source terms g, S_b and C_f represent a gravitational constant, the bottom slope of the channel and a friction coefficient, respectively with a parameter a, that encompasses porosity and viscosity effects on the sediment dynamics.

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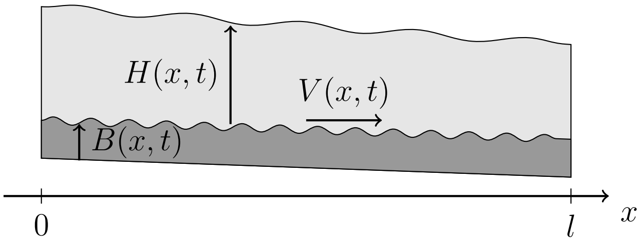

Figure 2.6: Lateral view of a prismatic channel with rectangular cross-section, constant bottom slope and a sediment bed.

Initial and boundary conditions for the system (2.63) can be specified in the following form

$$\begin{bmatrix} H(x,0) \\ V(x,0) \\ B(x,0) \end{bmatrix} = \begin{bmatrix} H_0(x) \\ V_0(x) \\ B_0(x) \end{bmatrix}, \quad x \in [0,l], \tag{2.64}$$

and

$$V(0,t) = -k_0 H(0,t), \quad V(l,t) = -k_l \left(H(l,t) + B(l,t)\right) \quad \text{and} \quad B(0,t) = 0, \quad t \geq 0, \tag{2.65}$$

where k_0 and k_l are parameters.

The system (2.63) in quasilinear form can be written as follows

$$\partial_t Y + A(Y)\partial_x Y + G(Y) = 0, \tag{2.66}$$

with

$$Y := \begin{bmatrix} H \\ V \\ B \end{bmatrix}, \quad A(Y) := \begin{bmatrix} V & H & 0 \\ g & V & g \\ 0 & aV^2 & 0 \end{bmatrix} \quad \text{and} \quad G(Y) := \begin{bmatrix} 0 \\ C_f \frac{V^2}{H} - gS_b \\ 0 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix, $A(Y)$ can be obtained by Cardano-Vieta method, which is discussed in Appendix B. To this end, we consider the characteristic equation of $A(Y)$, which is a cubic equation

$$\lambda^3 - 2V\lambda^2 + \left(V^2 - g\left(H + V^2 a\right)\right)\lambda + V^3 ag = 0.$$

Let

$$Q = -\frac{1}{9}\left[3gH + (1+3ag)V^2\right] \quad \text{and} \quad R = -\frac{1}{54}\left[(2+9ag)V^3 - 18gHV\right].$$

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Then we obtain distinct real eigenvalues (or characteristic speeds) if

$$\begin{aligned} D &= Q^3 + R^2, \\ &= -\frac{1}{108}\left(4ag+1\right)a^2g^2V^6 - \frac{1}{27}\left(3a^2g^2+5ag+1\right)HgV^4 \\ &\quad -\frac{1}{27}\left(3ag^2-2\right)H^2g^2V^2 - \frac{1}{27}H^3g^3 < 0. \end{aligned}$$

This is satisfied if

$$8H^2gV^2 < \left(4ag+1\right)a^2gV^6 + 4\left(3a^2g^2+5ag+1\right)HV^4 + 12H^2ag^2V^2 + 8H^3g^2,$$

which is always satisfied since $H > 0$ and $8H^2gV^2 \leq 4HV^4 + 4H^3g^2$. To determine the signs of the characteristic speeds, we consider a positive flow (or $V > 0$). Since $|A| = -V^3ag$, the characteristic speeds either all are negative or one of them is negative. However, since two of them, say λ_1 and λ_3, which are characteristic speeds of the water flow, have opposite signs from Saint-Venant equations and the water flow is much faster than the motion of the sediment, one can write the signs of the characteristic speeds in the following order,

$$\lambda_3 < 0 < \lambda_2 < \lambda_1,$$

where λ_2 is characteristic speed of sediment.

The left eigenvectors of the $A(Y)$ can be computed by using the corresponding eigenvalues as suggested in [22]. For simplicity, we adopt as it is described there by

$$L_i = \frac{1}{\left(\lambda_i-\lambda_j\right)\left(\lambda_i-\lambda_k\right)}\begin{bmatrix}\left(V-\lambda_j\right)\left(V-\lambda_k\right)+gH^* \\ H\lambda_i \\ gH\end{bmatrix}^T, \quad i \neq j \neq k \in \{1,2,3\}. \tag{2.67}$$

Since we are interested in a linear system, we move now into the linearization of the system (2.66). Thus, let $Y^* = [H^*, V^*, B^*]^T$ be an equilibrium solution of the system (2.66) such that $C_f(V^*)^2 = gS_bH^*$. Then, the linearization of the system (2.66) around the equilibrium can be computed as discussed above and is given by

$$\partial_t Z + A\partial_x Z + EZ = 0, \tag{2.68}$$

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with

$$Z := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} := \begin{bmatrix} H - H^* \\ V - V^* \\ B - B^* \end{bmatrix}, \quad A := A(Y^*) \quad \text{and}$$

$$E := \begin{bmatrix} 0 & 0 & 0 \\ -C_f \frac{V^{*2}}{H^{*2}} & 2C_f \frac{V^*}{H^*} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The characteristic variables of the linearized system (2.68) can be obtained in the following form

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} := \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},$$

where $L_i := L_i(Y^*), i = 1, 2, 3$ and the inverse transformation gives

$$z_1 := \xi_1 + \xi_2 + \xi_3, \tag{2.69a}$$

$$z_2 := \frac{1}{H^*} \left[(\lambda_1 - V^*) \xi_1 + (\lambda_2 - V^*) \xi_2 + (\lambda_3 - V^*) \xi_3 \right], \tag{2.69b}$$

$$z_3 := \frac{1}{gH^*} \left[\left((\lambda_1 - V^*)^2 - gH^* \right) \xi_1 + \left((\lambda_2 - V^*)^2 - gH^* \right) \xi_2 + \left((\lambda_3 - V^*)^2 - gH^* \right) \xi_3 \right]. \tag{2.69c}$$

The linearized system (2.68) in characteristic variables can be expressed as follows

$$\partial_t \xi_i + \lambda_i \partial_x \xi_i + L_i E Z = 0, \quad i = 1, 2, 3. \tag{2.70}$$

Substituting (2.69) into (2.70), we have

$$\partial_t \xi_i + \lambda_i \partial_x \xi_i + \left(C_f \frac{V^*}{H^*} \right) \left(\frac{\lambda_i}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \right) \sum_{l=1}^{3} (2\lambda_l - 3V^*) \xi_l = 0,$$
$$i \neq j \neq k \in \{1, 2, 3\}. \tag{2.71}$$

To simplify the expressions in the equation (2.71), we let

$$\delta_i := \left(C_f \frac{V^*}{H^*} \right) \left(\frac{\lambda_i}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \right), \quad i \neq j \neq k \in \{1, 2, 3\}.$$

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In that case, equation (2.71) can be written as

$$\partial_t w_i + \lambda_i \partial_x w_i + \sum_{l=1}^{3} (2\lambda_l - 3V^*)\, \delta_l \xi_l = 0, \quad i = 1, 2, 3, \tag{2.72}$$

where $w_i := \frac{1}{\delta_i}\xi_i$. Therefore, equations in (2.66) can be written in the following characteristic form

$$\partial_t W + \Lambda \partial_x W + \Pi W = 0, \tag{2.73}$$

with

$$W := \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{and} \quad \Pi := \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix},$$

where $\gamma_i := (2\lambda_i - 3V^*)\, \delta_i$.

One can determine the signs of coefficients, $\delta_i, i = 1, 2, 3,$ from the signs of eigenvalues [7]. These are going to be determined as follows: for

$$\gamma_1 = \frac{\lambda_1 (2\lambda_1 - 3V^*)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \left(C_f \frac{V^*}{H^*} \right).$$

Since $\lambda_1 > 0$, $\lambda_1 - \lambda_2 > 0$ and $\lambda_1 - \lambda_3 > 0$, γ_1 has the same sign as $2\lambda_1 - 3V^*$. From trace of the Jacobian matrix, we have

$$2\lambda_1 - 3V^* = V^* - 2\lambda_2 - 2\lambda_3 > 0,$$

since $\lambda_3 < 0$ and λ_2 is small.

Similarly, for

$$\gamma_2 = \frac{\lambda_2 (2\lambda_2 - 3V^*)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \left(C_f \frac{V^*}{H^*} \right).$$

Since $\lambda_2 > 0$, $\lambda_2 - \lambda_1 < 0$, $\lambda_2 - \lambda_3 > 0$ and the speed of water flow is greater than the motion of the sediment, γ_2 has the opposite sign of $2\lambda_2 - 3V^* < 0$.

Finally, for

$$\gamma_3 = \frac{\lambda_3 (2\lambda_3 - 3V^*)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \left(C_f \frac{V^*}{H^*} \right).$$

Since $\lambda_3 < 0$, $\lambda_3 - \lambda_1 < 0$ and $\lambda_3 - \lambda_2 < 0$, γ_3 has the opposite sign of $2\lambda_3 - 3V^* < 0$. Therefore, all the coefficients are strictly positive.

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The initial condition (2.64) and the boundary conditions (2.65) in characteristic variables can be expressed in the following form

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \\ w_3(x,0) \end{bmatrix} = \begin{bmatrix} \frac{1}{\delta_1} L_1 \\ \frac{1}{\delta_2} L_2 \\ \frac{1}{\delta_3} L_3 \end{bmatrix} \begin{bmatrix} H(x,0) - H^* \\ V(x,0) - V^* \\ B(x,0) - B^* \end{bmatrix}, \quad x \in [0,l], \tag{2.74}$$

and

$$\begin{bmatrix} w_1(0,t) \\ w_2(0,t) \\ w_3(l,t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_{13} \\ 0 & 0 & k_{23} \\ k_{31} & k_{k2} & 0 \end{bmatrix} \begin{bmatrix} w_1(l,t) \\ w_2(l,t) \\ w_3(0,t) \end{bmatrix}, \quad t \geq 0, \tag{2.75}$$

with

$$k_{13} := \frac{\delta_3 L_1 \zeta_1}{\delta_1 L_3 \zeta_1}, \quad k_{23} := \frac{\delta_3 L_2 \zeta_1}{\delta_2 L_3 \zeta_1}, \quad k_{31} := \frac{\delta_1 \left(L_2 \zeta_2 L_3 \zeta_3 - L_3 \zeta_2 L_2 \zeta_3 \right)}{\delta_3 \left(L_2 \zeta_2 L_1 \zeta_3 - L_1 \zeta_2 L_2 \zeta_3 \right)} \quad \text{and}$$
$$k_{32} := \frac{-\delta_2 \left(L_1 \zeta_2 L_3 \zeta_3 - L_3 \zeta_2 L_1 \zeta_3 \right)}{\delta_3 \left(L_2 \zeta_2 L_1 \zeta_3 - L_1 \zeta_2 L_2 \zeta_3 \right)},$$

where

$$\zeta_1 := \begin{bmatrix} 1 \\ -k_0 \\ 0 \end{bmatrix}, \quad \zeta_2 := \begin{bmatrix} 0 \\ -k_l \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_3 := \begin{bmatrix} 1 \\ -k_l \\ 0 \end{bmatrix}.$$

2.8 Summary

In this chapter, the hyperbolic systems of balance laws in one space dimension has been summarized. Some well-known definitions and fluid dynamics models have been included in the discussion of this chapter.

In the next chapter, we will discuss boundary stabilization of a linear hyperbolic system of balance laws and apply it to examples presented in this chapter.

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Chapter 3

Boundary Stabilization of Linear Hyperbolic Systems of Balance Laws

The main purpose of this chapter is to discuss a stability analysis of linear hyperbolic systems of balance laws in one spatial dimension. For this reason, we use a Lyapunov function to investigate conditions for exponential stability of a Cauchy problem associated with a linear system.

3.1 Lyapunov Exponential Stability Analysis

In this section, we investigate conditions for exponential stability of the Cauchy problem (2.31) under steady state solution $W \equiv 0$. For this reason, the definition of exponential stability follows.

Definition 6 (Exponentially stable [7]). *The system* (2.31a) *(or* (2.32a)*) with boundary condition* (2.31b) *(or* (2.32b)*) is exponentially stable for the* $\mathbb{L}^2-$*norm if there exist* $\eta > 0$ *and* $C > 0$ *such that, for every initial condition* $W_0(x) \in \mathbb{L}^2((0,l);\mathbb{R}^k)$*, the* $\mathbb{L}^2$ *solution to the Cauchy problem* (2.31) *(or* (2.32)*) satisfies*

$$\|W(\cdot,t)\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)} \leq Ce^{-\eta t}\|W_0\|_{\mathbb{L}^2((0,l);\mathbb{R}^k)}, \quad t \geq 0.$$

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In order to investigate conditions for exponential stability in sense of the Definition 6, we need to define the following candidate $\mathbb{L}^2$ Lyapunov function

$$L(t) := \int_0^l W^T(x,t)\Phi(x)W(x,t)dx, \quad t \geq 0, \tag{3.1}$$

where $\Phi(x) :=$ diag$\{P^+e^{-\mu x}, P^-e^{\mu x}\}$, with $P^+ :=$ diag$\{p_1, \ldots, p_m\}$, $P^- :=$ diag$\{p_{m+1}, \ldots, p_k\}$ and $p_i > 0, i = 1, \ldots, k$.

Theorem 3 ([7])**.** *The system* (2.31a) *with boundary condition* (2.31b) *is exponentially stable for the* $\mathbb{L}^2-$*norm if there exists* $\mu > 0$ *and* $p_i > 0, i = 1, \ldots, k$, *such that the following matrices*

$$\begin{bmatrix} \Lambda^+(l) & 0 \\ 0 & \Lambda^-(0) \end{bmatrix} \begin{bmatrix} P^+e^{-\mu l} & 0 \\ 0 & P^- \end{bmatrix} - K^T \begin{bmatrix} \Lambda^+(0) & 0 \\ 0 & \Lambda^-(l) \end{bmatrix} \begin{bmatrix} P^+ & 0 \\ 0 & P^-e^{\mu l} \end{bmatrix} K \tag{3.2}$$

and

$$\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x), \quad x \in [0,l] \tag{3.3}$$

are positive definite.

Proof. *The time derivative of the Lyapunov function* (3.1) *by using the Cauchy problem* (2.31) *is obtained as follows*

$$\begin{aligned} L'(t) &= \int_0^l \partial_t \left(W^T\Phi(x)W\right) dx, \\ &= \int_0^l \left[(\partial_t W)^T \Phi(x)W + W^T\Phi(x)\left(\partial_t W\right)\right] dx, \\ &= \int_0^l \left[(-\Lambda(x)\partial_x W - \Pi(x)W)^T \Phi(x)W + W^T\Phi(x)\left(-\Lambda(x)\partial_x W - \Pi(x)W\right)\right] dx, \\ &= -\int_0^l \left[(\partial_x W)^T \Lambda(x)\Phi(x)W + W^T\Phi(x)\Lambda(x)\partial_x W\right] dx \\ &\quad - \int_0^l \left[W^T\Pi^T(x)\Phi(x)W + W^T\Phi(x)\Pi(x)W\right] dx. \end{aligned}$$

Since

$$\begin{aligned} (\partial_x W)^T \Lambda(x)\Phi(x)W + W^T\Phi(x)\Lambda(x)\partial_x W = \partial_x \left(W^T\Lambda(x)\Phi(x)W\right) - W^T\Lambda'(x)\Phi(x)W \\ - W^T\Lambda(x)\Phi'(x)W \end{aligned}$$

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and

$$\Lambda(x)\Phi'(x) = -\mu|\Lambda(x)|\Phi(x),$$

we have the following

$$\begin{aligned}
L'(t) =& -\int_0^l \partial_x \left(W^T\Lambda(x)\Phi(x)W\right)dx \\
&- \int_0^l W^T\left[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)\right]W dx, \\
=& -\left[W^T\Lambda(x)\Phi(x)W\right]_0^l \\
&- \int_0^l W^T\left[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)\right]W dx, \\
=& -\left[W^T(l,t)\Lambda(l)\Phi(l)W(l,t) - W^T(0,t)\Lambda(0)\Phi(0)W(0,t)\right] \\
&- \int_0^l W^T\left[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)\right]W dx, \\
=& -\begin{bmatrix} W^+(l,t) \\ W^-(l,t)\end{bmatrix}^T \begin{bmatrix}\Lambda^+(l) & 0 \\ 0 & \Lambda^-(l)\end{bmatrix}\begin{bmatrix}P^+e^{-\mu l} & 0 \\ 0 & P^-e^{\mu l}\end{bmatrix}\begin{bmatrix} W^+(l,t) \\ W^-(l,t)\end{bmatrix} \\
&+ \begin{bmatrix} W^+(0,t) \\ W^-(0,t)\end{bmatrix}^T \begin{bmatrix}\Lambda^+(0) & 0 \\ 0 & \Lambda^-(0)\end{bmatrix}\begin{bmatrix}P^+ & 0 \\ 0 & P^-\end{bmatrix}\begin{bmatrix} W^+(0,t) \\ W^-(0,t)\end{bmatrix} \\
&- \int_0^l W^T\left[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)\right]W dx, \\
=& -\begin{bmatrix} W^+(l,t) \\ W^-(0,t)\end{bmatrix}^T \begin{bmatrix}\Lambda^+(l) & 0 \\ 0 & \Lambda^-(0)\end{bmatrix}\begin{bmatrix}P^+e^{-\mu l} & 0 \\ 0 & P^-\end{bmatrix}\begin{bmatrix} W^+(l,t) \\ W^-(0,t)\end{bmatrix} \\
&+ \begin{bmatrix} W^+(0,t) \\ W^-(l,t)\end{bmatrix}^T \begin{bmatrix}\Lambda^+(0) & 0 \\ 0 & \Lambda^-(l)\end{bmatrix}\begin{bmatrix}P^+ & 0 \\ 0 & P^-e^{\mu l}\end{bmatrix}\begin{bmatrix} W^+(0,t) \\ W^-(l,t)\end{bmatrix} \\
&- \int_0^l W^T\left[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)\right]W dx, \\
=& -\begin{bmatrix} W^+(l,t) \\ W^-(0,t)\end{bmatrix}^T \begin{bmatrix}\Lambda^+(l) & 0 \\ 0 & \Lambda^-(0)\end{bmatrix}\begin{bmatrix}P^+e^{-\mu l} & 0 \\ 0 & P^-\end{bmatrix}\begin{bmatrix} W^+(l,t) \\ W^-(0,t)\end{bmatrix} \\
&+ \begin{bmatrix} W^+(l,t) \\ W^-(0,t)\end{bmatrix}^T K^T \begin{bmatrix}\Lambda^+(0) & 0 \\ 0 & \Lambda^-(l)\end{bmatrix}\begin{bmatrix}P^+ & 0 \\ 0 & P^-e^{\mu l}\end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t)\end{bmatrix} \\
&- \int_0^l W^T\left[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)\right]W dx < 0,
\end{aligned}$$

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by using the assumptions made for the matrices (3.2) *and* (3.3).

Since the matrices (3.3) *and* $\mu|\Lambda(x)|\Phi(x), \forall x$ *are positive definite, for any vector* $W \in \mathbb{R}^k$ *it implies that*

$$\begin{aligned} W^T[\mu|\Lambda(x)|\Phi(x) - \Lambda'(x)\Phi(x) + \Pi^T(x)\Phi(x) + \Phi(x)\Pi(x)]W &> \mu W^T|\Lambda(x)|\Phi(x)W \\ &> \mu\alpha W^T\Phi(x)W, \end{aligned}$$

with $\alpha := \min_{\substack{1\le i\le k \\ 0\le x\le l}} |\lambda_i(x)|$. *Therefore, we have*

$$L'(t) < -\mu\alpha \int_0^l W^T\Phi(x)W dx = -\eta L(t), \quad \eta := \mu\alpha. \tag{3.4}$$

Hence, the Cauchy problem (2.31) *is exponentially stable for the* $\mathbb{L}^2-$*norm.*

In the special case, we deal with the Cauchy problem (2.32) under the steady state solution $W \equiv 0$. Here, the candidate $\mathbb{L}^2$ Lyapunov function (3.1) can be used to investigate conditions for exponential stability in the sense of Definition 6.

Corollary 1 ([22]). *The system* (2.32a) *with boundary condition* (2.32b) *is exponentially stable for the* $\mathbb{L}^2-$*norm if there exist* $\mu > 0$ *and* $p_i > 0, i = 1, 2, \ldots, k$, *such that the following matrices*

$$\begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+e^{-\mu l} & 0 \\ 0 & P^- \end{bmatrix} - K^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+ & 0 \\ 0 & P^-e^{\mu l} \end{bmatrix} K \tag{3.5}$$

and

$$\mu|\Lambda|\Phi(x) + \Pi^T\Phi(x) + \Phi(x)\Pi, \quad x \in [0, l] \tag{3.6}$$

are positive definite.

