A structure theorem for asymptotically abelian W*-dynamical systems

by

Malcolm Bruce King

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Declaration

I, Malcolm Bruce King, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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DATE:

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Supervisor: Prof RDV Duvenhage
Co-supervisor: Prof A Ströh
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Abstract

We prove a partial non-commutative analogue of the Furstenberg-Zimmerman Structure Theorem, originally proved by Tim Austin, Tanya Eisner and Terence Tao.

In Chapter 1, we review the GNS construction for states on von Neumann algebras and the related semicyclic representation for tracial weights. We look at Tomita-Takasaki theory in the special case of traces. This will allow us to introduce the Jones projection and conditional expectations of von Neumann algebras. We then define the basic construction and its associated finite lifted trace. We also introduce the notion of projections of finite lifted trace and how they relate to right submodules.

Chapter 2 introduces dynamics in the form of automorphisms on von Neumamnn algebras. We will see how the dynamics is represented on the GNS Hilbert space using a cyclic and separating vector. It is then shown how the dynamics is extended to the basic construction and the semicyclic representation.

The last three chapters form the "core". At the beginning of each aforementioned chapter, we present a summary of the required theory, before providing detailed proofs.

In Chapter 3, we prove one of two "fundamental lemmas" where we introduce some non-commutative integration theory. We use a version of the spectral theorem expressed in terms of a spectral measure to produce a certain projection of finite lifted trace. In Chapter 4, we prove our next fundamental lemma. We use direct integral theory in order to obtain a representation of the dynamics, in terms of a module basis, on the image of the projection of finite lifted trace. In Chapter 5, we apply our previous results to asymptotically abelian W*-dynamical systems, culminating in the proof of the titular theorem.

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Contents

Introduction

Ergodic theory has its origins in statistical mechanics, a subfield of physics (see, for instance, [TKS92]). In part, it comprises the study of the long term behaviour of abstract dynamical systems and abstracting, into various notions, the physically intuitive idea of "mixing".

Ergodic ideas have increasingly made their presence known within mathematics itself, for instance, the work of Furstenberg in number theory ([Fur14]).

In this dissertation, we will prove a partial non-commutative analogue of the Furstenberg-Zimmerman Structure Theorem ([Fur77], [Z$^+$76]), originally proved in [AET11]:

Theorem 5.2.3 ([AET11] Theorem 1.14)**.** If $(\mathcal{M}, \tau, \alpha)$ is an asymptotically abelian W^*-dynamical system, then α is weakly mixing relative to the centre $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$. □

The following list provides an overview of the mathematical topics used in this dissertation:

- von Neumann algebras, their normal states and tracial weights
- The GNS- and semicyclic representations
- Tomita Takesaki theory for traces
- Conditional Expectation of von Neumann algebras
- The basic construction and finite lifted trace
- Unbounded linear operators and operators affiliated to a von Neumann algebra
- Measurable operators in the theory of non-commutative integration
- Spectral theory (for bounded and unbounded linear operators)
- Direct integrals (of Hilbert spaces and von Neumann algebras)
- A Gram-Schmidt process for modules
- Composition Operators on L^∞.

At a bare minimum, the reader should be acquainted with the basics of von Neumann algebras and unbounded linear operators. The content of [Zhu93] (up to and including Chapter 18) and [Kre78] Chapter 10 (up to and including §10.3) should serve as an example of prerequisites.

This dissertation consists of five chapters. The first two chapters provides background. To keep the length of this dissertation reasonable, we have limited the number of proofs in Chapters 1 and 2, providing references instead.

In Chapter 1, we review the GNS construction for states on von Neumann algebras and the related semicyclic representation for tracial weights. We shall look at Tomita-Takasaki theory in the special case of traces. This will allow us to introduce the Jones projection and conditional expectations of von Neumann algebras. We then define the basic construction and its associated finite lifted trace. We also introduce the notion of projections of finite lifted trace and how they relate to right submodules.