Proof. *The time derivative of the Lyapunov function* (3.1) *by using the Cauchy problem* (2.32) *is obtained as follows*

$$\begin{aligned} L'(t) &= \int_0^l \partial_t \left(W^T\Phi(x)W\right) dx, \\ &= \int_0^l \left[(\partial_t W)^T \Phi(x)W + W^T\Phi(x) (\partial_t W)\right] dx, \end{aligned}$$

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$$
\begin{aligned}
&= \int_0^l \left[\left(-\Lambda \partial_x W - \Pi W\right)^T \Phi(x) W + W^T \Phi(x) \left(-\Lambda \partial_x W - \Pi W\right) \right] dx, \\
&= -\int_0^l \left[\left(\partial_x W\right)^T \Lambda \Phi(x) W + W^T \Phi(x) \Lambda \partial_x W \right] dx \\
&\quad - \int_0^l \left[W^T \Pi^T \Phi(x) W + W^T \Phi(x) \Pi W \right] dx.
\end{aligned}
$$

Since

$$
\left(\partial_x W\right)^T \Lambda \Phi(x) W + W^T \Phi(x) \Lambda \partial_x W = \partial_x \left(W^T \Lambda \Phi(x) W \right) - W^T \Lambda \Phi'(x) W
$$

and

$$
\Lambda(x) \Phi'(x) = -\mu |\Lambda| \Phi(x),
$$

we have the following

$$
\begin{aligned}
L'(t) &= -\int_0^l \partial_x \left(W^T \Lambda \Phi(x) W \right) dx \\
&\quad - \int_0^l W^T \left[\mu |\Lambda| \Phi(x) + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx, \\
&= -\left[W^T \Lambda \Phi(x) W \right]_0^l \\
&\quad - \int_0^l W^T \left[\mu |\Lambda| \Phi(x) + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx, \\
&= -\left[W^T(l,t) \Lambda \Phi(l) W(l,t) - W^T(0,t) \Lambda \Phi(0) W(0,t) \right] \\
&\quad - \int_0^l W^T \left[\mu |\Lambda| \Phi(x) + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx, \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(l,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+ e^{-\mu l} & 0 \\ 0 & P^- e^{\mu l} \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(l,t) \end{bmatrix} \\
&\quad + \begin{bmatrix} W^+(0,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+ & 0 \\ 0 & P^- \end{bmatrix} \begin{bmatrix} W^+(0,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad - \int_0^l W^T \left[\mu |\Lambda| \Phi(x) + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx, \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+ e^{-\mu l} & 0 \\ 0 & P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+ & 0 \\ 0 & P^- e^{\mu l} \end{bmatrix} \begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix}
\end{aligned}
$$

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$$
\begin{aligned}
&- \int_0^l W^T \left[\mu|\Lambda|\Phi(x) + \Pi^T\Phi(x) + \Phi(x)\Pi\right] W dx, \\
&= - \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+e^{-\mu l} & 0 \\ 0 & P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&+ \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T K^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} P^+ & 0 \\ 0 & P^-e^{\mu l} \end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&- \int_0^l W^T \left[\mu|\Lambda|\Phi(x) + \Pi^T\Phi(x) + \Phi(x)\Pi\right] W dx < 0,
\end{aligned} \tag{3.7}
$$

by using the assumptions for matrices (3.5) *and* (3.6). *Since the matrices* (3.6) *and* $\mu|\Lambda|\Phi(x), \forall x$, *are positive definite, we have the following implication for any* $W \in \mathbb{R}^k$,

$$
W^T[\mu|\Lambda|\Phi(x) + \Pi^T\Phi(x) + \Phi(x)\Pi]W > \mu W^T|\Lambda|\Phi(x)W > \mu\alpha W^T\Phi(x)W,
$$

with $\alpha := \min_{1\leq i\leq k}|\lambda_i|$. *Therefore, we have*

$$
L'(t) < -\mu\alpha \int_0^l W^T\Phi(x)W dx = -\eta L(t), \quad \eta := \mu\alpha. \tag{3.8}
$$

Hence, the Cauchy problem (2.32) *is exponentially stable for the* $\mathbb{L}^2-$*norm.*

Remark 7. *Boundary conditions that satisfy conditions* (3.2) *(or* (3.5)*) are called* **Dissipative Boundary Conditions** *[22]. In order to show that conditions* (3.2)*(or* (3.5)*) and* (3.3)*(or* (3.6)*) are positive definite, it suffices to show that the determinant of every principal submatrix is positive (see Appendix A).*

One of the conditions in the Corollary 1 is to show that the matrix (3.5) is positive definite, which alternatively is expressed in the following corollary.

Corollary 2 ([7])**.** *If there exists* $\mu > 0$ *and* $p_i > 0, i = 1, \ldots, k$ *such that*

$$
\mu|\Lambda|\Phi(x) + \Pi^T\Phi(x) + \Phi(x)\Pi, \quad x \in [0, l], \tag{3.9}
$$

is positive definite and

$$
\|\Delta K \Delta^{-1}\| < 1,
$$

with $\Delta := \sqrt{P|\Lambda|}$, *where* $P :=$ *diag*$\{P^+, P^-\}$, *then the system* (2.32a) *with boundary condition* (2.32b) *is exponentially stable for the* $\mathbb{L}^2-$*norm.*

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Proof. *From equation (3.7), the time derivative of the Lyapunov function (3.1) can be expressed as follows*

$$L'(t) = L'_1(t) + L'_2(t), \tag{3.10}$$

with

$$\begin{aligned} L'_1(t) := &- \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ e^{-\mu l} & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\ &+ \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T K^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- e^{\mu l} \end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \end{aligned}$$

and

$$L'_2(t) := -\int_0^l W^T \left[\mu \Phi(x) |\Lambda| + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx.$$

By using the assumption for the matrix (3.9), *for any* $\mu > 0$, *we have*

$$L'_2(t) < 0, \forall t \geq 0.$$

Then, positive definiteness of the matrix(3.9) *implies that for any vector* $W \in \mathbb{R}^k$,

$$W[\mu \Phi(x)|\Lambda| + \Pi^T \Phi(x) + \Phi(x)\Pi]W > \alpha \mu W^T \Phi(x) W, \quad \forall x \in [0, l], \quad \alpha := \min_{1 \leq i \leq k} |\lambda_i|.$$

Therefore,

$$L'_2(t) < -\eta L(t), \quad t \geq 0, \quad \eta := \alpha \mu.$$

Consider the following quadratic form, which is a special case ($L'_1(t)$ *at* $\mu \equiv 0$*)*

$$\begin{aligned} &- \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\ &+ \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T K^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \\ = &- \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\ &+ \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T K^T \left(\sqrt{|\Lambda| P} \right)^T \left(\sqrt{|\Lambda| P} \right) K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \end{aligned}$$

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$$\begin{aligned}
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \left(\sqrt{|\Lambda|P}K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}\right)^T \left(\sqrt{|\Lambda|P}K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}\right), \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + \left\| \sqrt{|\Lambda|P}K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \right\|^2, \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \left\| \left(\sqrt{|\Lambda|P}\right) K \left(\sqrt{|\Lambda|P}\right)^{-1} \left(\sqrt{|\Lambda|P}\right) \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \right\|^2, \\
&\leq -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \left\| \left(\sqrt{|\Lambda|P}\right) K \left(\sqrt{|\Lambda|P}\right)^{-1} \right\|^2 \left\| \left(\sqrt{|\Lambda|P}\right) \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \right\|^2,
\end{aligned}$$

where in the last line, the parameters $p_i > 0, i = 1, \ldots, k$, *can be selected such that*

$$\left\| \left(\sqrt{|\Lambda|P}\right) \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \right\| = 1 \quad \textit{and} \quad \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \neq 0, \quad t \geq 0.$$

If

$$\left\| \left(\sqrt{|\Lambda|P}\right) K \left(\sqrt{|\Lambda|P}\right)^{-1} \right\| < 1,$$

then

$$\begin{aligned}
&-\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^- \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&+ \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T K^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^- \end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \\
< &-\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T |\Lambda|P \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + \left\| \sqrt{|\Lambda|P} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \right\|^2,
\end{aligned}$$

$$= - \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T |\Lambda| P \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^T |\Lambda| P \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} = 0.$$

Therefore, by assumption $\|\Delta K \Delta^{-1}\| < 1$*, a sufficiently small* $\mu > 0$ *can be chosen such that* $L_1'(t) < 0$ *and then*

$$L'(t) < -\eta L(t), \quad t \geq 0.$$

Hence, the Cauchy problem (2.32) *is exponentially stable for the* $\mathbb{L}^2-$*norm.*

3.2 Analytical Results

In this section, we look at the application of the Lyapunov exponential stability analysis to some examples of linear (or linearized) hyperbolic systems of balance laws, such as the telegrapher equations, isentropic Euler equations, Saint-Venant equations and Saint-Venant-Exner equations. For each of these physical problems, we investigate conditions for the stability analysis.

3.2.1 Application to Telegrapher Equations

The Lyapunov function for the telegrapher equations (2.35) in decoupled characteristic form (2.42) can be defined by (3.1) with

$$W := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{and} \quad \Phi(x) := \begin{bmatrix} p_1 e^{-\mu x} & 0 \\ 0 & p_2 e^{\mu x} \end{bmatrix}, \quad \mu > 0, p_1 > 0, p_2 > 0.$$

The time derivative of the Lyapunov function (3.1) is obtained by (3.10) with

$$L_1'(t) := - \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix}^T \left(\begin{bmatrix} |\lambda_1| p_1 e^{-\mu l} & 0 \\ 0 & |\lambda_2| p_2 \end{bmatrix} - K^T \begin{bmatrix} |\lambda_1| p_1 & 0 \\ 0 & |\lambda_2| p_2 e^{\mu l} \end{bmatrix} K \right) \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix},$$

and

$$L_2'(t) := - \int_0^L W^T \left[\mu \Phi(x) |\Lambda| + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx.$$

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In order to show that the time derivative of the Lyapunov function, (3.10), is negative, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} |\lambda_1|p_1e^{-\mu l} - k_{21}^2|\lambda_2|p_2e^{\mu l} & 0 \\ 0 & |\lambda_2|p_2 - k_{12}^2|\lambda_1|p_1 \end{bmatrix}, \tag{3.11}$$

and

$$\begin{bmatrix} \mu|\lambda_1|p_1e^{-\mu x} + 2\gamma_1p_1e^{-\mu x} & \gamma_2p_2e^{\mu x} + \gamma_2p_1e^{-\mu x} \\ \gamma_2p_2e^{\mu x} + \gamma_2p_1e^{-\mu x} & \mu|\lambda_2|p_2e^{\mu x} + 2\gamma_1p_2e^{\mu x} \end{bmatrix}, \tag{3.12}$$

is positive.

The determinant of the sub-matrices of the matrix (3.12) are

$$\mu|\lambda_1|p_1e^{-\mu x} + 2\gamma_1p_1e^{-\mu x} = (\mu|\lambda_1| + 2\gamma_1)\, p_1e^{-\mu x}, \tag{3.13}$$

and

$$\begin{aligned} & p_1p_2\mu^2|\lambda_1\lambda_2| + 2|\lambda_2|\mu\gamma_1p_1p_2 + 2|\lambda_1|\mu\gamma_1p_1p_2 + 4\gamma_1^2p_1p_2 \\ & \quad - e^{2\mu x}\gamma_2^2p_2^2 - 2\gamma_2^2p_1p_2 - e^{-2\mu x}\gamma_2^2p_1^2, \\ =& p_1p_2\mu^2|\lambda_1\lambda_2| + 2|\lambda_2|\mu\gamma_1p_1p_2 + 2|\lambda_1|\mu\gamma_1p_1p_2 + 4\gamma_1^2p_1p_2 - 4\gamma_2^2p_1p_2 \\ & \quad - e^{2\mu x}\gamma_2^2p_2^2 + 2\gamma_2^2p_1p_2 - e^{-2\mu x}\gamma_2^2p_1^2, \\ =& \left(\mu^2|\lambda_1\lambda_2| + 2\mu\gamma_1\left(|\lambda_1| + |\lambda_2|\right) + 4\left(\gamma_1^2 - \gamma_2^2\right)\right)p_1p_2 - \left(\gamma_2p_2e^{\mu x} - \gamma_2p_1e^{-\mu x}\right)^2, \\ =& \left[(\mu|\lambda_1| + 2\gamma_1)(\mu|\lambda_2| + 2\gamma_1) - 4\gamma_2^2\right]p_1p_2 - \left(\gamma_2p_2e^{\mu x} - \gamma_2p_1e^{-\mu x}\right)^2. \end{aligned} \tag{3.14}$$

If $\mu|\lambda_1| + 2\gamma_1 > 0$, $\mu|\lambda_2| + 2\gamma_1 > 0$, $\gamma_1^2 > \gamma_2^2$ and if p_1 and p_2 can be chosen such that $p_1 = p_2$, then for sufficiently small $\mu > 0$,

$$\max_{0\le x\le l}\{\gamma_2p_2e^{\mu x} - \gamma_2p_1e^{-\mu x}\} \approx 0,$$

and both determinants (3.13) and (3.14) are positive.

Therefore, with the choice of p_1 and p_2, the matrix (3.11) is positive definite if k_{12} and k_{21} satisfy

$$|k_{12}| < \sqrt{\left|\frac{\lambda_2}{\lambda_1}\right|} \quad \text{and} \quad |k_{21}| < \sqrt{\left|\frac{\lambda_1}{\lambda_2}\right|}e^{-\mu l},$$

respectively.

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3.2.2 Application to Isentropic Euler Equations

The Lyapunov function for the isentropic Euler equations (2.45) in decoupled characteristic form (2.51) is defined by (3.1) with

$$W := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{and} \quad \Phi(x) := \begin{bmatrix} p_1 e^{-\mu x} & 0 \\ 0 & p_2 e^{\mu x} \end{bmatrix}, \quad \mu > 0, p_1 > 0, p_2 > 0.$$

The time derivative of the Lyapunov function (3.1) is obtained by (3.10) with

$$L_1'(t) := -\begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix}^T \left(\begin{bmatrix} |\lambda_1(l)| p_1 e^{-\mu l} & 0 \\ 0 & |\lambda_2(0)| p_2 \end{bmatrix} - K^T \begin{bmatrix} |\lambda_1(0)| p_1 & 0 \\ 0 & |\lambda_2(l)| p_2 e^{\mu l} \end{bmatrix} K \right) \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix},$$

and

$$L_2'(t) := -\int_0^L W^T \left[\mu |\Lambda(x)| \Phi(x) - \Lambda'(x)\Phi(x) + \Pi(x)^T \Phi(x) + \Phi(x)\Pi(x) \right] W dx.$$

In order to show that the time derivative of the Lyapunov function, (3.10), is negative, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} |\lambda_1(l)| p_1 e^{-\mu l} - k_{21}^2 |\lambda_2(l)| p_2 e^{\mu l} & 0 \\ 0 & |\lambda_2(0)| p_2 - k_{12}^2 |\lambda_1(0)| p_1 \end{bmatrix}, \tag{3.15}$$

and

$$\begin{bmatrix} \mu|\lambda_1(x)| p_1 e^{-\mu x} - \lambda_1'(x) p_1 e^{-\mu x} + 2\gamma_{11}(x) p_1 e^{-\mu x} & \gamma_{21}(x) p_2 e^{\mu x} + \gamma_{12}(x) p_1 e^{-\mu x} \\ \gamma_{21}(x) p_2 e^{\mu x} + \gamma_{12}(x) p_1 e^{-\mu x} & \mu|\lambda_2(x)| p_2 e^{\mu x} - \lambda_2'(x) p_2 e^{\mu x} + 2\gamma_{22}(x) p_2 e^{\mu x} \end{bmatrix}, \tag{3.16}$$

is positive.

The determinant of the sub-matrices of the matrix (3.16) are

$$\mu|\lambda_1(x)| p_1 e^{-\mu x} - \lambda_1'(x) p_1 e^{-\mu x} + 2\gamma_{11}(x) p_1 e^{-\mu x} = \left(\mu|\lambda_1(x)| - \lambda_1'(x) + 2\gamma_{11}(x) \right) p_1 e^{-\mu x}, \tag{3.17}$$

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and

$$\begin{aligned}
&p_1p_2\mu^2|\lambda_1(x)\lambda_2(x)| - |\lambda_1(x)|\mu\lambda_2'(x)p_1p_2 - |\lambda_2(x)|\mu\lambda_1'(x)p_1p_2 \\
&+ 2|\lambda_1(x)|\mu\gamma_{22}(x)p_1p_2 + 2|\lambda_2(x)|\mu\gamma_{11}(x)p_1p_2 + \lambda_1'(x)\lambda_2'(x)p_1p_2 \\
&- 2\lambda_1'(x)\gamma_{22}(x)p_1p_2 - 2\lambda_2'(x)\gamma_{11}(x)p_1p_2 \\
&- e^{2\mu x}\gamma_{21}^2(x)p_2^2 + 2\gamma_{12}(x)\gamma_{21}(x)p_1p_2 - e^{-2\mu x}\gamma_{12}(x)p_1^2, \\
=& \left(\mu^2|\lambda_1(x)\lambda_2(x)| + \lambda_1'(x)\lambda_2'(x) - \mu|\lambda_1(x)|\lambda_2'(x) - \mu|\lambda_2(x)|\lambda_1'(x)\right)p_1p_2 \\
&+ 2\left(\left(\mu|\lambda_1(x)| - \lambda_1'(x)\right)\gamma_{22}(x) + \left(\mu|\lambda_2(x)| - \lambda_2'(x)\right)\gamma_{11}(x)\right)p_1p_2 \\
&- \left(\gamma_{21}(x)p_2e^{\mu x} - \gamma_{12}(x)p_1e^{-\mu x}\right)^2, \\
=& \left(\mu|\lambda_1(x)| - \lambda_1'(x)\right)\left(\mu|\lambda_2(x)| - \lambda_2'(x)\right)p_1p_2 \\
&+ 2\left(\left(\mu|\lambda_1(x)| - \lambda_1'(x)\right)\gamma_{22}(x) + \left(\mu|\lambda_2(x)| - \lambda_2'(x)\right)\gamma_{11}(x)\right)p_1p_2 \\
&- \left(\gamma_{21}(x)p_2e^{\mu x} - \gamma_{12}(x)p_1e^{-\mu x}\right)^2. \qquad (3.18)
\end{aligned}$$

If the following holds for all $x \in [0, l]$ and $\mu > 0$,

$$\begin{aligned}
&\mu|\lambda_1(x)| - \lambda_1'(x) > 0, \quad \mu|\lambda_2(x)| - \lambda_2'(x) > 0, \\
&\mu|\lambda_1(x)| - \lambda_1'(x) + 2\gamma_{11}(x) > 0, \\
&\mu|\lambda_2(x)| - \lambda_2'(x) + 2\gamma_{22}(x) > 0,
\end{aligned}$$

and if p_1 and p_2 can be chosen such that

$$\frac{p_1}{p_2} = \max_{0\le x\le l} \frac{\gamma_{21}(x)}{\gamma_{12}(x)},$$

then for sufficiently small $\mu > 0$,

$$\max_{0\le x\le l}\{\gamma_{21}(x)p_2e^{\mu x} - \gamma_{12}(x)p_1e^{-\mu x}\} \approx 0,$$

and both (3.17) and (3.18) are positive.

With the choice of p_1 and p_2, the matrix (3.15) is positive definite if k_{12} and k_{21} satisfy

$$|k_{12}| < \sqrt{\left|\frac{\lambda_2(0)}{\lambda_1(0)}\right| \frac{1}{p_1/p_2}} \quad \text{and} \quad |k_{21}| < \sqrt{\left|\frac{\lambda_1(l)}{\lambda_2(l)}\right| \frac{p_1}{p_2}} e^{-\mu l},$$

respectively.

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3.2.3 Application to Saint-Venant Equations

The Lyapunov function for the Saint-Venant Equations (2.54) in decoupled characteristic form (2.60) is defined by (3.1) with

$$W := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{and} \quad \Phi(x) := \begin{bmatrix} p_1 e^{-\mu x} & 0 \\ 0 & p_2 e^{\mu x} \end{bmatrix}, \quad \mu > 0, p_1 > 0, p_2 > 0.$$

The time derivative of the Lyapunov function (3.1) is obtained by (3.10) with

$$L_1'(t) := -\begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix}^T \left(\begin{bmatrix} |\lambda_1(l)| p_1 e^{-\mu l} & 0 \\ 0 & |\lambda_2(0)| p_2 \end{bmatrix} \right. \\ \left. -K^T \begin{bmatrix} |\lambda_1(0)| p_1 & 0 \\ 0 & |\lambda_2(l)| p_2 e^{\mu l} \end{bmatrix} K \right) \begin{bmatrix} w_1(l,t) \\ w_2(0,t) \end{bmatrix},$$

and

$$L_2'(t) := -\int_0^l W^T \left[\mu |\Lambda(x)| \Phi(x) - \Lambda'(x)\Phi(x) + \Pi(x)^T \Phi(x) + \Phi(x)\Pi(x) \right] W dx.$$

In order to show that the time derivative of the Lyapunov function, (3.10), is negative, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} |\lambda_1(l)| p_1 e^{-\mu l} - k_{21}^2 |\lambda_2(l)| p_2 e^{\mu l} & 0 \\ 0 & |\lambda_2(0)| p_2 - k_{12}^2 |\lambda_1(0)| p_1 \end{bmatrix}, \tag{3.19}$$

and

$$\begin{bmatrix} \mu|\lambda_1(x)| p_1 e^{-\mu x} - \lambda_1'(x) p_1 e^{-\mu x} + 2\gamma_{11}(x) p_1 e^{-\mu x} & \gamma_{21}(x) p_2 e^{\mu x} + \gamma_{12}(x) p_1 e^{-\mu x} \\ \gamma_{21}(x) p_2 e^{\mu x} + \gamma_{12}(x) p_1 e^{-\mu x} & \mu|\lambda_2(x)| p_2 e^{\mu x} - \lambda_2'(x) p_2 e^{\mu x} + 2\gamma_{22}(x) p_2 e^{\mu x} \end{bmatrix}, \tag{3.20}$$

is positive.

The determinant of the sub-matrices of the matrix (3.20) are

$$\mu|\lambda_1(x)| p_1 e^{-\mu x} - \lambda_1'(x) p_1 e^{-\mu x} + 2\gamma_{11}(x) p_1 e^{-\mu x} = \left(\mu|\lambda_1(x)| - \lambda_1'(x) + 2\gamma_{11}(x)\right) p_1 e^{-\mu x}, \tag{3.21}$$

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and

$$\begin{aligned}
&p_1p_2\mu^2|\lambda_1(x)\lambda_2(x)| - |\lambda_1(x)|\mu\lambda_2'(x)p_1p_2 - |\lambda_2(x)|\mu\lambda_1'(x)p_1p_2\\
&\quad + 2|\lambda_1(x)|\mu\gamma_{22}(x)p_1p_2 + 2|\lambda_2(x)|\mu\gamma_{11}(x)p_1p_2 + \lambda_1'(x)\lambda_2'(x)p_1p_2\\
&\quad - 2\lambda_1'(x)\gamma_{22}(x)p_1p_2 - 2\lambda_2'(x)\gamma_{11}(x)p_1p_2\\
&\quad - e^{2\mu x}\gamma_{21}^2(x)p_2^2 + 2\gamma_{12}(x)\gamma_{21}(x)p_1p_2 - e^{-2\mu x}\gamma_{12}(x)p_1^2,\\
&= \left(\mu^2|\lambda_1(x)\lambda_2(x)| + \lambda_1'(x)\lambda_2'(x) - \mu|\lambda_1(x)|\lambda_2'(x) - \mu|\lambda_2(x)|\lambda_1'(x)\right)p_1p_2\\
&\quad + 2\left(\left(\mu|\lambda_1(x)| - \lambda_1'(x)\right)\gamma_{22}(x) + \left(\mu|\lambda_2(x)| - \lambda_2'(x)\right)\gamma_{11}(x)\right)p_1p_2\\
&\quad - \left(\gamma_{21}(x)p_2e^{\mu x} - \gamma_{12}(x)p_1e^{-\mu x}\right)^2,\\
&= \left(\mu|\lambda_1(x)| - \lambda_1'(x)\right)\left(\mu|\lambda_2(x)| - \lambda_2'(x)\right)p_1p_2\\
&\quad + 2\left(\left(\mu|\lambda_1(x)| - \lambda_1'(x)\right)\gamma_{22}(x) + \left(\mu|\lambda_2(x)| - \lambda_2'(x)\right)\gamma_{11}(x)\right)p_1p_2\\
&\quad - \left(\gamma_{21}(x)p_2e^{\mu x} - \gamma_{12}(x)p_1e^{-\mu x}\right)^2. \qquad (3.22)
\end{aligned}$$

If the following holds for all $x \in [0, l]$ and $\mu > 0$,

$$\begin{aligned}
&\mu|\lambda_1(x)| - \lambda_1'(x) > 0, \quad \mu|\lambda_2(x)| - \lambda_2'(x) > 0,\\
&\mu|\lambda_1(x)| - \lambda_1'(x) + 2\gamma_{11}(x) > 0,\\
&\mu|\lambda_2(x)| - \lambda_2'(x) + 2\gamma_{22}(x) > 0,
\end{aligned}$$

and if p_1 and p_2 can be chosen such that

$$\frac{p_1}{p_2} = \max_{0\le x\le l}\frac{\gamma_{21}(x)}{\gamma_{12}(x)},$$

then for sufficiently small $\mu > 0$,

$$\max_{0\le x\le l}\{\gamma_{21}(x)p_2e^{\mu x} - \gamma_{12}(x)p_1e^{-\mu x}\} \approx 0,$$

and both (3.21) and (3.22) are positive.

With the choice of p_1 and p_2, the matrix (3.19) is positive definite if k_{12} and k_{21} satisfy

$$|k_{12}| < \sqrt{\left|\frac{\lambda_2(0)}{\lambda_1(0)}\right|\frac{1}{p_1/p_2}} \quad \text{and} \quad |k_{21}| < \sqrt{\left|\frac{\lambda_1(l)}{\lambda_2(l)}\right|\frac{p_1}{p_2}}e^{-\mu l},$$

respectively.

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3.2.4 Application to Saint-Venant-Exner Equations

The Lyapunov function for the Saint-Venant-Exner equations (2.63) in decoupled characteristic form (2.73) is defined by (3.1) with

$$W := \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \text{and} \quad \Phi(x) := \begin{bmatrix} p_1 e^{-\mu x} & 0 & 0 \\ 0 & p_2 e^{-\mu x} & 0 \\ 0 & 0 & p_3 e^{\mu x} \end{bmatrix}, \quad \mu > 0, p_1 > 0, p_2 > 0, p_3 > 0.$$

The time derivative of the Lyapunov function (3.1) is obtained by (3.10) with

$$L_1'(t) := -\begin{bmatrix} w_1(l,t) \\ w_2(l,t) \\ w_3(0,t) \end{bmatrix}^T \left(\begin{bmatrix} |\lambda_1| p_1 e^{-\mu l} & 0 & 0 \\ 0 & |\lambda_2| p_2 e^{-\mu l} & 0 \\ 0 & 0 & |\lambda_3| p_3 \end{bmatrix} - K^T \begin{bmatrix} |\lambda_1| p_1 & 0 & 0 \\ 0 & |\lambda_2| p_2 & 0 \\ 0 & 0 & |\lambda_3| p_3 e^{\mu l} \end{bmatrix} K \right) \begin{bmatrix} w_1(l,t) \\ w_2(l,t) \\ w_3(0,t) \end{bmatrix},$$

and

$$L_2'(t) := -\int_0^l W^T \left[\mu \Phi(x) |\Lambda| + \Pi^T \Phi(x) + \Phi(x) \Pi \right] W dx.$$

In order to show that the time derivative of the Lyapunov function, (3.10), is negative, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} |\lambda_1| p_1 e^{-\mu l} - k_{31}^2 |\lambda_3| p_3 e^{\mu l} & -k_{31} k_{32} |\lambda_3| p_3 e^{\mu l} & 0 \\ -k_{31} k_{32} |\lambda_3| p_3 e^{\mu l} & |\lambda_2| p_2 e^{-\mu l} - k_{32}^2 |\lambda_3| p_3 e^{\mu l} & 0 \\ 0 & 0 & |\lambda_3| p_3 - k_{13}^2 |\lambda_1| p_1 - k_{23}^2 |\lambda_2| p_2 \end{bmatrix}, \tag{3.23}$$

and

$$\begin{bmatrix} \mu |\lambda_1| p_1 e^{-\mu x} + 2\gamma_1 p_1 e^{-\mu x} & \gamma_2 p_1 e^{-\mu x} + \gamma_1 p_2 e^{-\mu x} & \gamma_3 p_1 e^{-\mu x} + \gamma_1 p_3 e^{\mu x} \\ \gamma_2 p_1 e^{-\mu x} + \gamma_1 p_2 e^{-\mu x} & \mu |\lambda_2| p_2 e^{-\mu x} + 2\gamma_2 p_2 e^{-\mu x} & \gamma_3 p_2 e^{-\mu x} + \gamma_2 p_3 e^{\mu x} \\ \gamma_3 p_1 e^{-\mu x} + \gamma_1 p_3 e^{\mu x} & \gamma_3 p_2 e^{-\mu x} + \gamma_2 p_3 e^{\mu x} & \mu |\lambda_3| p_3 e^{\mu x} + 2\gamma_3 p_3 e^{\mu x} \end{bmatrix}, \tag{3.24}$$

is positive.

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The determinants of every sub-matrix of the matrix (3.24) are

$$\mu|\lambda_1|p_1e^{-\mu x} + 2\gamma_1 p_1 e^{-\mu x} = (\mu|\lambda_1| + 2\gamma_1)\, p_1 e^{-\mu x}, \tag{3.25}$$

$$\begin{aligned}&\left(2\,|\lambda_1|\,\mu\gamma_2 p_1 p_2 + 2\,|\lambda_2|\,\mu\gamma_1 p_1 p_2 + |\lambda_1\lambda_2|\,\mu^2 p_1 p_2 - {\gamma_1}^2{p_2}^2 + 2\gamma_1\gamma_2 p_1 p_2 - {\gamma_2}^2{p_1}^2\right) e^{-2\mu x}\\ &= \left(2\,|\lambda_1|\,\mu\gamma_2 p_1 p_2 + 2\,|\lambda_2|\,\mu\gamma_1 p_1 p_2 + |\lambda_1\lambda_2|\,\mu^2 p_1 p_2 - (\gamma_1 p_2 - \gamma_2 p_1)^2\right) e^{-2\mu x},\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}&\mu^3|\lambda_1\lambda_2\lambda_3|p_1p_2p_3e^{-\mu x} + 2\mu^2|\lambda_2\lambda_3|\gamma_1p_1p_2p_3e^{-\mu x} + 2\mu^2|\lambda_1\lambda_3|\gamma_2p_1p_2p_3e^{-\mu x}\\
&+ 2\mu^2|\lambda_1\lambda_2|\gamma_3p_1p_2p_3e^{-\mu x} + 2\mu|\lambda_3|\gamma_1\gamma_2p_1p_2p_3e^{-\mu x} + 2\mu|\lambda_2|\gamma_1\gamma_3p_1p_2p_3e^{-\mu x}\\
&+ 2\mu|\lambda_1|\gamma_2\gamma_3p_1p_2p_3e^{-\mu x} - \mu|\lambda_3|\gamma_1^2p_2^2p_3e^{-\mu x} - \mu|\lambda_3|\gamma_2^2p_1^2p_3e^{-\mu x}\\
&- \mu|\lambda_2|\gamma_1^2p_2p_3^2e^{\mu x} - \mu|\lambda_2|\gamma_3^2p_1^2p_2e^{-3\mu x} - \mu|\lambda_1|\gamma_2^2p_1p_3^2e^{\mu x} - \mu|\lambda_1|\gamma_3^2p_1p_2^2e^{-3\mu x},\\
&= \mu^2p_1p_2p_3\left(\mu|\lambda_1\lambda_2\lambda_3| + 2|\lambda_2\lambda_3|\gamma_1 + 2|\lambda_1\lambda_3|\gamma_2 + 2|\lambda_1\lambda_2|\gamma_3\right)e^{-\mu x}\\
&+ 2\mu|\lambda_3|\gamma_1\gamma_2p_1p_2p_3e^{-\mu x} + 2\mu|\lambda_2|\gamma_1\gamma_3p_1p_2p_3e^{-\mu x} + 2\mu|\lambda_1|\gamma_2\gamma_3p_1p_2p_3e^{-\mu x}\\
&- \mu|\lambda_3|p_3\left(\gamma_1^2p_2^2 + \gamma_2^2p_1^2\right)e^{-\mu x} - \mu|\lambda_2|p_2\left(\gamma_1^2p_3^2e^{2\mu x} + \gamma_3^2p_1^2e^{-2\mu x}\right)e^{-\mu x}\\
&- \mu|\lambda_1|p_1\left(\gamma_2^2p_3^2e^{2\mu x} + \gamma_3^2p_2^2e^{-2\mu x}\right)e^{-\mu x},\\
&= \mu^2p_1p_2p_3\left(\mu|\lambda_1\lambda_2\lambda_3| + 2|\lambda_2\lambda_3|\gamma_1 + 2|\lambda_1\lambda_3|\gamma_2 + 2|\lambda_1\lambda_2|\gamma_3\right)e^{-\mu x}\\
&- \mu|\lambda_3|p_3\left(\gamma_1^2p_2^2 - 2\gamma_1\gamma_2p_1p_2 + \gamma_2^2p_1^2\right)e^{-\mu x}\\
&- \mu|\lambda_2|p_2\left(\gamma_1^2p_3^2e^{2\mu x} - 2\gamma_1\gamma_3p_1p_3 + \gamma_3^2p_1^2e^{-2\mu x}\right)e^{-\mu x}\\
&- \mu|\lambda_1|p_1\left(\gamma_2^2p_3^2e^{2\mu x} - 2|\lambda_1|\gamma_2\gamma_3p_2p_3 + \gamma_3^2p_2^2e^{-2\mu x}\right)e^{-\mu x},\\
&= \mu^2p_1p_2p_3\left(\mu|\lambda_1\lambda_2\lambda_3| + 2|\lambda_2\lambda_3|\gamma_1 + 2|\lambda_1\lambda_3|\gamma_2 + 2|\lambda_1\lambda_2|\gamma_3\right)e^{-\mu x}\\
&- \mu|\lambda_3|p_3\left(\gamma_1p_2 - \gamma_2p_1\right)^2e^{-\mu x} - \mu|\lambda_2|p_2\left(\gamma_1p_3e^{\mu x} - \gamma_3p_1e^{-\mu x}\right)^2e^{-\mu x}\\
&- \mu|\lambda_1|p_1\left(\gamma_2p_3e^{\mu x} - \gamma_3p_2e^{-\mu x}\right)^2e^{-\mu x}.\end{aligned} \tag{3.27}$$

Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $|\lambda_3| > 0$, if one can choose p_1, p_2 and p_3 such that $p_1 = \gamma_1$, $p_2 = \gamma_2$ and $p_3 = \gamma_3$, then for sufficiently small $\mu > 0$,

$$\max_{0\le x\le l}\{\gamma_1p_3e^{\mu x} - \gamma_3p_1e^{-\mu x}\} \approx 0, \quad \max_{0\le x\le l}\{\gamma_2p_3e^{\mu x} - \gamma_3p_2e^{-\mu x}\} \approx 0,$$

and the determinants (3.25), (3.26) and (3.27) are positive.

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Therefore, with the choice of p_1, p_2 and p_3, the matrix (3.23) is positive definite if k_{13}, k_{23}, k_{31} and k_{32} satisfy

$$|k_{13}| < \sqrt{\left|\frac{\lambda_3}{\lambda_1}\right| \frac{\gamma_3}{\gamma_1}}, \quad |k_{23}| < \sqrt{\left|\frac{\lambda_3}{\lambda_2}\right| \frac{\gamma_3}{\gamma_2}}, \quad |k_{31}| < \sqrt{\left|\frac{\lambda_1}{\lambda_3}\right| \frac{\gamma_1}{\gamma_3} e^{-\mu l}} \quad \text{and} \quad |k_{32}| < \sqrt{\left|\frac{\lambda_2}{\lambda_3}\right| \frac{\gamma_2}{\gamma_3} e^{-\mu l}}.$$

3.3 Summary

In this chapter, we have discussed the exponential stability analysis of linear hyperbolic systems of balance laws in general and particular cases. Furthermore, it has been applied into some relevant examples of linear or linearized hyperbolic systems of balance laws.

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Chapter 4

Numerical Methods for Linear Hyperbolic Systems of Balance Laws

In this chapter, we mainly focus on first-order numerical schemes for the solution of linear hyperbolic systems of balance laws in one-space dimension, in particular upwind scheme for conservation part and centered discretization for source terms by considering two main approaches; direct and splitting methods. For this reason, we study finite volume methods under a uniform grid and then we give a first-order discrete linear hyperbolic systems of balance laws.

4.1 Finite volume methods in one space dimension

Consider the Cauchy problem (2.31) associated with a linear hyperbolic system of balance laws in one-space dimension and the independent variables x and t are defined on finite intervals $[0, l]$ and $[0, T]$, respectively for $l > 0$ and $T > 0$. In order to obtain numerical solution for this system, the finite volume methods (**FVM**) will be used [33]. Therefore, the spatial and temporal interval will be discretized into subintervals (also called grid cells). For simplicity, we use a uniform grid illustrated by Figure 4.1 below with grid sizes for the temporal and spatial intervals denoted by Δt and Δx, respectively. Hence, there are a finite number of grid points

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along the time and the space directions, which are denoted by

$$t^n := n\Delta t, \quad n = 0, \dots, N, \quad \text{and} \quad x_{j-\frac{1}{2}} = j\Delta x, \quad j = 0, \dots, J,$$

with $N\Delta t = T$ and $J\Delta x = l$, respectively. Moreover, the left and right boundary points are denoted by $x_{-\frac{1}{2}}$ and $x_{J-\frac{1}{2}}$, respectively. Let $x_j = (j + \frac{1}{2})\Delta x$ for $j = 0, \dots, J-1$ denote cell centers.

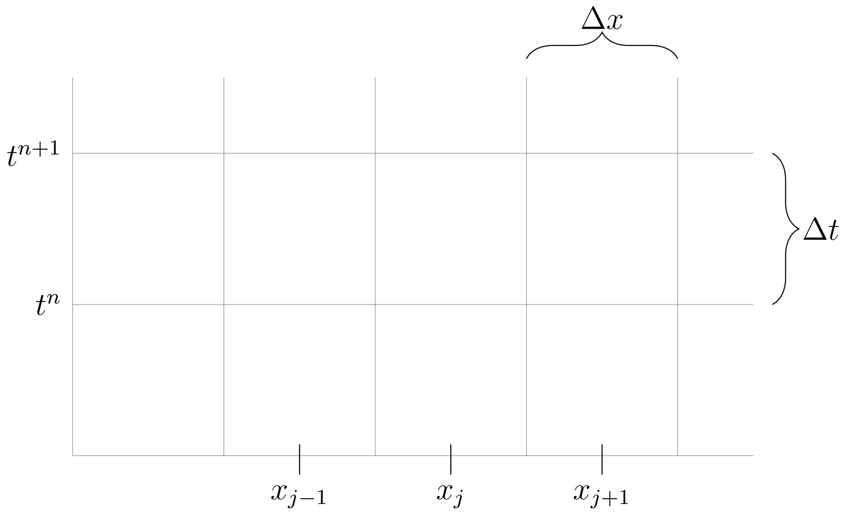

Figure 4.1: Example for a uniform grid

The next step in FVM here is approximating the integral of W over each grid cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. Therefore, the value W_j^n approximates the jth cell average at time $t^n, n = 0, \dots, N$ and it can be expressed as

$$W_j^n = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W(x, t^n) dx, \quad j = 0, \dots, J-1. \tag{4.1}$$

Since W is a smooth function, an approximate value W_j^n can be obtained by the midpoint value of W at each midpoint of the cells and then the integral of W over the spatial domain $[0, l]$ is approximated by $\sum_{j=0}^{J-1} W_j^n \Delta x$ at each time step $t^n, n = 0, \dots, N$.

In order to obtain a numerical solution for the Cauchy problem (2.31), we use the integral equation form of an equivalent expression for the system (2.31a). Since

$$\partial_x (\Lambda(x)W) = \Lambda(x)\partial_x W + \Lambda'(x)W,$$

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the system (2.31a) can written in the following form

$$\partial_t W + \partial_x \left(\Lambda(x)W\right) + \left(\Pi(x) - \Lambda'(x)\right) W = 0.$$

Therefore, the integral form is expressed as follows

$$\begin{aligned}\partial_t \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W(x,t)dx &= \Lambda(x_{j-\frac{1}{2}})W(x_{j-\frac{1}{2}},t) - \Lambda(x_{j+\frac{1}{2}})W(x_{j+\frac{1}{2}},t) \\ &\quad - \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\Pi(x) - \Lambda'(x)\right) W(x,t)dx, \quad j = 0,\dots,J-1.\end{aligned} \tag{4.2}$$

We integrate now the semi-discrete integral equation (4.2) over $[t^n, t^{n+1}], n = 0,\dots,N-1$ to obtain

$$\begin{aligned}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W(x,t^{n+1})dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W(x,t^n)dx &= \int_{t^n}^{t^{n+1}} \Lambda(x_{j-\frac{1}{2}})W(x_{j-\frac{1}{2}},t)dt \\ &\quad - \int_{t^n}^{t^{n+1}} \Lambda(x_{j+\frac{1}{2}})W(x_{j+\frac{1}{2}},t)dt \\ &\quad - \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\Pi(x) - \Lambda'(x)\right) W(x,t)dxdt, \\ &\quad j = 0,\dots,J-1.\end{aligned} \tag{4.3}$$

Dividing the fully discrete integral equation (4.3) by Δx and using equation (4.1), we obtain

$$W_j^{n+1} = W_j^n - \frac{\Delta t}{\Delta x}\left(O_{j+\frac{1}{2}} - O_{j-\frac{1}{2}}\right) - \Delta t S_j, \quad j = 0,\dots,J-1, \quad n = 0,\dots,N-1, \tag{4.4}$$

with

$$O_{j-\frac{1}{2}} := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \Lambda(x_{j-\frac{1}{2}})W(x_{j-\frac{1}{2}},t)dt \quad \text{and}$$

$$S_j := \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\Pi(x) - \Lambda'(x)\right) W(x,t)dxdt.$$

Since the system (2.31a) is strictly hyperbolic this fully discrete system (4.4) can be obtained by means of a numerical flux $\mathcal{O}$ and a numerical source function $\mathcal{S}$ [10, 33]. It was shown in Chapter 2 that the characteristic variables $w_i, i = 1,\dots,k$ are constant along the characteristic

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curves and represent the propagation of information with positive and negative characteristic speeds $\lambda_{i,j}$. With this, for positive $\lambda_{i,j}, i = 1, \ldots, m, j = 0, \ldots, J-1$, the corresponding values $w_i(x_j, t^{n+1}), i = 1, \ldots, m$ depend only on the values of $w_i, i = 1, \ldots, m$ at time t^n to the left of x_j and $w_{i,j}^{n+1}$ depends only on $w_{i,j}^n$. Similarly, for negative $\lambda_{i,j}, i = m+1, \ldots, k, j = 0, \ldots, J-1$, the corresponding values $w_i(x_j, t^{n+1}), i = m+1, \ldots, k$ depend only on the right of x_j and $w_{i,j}^{n+1}, i = m+1, \ldots, k$ depends only on $w_{i,j}^n$. In that case, the flux $O_{j-\frac{1}{2}}$ and the source term S_j can be written in the following form

$$O_{j-\frac{1}{2}} := \left[O^+_{j-\frac{1}{2}}, O^-_{j-\frac{1}{2}}\right]^T \quad \text{and} \quad S_j := S_j^+ + S_j^-,$$

with

$$\begin{aligned} O^{\pm}_{j-\frac{1}{2}} &:= \pm\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \Lambda^{\pm}(x_{j-\frac{1}{2}})W^{\pm}(x_{j-\frac{1}{2}}, t)dt \quad \text{and} \\ S_j^{\pm} &:= \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathcal{G}^{\pm}(x)W^{\pm}(x,t)dxdt, \end{aligned}$$

with

$$\mathcal{G}^+(x) = \Pi^+(x) - \begin{bmatrix} \Lambda^{+\prime}(x) \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{G}^-(x) = \Pi^-(x) - \begin{bmatrix} 0 \\ -\Lambda^{-\prime}(x) \end{bmatrix},$$

where matrices $\Pi^+(x) \in \mathbb{R}^{k\times m}$ and $\Pi^-(x) \in \mathbb{R}^{k\times(k-m)}$. Therefore, the flux $O^{\pm}_{j-\frac{1}{2}}$ depends only on the values x_{j-1}, x_j, $W^{\pm}{}^n_{j-1}$ and $W^{\pm}{}^n_j$ and the source term S_j^+ and S_j^- depend only on the values x_{j-1}, x_j, $W^{+}{}^n_{j-1}$ and $W^{+}{}^n_j$ and x_j, x_{j+1}, , $W^{-}{}^n_j$ and $W^{-}{}^n_{j+1}$, respectively and then for each $j = 0, \ldots, J-1$ and $n = 0, \ldots, N-1$, we have

$$\begin{aligned} O^{\pm}{}^n_{j-\frac{1}{2}} &:= \mathcal{O}^{\pm}(x_{j-1}, x_j, W^{\pm}{}^n_{j-1}, W^{\pm}{}^n_j), \\ S^{+}{}^n_j &:= \mathcal{S}^+\left(x_{j-1}, x_j, W^{+}{}^n_{j-1}, W^{+}{}^n_j\right) \quad \text{and} \\ S^{-}{}^n_j &:= \mathcal{S}^-\left(x_j, x_{j+1}, W^{-}{}^n_j, W^{-}{}^n_{j+1}\right). \end{aligned}$$

Hence, the discretization of the system (2.31a) can now be expressed as follows

$$\begin{aligned} \begin{bmatrix} W^{+}{}^{n+1}_j \\ W^{-}{}^{n+1}_j \end{bmatrix} = \begin{bmatrix} W^{+}{}^n_j \\ W^{-}{}^n_j \end{bmatrix} - \frac{\Delta t}{\Delta x}\begin{bmatrix} \mathcal{O}^+(x_j, x_{j+1}, W^{+}{}^n_j, W^{+}{}^n_{j+1}) - \mathcal{O}^+(x_{j-1}, x_j, W^{+}{}^n_{j-1}, W^{+}{}^n_j) \\ \mathcal{O}^-(x_j, x_{j+1}, W^{-}{}^n_j, W^{-}{}^n_{j+1}) - \mathcal{O}^-(x_{j-1}, x_j, W^{-}{}^n_{j-1}, W^{-}{}^n_j) \end{bmatrix} \\ - \Delta t\left[\mathcal{S}^+\left(x_{j-1}, x_j, W^{+}{}^n_{j-1}, W^{+}{}^n_j\right) + \mathcal{S}^-\left(x_j, x_{j+1}, W^{-}{}^n_j, W^{-}{}^n_{j+1}\right)\right], \\ j = 0, \ldots, J-1, \quad n = 0, \ldots, N-1. \quad (4.5) \end{aligned}$$