Chapter 2 introduces dynamics in the form of automorphisms on von Neumamnn algebras. We will see how the dynamics is represented on the GNS Hilbert space using a cyclic and separating vector. It is then shown how the dynamics is extended to the basic construction and the semicyclic representation.

The reader should view the last three chapters as the "core" of this dissertation. At the beginning of each aforementioned chapter, we present a summary of the required theory, before providing detailed proofs to results presented in [AET11].

In Chapter 3, we prove one of two "fundamental lemmas" (Lemma 3.0.3), where we introduce some non-commutative integration theory. We use a version of the spectral theorem expressed in terms of a spectral measure to produce a certain projection of finite lifted trace. In Chapter 4, we prove our next fundamental lemma (Lemma 4.0.2). We use direct integral theory in order to obtain a representation of the dynamics, in terms of a module basis, on the image of the projection of finite lifted trace. In Chapter 5, we apply our previous results to asymptotically abelian W^*-dynamical systems, culminating in the proof of Theorem 5.2.3.

Chapter 1

Background

1.1 The GNS construction

Though we assume that the reader is familiar with the GNS Construction, we will present a summary of results and definitions that will be useful to us. For a more in depth discussion, we refer the reader to a number of books, for instance [Zhu93] §14.1, [Mur90] §§3.4 and 5.1, [Ave02] §4.7 and [KR97a] §4.1. We follow the presentation of [JS97] §1.2 below.

For any subset $\mathfrak{S}$ of a Hilbert space $\mathfrak{H}$ (over $\mathbb{C}$), denote by $[\mathfrak{S}]$ the closed subspace spanned by $\mathfrak{S}$. By a representation $(\pi, \mathfrak{H})$ of a von Neumann algebra $\mathcal{A}$, we mean a 2-tuple consisting of a Hilbert space $\mathfrak{H}$ and an algebra homomorphism $\pi : \mathcal{A} \to B(\mathfrak{H})$ such that π is $*$-preserving i.e. for all $a \in \mathcal{A}$, $\pi(a)^* = \pi(a^*)$. If the underlying Hilbert space is understood, we usually refer to π as a representation of $\mathcal{A}$, instead of $(\pi, \mathfrak{H})$.

Definition 1.1.1. Suppose $\mathcal{A}$ is a von Neumann algebra with representation $\pi : \mathcal{A} \to B(\mathfrak{H})$. We call a vector $x \in \mathfrak{H}$, a **cyclic vector** for π if $[\pi(\mathcal{A})x] = \mathfrak{H}$. The representation π is then referred to as a **cyclic representation of $\mathcal{A}$** . □

Definition 1.1.2. Let $\mathcal{A}$ be a von Neumann algebra. A **state** of $\mathcal{A}$, is a linear functional $\psi : \mathcal{A} \to \mathbb{C}$ with the following properties: For all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$

(a) $\psi(\mathbb{1}) = 1$

(b) $\psi(x^*) = \overline{\psi(x)}$

(c) $\psi(x^*x) \geq 0$ (ψ is **positive**)

We call ψ faithful if $\psi(x^*x) = 0$ if and only if $x = 0$ and **normal** if for all increasing nets (x_α) of positive elements with supremum $\sup_\alpha\{x_\alpha\}$, we have $\sup_\alpha\{\psi(x_\alpha)\} = \psi(\sup_\alpha\{x_\alpha\})$.