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In order to obtain a numerical solution for the system (2.31a), we should use a convergent, consistent and stable numerical scheme for the discretized system (4.5) as $\Delta x, \Delta t \to 0$ [33]. For this reason, we choose the first-order upwind method and then the numerical flux can be expressed in the following form

$$\begin{aligned} \mathcal{O}^+(x_{j-1}, x_j, W^{+n}_{j-1}, W^{+n}_{j}) &:= \Lambda^+(x_{j-1}) W^{+n}_{j-1} \quad \text{and} \\ \mathcal{O}^-(x_{j-1}, x_j, W^{-n}_{j-1}, W^{-n}_{j}) &:= \Lambda^-(x_{j}) W^{-n}_{j}. \end{aligned}$$

The numerical source term by using two possible approximation methods, which are called centered discretization and upwinded approximation [36], can be expressed as follows

$$\begin{aligned} \mathcal{S}^+\left(x_{j-1}, x_j, W^{+n}_{j-1}, W^{+n}_{j}\right) &:= \mathcal{G}^+(x_j) W^{+n}_{j} \quad \text{and} \\ \mathcal{S}^-\left(x_{j}, x_{j+1}, W^{-n}_{j}, W^{-n}_{j+1}\right) &:= \mathcal{G}^-(x_j) W^{-n}_{j}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}^+\left(x_{j-1}, x_j, W^{+n}_{j-1}, W^{+n}_{j}\right) &:= \frac{1}{2}\left(\mathcal{G}^+(x_{j-1}) W^{+n}_{j-1} + \mathcal{G}^+(x_j) W^{+n}_{j}\right) \quad \text{and} \\ \mathcal{S}^-\left(x_{j}, x_{j+1}, W^{-n}_{j}, W^{-n}_{j+1}\right) &:= \frac{1}{2}\left(\mathcal{G}^-(x_j) W^{-n}_{j} + \mathcal{G}^-(x_{j+1}) W^{-n}_{j+1}\right), \end{aligned}$$

respectively. It must be noted that the stability holds under CFL condition

$$\frac{\Delta t}{\Delta x} \max_{\substack{1 \le i \le k \\ 1 \le j \le J-1}} |\lambda_{i,j}| \le 1.$$

Hence, the first-order discretization of the system (2.31a) with centered discretization can be written in the following form

$$\begin{bmatrix} W^{+n+1}_{j} \\ W^{-n+1}_{j} \end{bmatrix} = \begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \Lambda^+_j W^{+n}_{j} - \Lambda^+_{j-1} W^{+n}_{j-1} \\ \Lambda^-_j W^{-n}_{j} - \Lambda^-_{j+1} W^{-n}_{j+1} \end{bmatrix} - \Delta t \left(\Pi_j - \Lambda'_j\right) \begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix},$$
$$j = 0, \dots, J-1, \quad n = 0, \dots, N-1, \quad (4.6)$$

with

$$\Lambda^+_{j-\frac{1}{2}} := \Lambda^+(x_{j-\frac{1}{2}}), \quad \Lambda^-_{j-\frac{1}{2}} := \Lambda^-(x_{j-\frac{1}{2}}) \quad \text{and} \quad \Pi_j - \Lambda'_j := \Pi(x_j) - \Lambda'(x_j).$$

However, since

$$\frac{1}{\Delta x} \begin{bmatrix} \Lambda^+_j W^{+n}_{j} - \Lambda^+_{j-1} W^{+n}_{j-1} \\ \Lambda^-_j W^{-n}_{j} - \Lambda^-_{j+1} W^{-n}_{j+1} \end{bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} \Lambda^+_{j-1} & 0 \\ 0 & \Lambda^-_{j+1} \end{bmatrix} \begin{bmatrix} W^{+n}_{j} - W^{+n}_{j-1} \\ W^{-n}_{j} - W^{-n}_{j+1} \end{bmatrix}$$

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$$+\frac{1}{\Delta x}\begin{bmatrix}\Lambda_j^+ - \Lambda_{j-1}^+ & 0 \\ 0 & -\left(\Lambda_{j+1}^- - \Lambda_j^-\right)\end{bmatrix} W_j^n,$$
$$=\frac{1}{\Delta x}\begin{bmatrix}\Lambda_{j-1}^+ & 0 \\ 0 & \Lambda_{j+1}^-\end{bmatrix}\begin{bmatrix}W^{+}{}_j^n - W^{+}{}_{j-1}^n \\ W^{-}{}_j^n - W^{-}{}_{j+1}^n\end{bmatrix} + \Lambda_j' W_j^n,$$

the fully discretized system (4.6) can be expressed as follows

$$\begin{bmatrix}W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1}\end{bmatrix} = \begin{bmatrix}W^{+}{}_j^{n} \\ W^{-}{}_j^{n}\end{bmatrix} - \frac{\Delta t}{\Delta x}\begin{bmatrix}\Lambda_{j-1}^+ & 0 \\ 0 & \Lambda_{j+1}^-\end{bmatrix}\begin{bmatrix}W^{+}{}_j^n - W^{+}{}_{j-1}^n \\ W^{-}{}_j^n - W^{-}{}_{j+1}^n\end{bmatrix} - \Delta t \Pi_j \begin{bmatrix}W^{+}{}_j^{n} \\ W^{-}{}_j^{n}\end{bmatrix},$$
$$j = 0, \ldots, J-1, \quad n = 0, \ldots, N-1. \tag{4.7}$$

The discretized system (4.7) is an explicit discretization. Then an explicit with implicit source terms discretization can be expressed in the following form

$$\begin{bmatrix}W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1}\end{bmatrix} = \begin{bmatrix}W^{+}{}_j^{n} \\ W^{-}{}_j^{n}\end{bmatrix} - \frac{\Delta t}{\Delta x}\begin{bmatrix}\Lambda_j^+ W^{+}{}_j^n - \Lambda_{j-1}^+ W^{+}{}_{j-1}^n \\ \Lambda_j^- W^{-}{}_j^n - \Lambda_{j+1}^- W^{-}{}_{j+1}^n\end{bmatrix} - \Delta t\left(\Pi_j - \Lambda_j'\right)\begin{bmatrix}W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1}\end{bmatrix},$$
$$j = 0, \ldots, J-1, \quad n = 0, \ldots, N-1. \tag{4.8}$$

In the special case, the linear system with constant coefficients (2.32a) can be discretized using similar method of discretization for the system (4.7). Therefore, it is obtained as follows

$$\begin{bmatrix}W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1}\end{bmatrix} = \begin{bmatrix}W^{+}{}_j^{n} \\ W^{-}{}_j^{n}\end{bmatrix} - \frac{\Delta t}{\Delta x}\begin{bmatrix}\Lambda^+ & 0 \\ 0 & \Lambda^-\end{bmatrix}\begin{bmatrix}W^{+}{}_j^n - W^{+}{}_{j-1}^n \\ W^{-}{}_j^n - W^{-}{}_{j+1}^n\end{bmatrix} - \Delta t \Pi \begin{bmatrix}W^{+}{}_j^{n} \\ W^{-}{}_j^{n}\end{bmatrix},$$
$$j = 0, \ldots, J-1, \quad n = 0, \ldots, N-1. \tag{4.9}$$

The explicit with implicit source terms discretization for the linear system with constant coefficients can be expressed in the following form

$$\begin{bmatrix}W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1}\end{bmatrix} = \begin{bmatrix}W^{+}{}_j^{n} \\ W^{-}{}_j^{n}\end{bmatrix} - \frac{\Delta t}{\Delta x}\begin{bmatrix}\Lambda^+ & 0 \\ 0 & \Lambda^-\end{bmatrix}\begin{bmatrix}W^{+}{}_j^n - W^{+}{}_{j-1}^n \\ W^{-}{}_j^n - W^{-}{}_{j+1}^n\end{bmatrix} - \Delta t \Pi \begin{bmatrix}W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1}\end{bmatrix},$$
$$j = 0, \ldots, J-1, \quad n = 0, \ldots, N-1. \tag{4.10}$$

In general, the direct method approach, which has been used to discretize the systems (2.31a) and (2.32a) might have some problems in the analysis of convergence and stability. For

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instance, in [44] the direct method of discretization fails in the analysis of convergence. For this reason and to obtain better results, we prefer to use the splitting method approach discretization. Therefore, by using this method the following discretized linear systems with variable coefficients

$$\begin{bmatrix} \widetilde{W^+}_j^n \\ \widetilde{W^-}_j^n \end{bmatrix} = \begin{bmatrix} W^{+n}_j \\ W^{-n}_j \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \Lambda^+_{j-1} & 0 \\ 0 & \Lambda^-_{j+1} \end{bmatrix} \begin{bmatrix} W^{+n}_j - W^{+n}_{j-1} \\ W^{-n}_j - W^{-n}_{j+1} \end{bmatrix}, \tag{4.11a}$$

$$\begin{bmatrix} W^{+n+1}_j \\ W^{-n+1}_j \end{bmatrix} = \begin{bmatrix} \widetilde{W^+}_j^n \\ \widetilde{W^-}_j^n \end{bmatrix} - \Delta t \Pi_j \begin{bmatrix} \widetilde{W^+}_j^n \\ \widetilde{W^-}_j^n \end{bmatrix}, \tag{4.11b}$$

$$n = 0, \dots, N-1, \quad j = 0, \dots, J-1,$$

and with constant coefficients

$$\begin{bmatrix} \widetilde{W^+}_j^n \\ \widetilde{W^-}_j^n \end{bmatrix} = \begin{bmatrix} W^{+n}_j \\ W^{-n}_j \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} W^{+n}_j - W^{+n}_{j-1} \\ W^{-n}_j - W^{-n}_{j+1} \end{bmatrix}, \tag{4.12a}$$

$$\begin{bmatrix} W^{+n+1}_j \\ W^{-n+1}_j \end{bmatrix} = \begin{bmatrix} \widetilde{W^+}_j^n \\ \widetilde{W^-}_j^n \end{bmatrix} - \Delta t \Pi \begin{bmatrix} \widetilde{W^+}_j^n \\ \widetilde{W^-}_j^n \end{bmatrix}, \tag{4.12b}$$

$$n = 0, \dots, N-1, \quad j = 0, \dots, J-1,$$

are obtained.

4.2 The discrete Cauchy problem

The discretized initial condition for the discretized system (4.7) can be specified by using equation (4.1) and (2.31c) as follows

$$W_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_0(x) dx, \quad j = 0, \dots, J-1. \tag{4.13}$$

In order to discretize the boundary condition (2.31b), there must be extra cells (also called ghost cells) to approximate the cell average at cell center. Since we are interested in first-order accurate approximation, we need only one extra cell to specify boundary values for each characteristic variable. Therefore, for each characteristic variable traveling to the right

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direction, we set boundary condition at the left boundary point and for each characteristic variables traveling to the left direction, we set boundary condition at the right boundary point. Hence, the boundary values at boundary points $x_{-\frac{1}{2}}$ and $x_{J-\frac{1}{2}}$ can be approximated by W^{+n}_{-1} and W^{-n}_{J}, respectively and then by using the algebraic relation of boundary values (2.31b), we have

$$\begin{bmatrix} W^{+n+1}_{-1} \\ W^{-n+1}_{J} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} W^{+n+1}_{J-1} \\ W^{-n+1}_{0} \end{bmatrix}, \quad n = 0, \dots, N-1. \tag{4.14}$$

We have now the discrete Cauchy problem associated with the discretized system (4.7), initial (4.13) and boundary (4.14) conditions.

In general, the numerical solution, W^n_j, for the discretized system (4.7) is now used to approximate the smooth solution for the Cauchy problem (2.31), which is expressed in the following form

$$W(x,t) = \sum_{n=0}^{N-1} \sum_{j=0}^{J-1} W^n_j \chi_{[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1}]}, \tag{4.15}$$

where $\chi_{[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1}]}$ is a step function over $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1}]$.

4.3 Application to some hyperbolic systems of balance laws

4.3.1 The Telegrapher Equations

The telegrapher equations in the decoupled linear system with constant coefficients form (2.42) can be discretized by considering the splitting method approach and then it can be expressed as follows

$$\begin{bmatrix} \widetilde{w_1}^n_j \\ \widetilde{w_2}^n_j \end{bmatrix} = \begin{bmatrix} w_1{}^n_j \\ w_2{}^n_j \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \lambda_1 & 0 \\ 0 & |\lambda_2| \end{bmatrix} \begin{bmatrix} w_1{}^n_j - w_1{}^n_{j-1} \\ w_2{}^n_j - w_2{}^n_{j+1} \end{bmatrix}, \quad n = 0, \dots, N-1, \tag{4.16a}$$

$$\begin{bmatrix} w_1{}^{n+1}_j \\ w_2{}^{n+1}_j \end{bmatrix} = \begin{bmatrix} \widetilde{w_1}^n_j \\ \widetilde{w_2}^n_j \end{bmatrix} - \Delta t \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{bmatrix} \begin{bmatrix} \widetilde{w_1}^n_j \\ \widetilde{w_2}^n_j \end{bmatrix}, \quad j = 0, \dots, J-1. \tag{4.16b}$$

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The discretization of the initial condition (2.43) and the boundary conditions (2.44) can be obtained by using (4.13) and (4.14), respectively and then it can be expressed as follows

$$W_j^0 = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_0(x)dx, \quad j = 0, \ldots, J-1 \tag{4.17}$$

and

$$\begin{bmatrix} {w_1}_{-1}^{n+1} \\ {w_2}_{J}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} {w_1}_{J-1}^{n+1} \\ {w_2}_{0}^{n+1} \end{bmatrix}, \quad n = 0, \ldots, N-1. \tag{4.18}$$

4.3.2 The Isentropic Euler Equations

The isentropic Euler equations in the decoupled linear system with variable coefficients form (2.51) can be discretized by considering the splitting method approach and then it can be expressed as follows

$$\begin{bmatrix} \widetilde{{w_1}_j^n} \\ \widetilde{{w_2}_j^n} \end{bmatrix} = \begin{bmatrix} {w_1}_j^n \\ {w_2}_j^n \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \lambda_{1,j-1} & 0 \\ 0 & |\lambda_{2,j+1}| \end{bmatrix} \begin{bmatrix} {w_1}_j^n - {w_1}_{j-1}^n \\ {w_2}_j^n - {w_2}_{j+1}^n \end{bmatrix}, \quad n = 0, \ldots, N-1, \tag{4.19a}$$

$$\begin{bmatrix} {w_1}_j^{n+1} \\ {w_2}_j^{n+1} \end{bmatrix} = \begin{bmatrix} \widetilde{{w_1}_j^n} \\ \widetilde{{w_2}_j^n} \end{bmatrix} - \Delta t \begin{bmatrix} \gamma_{11,j} & \gamma_{12,j} \\ \gamma_{21,j} & \gamma_{22,j} \end{bmatrix} \begin{bmatrix} \widetilde{{w_1}_j^n} \\ \widetilde{{w_2}_j^n} \end{bmatrix}, \quad j = 0, \ldots, J-1. \tag{4.19b}$$

The discretization of the initial condition (2.52) and the boundary conditions (2.53) can be obtained by using (4.13) and (4.14), respectively and then it can be expressed as follows

$$W_j^0 = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_0(x)dx, \quad j = 0, \ldots, J-1 \tag{4.20}$$

and

$$\begin{bmatrix} {w_1}_{-1}^{n+1} \\ {w_2}_{J}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} {w_1}_{J-1}^{n+1} \\ {w_2}_{0}^{n+1} \end{bmatrix}, \quad n = 0, \ldots, N-1. \tag{4.21}$$

4.3.3 The Saint-Venant Equations

The Saint-Venant equations in the decoupled linear system with variable coefficients form (2.60) can be discretized by considering the splitting method approach and then it can be

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expressed as follows

$$\begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \end{bmatrix} = \begin{bmatrix} {w_1}_j^n \\ {w_2}_j^n \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \lambda_{1,j-1} & 0 \\ 0 & |\lambda_{2,j+1}| \end{bmatrix} \begin{bmatrix} {w_1}_j^n - {w_1}_{j-1}^n \\ {w_2}_j^n - {w_2}_{j+1}^n \end{bmatrix}, \quad n = 0, \ldots, N-1, \tag{4.22a}$$

$$\begin{bmatrix} {w_1}_j^{n+1} \\ {w_2}_j^{n+1} \end{bmatrix} = \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \end{bmatrix} - \Delta t \begin{bmatrix} \gamma_{11,j} & \gamma_{12,j} \\ \gamma_{21,j} & \gamma_{22,j} \end{bmatrix} \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \end{bmatrix}, \quad j = 0, \ldots, J-1. \tag{4.22b}$$

The discretization of the initial condition (2.61) and the boundary conditions (2.62) can be obtained by using (4.13) and (4.14), respectively and then it can be expressed as follows

$$W_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_0(x)dx, \quad j = 0, \ldots, J-1 \tag{4.23}$$

and

$$\begin{bmatrix} {w_1}_{-1}^{n+1} \\ {w_2}_J^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} {w_1}_{J-1}^{n+1} \\ {w_2}_0^{n+1} \end{bmatrix}, \quad n = 0, \ldots, N-1. \tag{4.24}$$

4.3.4 The Saint-Venant-Exner Equations

The Saint-Venant-Exner equations in the decoupled linear system with constant coefficients form (2.73) can be discretized by considering the splitting method approach and then it can be expressed as follows

$$\begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \\ \widetilde{w_3}_j^n \end{bmatrix} = \begin{bmatrix} {w_1}_j^n \\ {w_2}_j^n \\ {w_3}_j^n \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & |\lambda_3| \end{bmatrix} \begin{bmatrix} {w_1}_j^n - {w_1}_{j-1}^n \\ {w_2}_j^n - {w_2}_{j-1}^n \\ {w_3}_j^n - {w_3}_{j+1}^n \end{bmatrix}, \quad n = 0, \ldots, N-1, \tag{4.25a}$$

$$\begin{bmatrix} {w_1}_j^{n+1} \\ {w_2}_j^{n+1} \\ {w_3}_j^{n+1} \end{bmatrix} = \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \\ \widetilde{w_3}_j^n \end{bmatrix} - \Delta t \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \\ \widetilde{w_3}_j^n \end{bmatrix}, \quad j = 0, \ldots, J-1. \tag{4.25b}$$

The discretization of the initial condition (2.74) and the boundary conditions (2.75) can be obtained by using (4.13) and (4.14), respectively and then it can be expressed as follows

$$W_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W_0(x)dx, \quad j = 0, \ldots, J-1 \tag{4.26}$$

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and

$$\begin{bmatrix} {w_1}_{-1}^{n+1} \\ {w_2}_{-1}^{n+1} \\ {w_3}_{J}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_{13} \\ 0 & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{bmatrix} \begin{bmatrix} {w_1}_{J-1}^{n+1} \\ {w_2}_{J-1}^{n+1} \\ {w_3}_{0}^{n+1} \end{bmatrix}, \quad n = 0, \ldots, N-1. \tag{4.27}$$

4.4 Summary

In this chapter, the numerical methods for a linear hyperbolic system of balance laws has been discussed and used to discretize the Cauchy problem of examples of hyperbolic system of balance laws presented in Chapter 2.

In the next chapter, numerical boundary stabilization for a linear hyperbolic system of balance laws will be presented.

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Chapter 5

Numerical Boundary Stabilization of Linear Hyperbolic Systems of Balance Laws

In this chapter, we investigate conditions for a numerical exponential stability analysis of discretized Cauchy problem associated with discretized linear hyperbolic systems of balance laws (4.7) and (4.9) by considering the time splitting method with the discretized initial (4.13) and boundary (4.14) conditions under steady state solution $W_j^n \equiv 0, \forall j = 0, \ldots, J-1, \forall$ $n = 0, \ldots, N-1$. For this reason, we introduce a discrete version of the $\mathbb{L}^2$ Lyapunov function (3.1) and exponential stability definition for $\mathbb{L}^2-$norm.

5.1 Discrete Lyapunov Exponential Stability Analysis

In this section, our aim is to investigate conditions for the exponential decay of the numerical solution W_j^n. First, we define exponential stability as follows.

Definition 7. *The discretized linear hyperbolic system* (4.11) *(or* (4.12)*) with the discretized boundary conditions* (4.14) *is exponentially stable if there exist* $\eta > 0$ *and* $C > 0$ *such that, for every* $W_j^0 \in \mathbb{L}^2((x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}); \mathbb{R}^k)$, $j = 0, \ldots, J-1$, *the solution to the discretized Cauchy*

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problem (4.11) *(or* (4.12)*),* (4.13), (4.14) *satisfies*

$$\Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \leq Ce^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \ldots, N.$$

Consider a Cauchy problem defined by the discretized system (4.11) with the discretized initial (4.13) and boundary (4.14) conditions under steady state solution $W_j^n \equiv 0, \forall j = 0, \ldots, J-1, \forall\, n = 0, \ldots, N-1$.

In order to obtain conditions for the stability of the Cauchy problem, we define the following discrete candidate Lyapunov function

$$L^n = \Delta x \sum_{j=0}^{J-1} \left(W_j^n\right)^T \Phi_j W_j^n, \quad n = 0, \ldots, N, \tag{5.1}$$

where $\Phi_j := \text{diag}\{P^+ e^{-\mu x_j}, P^- e^{\mu x_j}\}$, with

$$P^+(x) := \text{diag}\{p_1, \ldots, p_m\}, \quad P^-(x) := \text{diag}\{p_{m+1}, \ldots, p_k\} \quad \text{and} \quad p_i > 0, \quad i = 1, \ldots, k.$$

Theorem 4. *Let $T > 0$ be fixed and for each $j = 0, \ldots, J-1$ assume $\lambda_{i,j} > 0, i = 1, \ldots, m$ and $\lambda_{i,j} < 0, i = m+1, \ldots, k$. Let the discrete Lyapunov function be defined by* (5.1)*. If the CFL condition $\frac{\Delta t}{\Delta x} \max_{\substack{1 \leq i \leq k \\ 0 \leq j \leq J-1}} |\lambda_{i,j}| \leq 1$ holds and there exists a real number $\mu > 0$, $p_i > 0, i = 1, \ldots, k$, such that $0 < \mu\alpha e^{-\mu \Delta x} - \beta < 1$, where*

$$\alpha = \min_{\substack{1 \leq i \leq k \\ 0 \leq j \leq J-1}} |\lambda_{i,j}| \quad \textit{and} \quad \beta = \max_{\substack{1 \leq i \leq k \\ 0 \leq j \leq J-1}} \lambda'_{i,j},$$

$$\Pi_j^T \Phi_j + \Phi_j \Pi_j - \Delta t \Pi_j^T \Phi_j \Pi_j, \quad j = 0, \ldots, J-1, \quad \textit{is positive semi-definite} \tag{5.2}$$

and

$$\begin{bmatrix} P^+ e^{-\mu x_J} \Lambda_{J-1}^+ & 0 \\ 0 & P^- e^{\mu x_{-1}} \Lambda_0^- \end{bmatrix} - K^T \begin{bmatrix} P^+ e^{-\mu x_0} \Lambda_{-1}^+ & 0 \\ 0 & P^- e^{\mu x_{J-1}} \Lambda_J^- \end{bmatrix} K, \tag{5.3}$$

is positive definite, then the numerical solution W_j^n of the Cauchy problem (4.11), (4.13), (4.14) *converges to a steady-state solution $W_j^* = 0$ for the $\mathbb{L}^2-$norm.*

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Proof. *The time derivative of the Lyapunov function* (3.1) *is approximated by using the discrete Lyapunov function* (5.1), *which is obtained in the following form*

$$\frac{L^{n+1}-L^n}{\Delta t}=\frac{L^{n+1}-\widetilde{L}^n}{\Delta t}+\frac{\widetilde{L}^n-L^n}{\Delta t}, \tag{5.4}$$

with

$$\widetilde{L}^n:=\Delta x\sum_{j=0}^{J-1}\left(\widetilde{W_j^n}\right)^T\Phi_j\widetilde{W_j^n},\quad n=0,\dots,N.$$

In order to show that the time derivative (5.4) *is a negative definite quadratic form, it suffices to show that both approximation of the time derivatives*

$$\frac{L^{n+1}-\widetilde{L}^n}{\Delta t}=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[\left(W_j^{n+1}\right)^T\Phi_jW_j^{n+1}-\left(\widetilde{W_j^n}\right)^T\Phi_j\widetilde{W_j^n}\right], \tag{5.5}$$

and

$$\frac{\widetilde{L}^n-L^n}{\Delta t}=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[\left(\widetilde{W_j^n}\right)^T\Phi_j\widetilde{W_j^n}-\left(W_j^n\right)^T\Phi_jW_j^n\right] \tag{5.6}$$

are negative definite quadratic forms.