A state ψ with the tracial property (i.e. $\psi(xy) = \psi(yx)$ for all $x, y \in \mathcal{A}$) is called a **trace**. □

Let $\mathcal{A}$ be any von Neumann algebra with a normal state ψ. Consider the following sesquilinear form on $\mathcal{A}$: for all $x, y \in \mathcal{A}$, $(x, y) \mapsto \langle x, y\rangle := \psi(y^*x)$. We almost have an inner product. However, we may not have positive-definiteness, i.e. there may exist an $x \neq 0$ such that $\psi(x^*x) = 0$. We thus consider the set

$$K_\psi := \{x \in \mathcal{A} : \psi(x^*x) = 0\}. \tag{1.1.1}$$

A consequence of a Cauchy-Schwarz type inequality implies that K_ψ is a left ideal of $\mathcal{A}$. Thus, we have an inner product on the quotient space $\mathcal{A}/K_\psi$. Hence, the equation $\pi(x)(y+K_\psi) = xy+K_\psi$ defines a bounded linear operator $\pi(x)$ on $\mathcal{A}/K_\psi$, with respect to the norm $\|y + K_\psi\|_2 = \sqrt{\psi(y^*y)}$. Denoting the Hilbert space completion of $\mathcal{A}/K_\psi$ by $\mathfrak{H}_\psi$, $\pi_\psi(x)$, the extension of $\pi(x)$ to $\mathfrak{H}_\psi$, defines a normal (σ-weakly continuous) representation of $\mathcal{A}$ on $\mathfrak{H}_\psi$ ([Tak03a] Proposition 3.12). Further, if $\Omega_\psi := \mathbb{1}_\mathcal{A} + K_\psi$, where $\mathbb{1}_\mathcal{A}$ denotes the multiplicative identity of $\mathcal{A}$, then Ω_ψ is a cyclic vector for $\mathfrak{H}_\psi$. Given the triple $(\mathfrak{H}_\psi, \pi_\psi, \Omega_\psi)$ we can recover ψ by setting

$$\psi(x) := \langle \pi_\psi(x)\Omega_\psi, \Omega_\psi\rangle.$$

This concludes the summary of the GNS representation with respect to ψ, after Gelfand, Naimark and Segal:

Proposition 1.1.3. *Let ψ be a normal state on a von Neumann algebra $\mathcal{A}$. Then there exists a triple $(\mathfrak{H}_\psi, \pi_\psi, \Omega_\psi)$ consisting of a Hilbert space $\mathfrak{H}_\psi$ carrying a normal representation π_ψ of $\mathcal{A}$ and a distinguished cyclic vector Ω_ψ of the representation satisfying $\psi(x) = \langle \pi_\psi(x)\Omega_\psi, \Omega_\psi\rangle$.*

Remark 1.1.4. If there is no confusion, we usually do not mention the underlying state, dropping the subscripts and writing π and Ω, instead of π_ψ and Ω_ψ. □

Definition 1.1.5. Suppose $\mathcal{A}$ is a von Neumann algebra with representation $\pi : \mathcal{A} \to B(\mathfrak{H})$. We call a vector $\xi \in \mathfrak{H}$ a **separating vector** for $\mathcal{A}$ if the map $\mathcal{A} \ni x \mapsto \pi(x)\xi \in \pi(\mathcal{A})\Omega$ is injective. In other words, for every $x \in \mathcal{A}$, $\pi(x)\xi = 0 \implies x = 0$. □

Proposition 1.1.6 ([JS97] Lemma 1.2.3, [BR02] Proposition 2.5.3)**.** *If $\mathcal{A} \subseteq B(\mathfrak{H})$ is a von Neumann algebra, a vector $\xi \in \mathfrak{H}$ is cyclic for $\mathcal{A}$ if and only if it is separating for $\mathcal{A}'$.* □

Remark 1.1.7. For the remainder of this work, we reserve the symbols $\mathcal{M}$ and τ. We let $\mathcal{M}$ denote a von Neumann algebra admitting a faithful normal trace τ.