We use now the discretized system (4.11) *with boundary condition* (4.14) *to obtain the following*

$$\begin{aligned}\frac{L^{n+1}-\widetilde{L}^n}{\Delta t}=&\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[\left(\widetilde{W_j^n}-\Delta t\Pi_j\widetilde{W_j^n}\right)^T\Phi_j\left(\widetilde{W_j^n}-\Delta t\Pi_j\widetilde{W_j^n}\right)-\left(\widetilde{W_j^n}\right)^T\Phi_j\left(\widetilde{W_j^n}\right)\right],\\ =&\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[\left(\widetilde{W_j^n}\right)^T\Phi_j\left(\widetilde{W_j^n}\right)-\left(\widetilde{W_j^n}\right)^T\Phi_j\left(\widetilde{W_j^n}\right)\right]\\ &-\Delta x\sum_{j=0}^{J-1}\left[\left(\widetilde{W_j^n}\right)^T\Pi_j^T\Phi_j\left(\widetilde{W_j^n}\right)+\left(\widetilde{W_j^n}\right)^T\Phi_j\Pi_j\left(\widetilde{W_j^n}\right)-\Delta t\left(\widetilde{W_j^n}\right)^T\Pi_j^T\Phi_j\Pi_j\left(\widetilde{W_j^n}\right)\right],\\ =&-\Delta x\sum_{j=0}^{J-1}\left(\widetilde{W_j^n}\right)^T\left[\Pi_j^T\Phi_j+\Phi_j\Pi_j-\Delta t\Pi_j^T\Phi_j\Pi_j\right]\left(\widetilde{W_j^n}\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\widetilde{L}^n-L^n}{\Delta t}=&\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[\left(W_j^n-\frac{\Delta t}{\Delta x}\begin{bmatrix}\Lambda_{j-1}^+ & 0\\ 0 & \Lambda_{j+1}^-\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\ W^{-n}_j-W^{-n}_{j+1}\end{bmatrix}\right)^T\Phi_j\right.\\ &\left.\left(W_j^n-\frac{\Delta t}{\Delta x}\begin{bmatrix}\Lambda_{j-1}^+ & 0\\ 0 & \Lambda_{j+1}^-\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\ W^{-n}_j-W^{-n}_{j+1}\end{bmatrix}\right)-\left(W_j^n\right)^T\Phi_j\left(W_j^n\right)\right],\end{aligned}$$

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$$=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[\left(W_j^n - D_j\left(W_j^n-\Xi_j^n\right)\right)^T\Phi_j\left(W_j^n - D_j\left(W_j^n-\Xi_j^n\right)\right)-\left(W_j^n\right)^T\Phi_j\left(W_j^n\right)\right],$$

with

$$D_j := \begin{bmatrix} D_{j-1}^+ & 0 \\ 0 & D_{j+1}^- \end{bmatrix} \quad \text{and} \quad \Xi_j^n := \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix},$$

where $D_{j-1}^+ := \frac{\Delta t}{\Delta x}\Lambda_{j-1}^+$ *and* $D_{j+1}^- := \frac{\Delta t}{\Delta x}\Lambda_{j+1}^-$. *Then,*

$$\begin{aligned}\frac{\widetilde{L}^n - L^n}{\Delta t} =& \frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\Big[\left(W_j^n\right)^T\Phi_j\left(W_j^n\right) - 2\left(W_j^n\right)^T\Phi_jD_j\left(W_j^n\right) + 2\left(W_j^n\right)^T\Phi_jD_j\left(\Xi_j^n\right) \\ &+\left(W_j^n\right)^TD_j\Phi_jD_j\left(W_j^n\right)+\left(\Xi_j^n\right)^TD_j\Phi_jD_j\left(\Xi_j^n\right) \\ &-2\left(W_j^n\right)^TD_j\Phi_jD_j\left(\Xi_j^n\right)-\left(W_j^n\right)^T\Phi_j\left(W_j^n\right)\Big], \\ =& \frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\Big[-2\left(W_j^n\right)^T\Phi_jD_j\left(W_j^n\right)+2\left(W_j^n\right)^T\left(I-D_j\right)\Phi_jD_j\left(\Xi_j^n\right) \\ &+\left(W_j^n\right)^TD_j\Phi_jD_j\left(W_j^n\right)+\left(\Xi_j^n\right)^TD_j\Phi_jD_j\left(\Xi_j^n\right)\Big].\end{aligned}$$

By the CFL condition and since $(I-D_j)\Phi_jD_j$ *is a positive definite diagonal matrix, we have* $I-D_j \geq 0$ *and*

$$\begin{aligned}2\left(W_j^n\right)^T\left(I-D_j\right)\Phi_jD_j\left(\Xi_j^n\right) \leq& \left(W_j^n\right)^T\left(I-D_j\right)\Phi_jD_j\left(W_j^n\right)+\left(\Xi_j^n\right)^T\left(I-D_j\right)\Phi_jD_j\left(\Xi_j^n\right), \\ =& \left(W_j^n\right)^T\Phi_jD_j\left(W_j^n\right)-\left(W_j^n\right)^TD_j\Phi_jD_j\left(W_j^n\right) \\ &+\left(\Xi_j^n\right)^T\Phi_jD_j\left(\Xi_j^n\right)-\left(\Xi_j^n\right)^TD_j\Phi_jD_j\left(\Xi_j^n\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\widetilde{L}^n - L^n}{\Delta t} \leq& \frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left[-\left(W_j^n\right)^T\Phi_jD_j\left(W_j^n\right)+\left(\Xi_j^n\right)^T\Phi_jD_j\left(\Xi_j^n\right)\right], \\ =& -\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_{j-1}^+ & 0 \\ 0 & \Lambda_{j+1}^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix} \\ &+\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_{j-1}^+ & 0 \\ 0 & \Lambda_{j+1}^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}.\end{aligned}$$

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Consider the second term on the right hand side of the above equation

$$
\begin{aligned}
&\sum_{j=0}^{J-1} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda^+_{j-1} & 0 \\ 0 & \Lambda^-_{j+1} \end{bmatrix} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix} \\
&= \sum_{j=0}^{J-1} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}^T \begin{bmatrix} P^+ e^{-\mu x_j} \Lambda^+_{j-1} & 0 \\ 0 & P^- e^{\mu x_j} \Lambda^-_{j+1} \end{bmatrix} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}, \\
&= \sum_{j=0}^{J-1} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}^T \begin{bmatrix} e^{-\mu\Delta x} P^+ e^{-\mu x_{j-1}} \Lambda^+_{j-1} & 0 \\ 0 & e^{-\mu\Delta x} P^- e^{\mu x_{j+1}} \Lambda^-_{j+1} \end{bmatrix} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}, \\
&= e^{-\mu\Delta x} \sum_{j=0}^{J-1} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}^T \begin{bmatrix} P^+ e^{-\mu x_{j-1}} \Lambda^+_{j-1} & 0 \\ 0 & P^- e^{\mu x_{j+1}} \Lambda^-_{j+1} \end{bmatrix} \begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1} \end{bmatrix}, \\
&= e^{-\mu\Delta x} \sum_{j=0}^{J-1} \big(W^{+n}_{j-1}\big)^T P^+ e^{-\mu x_{j-1}} \Lambda^+_{j-1} \big(W^{+n}_{j-1}\big) \\
&+ e^{-\mu\Delta x} \sum_{j=0}^{J-1} \big(W^{-n}_{j+1}\big)^T P^- e^{\mu x_{j+1}} \Lambda^-_{j+1} \big(W^{-n}_{j+1}\big), \\
&= e^{-\mu\Delta x} \sum_{j=0}^{J-1} \big(W^{+n}_{j}\big)^T P^+ e^{-\mu x_j} \Lambda^+_j \big(W^{+n}_j\big) + e^{-\mu\Delta x} \sum_{j=0}^{J-1} \big(W^{-n}_j\big)^T P^- e^{\mu x_j} \Lambda^-_j \big(W^{-n}_j\big) \\
&+ e^{-\mu\Delta x} \big(W^{+n}_{-1}\big)^T P^+ e^{-\mu x_{-1}} \Lambda^+_{-1} \big(W^{+n}_{-1}\big) - e^{-\mu\Delta x} \big(W^{+n}_{J-1}\big)^T P^+ e^{-\mu x_{J-1}} \Lambda^+_{J-1} \big(W^{+n}_{J-1}\big) \\
&- e^{-\mu\Delta x} \big(W^{-n}_0\big)^T P^- e^{\mu x_0} \Lambda^-_0 \big(W^{-n}_0\big) + e^{-\mu\Delta x} \big(W^{-n}_J\big)^T P^- e^{\mu x_J} \Lambda^-_J \big(W^{-n}_J\big), \\
&= e^{-\mu\Delta x} \sum_{j=0}^{J-1} \begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix}^T \begin{bmatrix} P^+ e^{-\mu x_j} \Lambda^+_{j} & 0 \\ 0 & P^- e^{\mu x_j} \Lambda^-_{j} \end{bmatrix} \begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix} \\
&+ \big(W^{+n}_{-1}\big)^T P^+ e^{-\mu x_0} \Lambda^+_{-1} \big(W^{+n}_{-1}\big) - \big(W^{+n}_{J-1}\big)^T P^+ e^{-\mu x_J} \Lambda^+_{J-1} \big(W^{+n}_{J-1}\big) \\
&- \big(W^{-n}_0\big)^T P^- e^{\mu x_{-1}} \Lambda^-_0 \big(W^{-n}_0\big) + \big(W^{-n}_J\big)^T P^- e^{\mu x_{J-1}} \Lambda^-_J \big(W^{-n}_J\big), \\
&= e^{-\mu\Delta x} \sum_{j=0}^{J-1} \begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda^+_j & 0 \\ 0 & \Lambda^-_j \end{bmatrix} \begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j} \end{bmatrix} \\
&+ \begin{bmatrix} W^{+n}_{-1} \\ W^{-n}_{J} \end{bmatrix}^T \begin{bmatrix} P^+ e^{-\mu x_0} \Lambda^+_{-1} & 0 \\ 0 & P^- e^{\mu x_{J-1}} \Lambda^-_{J} \end{bmatrix} \begin{bmatrix} W^{+n}_{-1} \\ W^{-n}_{J} \end{bmatrix} \\
&- \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T \begin{bmatrix} P^+ e^{-\mu x_J} \Lambda^+_{J-1} & 0 \\ 0 & P^- e^{\mu x_{-1}} \Lambda^-_{0} \end{bmatrix} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix},
\end{aligned}
$$

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$$= e^{-\mu\Delta x}\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_j^+ & 0 \\ 0 & \Lambda_j^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}$$
$$+ \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0}\end{bmatrix}^T K^T \begin{bmatrix} P^+ e^{-\mu x_0}\Lambda_{-1}^+ & 0 \\ 0 & P^- e^{\mu x_{J-1}}\Lambda_J^- \end{bmatrix} K \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0}\end{bmatrix}$$
$$- \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0}\end{bmatrix}^T \begin{bmatrix} P^+ e^{-\mu x_J}\Lambda_{J-1}^+ & 0 \\ 0 & P^- e^{\mu x_{-1}}\Lambda_0^- \end{bmatrix}\begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0}\end{bmatrix}.$$

By using assumption (5.3), *we have*

$$\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1}\end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_{j-1}^+ & 0 \\ 0 & \Lambda_{j+1}^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1}\end{bmatrix}$$
$$\leq e^{-\mu\Delta x}\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_j^+ & 0 \\ 0 & \Lambda_j^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}.$$

Therefore,

$$\frac{\widetilde{L}^n - L^n}{\Delta t} \leq -\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_{j-1}^+ & 0 \\ 0 & \Lambda_{j+1}^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}$$
$$+ e^{-\mu\Delta x}\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} \Lambda_j^+ & 0 \\ 0 & \Lambda_j^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix},$$
$$= \sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} -\Lambda_{j-1}^+ + e^{-\mu\Delta x}\Lambda_j^+ & 0 \\ 0 & -\Lambda_{j+1}^- + e^{-\mu\Delta x}\Lambda_j^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix},$$
$$= \sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} e^{-\mu\Delta x}\Lambda_j^+ - \Lambda_j^+ & 0 \\ 0 & e^{-\mu\Delta x}\Lambda_j^- - \Lambda_j^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}$$
$$+ \sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^T \Phi_j \begin{bmatrix} -\Lambda_{j-1}^+ + \Lambda_j^+ & 0 \\ 0 & -\Lambda_{j+1}^- + \Lambda_j^- \end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix},$$
$$= -\mu e^{-\mu\Delta x}\Delta x\sum_{j=0}^{J-1}\left(W_j^n\right)^T \Phi_j |\Lambda_j| \left(W_j^n\right) + \Delta x\sum_{j=0}^{J-1}\left(W_j^n\right)^T \Phi_j \Lambda_j' \left(W_j^n\right).$$

Since

$$\Pi_j^T\Phi_j + \Phi_j\Pi_j - \Delta t\Pi_j^T\Phi_j\Pi_j, \quad j = 0,\ldots,J-1,$$

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is positive semi-definite and there are constants

$$\alpha = \min_{\substack{1\le i\le k\\ 0\le j\le J-1}} |\lambda_{i,j}| \quad \textit{and} \quad \beta = \max_{\substack{1\le i\le k\\ 0\le j\le J-1}} \lambda'_{i,j},$$

such that $0 < \mu\alpha e^{-\mu\Delta x} - \beta < 1$, *we have*

$$\frac{L^{n+1} - \widetilde{L}^n}{\Delta t} \le 0 \quad \textit{and} \quad \frac{\widetilde{L}^n - L^n}{\Delta t} < -\eta L^n,$$

with $\eta := \mu\alpha e^{-\mu\Delta x} - \beta$. *Thus,*

$$\frac{L^{n+1} - L^n}{\Delta t} < -\eta L^n. \tag{5.7}$$

Recursively applying inequality (5.7) *yields*

$$L^{n+1} < (1 - \Delta t\eta)^{n+1} L^0 \le e^{-\eta\Delta t(n+1)} L^0 = e^{-\eta t^{n+1}} L^0, \quad n = 0, \dots, N-1. \tag{5.8}$$

We use now the inequality (5.8) *to conclude the proof. To this end, let*

$$C_1 := \min_{\substack{1\le i\le k\\ 0\le j\le J-1}} \{\sigma_{i,j} : |\Phi_j - \sigma_{i,j} I| = 0\} \quad \textit{and} \quad C_2 := \max_{\substack{1\le i\le k\\ 0\le j\le J-1}} \{\sigma_{i,j} : |\Phi_j - \sigma_{i,j} I| = 0\}.$$

Then,

$$C_1 I \le \Phi_j \le C_2 I, \quad j = 0, \dots, J-1. \tag{5.9}$$

From inequalities (5.8) *and* (5.9), *one obtains*

$$C_1 \Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \le L^n \le C_2 e^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \dots, N.$$

It follows that

$$\Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \le L^n \le C e^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \dots, N$$

where $C := C_2/C_1$. *Hence, the numerical solution* W_j^n *of the Cauchy problem* (4.11), (4.13), (4.14) *is exponentially stable for the* $\mathbb{L}^2-$*norm.*

Remark 8. *Stability conditions in the above Theorem 4 for linear hyperbolic system with variable coefficients are analyzed for examples, which will be presented in the next section.*

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In special cases, we consider the Cauchy problem (4.12), (4.13), (4.14) under the steady state solution, $W_j^n \equiv 0, \forall j = 0, \ldots, J-1, \forall n = 0, \ldots, N-1$. The discrete Lyapunov function (5.1) can be used to investigate conditions for stability of this Cauchy problem.

Corollary 3 ([26]). *Let* $T > 0$ *and assume that* $\lambda_i > 0$, $i = 1, \ldots, m$ *and* $\lambda_i < 0$, $i = m+1, \ldots, k$. *Let the Lyapunov function be given by* (5.1). *If the CFL condition* $\frac{\Delta t}{\Delta x} \max_{1 \le i \le k} |\lambda_i| \le 1$ *holds and if there exists* $\mu > 0$ *such that*

$$\Phi_j \Pi + \Pi^T \Phi_j - \Delta t \Pi^T \Phi_j \Pi \quad \text{is positive semi-definite} \tag{5.10}$$

and

$$\begin{bmatrix} P^+ e^{-\mu x_J} \Lambda^+ & 0 \\ 0 & P^- e^{\mu x_{-1}} \Lambda^- \end{bmatrix} - K^T \begin{bmatrix} P^+ e^{-\mu x_0} \Lambda^+ & 0 \\ 0 & P^- e^{\mu x_{J-1}} \Lambda^- \end{bmatrix} K, \tag{5.11}$$

is positive definite for each $j = 0, \ldots, J-1$, *then the numerical solution* W_j^n *of the Cauchy problem* (4.12), (4.13), (4.14) *satisfies*

$$L^n \le e^{-\eta t^n} L^0, \tag{5.12}$$

for some $\eta > 0$. *Moreover,* W_j^n *is exponentially stable for the* $\mathbb{L}^2-$ *norm.*

Proof. *The time derivative of the Lyapunov funtion* (3.1) *is approximated by using the discrete Lyapunov function* (5.1), *which is given by*

$$\frac{L^{n+1} - L^n}{\Delta t} = \frac{L^{n+1} - \widetilde{L}^n}{\Delta t} + \frac{\widetilde{L}^n - L^n}{\Delta t}, \tag{5.13}$$

with

$$\frac{L^{n+1} - \widetilde{L}^n}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=0}^{J-1} \left(\left[W_j^{n+1}\right]^T \Phi_j \left[W_j^{n+1}\right] - \left[\widetilde{W}_j^n\right]^T \Phi_j \left[\widetilde{W}_j^n\right] \right), \tag{5.14}$$

and

$$\frac{\widetilde{L}^n - L^n}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=0}^{J-1} \left(\left[\widetilde{W}_j^n\right]^T \Phi_j \left[\widetilde{W}_j^n\right] - \left[W_j^n\right]^T \Phi_j \left[W_j^n\right] \right). \tag{5.15}$$

We need now to show that both equations (5.14) *and* (5.15) *are negative. It suffices to show that*

$$\frac{L^{n+1} - \widetilde{L}^n}{\Delta t} \le 0, \quad \text{and} \quad \frac{\widetilde{L}^n - L^n}{\Delta t} < -\eta L^n, \quad \eta > 0. \tag{5.16}$$

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By using the discretized system (4.12) *and assumption* (5.10)*, we have*

$$\begin{aligned}\frac{L^{n+1}-\widetilde{L}^{n}}{\Delta t} &= \frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\left[\widetilde{W}_j^n-\Delta t\Pi\widetilde{W}_j^n\right]^T\Phi_j\left[\widetilde{W}_j^n-\Delta t\Pi\widetilde{W}_j^n\right]-\left[\widetilde{W}_j^n\right]^T\Phi_j\left[\widetilde{W}_j^n\right]\right),\\ &= \frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\left[\widetilde{W}_j^n\right]^T\Phi_j\left[\widetilde{W}_j^n\right]-\left[\widetilde{W}_j^n\right]^T\Phi_j\left[\widetilde{W}_j^n\right]\right)\\ &-\Delta x\sum_{j=0}^{J-1}\left(\left[\widetilde{W}_j^n\right]^T\left[\Phi_j\Pi+\Pi^T\Phi_j-\Delta t\Pi^T\Phi_j\Pi\right]\left[\widetilde{W}_j^n\right]\right)\le 0.\end{aligned}$$

The discretized system (4.12) *with the discretized boundary condition* (4.14) *can be used into equation* (5.15) *to obtain the following*

$$\frac{\widetilde{L}^{n}-L^{n}}{\Delta t}=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\left[W_j^n-D\left(W_j^n-\Xi_j^n\right)\right]^T\Phi_j\left[W_j^n-D\left(W_j^n-\Xi_j^n\right)\right]-\left[W_j^n\right]\Phi_j\left[W_j^n\right]\right),$$

with

$$D:=\textit{diag}\{D^+,D^-\}\quad\textit{and}\quad\Xi_j^n:=\left[W^{+}{}_{j-1}^{n}\quad W^{-}{}_{j+1}^{n}\right]^T,$$

where $D^+:=\frac{\Delta t}{\Delta x}\Lambda^+$ *and* $D^-:=\frac{\Delta t}{\Delta x}\Lambda^-$*. Then,*

$$\begin{aligned}\frac{\widetilde{L}^{n}-L^{n}}{\Delta t} =&\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\left[W_j^n\right]^T\Phi_j\left[W_j^n\right]-2\left[W_j^n\right]^T\Phi_jD\left[W_j^n\right]+2\left[W_j^n\right]^T\Phi_jD\left[\Xi_j^n\right]\right.\\ &+\left[W_j^n\right]^TD\Phi_jD\left[W_j^n\right]+\left[\Xi_j^n\right]^TD\Phi_jD\left[\Xi_j^n\right]\\ &\left.-2\left[W_j^n\right]^TD\Phi_jD\left[\Xi_j^n\right]-\left[W_j^n\right]^T\Phi_j\left[W_j^n\right]\right),\\ =&\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(-2\left[W_j^n\right]^T\Phi_jD\left[W_j^n\right]+2\left[W_j^n\right]^T(I-D)\Phi_jD\left[\Xi_j^n\right]\right.\\ &\left.+\left[W_j^n\right]^TD\Phi_jD\left[W_j^n\right]+\left[\Xi_j^n\right]^TD\Phi_jD\left[\Xi_j^n\right]\right).\end{aligned}$$