The GNS representation π is then faithful (i.e injective), since, if $x \in \mathcal{M}$, then

$$\pi(x) = 0 \implies 0 = \|\pi(x)\Omega_\tau\|^2 = \tau(x^*x) = 0 \implies x = 0.$$

And now Ω_τ is separating:

$$\pi(a)\Omega_\tau = 0 \implies \tau(a^*a) = \|\pi(a)\Omega_\tau\|^2 = 0 \implies a = 0.$$

Thus, without loss of generality, we always assume that $\mathcal{M}$ is in its cyclic representation, identifying x with $\pi(x)$. In contrast, for other algebras ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$, etc.) we only assume this when we explicitly say so.

We emphasize that there are two ways of viewing $\mathcal{M}$ (which we will keep distinct in this dissertation). One is as a subalgebra of $B(\mathfrak{H}_\tau)$ (using the GNS representation). The other is as a dense set in $\mathfrak{H}_\tau = [\mathcal{M}\Omega]$ via the map $x \mapsto x\Omega$. When referring to the embedded copy of an element $x \in \mathcal{M}$ in $\mathfrak{H}_\tau$, some authors use $\hat{x}$ instead of $x\Omega$. The notation $x\hat{\mathbb{1}}$, with $\hat{\mathbb{1}} := \Omega$ is also used. □

1.2 Modules

Though we do not use any deep results with regards to modules, we will find it a useful way of thinking. We introduce some terminology which make precise what is meant by "a right (or left) action of an algebra on a Hilbert space".

Definition 1.2.1 ([JS97] p.19)**.** Let $\mathcal{A}$ be an arbitrary von Neumann algebra. By a **left $\mathcal{A}$-module**, we mean a Hilbert space $\mathfrak{H}$ equipped with an **action** (more precisely a **left action**) i.e a unital σ-weakly continuous representation $\pi_l : \mathcal{A} \to B(\mathfrak{H})$. □

Definition 1.2.2 ([JS97] p.23)**.** Let $\mathcal{A}$ be an arbitrary von Neumann algebra. A Hilbert space $\mathfrak{H}$ is called a **right $\mathcal{A}$-module** if there is a σ-weakly continuous representation $\pi_r : \mathcal{A} \to B(\mathfrak{H})$, satisfying, for all $x, y \in \mathcal{A}$

$$\pi_r(xy) = \pi_r(y)\pi_r(x). \tag{1.2.1}$$

We refer to π_r as the **right action of $\mathcal{A}$ on $\mathfrak{H}$**. □

Remark 1.2.3. The σ-weak topology is also referred to as the weak* topology (p.95 [Con00]). From [Con00] Corollary 46.5, a representation is σ-weakly continuous if and only if it is normal. □

Definition 1.2.4. Let $\mathcal{A}$ and $\mathcal{B}$ are two arbitrary von Neumann algebras. Then a Hilbert space $\mathfrak{H}$ is called an **$\mathcal{A}$-$\mathcal{B}$-module** if

(a) $\mathfrak{H}$ is a left $\mathcal{A}$-module;

(b) $\mathfrak{H}$ is a right $\mathcal{B}$-module;

(c) For all $m \in \mathcal{A}$, $n \in \mathcal{B}$ and $\xi \in \mathfrak{H}$, $\pi_l(m)\pi_r(n)\xi = \pi_r(n)\pi_l(m)\xi$.

□

Remark 1.2.5. The definition of a right action is consistent with how right multiplication is understood informally. Suppose $\mathfrak{H}$ is a $\mathcal{A}$-$\mathcal{B}$-bimodule for some von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ represented on $\mathfrak{H}$. Then, in terms of the notation $ax \equiv \pi_l(a)x$ and $xb \equiv \pi_r(b)x$, (c) in Definition 1.2.4 can be expressed as

(1.2.2) $$x(\xi y) = (x\xi)y \qquad \text{for all } x \in \mathcal{A}\,, b \in \mathcal{B} \text{ and } \xi \in \mathfrak{H}.$$

□

Remark 1.2.6. Using Definition 1.2.1, $\mathfrak{H}_\psi$ is a left $\mathcal{A}$-module. Later we will see how we can view $\mathfrak{H}_\psi$ as a right-$\mathcal{A}$-module (Proposition 1.4.3). □

We also take note of what it means for a subspace to be a submodule.