Since a diagonal matrix $(I-D)\,\Phi_jD$ *is positive definite and* $I-D\ge 0$ *by the CFL condition, we have the following matrix inequality*

$$\begin{aligned}2\left[W_j^n\right]^T(I-D)\Phi_jD\left[\Xi_j^n\right]&\le\left[W_j^n\right]^T(I-D)\Phi_jD\left[W_j^n\right]+\left[\Xi_j^n\right]^T(I-D)\Phi_jD\left[\Xi_j^n\right],\\ &=\left[W_j^n\right]^T\Phi_jD\left[W_j^n\right]-\left[W_j^n\right]^TD\Phi_jD\left[W_j^n\right]\\ &+\left[\Xi_j^n\right]^T\Phi_jD\left[\Xi_j^n\right]-\left[\Xi_j^n\right]^TD\Phi_jD\left[\Xi_j^n\right].\end{aligned}$$

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Therefore,

$$\begin{aligned}\frac{\widetilde{L}^n - L^n}{\Delta t} &\leq \frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\Big(-\left[W_j^n\right]^T\Phi_j D\left[W_j^n\right] + \left[\Xi_j^n\right]^T\Phi_j D\left[\Xi_j^n\right]\Big),\\ &= \sum_{j=0}^{J-1}\Big(-\left[W_j^n\right]^T\Phi_j|\Lambda|\left[W_j^n\right] + \left[\Xi_j^n\right]^T\Phi_j|\Lambda|\left[\Xi_j^n\right]\Big)\end{aligned}$$

Consider the second term on the right hand side of the above inequality

$$\begin{aligned}\sum_{j=0}^{J-1}\left[\Xi_j^n\right]^T\Phi_j|\Lambda|\left[\Xi_j^n\right] &= \sum_{j=0}^{J-1}\begin{bmatrix} {W^+}_{j-1}^n \\ {W^-}_{j+1}^n\end{bmatrix}^T\begin{bmatrix} P^+e^{-\mu x_j}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_j}\Lambda^-\end{bmatrix}\begin{bmatrix} {W^+}_{j-1}^n \\ {W^-}_{j+1}^n\end{bmatrix},\\
&= \sum_{j=0}^{J-1}\left[{W^+}_{j-1}^n\right]^T P^+e^{-\mu x_j}\Lambda^+\left[{W^+}_{j-1}^n\right] + \sum_{j=0}^{J-1}\left[{W^-}_{j+1}^n\right]^T P^-e^{\mu x_j}\Lambda^-\left[{W^-}_{j+1}^n\right],\\
&= e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[{W^+}_{j-1}^n\right]^T P^+e^{-\mu x_{j-1}}\Lambda^+\left[{W^+}_{j-1}^n\right]\\
&+ e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[{W^-}_{j+1}^n\right]^T P^-e^{\mu x_{j+1}}\Lambda^-\left[{W^-}_{j+1}^n\right],\\
&= e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[{W^+}_{j}^n\right]^T P^+e^{-\mu x_j}\Lambda^+\left[{W^+}_{j}^n\right] + e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[{W^-}_{j}^n\right]^T P^-e^{\mu x_j}\Lambda^-\left[{W^-}_{j}^n\right]\\
&+ e^{-\mu\Delta x}\left[{W^+}_{-1}^n\right]^T P^+e^{-\mu x_{-1}}\Lambda^+\left[{W^+}_{-1}^n\right] - e^{-\mu\Delta x}\left[{W^+}_{J-1}^n\right]^T P^+e^{-\mu x_{J-1}}\Lambda^+\left[{W^+}_{J-1}^n\right]\\
&+ e^{-\mu\Delta x}\left[{W^-}_{J}^n\right]^T P^-e^{\mu x_J}\Lambda^-\left[{W^-}_{J}^n\right] - e^{-\mu\Delta x}\left[{W^-}_{0}^n\right]^T P^-e^{\mu x_0}\Lambda^-\left[{W^-}_{0}^n\right],\\
&= e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j^+|\Lambda|\left[W_j^n\right]\\
&+ \left[{W^+}_{-1}^n\right]^T P^+e^{-\mu x_0}\Lambda^+\left[{W^+}_{-1}^n\right] - \left[{W^+}_{J-1}^n\right]^T P^+e^{-\mu x_J}\Lambda^+\left[{W^+}_{J-1}^n\right]\\
&+ \left[{W^-}_{J}^n\right]^T P^-e^{\mu x_{J-1}}\Lambda^-\left[{W^-}_{J}^n\right] - \left[{W^-}_{0}^n\right]^T P^-e^{\mu x_{-1}}\Lambda^-\left[{W^-}_{0}^n\right],\\
&= e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j^+|\Lambda|\left[W_j^n\right]\\
&+ \begin{bmatrix} {W^+}_{-1}^n \\ {W^-}_{J}^n\end{bmatrix}^T\begin{bmatrix} P^+e^{-\mu x_0}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_{J-1}}\Lambda^-\end{bmatrix}\begin{bmatrix} {W^+}_{-1}^n \\ {W^-}_{J}^n\end{bmatrix}\\
&- \begin{bmatrix} {W^+}_{J-1}^n \\ {W^-}_{0}^n\end{bmatrix}^T\begin{bmatrix} P^+e^{-\mu x_J}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_{-1}}\Lambda^-\end{bmatrix}\begin{bmatrix} {W^+}_{J-1}^n \\ {W^-}_{0}^n\end{bmatrix},\end{aligned}$$

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$$
\begin{aligned}
&= e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j^+|\Lambda|\left[W_j^n\right] \\
&+ \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T K^T \begin{bmatrix} P^+e^{-\mu x_0}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_{J-1}}\Lambda^- \end{bmatrix} K \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \\
&- \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T \begin{bmatrix} P^+e^{-\mu x_J}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_{-1}}\Lambda^- \end{bmatrix} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}.
\end{aligned}
$$

We use now assumption (5.11) *to obtain*

$$
\sum_{j=0}^{J-1}\left[\Xi_j^n\right]^T\Phi_j D\left[\Xi_j^n\right] \le e^{-\mu\Delta x}\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j^+|\Lambda|\left[W_j^n\right].
$$

Then,

$$
\begin{aligned}
\frac{\widetilde{L}^n - L^n}{\Delta t} &\le \left(-1 + e^{-\mu\Delta x}\right)\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j|\Lambda|\left[W_j^n\right], \\
&= -\mu e^{-\mu\Delta x}\Delta x\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j|\Lambda|\left[W_j^n\right], \\
&< -\mu\alpha e^{-\mu\Delta x}\Delta x\sum_{j=0}^{J-1}\left[W_j^n\right]^T\Phi_j\left[W_j^n\right] = -\eta L^n,
\end{aligned}
$$

where $\alpha := \min_{1\le i\le k}|\lambda_i|$ *and* $\eta := \mu\alpha e^{-\mu\Delta x}$. *Therefore,*

$$
\frac{L^{n+1} - L^n}{\Delta t} < -\eta L^n. \tag{5.17}
$$

It follows that

$$
L^{n+1} < (1 - \Delta t\eta)L^n \tag{5.18}
$$

Since $0 < \eta < 1$, *recursively applying inequality* (5.18) *yields*

$$
L^{n+1} < (1 - \Delta t\eta)^{n+1}L^0 \le e^{-\eta\Delta t(n+1)}L^0 = e^{-\eta t^{n+1}}L^0, \quad n = 0, \ldots, N-1. \tag{5.19}
$$

We use now the inequality (5.19) *to conclude the proof. To this end, let*

$$
C_1 := \min_{\substack{1\le i\le k \\ 0\le j\le J-1}}\{\sigma_{i,j} : |\Phi_j - \sigma_{i,j}I| = 0\} \quad \textit{and} \quad C_2 := \max_{\substack{1\le i\le k \\ 0\le j\le J-1}}\{\sigma_{i,j} : |\Phi_j - \sigma_{i,j}I| = 0\}.
$$

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Then,

$$C_1 I \leq \Phi_j \leq C_2 I, \quad j = 0, \ldots, J-1. \tag{5.20}$$

From inequalities (5.19) *and* (5.20), *one obtains*

$$C_1 \Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \leq L^n \leq C_2 e^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \ldots, N.$$

It follows that

$$\Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \leq L^n \leq C e^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \ldots, N,$$

where $C := C_2/C_1$. *Hence,* W_j^n *is exponentially stable for the* $\mathbb{L}^2-$ *norm.*

One of the conditions in the Corollary 3 is to show that the matrix (5.11) is positive definite. However, it has an alternative condition to show, which is described in the following corollary.

Corollary 4. *Let* $T > 0$ *and assume that* $\lambda_i > 0$, $i = 1, \ldots, m$ *and* $\lambda_i < 0$, $i = m+1, \ldots, k$. *Let the Lyapunov function be given by* (5.1). *If the CFL condition* $\frac{\Delta t}{\Delta x} \max_{1 \leq i \leq k} |\lambda_i| \leq 1$ *holds and if there exists* $\mu > 0$ *such that*

$$\Phi_j \Pi + \Pi^T \Phi_j - \Delta t \Pi^T \Phi_j \Pi, \quad j = 0, \ldots, J-1, \quad \textit{is positive semi-definite} \tag{5.21}$$

and

$$\|\Delta K \Delta^{-1}\| < 1, \tag{5.22}$$

with $\Delta := \sqrt{P|\Lambda|}$, *where* $P :=$ *diag*$\{P^+, P^-\}$, *then the numerical solution* W_j^n *of the Cauchy problem* (4.12),(4.13),(4.14) *satisfies*

$$L^n \leq e^{-\eta t^n} L^0,$$

for some $\eta > 0$. *Moreover,* W_j^n *is exponentially stable for the* $\mathbb{L}^2-$*norm.*

Proof. *The approximation of the time derivative of the Lyapunov function* (3.1) *under the CFL condition* $\frac{\Delta t}{\Delta x} \max_{1 \leq i \leq k} |\lambda_i| \leq 1$ *is given by*

$$\frac{L^{n+1} - L^n}{\Delta t} = \frac{L^{n+1} - \widetilde{L}^n}{\Delta t} + \frac{\widetilde{L}^n - L^n}{\Delta t}, \tag{5.23}$$

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with

$$\frac{L^{n+1}-\widetilde{L}^n}{\Delta t} = -\Delta x \sum_{j=0}^{J-1}\left(\left[\widetilde{W}_j^n\right]^T\left[\Phi_j\Pi+\Pi^T\Phi_j-\Delta t\Pi^T\Phi_j\Pi\right]\left[\widetilde{W}_j^n\right]\right), \tag{5.24}$$

and

$$\begin{aligned}\frac{\widetilde{L}^n-L^n}{\Delta t} &< -\mu\alpha e^{-\mu\Delta x}L^n \\ &+\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T K^T \begin{bmatrix} P^+e^{-\mu x_0}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_{J-1}}\Lambda^-\end{bmatrix} K \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix} \\ &-\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T \begin{bmatrix} P^+e^{-\mu x_J}\Lambda^+ & 0 \\ 0 & P^-e^{\mu x_{-1}}\Lambda^-\end{bmatrix} \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix},\end{aligned} \tag{5.25}$$

where $\alpha := \min_{1\le i\le k}|\lambda_i|$.

Since $\Phi_j\Pi+\Pi^T\Phi_j-\Delta t\Pi^T\Phi_j\Pi, \forall j$ *is positive semi-definite by assumption* (5.21), *we have*

$$\frac{L^{n+1}-\widetilde{L}^n}{\Delta t} \le 0.$$

In order to show that the remaining approximation of the time derivative is negative, consider the following quadratic form

$$\begin{aligned} &-\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^-\end{bmatrix} \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix} \\ &+\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T K^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^-\end{bmatrix} K \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}, \\ =&-\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^-\end{bmatrix} \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix} \\ &+\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T K^T\left(\sqrt{|\Lambda|P}\right)^T\left(\sqrt{|\Lambda|P}\right)K \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}, \\ =&-\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}^T \begin{bmatrix} \Lambda^+P^+ & 0 \\ 0 & \Lambda^-P^-\end{bmatrix} \begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix} \\ &+\left(\sqrt{|\Lambda|P}K\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}\right)^T\left(\sqrt{|\Lambda|P}K\begin{bmatrix} {W^+}^n_{J-1} \\ {W^-}^n_0\end{bmatrix}\right),\end{aligned}$$

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$$
\begin{aligned}
&= - \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} + \left\| \sqrt{|\Lambda| P} K \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \right\|^2, \\
&= - \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \\
&\quad + \left\| \left(\sqrt{|\Lambda| P}\right) K \left(\sqrt{|\Lambda| P}\right)^{-1} \left(\sqrt{|\Lambda| P}\right) \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \right\|^2, \\
&\leq - \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \\
&\quad + \left\| \left(\sqrt{|\Lambda| P}\right) K \left(\sqrt{|\Lambda| P}\right)^{-1} \right\|^2 \left\| \left(\sqrt{|\Lambda| P}\right) \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \right\|^2,
\end{aligned}
$$

where in the last line, $p_i > 0, i = 1, \dots, k$, *can be selected such that*

$$
\left\| \left(\sqrt{|\Lambda| P}\right) \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \right\| = 1 \quad \textit{and} \quad \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \neq 0, \quad n = 0, \dots, N-1.
$$

If

$$
\left\| \left(\sqrt{|\Lambda| P}\right) K \left(\sqrt{|\Lambda| P}\right)^{-1} \right\| < 1,
$$

then

$$
\begin{aligned}
&- \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \\
&+ \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T K^T \begin{bmatrix} \Lambda^+ P^+ & 0 \\ 0 & \Lambda^- P^- \end{bmatrix} K \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}, \\
&< - \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T |\Lambda| P \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} + \left\| \sqrt{|\Lambda| P} \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} \right\|^2, \\
&= - \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T |\Lambda| P \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} + \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix}^T |\Lambda| P \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0} \end{bmatrix} = 0.
\end{aligned}
$$

Therefore, by assumption (5.22), *a sufficiently small* $\mu > 0$ *can be chosen such that*

$$
\frac{\widetilde{L}^n - L^n}{\Delta t} < -\eta L^n, \quad \eta := \mu \alpha e^{-\mu \Delta x},
$$

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and then

$$\frac{L^{n+1} - L^n}{\Delta t} < -\eta L^n, \quad n = 0, \dots, N-1. \tag{5.26}$$

Inequality (5.27) *can be expressed by*

$$L^{n+1} < (1 - \Delta t \eta) L^n \tag{5.27}$$

Since $0 < \eta < 1$, *recursively applying inequality* (5.27) *yields*

$$L^{n+1} < (1 - \Delta t \eta)^{n+1} L^0 \leq e^{-\eta \Delta t (n+1)} L^0 = e^{-\eta t^{n+1}} L^0, \quad n = 0, \dots, N-1. \tag{5.28}$$

We use now the inequality (5.28) *to conclude the proof. To this end, let*

$$C_1 := \min_{\substack{1 \leq i \leq k \\ 0 \leq j \leq J-1}} \{\sigma_{i,j} : |\Phi_j - \sigma_{i,j} I| = 0\} \quad \textit{and} \quad C_2 := \max_{\substack{1 \leq i \leq k \\ 0 \leq j \leq J-1}} \{\sigma_{i,j} : |\Phi_j - \sigma_{i,j} I| = 0\}.$$

Then,

$$C_1 I \leq \Phi_j \leq C_2 I, \quad j = 0, \dots, J-1. \tag{5.29}$$

From inequalities (5.28) *and* (5.29), *one obtains*

$$C_1 \Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \leq L^n \leq C_2 e^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \dots, N.$$

It follows that

$$\Delta x \sum_{j=0}^{J-1} \left[W_j^n\right]^T \left[W_j^n\right] \leq L^n \leq C e^{-\eta t^n} \Delta x \sum_{j=0}^{J-1} \left[W_j^0\right]^T \left[W_j^0\right], \quad n = 0, \dots, N,$$

where $C := C_2 / C_1$. *Hence,* W_j^n *is exponentially stable for the* $\mathbb{L}^2-$ *norm.*

5.2 Numerical Results

In this section, we apply the numerical Lyapunov exponential stability analysis to the examples of linear hyperbolic systems of balance laws discussed in Chapter 4.

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5.2.1 A Linear Hyperbolic System of Balance Laws with Constant Coefficients

Consider the following linear hyperbolic system of balance laws in one-space dimension

$$\partial_t \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 15 & 1 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0, \quad t \geq 0, \quad x \in [0,1], \tag{5.30}$$

with an initial condition

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, \tag{5.31}$$

and boundary conditions

$$\begin{bmatrix} w_1(0,t) \\ w_2(1,t) \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} w_1(1,t) \\ w_2(0,t) \end{bmatrix}. \tag{5.32}$$

The solution of the Cauchy problem (5.30), (5.31), (5.32) is

$$\begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix} = \begin{bmatrix} -0.5e^{-14t} \\ 0.5e^{-14t} \end{bmatrix}, \quad x \in [0,1], \quad t \geq 0. \tag{5.33}$$

For stability analysis, the Lyapunov function is defined by

$$L(t) = \frac{1}{\mu}\left(e^{\mu} - e^{-\mu}\right) e^{-28t}, \quad t \geq 0, \quad \mu > 0. \tag{5.34}$$

The discretization of the Cauchy problem (5.30), (5.31), (5.32) is expressed as

$$\begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \end{bmatrix} = \begin{bmatrix} {w_1}_j^n \\ {w_2}_j^n \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {w_1}_j^n - {w_1}_{j-1}^n \\ {w_2}_j^n - {w_2}_{j+1}^n \end{bmatrix}, \quad n = 0, \dots, N-1, \tag{5.35a}$$

$$\begin{bmatrix} {w_1}_j^{n+1} \\ {w_2}_j^{n+1} \end{bmatrix} = \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \end{bmatrix} - \Delta t \begin{bmatrix} 15 & 1 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \end{bmatrix}, \quad j = 0, \dots, J-1, \tag{5.35b}$$

$$\begin{bmatrix} {w_1}_j^0 \\ {w_2}_j^0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, \quad j = 0, \dots, J-1, \tag{5.35c}$$

$$\begin{bmatrix} {w_1}_{-1}^{n+1} \\ {w_2}_J^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \begin{bmatrix} {w_1}_{J-1}^{n+1} \\ {w_2}_0^{n+1} \end{bmatrix}, \quad n = 0, \dots, N-1. \tag{5.35d}$$

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The discrete Lyapunov function (5.1) for the discretized Cauchy problem (5.35) is defined by

$$L^n = \Delta x \sum_{j=0}^{J-1} \left[p_1 e^{-\mu x_j} \left(w_{1j}^{\,n} \right)^2 + p_2 e^{\mu x_j} \left(w_{2j}^{\,n} \right)^2 \right], \quad n = 0, \ldots, N. \tag{5.36}$$

With the above definition, the approximation of the time derivative of the Lyapunov function is given by

$$\frac{L^{n+1} - L^n}{\Delta t} = \frac{L^{n+1} - \widetilde{L}^n}{\Delta t} + \frac{\widetilde{L}^n - L^n}{\Delta t}, \tag{5.37}$$

with

$$\widetilde{L}^n = \Delta x \sum_{j=0}^{J-1} \left[p_1 e^{-\mu x_j} \left(\widetilde{w_{1j}^{\,n}} \right)^2 + p_2 e^{\mu x_j} \left(\widetilde{w_{2j}^{\,n}} \right)^2 \right], \quad n = 0, \ldots, N.$$

If the CFL condition holds with

$$\Delta t \gamma \leq 1,$$

where γ is any diagonal entry in a coefficient matrix of the source term and if the choice of parameters are

$$\mu = 0.575, \quad p_1 = p_2 = 1 \quad \text{and} \quad k_{12} = k_{21} = 0.75,$$

then Corollary 3 is satisfied with the following decay rate

$$\eta := \alpha \frac{1 - e^{-\mu \Delta x}}{\Delta x}, \quad \alpha := \min\{|-1|, |1|\} = 1.$$

Hence, the discrete Lyapunov function (5.36) is bounded above by

$$L^n < e^{-\eta t^n} L^0, \quad n = 1, \ldots, N,$$

and the error of the scheme for the Lyapunov function is shown in Table 5.1 and 5.2 below.

J	$\|L - L^n\|_\infty$	$\|L - L^n\|_2$	μ	η
100	0.018922	0.0048535	0.575	0.57335
200	0.011638	0.0030225	0.575	0.57417
400	0.008116	0.002138	0.575	0.57459
800	0.0063934	0.0017033	0.575	0.57479
1600	0.0055439	0.001488	0.575	0.5749

Table 5.1: The behavior of numerical values with number of cells, J, time $T = 1$ and CFL $=$ 1.

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J	$\|L-L^n\|_\infty$	$\|L-L^n\|_2$	μ	η
100	0.018798	0.0047198	0.575	0.57335
200	0.012462	0.0031774	0.575	0.57417
400	0.0091065	0.0023558	0.575	0.57459
800	0.0072704	0.0019062	0.575	0.57479
1600	0.0062423	0.0016541	0.575	0.5749

Table 5.2: The behavior of numerical values with number of cells, J, time $T = 1$ and CFL $=$ 0.75.

From Table 5.1 and 5.2 above, we observed that the rate of decay, η converges to μ and both $\mathbb{L}^\infty$ and $\mathbb{L}^2$ norm converge to 0.

In the following figures, we have tested the stability of the system for two different values of γ under CFL $= 1$ such that $\Delta t\gamma \leq 1$ and $\Delta t\gamma > 1$.

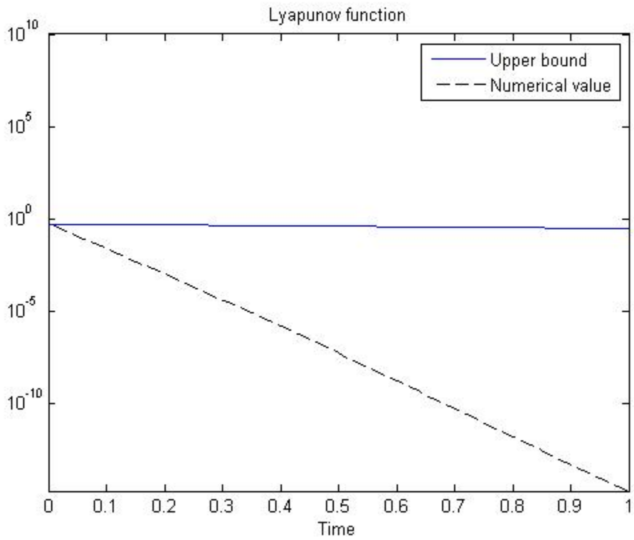

Figure 5.1: The log-scale of the discrete Lyapunov function for $\gamma = 15$ $(\Delta\gamma < 1)$, $J = 100$ and $T = 1$ under CFL $= 1$.

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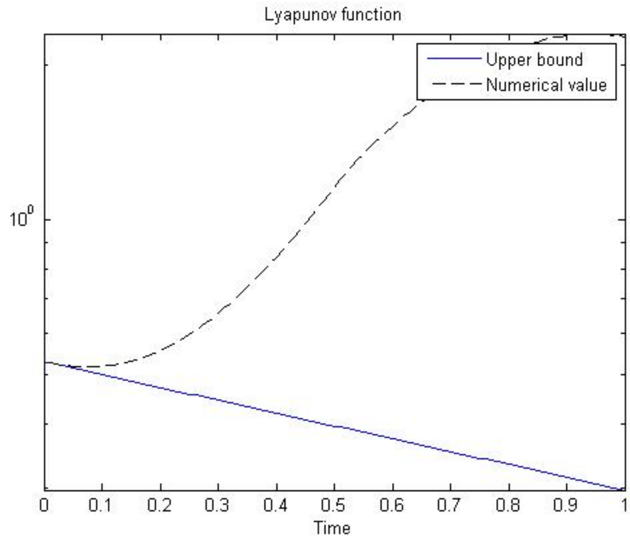

Figure 5.2: The log-scale of the discrete Lyapunov function for $\gamma = 201$ $(\Delta\gamma > 1)$, $J = 100$ and $T = 1$ under CFL = 1.

From Figure 5.1 and Figure 5.2, we observed that for the same number of cells the discrete Lyapunov function is bounded for $\gamma = 15$ and unbounded for $\gamma = 201$, respectively.

5.2.2 Application to Telegrapher Equations

The discrete Lyapunov function for the discretized telegrapher equations (4.16) is defined by (5.36) and the approximation of the time derivative of the Lyapunov function is given by (5.37).

If the CFL condition $\frac{\Delta t}{\Delta x} \max\{|\lambda_1|, |\lambda_2|\} \leq 1$ holds and if there exists $\mu > 0$, $p_1 > 0$ and $p_2 > 0$ such that

$$\Phi_j \Pi + \Pi^T \Phi_j - \Delta t \Pi^T \Phi_j \Pi, \quad j = 0, \ldots, J-1, \quad \text{is positive semi-definite} \tag{5.38}$$

and

$$\begin{bmatrix} p_1 e^{-\mu x_J} |\lambda_1| & 0 \\ 0 & p_2 e^{\mu x_{-1}} |\lambda_2| \end{bmatrix} - \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}^T \begin{bmatrix} p_1 e^{-\mu x_0} |\lambda_1| & 0 \\ 0 & p_2 e^{\mu x_{J-1}} |\lambda_2| \end{bmatrix} \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \tag{5.39}$$

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is positive definite, then

$$\frac{L^{n+1} - \widetilde{L}^n}{\Delta t} < 0,$$

and

$$\frac{\widetilde{L}^n - L^n}{\Delta t} < -\eta L^n, \quad 0 < \eta := \mu\alpha e^{-\mu\Delta x} < 1, \quad \alpha := \min\{|\lambda_1|, |\lambda_2|\}.$$

In order to show that both conditions (5.38) and (5.39) hold, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}, \tag{5.40}$$

with

$$\begin{aligned} M_{11} &:= 2p_1 e^{-\mu x_j}\gamma_1 - \Delta t\left(p_1 e^{-\mu x_j}\gamma_1^2 + p_2 e^{\mu x_j}\gamma_2^2\right), \\ M_{12} &:= p_2 e^{\mu x_j}\gamma_2 + p_1 e^{-\mu x_j}\gamma_2 - \Delta t\left(p_1 e^{-\mu x_j}\gamma_1\gamma_2 + p_2 e^{\mu x_j}\gamma_2\gamma_1\right), \\ M_{22} &:= 2p_2 e^{\mu x_j}\gamma_1 - \Delta t\left(p_1 e^{-\mu x_j}\gamma_2^2 + p_2 e^{\mu x_j}{\gamma_1}^2\right), \end{aligned}$$

and

$$\begin{bmatrix} p_1 e^{-\mu x_J}|\lambda_1| - k_{21}^2 p_2 e^{\mu x_{J-1}}|\lambda_2| & 0 \\ 0 & p_2 e^{\mu x_{-1}}|\lambda_2| - k_{12}^2 p_1 e^{-\mu x_0}|\lambda_1| \end{bmatrix}, \tag{5.41}$$

is non-negative and positive, respectively.

The determinant of the sub-matrices of the matrix (5.40) are

$$2p_1 e^{-\mu x_j}\gamma_1 - \Delta t\left(p_1 e^{-\mu x_j}\gamma_1^2 + p_2 e^{\mu x_j}\gamma_2^2\right), \tag{5.42}$$

and

$$\begin{aligned} &\Delta t^2{\gamma_1}^4 p_1 p_2 - 2\Delta t^2{\gamma_1}^2{\gamma_2}^2 p_1 p_2 + \Delta t^2{\gamma_2}^4 p_1 p_2 - 4\Delta t{\gamma_1}^3 p_1 p_2 + 4\Delta t\gamma_1{\gamma_2}^2 p_1 p_2 \\ &- e^{-2\mu x_j}{\gamma_2}^2{p_1}^2 + 4{\gamma_1}^2 p_1 p_2 - 2{\gamma_2}^2 p_1 p_2 - e^{2\mu x_j}{\gamma_2}^2{p_2}^2, \\ &= \Delta t^2\left({\gamma_1}^2 - {\gamma_2}^2\right) p_1 p_2 + 4\left({\gamma_1}^2 - {\gamma_2}^2\right) p_1 p_2 - 4\Delta t\gamma_1\left({\gamma_1}^2 - {\gamma_2}^2\right) p_1 p_2 \\ &- \left(p_1\gamma_2 e^{-\mu x_j} - p_2\gamma_2 e^{\mu x_j}\right)^2, \\ &= \Delta t^2\left({\gamma_1}^2 - {\gamma_2}^2\right) p_1 p_2 + 4\left(1 - \Delta t\gamma_1\right)\left({\gamma_1}^2 - {\gamma_2}^2\right) p_1 p_2 - \left(p_1\gamma_2 e^{-\mu x_j} - p_2\gamma_2 e^{\mu x_j}\right)^2. \end{aligned} \tag{5.43}$$

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If ${\gamma_1}^2 > {\gamma_2}^2$ and if p_1 and p_2 can be chosen such that $p_1 = p_2$, then for $\gamma_1 > 0$ and sufficiently small $\mu > 0$,

$$\Delta t \gamma_1 < \max_{0 \le x \le l} \left\{ \frac{2p_1 e^{-\mu x_j} \gamma_1^2}{(p_1 e^{-\mu x_j} \gamma_1^2 + p_2 e^{\mu x_j} \gamma_2^2)} \right\} \approx 1, \quad \max_{0 \le x \le l} \{ \gamma_2 p_2 e^{\mu x} - \gamma_2 p_1 e^{-\mu x} \} \approx 0,$$

and the determinants (5.42) and (5.43) are non-negative.

Therefore, with the choice of p_1 and p_2, the matrix (5.41) is positive definite if k_{12} and k_{21} satisfy

$$|k_{12}| < \sqrt{\left|\frac{\lambda_2}{\lambda_1}\right|} \quad \text{and} \quad |k_{21}| < \sqrt{\left|\frac{\lambda_1}{\lambda_2}\right|} e^{-\mu l},$$

respectively.

Consider the following steady state solution for the telegrapher equations, which is obtained by solving the system (2.40)

$$\begin{bmatrix} I^*(x) \\ V^*(x) \end{bmatrix} = \begin{bmatrix} 0.02\exp(0.15x) + 0.02\exp(-0.15x) \\ -\exp(0.15x) + \exp(-0.15x) \end{bmatrix}, \quad x \in [0, l], \tag{5.44}$$

with the numerical value taken from [31]. The length of the transmission line is 1m, $R = 7.5$Ohm/m, $L = 10$mH/m, $G = 0.003$S/m and $C = 0.4$F/m.

We take now an initial condition for the system (2.35) that is a perturbation of the steady state solution

$$\begin{bmatrix} I(x,0) \\ V(x,0) \end{bmatrix} = \begin{bmatrix} 0.02\exp(0.15x) + 0.02\exp(-0.15x) + \epsilon\sin(\pi x) \\ -\exp(0.15x) + \exp(-0.15x) + \epsilon\sin(\pi x) \end{bmatrix}, \quad x \in [0, l], \quad \epsilon = 0.01. \tag{5.45}$$

The decoupled system (2.42) has the following eigenvalues

$$\lambda_1 = 0.5 \quad \text{and} \quad \lambda_2 = -0.5,$$

the coefficients of source terms are

$$\gamma_1 = 0.3787 \quad \text{and} \quad \gamma_2 = -0.3713,$$

and the initial condition, which is obtained by substituting (5.44) and (5.45) into (2.43), is

$$w_1(x,0) = 0.06\sin(\pi x), \quad w_2(x,0) = -0.04\sin(\pi x), \quad x \in [0, l].$$

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Since the equilibrium solution for the decoupled system (2.42) is $W^*(x) \equiv 0$, it is possible the numerical solution converges to the initial condition in the sense of $\mathbb{L}^2-$norm.

Therefore, the convergence of the discrete Lyapunov function for different values of $\mu > 0$ is shown in Figure 5.3 below.

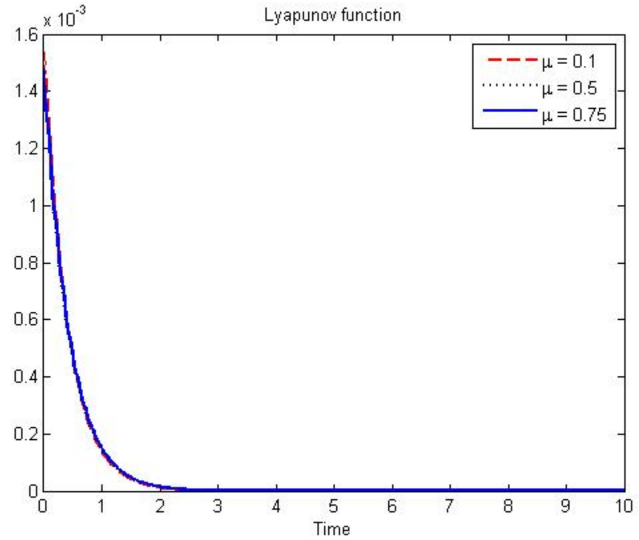

Figure 5.3: The decay of the Lyapunov function for telegrapher equations. The choice of parameters are $p_1 = p_2 = 0.6$ and $k_{12} = k_{21} = 0.9$ with $l = 1$, $J = 200$ and $T = 10$ under CFL $= 0.75$.

The three curves that are obtained for different values of $\mu > 0$, which are shown in the Figure 5.3, are nearly indistinguishable. Beside this, we observed that the discrete Lyapunov function converges to 0. This shows, in the sense of $\mathbb{L}^2-$norm, the Cauchy problem (4.16), (4.17), (4.18) is exponentially stable.

5.2.3 Application to Isentropic Euler Equations

The discrete Lyapunov function for the discretized isentropic Euler equations (4.19) is defined by (5.36) and the approximation of the time derivative of the Lyapunov function is obtained by (5.37).

If the CFL condition $\frac{\Delta t}{\Delta x} \max_{0 \leq j \leq J-1}\{|\lambda_{1,j}|, |\lambda_{2,j}|\} \leq 1$ holds and if there exists $\mu > 0$, $p_1 > 0$

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and $p_2 > 0$ such that

$$\Phi_j \Pi_j + {\Pi_j}^T \Phi_j - \Delta t {\Pi_j}^T \Phi_j \Pi_j, \quad j = 0, \dots, J-1, \quad \text{is positive semi-definite} \tag{5.46}$$

and

$$\begin{bmatrix} p_1 e^{-\mu x_J} |\lambda_{1,J-1}| & 0 \\ 0 & p_2 e^{\mu x_{-1}} |\lambda_{2,0}| \end{bmatrix} - \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}^T \begin{bmatrix} p_1 e^{-\mu x_0} |\lambda_{1,-1}| & 0 \\ 0 & p_2 e^{\mu x_{J-1}} |\lambda_{2,J}| \end{bmatrix} \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \tag{5.47}$$

is positive definite, then

$$\frac{L^{n+1} - \widetilde{L}^n}{\Delta t} \leq 0, \quad \text{and} \quad \frac{\widetilde{L}^n - L^n}{\Delta t} < -\eta L^n,$$

where $0 < \eta := \mu \alpha e^{-\mu \Delta x} - \beta < 1$,

$$\alpha := \min_{0 \leq j \leq J-1} \{|\lambda_{1,j}|, |\lambda_{2,j}|\}, \quad \text{and} \quad \beta := \max_{0 \leq j \leq J-1} \left\{ \frac{\lambda_{1,j} - \lambda_{1,j-1}}{\Delta x}, \frac{\lambda_{2,j+1} - \lambda_{2,j}}{\Delta x} \right\}.$$

In order to show that both conditions (5.46) and (5.47) hold, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} M_{11,j} & M_{12,j} \\ M_{12,j} & M_{22,j} \end{bmatrix}, \tag{5.48}$$

with

$$\begin{aligned} M_{11,j} &:= 2p_1 e^{-\mu x_j} \gamma_{11,j} - \Delta t \left(p_1 e^{-\mu x_j} \gamma_{11,j}^2 + p_2 e^{\mu x_j} \gamma_{21,j}^2 \right), \\ M_{12,j} &:= p_2 e^{\mu x_j} \gamma_{21,j} + p_1 e^{-\mu x_j} \gamma_{12,j} - \Delta t \left(p_1 e^{-\mu x_j} \gamma_{11,j} \gamma_{12,j} + p_2 e^{\mu x_j} \gamma_{21,j} \gamma_{22,j} \right), \\ M_{22,j} &:= 2p_2 e^{\mu x_j} \gamma_{22,j} - \Delta t \left(p_1 e^{-\mu x_j} \gamma_{12,j}^2 + p_2 e^{\mu x_j} \gamma_{22,j}^2 \right), \end{aligned}$$

and

$$\begin{bmatrix} p_1 e^{-\mu x_J} |\lambda_{1,J-1}| - k_{21}^2 p_2 e^{\mu x_{J-1}} |\lambda_{2,J}| & 0 \\ 0 & p_2 e^{\mu x_{-1}} |\lambda_{2,0}| - k_{12}^2 p_1 e^{-\mu x_0} |\lambda_{1,-1}| \end{bmatrix}, \tag{5.49}$$

is non-negative and positive, respectively.

The determinant of the sub-matrices of the matrix (5.48) are

$$2p_1 e^{-\mu x_j} \gamma_{11,j} - \Delta t \left(p_1 e^{-\mu x_j} \gamma_{11,j}^2 + p_2 e^{\mu x_j} \gamma_{21,j}^2 \right), \tag{5.50}$$

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and

$$\begin{aligned}
&\Delta t^2\gamma_{11,j}^2\gamma_{22,j}^2 p_1p_2 - 2\Delta t^2\gamma_{11,j}\gamma_{12,j}\gamma_{21,j}\gamma_{22,j}p_1p_2 + \Delta t^2\gamma_{12,j}^2\gamma_{21,j}^2 p_1p_2\\
&- 2\Delta t\gamma_{11,j}^2\gamma_{22,j}p_1p_2 + 2\Delta t\gamma_{11,j}\gamma_{12,j}\gamma_{21,j}p_1p_2 - 2\Delta t\gamma_{11,j}\gamma_{22,j}^2 p_1p_2\\
&+ 2\Delta t\gamma_{12,j}\gamma_{21,j}\gamma_{22,j}p_1p_2 + 4\gamma_{11,j}\gamma_{22,j}p_1p_2\\
&- e^{-2\mu x_j}\gamma_{12,j}^2 p_1^2 - 2\gamma_{12,j}\gamma_{21,j}p_1p_2 - e^{2\mu x_j}\gamma_{21,j}^2 p_2^2\\
&= \Delta t^2\left(\gamma_{11,j}\gamma_{22,j} - \gamma_{12,j}\gamma_{21,j}\right)^2 p_1p_2 - 2\Delta t\left(\gamma_{11,j}+\gamma_{22,j}\right)\left(\gamma_{11,j}\gamma_{22,j} - \gamma_{12,j}\gamma_{21,j}\right)p_1p_2\\
&+ 4\left(\gamma_{11,j}\gamma_{22,j} - \gamma_{12,j}\gamma_{21,j}\right)p_1p_2 - \left(p_1e^{-\mu x_j}\gamma_{12,j} - p_2e^{\mu x_j}\gamma_{21,j}\right)^2,\\
&= \Delta t^2\left(\gamma_{11,j}\gamma_{22,j} - \gamma_{12,j}\gamma_{21,j}\right)^2 p_1p_2 - \left(p_1e^{-\mu x_j}\gamma_{12,j} - p_2e^{\mu x_j}\gamma_{21,j}\right)^2\\
&+ \left(4 - 2\Delta t\left(\gamma_{11,j}+\gamma_{22,j}\right)\right)\left(\gamma_{11,j}\gamma_{22,j} - \gamma_{12,j}\gamma_{21,j}\right)p_1p_2. \qquad (5.51)
\end{aligned}$$

If $\gamma_{11,j}\gamma_{22,j} > \gamma_{12,j}\gamma_{21,j}, \forall j$ and if p_1 and p_2 can be chosen such that

$$\frac{p_1}{p_2} = \max_{0\le j\le J-1}\left\{\frac{\gamma_{21,j}}{\gamma_{12,j}}\right\},$$

then for $\gamma_{11,j} > 0, \gamma_{22,j} > 0, \forall j$ and sufficiently small $\mu > 0$,

$$\begin{aligned}
&\Delta t\gamma_{11,j} < \max_{0\le j\le J-1}\left\{\frac{2p_1e^{-\mu x_j}\gamma_{11,j}^2}{\left(p_1e^{-\mu x_j}\gamma_{11,j}^2 + p_2e^{\mu x_j}\gamma_{21,j}^2\right)}\right\} \approx 1,\\
&\Delta t\gamma_{22,j} < \max_{0\le j\le J-1}\left\{\frac{2p_2e^{\mu x_j}\gamma_{22,j}^2}{\left(p_1e^{-\mu x_j}\gamma_{12,j}^2 + p_2e^{\mu x_j}\gamma_{22,j}^2\right)}\right\} \approx 1,\\
&\max_{0\le x\le l}\{\gamma_2p_2e^{\mu x} - \gamma_2p_1e^{-\mu x}\} \approx 0,
\end{aligned}$$

and the determinants (5.50) and (5.51) are non-negative.

Therefore, with the choice of p_1 and p_2, the matrix (5.49) is positive definite if k_{12} and k_{21} satisfy

$$|k_{12}| < \sqrt{\left|\frac{\lambda_{2,0}}{\lambda_{1,-1}}\right|\frac{p_2}{p_1}} \quad \text{and} \quad |k_{21}| < \sqrt{\left|\frac{\lambda_{1,J-1}}{\lambda_{2,J}}\right|\frac{p_1}{p_2}e^{-\mu l}},$$

respectively.

For simplicity of the numerical tests for isentropic Euler equations, the steady state equations (2.49) are solved with gas pressure taken as suggested in [11]

$$P(\rho) = \kappa\rho^{\sigma}, \quad \kappa = 0.4, \quad \sigma = 1.4.$$

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Therefore, for $V^*(x) \geq 0, \forall x \in [0, l]$, we follow the calculation done in [29]. The first equation in (2.49) gives $\rho^*(x)V^*(x) = a$, where a constant a is chosen to satisfy the subsonic condition and then the second equation in (2.49) becomes

$$\frac{d}{dx}\rho^*(x) = \frac{a^2 C\rho^*(x)}{(a^2 - \gamma\kappa\rho^{*\gamma+1}(x))}.$$

With the following choices of parameters

$$C = 10^{-7}, \quad a = 0.3$$

and by setting $\rho^*(0) = 0.5$, we obtain the following numerical values of the steady state by using the help of Maple2015 [1]

$$\rho^*(x) = 0.5, \quad V^*(x) = 0.6, \quad x \in [0, l].$$

We set now an initial condition for the system (2.45) that is a perturbation of the steady state solution

$$\rho^*(x, 0) = 0.5 + \epsilon, \quad V^*(x, 0) = 0.6 + \epsilon, \quad \epsilon = 0.01.$$

The decoupled system (2.51) has the following eigenvalues

$$\lambda_1 = 1.2515 \quad \text{and} \quad \lambda_2 = -0.0515,$$

the coefficients of the source terms are:

$$\gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = 6 \times 10^{-8},$$

and the initial condition, which is obtained from (2.52), is

$$w_1(x, 0) = 0.023, \quad w_2(x, 0) = -0.003, \quad x \in [0, l].$$

Therefore, the convergence of the discrete Lyapunov function for different values of $\mu > 0$ and $\epsilon > 0$ is shown in Figure 5.4 below.

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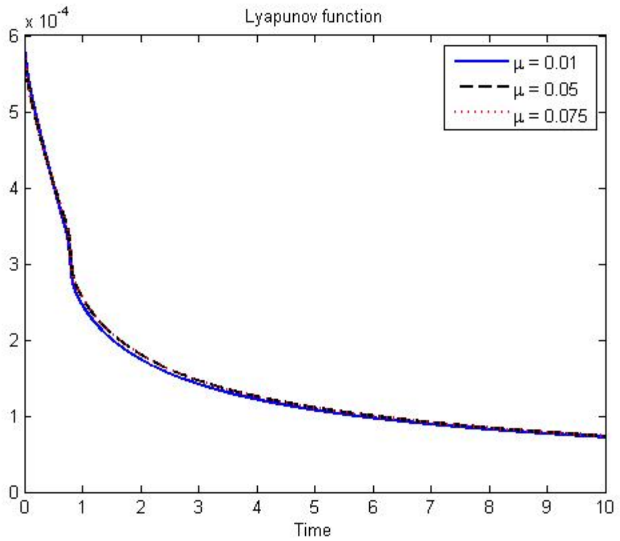

Figure 5.4: The decay of the Lyapunov function for isentropic Euler equations. The choice of parameters are $p_1 = p_2 = 1$, $k_{12} = 0.2$ and $k_{21} = 4.0$ with $l = 1$, $J = 200$ and $T = 10$ under CFL $=$ 0.75.

The three curves that are obtained for different values of $\mu > 0$, which are shown in the Figure 5.4, are nearly indistinguishable. We observed the convergence of the discrete Lyapunov function to 0. This shows that, in the sense of $\mathbb{L}^2-$norm, the Cauchy problem (4.19), (4.20), (4.21) is exponentially stable.