Definition 1.2.7. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{H}$ be an $\mathcal{A}$-$\mathcal{A}$-module with left action π_l and right action π_r. Let $\mathcal{C}$ be a von Neumann subalgebra of $\mathcal{A}$. We say that $\mathfrak{K} \subseteq \mathfrak{H}$ is a closed **right-$\mathcal{C}$-submodule** of $\mathfrak{H}$ provided

(a) $\mathfrak{K}$ is a closed subspace of $\mathfrak{H}$ and

(b) $\mathfrak{K}$ is invariant under the algebra $\pi_r(\mathcal{C})$, that is, $\pi_r(\mathcal{C})\mathfrak{K} \subseteq \mathfrak{K}$.

Similarly, $\mathfrak{K}$ is a closed **left-$\mathcal{C}$-submodule** of $\mathfrak{H}$ if $\mathfrak{K}$ is a closed subspace of $\mathfrak{H}$ and is invariant under the algebra $\pi_l(\mathcal{C})$. □

1.3 The Semicyclic Representation

We describe a generalisation of the GNS construction where a normal semifinite tracial weight is used instead of a state. Details can be found in [KR97b] §7.5. We lay out some useful notation that we will use in Chapters 2 and 3.

Let ρ be a normal semifinite tracial weight ([KR97b] Definition 7.5.1, [BR02] Definition 2.7.12) on a von Neumann algebra $\mathcal{R}$. We consider the quotient space $\mathcal{K}_\rho/K_\rho$, where

(1.3.1) $$\mathcal{K}_\rho := \{x \in \mathcal{R} \mid \rho(x^*x) < \infty\}$$

is a left $\mathcal{R}$-ideal ([KR97a] Lemma 7.5.2) and $K_\rho := \{x \in \mathcal{R} \mid \rho(x^*x) = 0\}$. The quotient map $g_\rho : \mathcal{K}_\rho \to \mathcal{K}_\rho/K_\rho$ sends elements $x \in \mathcal{K}_\rho$ to elements $g_\rho(x) = x + K_\rho$.

Though the tracial weight ρ is only defined on the positive elements of $\mathcal{R}$, it can be extended uniquely to positive hermitian functional on $\mathcal{S}_\rho := \operatorname{span}\{a \in \mathcal{R}^+ \mid \rho(a) < \infty\}$.

Thus, because $\mathcal{K}_\rho^*\mathcal{K}_\rho = \mathcal{S}_\rho$ ([KR97b] Lemma 7.5.2) the expression $\rho(y^*x)$ makes sense for every $x, y \in \mathcal{K}_\rho$. Hence, $\mathcal{K}_\rho/K_\rho$ has the inner product

$$\langle x + K_\rho, y + K_\rho\rangle := \rho(y^*x), \qquad \text{for all } x, y \in \mathcal{K}_\rho. \tag{1.3.2}$$

Completing $\mathcal{K}_\rho/K_\rho$ in the norm $\|x + K_\rho\|_\rho := \sqrt{\langle x + K_\rho, x + K_\rho\rangle}$ yields a Hilbert space which we denote by $\mathfrak{H}_\rho$.

The elements of $\mathcal{R}$ are represented as operators on $\mathfrak{H}_\rho$. We first define the action of $r \in \mathcal{R}$ on $\mathcal{K}_\rho\backslash K_\rho$

$$\pi_\rho(r)(x + K_\rho) := rx + K_\rho. \tag{1.3.3}$$

The arguments in the GNS construction -see for instance [KR97b] Theorem 4.5.2- give us a unique bounded linear extension with domain $\mathfrak{H}_\rho$, which we still denote by $\pi_\rho(r)$.

1.4 The Modular Conjugation Operator

Here, we briefly discuss the basics of Tomita-Takasaki theory that we need. We continue using the notation from Remark 1.1.7.

Consider the bijective conjugate linear map

$$J : \mathcal{M}\Omega \ni x\Omega \mapsto x^*\Omega \in \mathcal{M}\Omega \subseteq \mathfrak{H}_\tau. \tag{1.4.1}$$

Observe that because τ is tracial, J is an isometry on $\mathcal{M}\Omega$ (Proposition 1.4.1 below, or [BR02] p. 84) and is therefore continuous on $\mathcal{M}\Omega$. Similar to using the bounded linear extension theorem ([Kre78], Theorem 2.7-11) J can be extended to a unique conjugate linear operator with domain $[\mathcal{M}\Omega] \equiv \mathfrak{H}_\tau$. The extension obtained is called the **modular conjugation operator for $\mathcal{M}$** denoted by $J_\mathcal{M}$. If there is no confusion we usually drop the subscript, writing J, instead of $J_\mathcal{M}$.

Proposition 1.4.1 ([SS08] pp. 37-38)**.** *(Properties of J) We use the notation from Remark 1.1.7 and the paragraph above.*

(a) For all $\xi, \eta \in \mathfrak{H}_\tau$ we have, $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$; in particular, J is an isometry.

(b) J is continuous and bijective onto $\mathfrak{H}_\tau$.

(c) $J = J^* = J^{-1}$, where J^* is defined as follows: for every $x, y \in \mathfrak{H}_\tau$,

$$\langle J^*x, y\rangle = \overline{\langle x, Jy\rangle} = \langle Jy, x\rangle.$$

□

The following result relates $\mathcal{M}' := \{x' \in B(\mathfrak{H}_\tau) \mid \forall m \in \mathcal{M} \quad x'm = mx'\}$, the commutant of $\mathcal{M}$, in terms of the J operator.

Theorem 1.4.2 (([JS97] Theorem 1.2.4))**.** $J\mathcal{M}J = \mathcal{M}'$. □

Proposition 1.4.3. *Consider a von Neumann algebra $\mathcal{M}$ admitting a faithful normal tracial state τ.*

Then $\mathfrak{H}_\tau$ is an $\mathcal{M}$-$\mathcal{M}$-bimodule via $\pi_l := \mathrm{id}$ and $\pi_r(a) := Ja^*J$ (from Definitions 1.2.1 and 1.2.2).

Proof. We have remarked (Remark 1.2.6) that $\mathfrak{H}_\tau$ is a left $\mathcal{M}$-module. The map π_r is clearly linear and σ-weakly continuous because multiplication is separately continuous ([BR02] Proposition 2.4.2). If $x, y \in \mathcal{M}$, then the equality

$$Jx^*JJy^*J = J(yx)^*J,$$

shows that (1.2.1) holds. Using Theorem 1.4.2, $\pi_r(y) \in \mathcal{M}'$ for every $y \in \mathcal{M}$ and thus for every $x \in \mathcal{M}$, $\pi_l(x)$ commutes with $\pi_r(y)$.

□

Remark 1.4.4. The notation $\mathcal{M}_{\text{left}}$ is used for $\mathcal{M}$ acting on $\mathfrak{H}_\tau$ from the left and $\mathcal{M}_{\text{right}}$ for $\mathcal{M}$ acting on $\mathfrak{H}_\tau$ from the right, as in Proposition 1.4.3. Note that this essentially means that $\mathcal{M}_{\text{right}}$ is $\mathcal{M}'$, since $\pi_r(\mathcal{M}) = \mathcal{M}'$ (Theorem 1.4.2). □

1.5 The Conditional Expectation of von Neumann Algebras

We now further suppose that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with trace $\tau|_{\mathcal{N}}$. In order to avoid "significantly different formulations", we further assume that $\mathbb{1}_{\mathcal{M}} = \mathbb{1}_{\mathcal{N}}$ (see the remarks before [SS08] Definition 4.2.1).