5.2.4 Application to Saint-Venant Equations

A numerical boundary stabilization for the Saint-Venant equations has the same presentation as presented for isentropic Euler equations. Thus, we will only discuss here a numerical test on a specific example. To simplify the numerical computations, we take a constant steady state from [23]

$$H^*(x) = 2 \quad \text{and} \quad V^*(x) = 3, \quad x \in [0, l]$$

with physical parameters $g = 9.81$, $C_f = 0.1$ and $S_b = 0.0459$ and initial condition for the system (2.54)

$$H(x, 0) = 2.5 \quad \text{and} \quad V(x, 0) = 4\sin(\pi x), \quad x \in [0, l].$$

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The decoupled system (2.60) has the following eigenvalues

$$\lambda_1 = 7.4294 \quad \text{and} \quad \lambda_2 = -1.4294,$$

the coefficients of the source terms are:

$$\gamma_{11} = \gamma_{21} = 0.0992 \quad \text{and} \quad \gamma_{12} = \gamma_{22} = 0.2008,$$

and the initial condition, which is obtained from (2.61), is:

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} = \begin{bmatrix} -1.8926 + 4\sin(\pi x) \\ -4.1074 + 4\sin(\pi x) \end{bmatrix}, \quad x \in [0, l].$$

Therefore, the convergence of the discrete Lyapunov function for different values of $\mu > 0$ and $\epsilon > 0$ is shown in Figure 5.5 below.

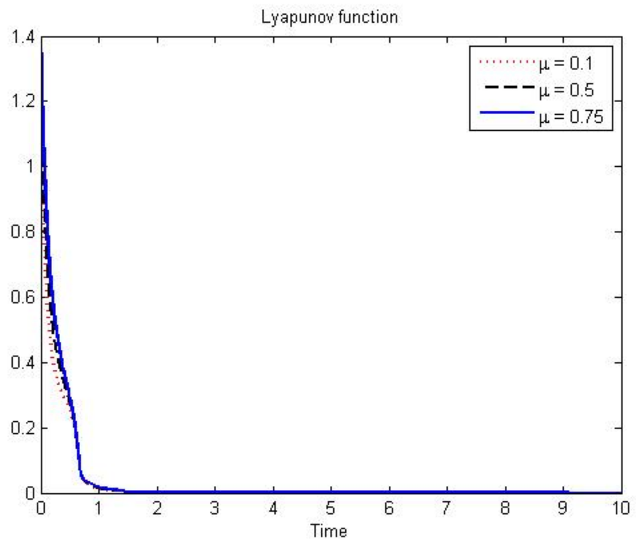

Figure 5.5: The decay of the Lyapunov function for Saint-Venant equations. The choice of parameters are $p_1 = 0.0992, p_2 = 0.2008$, $k_{12} = 0.3$ and $k_{21} = 0.8$ with $l = 1$, $J = 200$ and $T = 10$ under CFL $= 0.75$.

The three curves that are obtained for different values of $\mu > 0$, which are shown in the Figure 5.4, are nearly indistinguishable for time getting longer. We observed the convergence of the discrete Lyapunov function to 0. This shows, in the sense of $\mathbb{L}^2-$norm, the Cauchy problem (4.22), (4.23), (4.24) is exponentially stable.

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5.2.5 Application to Saint-Venant-Exner Equations

The discrete Lyapunov function for the discretized Saint-Venant-Exner equations (4.25) is defined by

$$L^n = \Delta x \sum_{j=0}^{J-1} \left(W_j^n\right)^T \Phi_j W_j^n, \quad n = 0, \ldots, N, \tag{5.52}$$

with

$$W_j^n := \begin{bmatrix} {w_1}_j^n \\ {w_2}_j^n \\ {w_3}_j^n \end{bmatrix}, \quad \text{and} \quad \Phi_j := \begin{bmatrix} p_1 e^{-\mu x_j} & 0 & 0 \\ 0 & p_2 e^{-\mu x_j} & 0 \\ 0 & 0 & p_3 e^{\mu x_j} \end{bmatrix}.$$

The approximation of the time derivative of the Lyapunov function is obtained by

$$\frac{L^{n+1} - L^n}{\Delta t} = \frac{L^{n+1} - \widetilde{L}^n}{\Delta t} + \frac{\widetilde{L}^n - L^n}{\Delta t},$$

where

$$\widetilde{L}^n = \Delta x \sum_{j=0}^{J-1} \left(\widetilde{W}_j^n\right)^T \Phi_j \widetilde{W}_j^n, \quad n = 0, \ldots, N, \quad \widetilde{W}_j^n := \begin{bmatrix} \widetilde{w_1}_j^n \\ \widetilde{w_2}_j^n \\ \widetilde{w_3}_j^n \end{bmatrix}.$$

If the CFL condition $\frac{\Delta t}{\Delta x} \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \leq 1$ holds and if there exists $\mu > 0$, $p_1 > 0$, $p_2 > 0$ and $p_3 > 0$ such that

$$\Phi_j \Pi + \Pi^T \Phi_j - \Delta t \Pi^T \Phi_j \Pi, \quad j = 0, \ldots, J-1, \quad \text{is positive semi-definite} \tag{5.53}$$

and

$$\begin{bmatrix} p_1 e^{-\mu x_J} |\lambda_1| & 0 & 0 \\ 0 & p_2 e^{-\mu x_J} |\lambda_2| & 0 \\ 0 & 0 & p_3 e^{\mu x_{-1}} |\lambda_3| \end{bmatrix} - \begin{bmatrix} 0 & 0 & k_{13} \\ 0 & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{bmatrix}^T \begin{bmatrix} p_1 e^{-\mu x_0} |\lambda_1| & 0 & 0 \\ 0 & p_2 e^{-\mu x_0} |\lambda_2| & 0 \\ 0 & 0 & p_3 e^{\mu x_{J-1}} |\lambda_3| \end{bmatrix} \begin{bmatrix} 0 & 0 & k_{13} \\ 0 & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{bmatrix} \tag{5.54}$$

is positive definite, then

$$\frac{L^{n+1} - \widetilde{L}^n}{\Delta t} \leq 0,$$

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and

$$\frac{\widetilde{L}^n - L^n}{\Delta t} < -\eta L^n, \quad 0 < \eta := \mu\alpha e^{-\mu\Delta x} < 1, \quad \alpha := \min\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}.$$

In order to show that both conditions (5.53) and (5.54) hold, it suffices to show that the determinant of every principal sub-matrix of the matrices

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix}, \tag{5.55}$$

with

$$\begin{aligned}
M_{11} &:= 2p_1e^{-\mu x_j}\gamma_1 - \Delta t\left(p_1e^{-\mu x_j}\gamma_1^2 + p_2e^{-\mu x_j}\gamma_1^2 + p_3e^{\mu x_j}\gamma_1^2\right), \\
M_{12} &:= p_2e^{-\mu x_j}\gamma_1 + p_1e^{-\mu x_j}\gamma_2 - \Delta t\left(p_1e^{-\mu x_j}\gamma_1\gamma_2 + p_2e^{-\mu x_j}\gamma_1\gamma_2 + p_3e^{\mu x_j}\gamma_1\gamma_2\right), \\
M_{13} &:= p_3e^{\mu x_j}\gamma_1 + p_1e^{-\mu x_j}\gamma_3 - \Delta t\left(p_1e^{-\mu x_j}\gamma_1\gamma_3 + p_2e^{-\mu x_j}\gamma_1\gamma_3 + p_3e^{\mu x_j}\gamma_1\gamma_3\right), \\
M_{22} &:= 2p_2e^{-\mu x_j}\gamma_2 - \Delta t\left(p_1e^{-\mu x_j}\gamma_2^2 + p_2e^{-\mu x_j}\gamma_2^2 + p_3e^{\mu x_j}\gamma_2^2\right), \\
M_{23} &:= p_3e^{\mu x_j}\gamma_2 + p_2e^{-\mu x_j}\gamma_3 - \Delta t\left(p_1e^{-\mu x_j}\gamma_2\gamma_3 + p_2e^{-\mu x_j}\gamma_2\gamma_3 + p_3e^{\mu x_j}\gamma_2\gamma_3\right), \\
M_{33} &:= 2p_3e^{\mu x_j}\gamma_3 - \Delta t\left(p_1e^{-\mu x_j}\gamma_3^2 + p_2e^{-\mu x_j}\gamma_3^2 + p_3e^{\mu x_j}\gamma_3^2\right),
\end{aligned}$$

and

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}, \tag{5.56}$$

with

$$\begin{aligned}
\sigma_{11} &:= |\lambda_1|p_1e^{-\mu x_J} - k_{31}^2|\lambda_3|p_3e^{\mu x_{J-1}}, \\
\sigma_{12} &:= -k_{31}k_{32}|\lambda_3|p_3e^{\mu x_{J-1}}, \\
\sigma_{22} &:= |\lambda_2|p_2e^{-\mu x_J} - k_{32}^2|\lambda_3|p_3e^{\mu x_{J-1}}, \quad \text{and} \\
\sigma_{33} &:= |\lambda_3|p_3e^{\mu x_{-1}} - k_{13}^2|\lambda_1|p_1e^{-\mu x_0} - k_{23}^2|\lambda_2|p_2e^{-\mu x_0},
\end{aligned}$$

is non-negative and positive, respectively.

The determinant of the sub-matrices of the matrix (5.55) are

$$2p_1e^{-\mu x_j}\gamma_1 - \Delta t\left(p_1e^{-\mu x_j}\gamma_1^2 + p_2e^{-\mu x_j}\gamma_1^2 + p_3e^{\mu x_j}\gamma_1^2\right), \tag{5.57}$$

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$$-\left(\gamma_1^2 p_2^2 - 2\gamma_1\gamma_2 p_1 p_2 + \gamma_2^2 p_1^2\right) e^{-2\mu x_j} = -\left(\gamma_1 p_2 - \gamma_2 p_1\right)^2 e^{-2\mu x_j}, \tag{5.58}$$

and

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{vmatrix} = 0. \tag{5.59}$$

If p_1, p_2 and p_3 can be chosen such that $p_1 = \gamma_1$, $p_2 = \gamma_2$ and $p_3 = \gamma_3$, then for $\gamma_1 > 0$ and sufficiently small $\mu > 0$,

$$\Delta t \gamma_1 < \max_{0 \le x \le l} \left\{ \frac{2p_1 e^{-\mu x_j}}{(p_1 e^{-\mu x_j} + p_2 e^{-\mu x_j} + p_3 e^{\mu x_j})} \right\} \approx 1,$$

and the determinants (5.57), (5.58) and (5.59) are non-negative.

Therefore, with the choice of p_1 and p_2, the matrix (5.56) is positive definite if k_{13}, k_{23}, k_{31} and k_{32} satisfy

$$|k_{13}| < \sqrt{\left|\frac{\lambda_3}{\lambda_1}\right| \frac{\gamma_3}{\gamma_1}}, \quad |k_{23}| < \sqrt{\left|\frac{\lambda_3}{\lambda_2}\right| \frac{\gamma_3}{\gamma_2}}, \quad |k_{31}| < \sqrt{\left|\frac{\lambda_1}{\lambda_3}\right| \frac{\gamma_1}{\gamma_3} e^{-\mu l}} \quad \text{and} \quad |k_{32}| < \sqrt{\left|\frac{\lambda_2}{\lambda_3}\right| \frac{\gamma_2}{\gamma_3} e^{-\mu l}}.$$

Consider a constant steady state [23]

$$H^*(x) = 2, \quad V^*(x) = 3 \quad \text{and} \quad B^*(x) = 0.4, \quad x \in [0, l]$$

with physical parameter values $g = 9.81$, $C_f = 0.1$, $a = 0.0184$ and $S_b = 0.0459$ in the system (2.63). The initial condition [23] for the system (2.63)

$$H(x,0) = 2.5 - B(x,0), \quad V(x,0) = \frac{10\sin(\pi x)}{H(x,0)} \quad \text{and}$$
$$B(x,0) = 0.4\left(1 + 0.25\exp\left(-\frac{(x-0.5)^2}{0.003}\right)\right), \quad x \in [0, l].$$

The decoupled system (2.73) has the following eigenvalues

$$\lambda_1 = 7.5383, \quad \lambda_2 = 0.3430 \quad \text{and} \quad \lambda_3 = -1.8813,$$

the coefficients of the source terms are:

$$\gamma_1 = 0.1014, \quad \gamma_2 = 0.0267 \quad \text{and} \quad \gamma_3 = 0.1719,$$

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and the initial condition, which is obtained from (2.74), is

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \\ w_3(x,0) \end{bmatrix} = \begin{bmatrix} -37.1179 - 1.147\exp\left(-\frac{(x-0.5)^2}{0.003}\right) + \frac{133.3333\sin(\pi x)}{2.1-0.1\exp\left(-\frac{(x-0.5)^2}{0.003}\right)} \\ -44.9229 + 43.0533\exp\left(-\frac{(x-0.5)^2}{0.003}\right) + \frac{133.3333\sin(\pi x)}{2.1-0.1\exp\left(-\frac{(x-0.5)^2}{0.003}\right)} \\ -42.6796 - 4.2729\exp\left(-\frac{(x-0.5)^2}{0.003}\right) + \frac{133.3333\sin(\pi x)}{2.1-0.1\exp\left(-\frac{(x-0.5)^2}{0.003}\right)} \end{bmatrix}, \quad x \in [0, l].$$

Therefore, the convergence of the discrete Lyapunov function for different values of $\mu > 0$ is shown in Figure 5.6 below.

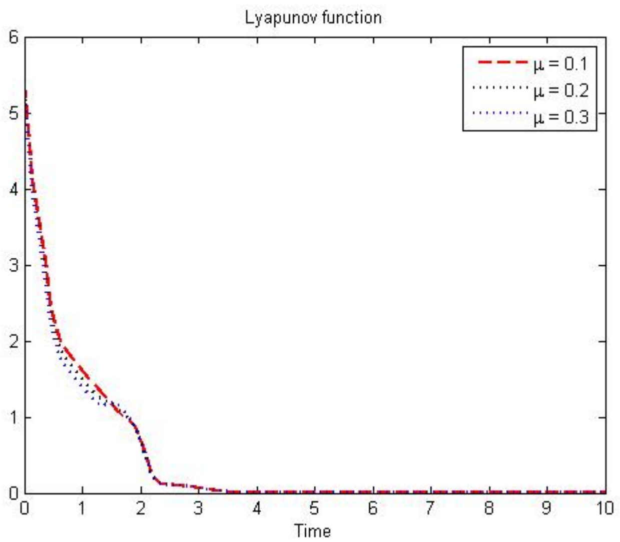

Figure 5.6: The decay of the Lyapunov function for Saint-Venant-Exner equations. The choice of parameters are $p_1 = 0.1014$, $p_2 = 0.0267$, $p_3 = 0.1719$, $k_{13} = 0.6$, $k_{23} = 0.5$, $k_{31} = 0.15$ and $k_{32} = 0.15$, with $l = 1$, $J = 200$ and $T = 10$ under CFL = 0.75.

The three curves that are obtained for different values of $\mu > 0$, which are shown in Figure 5.6, are nearly indistinguishable. We observed the convergence of the discrete Lyapunov function to 0. This shows, in the sense of $\mathbb{L}^2-$norm, the Cauchy problem (4.25), (4.26), (4.27) is exponentially stable.

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5.3 Summary

In this chapter, we have analyzed the numerical boundary stability of some examples of linear hyperbolic balance laws. In particular, conditions are given for the stability of these systems. Furthermore, numerical tests have been undertaken to show the implications of numerical analysis from analytical analysis.

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Conclusions and Future Work

In this dissertation, a linear hyperbolic system of balance laws has been considered and a finite volume method is used in the discretization of this linear system. In particular, the upwind scheme with splitting source term method is applied to obtain a fully discretized linear hyperbolic system of balance laws. Beside this, an $\mathbb{L}^2-$Lyapunov function is discretized and used to investigate conditions for exponential stability of the discretized system. Furthermore, the result was applied to some relevant physical problems such as the telegrapher equations, isentropic Euler equations, Saint-Venant equations and Saint-Venant-Exner equations. Finally, numerical simulations are computed in order to test the results and compare with analytical results.

As part of future work plans, the numerical stability analysis of the nonlinear hyperbolic systems of balance laws will be explored by using the $\mathbb{H}^2-$Lyapunov function.

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Appendix A

Quadratic Forms and Definite Matrices

The purpose of this appendix is to give short summary to the basic concepts related to quadratic forms and definite matrices. Special attention is given on some topics most relevant to the discussion of this dissertation.

Definition 8 (Linear Form [43]). *Let $V = (v_1, \dots, v_k)^T$ be an arbitrary vector in $\mathbb{R}^k$. For any vector $X = (x_1, \dots, x_k)^T$ in $\mathbb{R}^k$, a linear form is a function that is defined by*

$$V^T X = \sum_{i=1}^{k} v_i x_i = v_1 x_1 + \cdots + v_k x_k.$$

Notation 1. *If a vector $V = X$ in the definition above, then we have*

$$X^T X = \sum_{i=1}^{k} x_i^2 = \|X\|^2.$$

Example 1 ([43]). *A function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by*

$$4x_1 + 5x_2 - 3x_3$$

is a linear form with $X = (x_1, x_2, x_3)^T$ and $V = (4, 5, -3)^T$.

Definition 9 (Bilinear Form [43]). *Let $B = \{b_{ij}\}$ denote a $k \times k$ arbitrary matrix. For all vectors $X = (x_1, \dots, x_k)^T$ and $Y = (y_1, \dots, y_m)^T$ in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, a bilinear form is a function that is defined by*

$$X^T B Y = \sum_{i=1}^{k} \sum_{j=1}^{m} b_{ij} x_i y_j.$$

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Example 2 ([43]). *Given an expression*

$$x_1y_1 + 2x_1y_2 + 4x_2y_1 + 7x_2y_2 + 2x_3y_1 - 2x_3y_2,$$

its bilinear form is

$$X^TBY = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 7 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Definition 10 (Quadratic Form [43]). *Let $Q = \{q_{ij}\}$ be a $k \times k$ symmetric matrix with real entries. For any vector $X = (x_1, \ldots, x_k)^T$ in $\mathbb{R}^k$, a quadratic form is a function that is defined by*

$$X^TQX = \sum_{i=1}^{k}\sum_{j=1}^{k} q_{ij}x_ix_j.$$

Example 3 ([43]). *Consider the following expression*

$$x_1^2 + 7x_2^2 + 4x_3^2 + 4x_1x_2 + 10x_1x_3 - 4x_2x_3,$$

which can be written as a quadratic form

$$X^TQX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & -2 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Definition 11 (Definiteness [2]). *A quadratic form X^TQX is said to be:*

1. *positive definite if $X^TQX > 0$ for all $x \neq 0$.*

2. *positive semi-definite if $X^TQX \geq 0$ for all $x \neq 0$ and $X^TQX = 0$ for some $x \neq 0$.*

3. *negative definite if $X^TQX < 0$ for all $x \neq 0$.*

4. *negative semi-definite if $X^TQX \leq 0$ for all $x \neq 0$ and $X^TQX = 0$ for some $x \neq 0$.*

5. *indefinite if X^TQX has both positive and negative values.*

To determine whether a matrix Q and its associated quadratic form X^TQX are positive definite, negative definite or indefinite by using eigenvalues of a matrix Q, the following theorem can be refereed.

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Theorem 5 ([2]). *If Q is a symmetric matrix, then:*

1. *X^TQX is positive definite if and only if all eigenvalues of Q are positive.*
2. *X^TQX is negative definite if and only if all eigenvalues of Q are negative.*
3. *X^TQX is indefinite if and only if Q has at least one positive eigenvalue and at least one negative eigenvalue.*

Example 4 ([2]). *Consider a symmetric matrix*

$$Q = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

To tell that matrix Q and its associated quadratic form X^TQX are positive definite, negative definite or indefinite using Theorem 5, it is required to obtain eigenvalues of a matrix Q. Since eigenvalues of Q are $\lambda = -2, 1, 4$, a quadratic form

$$X^TQX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3,$$

is indefinite.

Definition 12 (Principal Sub-matrix [2]). *Let $Q = \{q_{ij}\}$ be a $k \times k$ matrix. The m^{th} principal sub-matrix of Q is a sub-matrix consisting of the first m rows and columns of Q.*

Example 5 ([2]). *For a $k \times k$ matrix $Q = \{q_{ij}\}$, the following are the principal sub-matrices of Q,*

$$\begin{bmatrix} q_{11} \end{bmatrix}, \quad \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \quad \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1k} \\ q_{21} & q_{22} & \cdots & q_{2k} \\ \vdots & \vdots & & \vdots \\ q_{k1} & q_{k2} & \cdots & q_{kk} \end{bmatrix}.$$

Whether a matrix Q and its associated quadratic form X^TQX is positive definite, negative definite or indefinite can be determined using the determinants of principal sub-matrices as by the following theorem.

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Theorem 6 ([2])**.** *If Q is a symmetric matrix, then:*

1. *Q is positive definite if and only if the determinant of every principal sub-matrix is positive.*
2. *Q is negative definite if and only if the determinants of the principal sub-matrices alternate between negative and positive values starting with negative value for the determinant of the first principal sub-matrix.*
3. *Q is indefinite if and only if it is neither positive nor negative definite and at least one principal sub-matrix has a positive determinant and at least one has a negative determinant.*

Example 6 ([2])**.** *Consider a symmetric matrix*

$$Q = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}.$$

Since the determinants

$$\begin{vmatrix} 2 \end{vmatrix} = 2, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \quad \begin{vmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{vmatrix} = 1,$$

are all positive, a quadratic form

$$X^TQX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1^2 - 2x_1x_2 - 6x_1x_3 + 2x_2^2 + 8x_2x_3 + 9x_3^2,$$

is positive definite.

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Appendix B

Determining the Eigenvalues of Cubic Equations

The purpose of this appendix is discuss how to determine the eigenvalues of the Jacobian matrix of order 3×3, which has to do with finding roots of cubic equation [30].

Consider a cubic equation

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where a_1, a_2 and a_3 are real coefficients. In order to find roots of the cubic equation, we let

$$Q = \frac{1}{9}\left(3a_2 - a_1^2\right) \quad \text{and} \quad R = \frac{1}{54}\left(9a_1a_2 - 27a_3 - 2a_1^3\right).$$

Then the discriminant is $D = Q^3 + R^2$ and if

1. $D > 0$ then the cubic equation has one real and two complex roots;
2. $D = 0$ then the cubic equation has all real and two of them are equal roots;
3. $D < 0$ then the cubic equation has real and unequal roots.

Thus, if $D < 0$ then the roots of the cubic equation can be determined by

$$\lambda_1 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta\right) - \frac{1}{3}a_1, \tag{B.1a}$$

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$$\lambda_2 = 2\sqrt{-Q}\cos\left(\frac{1}{3}(\theta + 2\pi)\right) - \frac{1}{3}a_1, \tag{B.1b}$$

and

$$\lambda_3 = 2\sqrt{-Q}\cos\left(\frac{1}{3}(\theta + 4\pi)\right) - \frac{1}{3}a_1, \tag{B.1c}$$

where $\cos\theta = \frac{R}{\sqrt{-Q^3}}$.

Note that before using (B.1) to determine the roots of the cubic equation, we need to check that $D < 0$.

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