Definition 1.5.1. Let $e_{\mathcal{N}}$ be the projection from $[\mathcal{M}\Omega]$ onto $[\mathcal{N}\Omega]$. We refer to $e_{\mathcal{N}}$ as the **Jones projection**. □

Using [SS08] Lemma 3.6.2, the projection $e_{\mathcal{N}}$ sends $\mathcal{M}\Omega$ to $\mathcal{N}\Omega$. Thus, we can consider:

Definition 1.5.2. Define the linear operator $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, via the equation:

$$E_{\mathcal{N}}(a)\Omega = e_{\mathcal{N}}(a\Omega) \qquad a \in \mathcal{M}. \tag{1.5.1}$$

The operator $E_{\mathcal{N}}$ is known as the **conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ associated with τ**. When there is no ambiguity, we usually write E in place of $E_{\mathcal{N}}$. □

Lemma 1.5.3 ([SS08] Lemma 3.6.2)**.** *Let $a \in \mathcal{M}$. The operator $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ has the following property: for all $\xi \in \mathfrak{H}_\tau$,*

$$e_{\mathcal{N}} a e_{\mathcal{N}} \xi = E_{\mathcal{N}}(a) e_{\mathcal{N}} \xi = e_{\mathcal{N}} E_{\mathcal{N}}(a) \xi.$$

□

1.6 The Basic Construction $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$

Definition 1.6.1. We continue to use the notation from the last section. In particular, recall that $\mathcal{M}$ is a von Neumann algebra with its distinguished faithful normal trace τ and $e_{\mathcal{N}}$ is the Jones projection. We consider $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$, the smallest von Neumann algebra (in $B(\mathfrak{H}_\tau)$) containing $\mathcal{M}$ and $e_{\mathcal{N}}$. We shall refer to $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ as the **basic construction**. □

The equality $J\mathcal{M}'J = \mathcal{M}$ (Theorem 1.4.2) leads us naturally to consider if $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ can be expressed similarly. We set

$$\mathcal{M}_{e_{\mathcal{N}}} := \text{span}\{x e_{\mathcal{N}} y : x, y \in \mathcal{M}\}.$$

Proposition 1.6.2. *([SS08] Lemma 4.2.3 and part of [JS97] Proposition 3.1.2) We continue from Definition 1.6.1.*

(a) $e_{\mathcal{N}} \in \mathcal{N}'$.

(b) The vector space $\mathcal{M}_{e_{\mathcal{N}}}$ is dense in $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ in both the weak- and strong operator topologies

(c) $\langle \mathcal{M}, e_{\mathcal{N}} \rangle = J\mathcal{N}'J = (J\mathcal{N}J)'$ *and* $\langle \mathcal{M}, e_{\mathcal{N}} \rangle' = J\mathcal{N}J$.

(d) $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ *is a semifinite von Neumann algebra.*

□

Remark 1.6.3. Consider part (c) in Proposition 1.6.2, above. In a manner similar to the notation $\mathcal{M}_{\text{left}}$ and $\mathcal{M}_{\text{right}}$ (Remark 1.4.4) we can view $\mathcal{N}$ as $\mathcal{N}_{\text{left}}$ and $J\mathcal{N}J$ as $\mathcal{N}_{\text{right}}$. Thus, $\langle \mathcal{M}, e_{\mathcal{N}} \rangle'$ may be viewed as $\mathcal{N}_{\text{right}}$ and $\langle \mathcal{M}, e_{\mathcal{N}} \rangle = \langle \mathcal{M}, e_{\mathcal{N}} \rangle''$ as $\mathcal{N}'_{\text{right}}$. □

It is a non-trivial fact that there exists a faithful semifinite normal tracial weight $\overline{\tau}$ on $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ (see §4.2 in [SS08]) referred to as the **lifted trace of $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$**. The weight $\overline{\tau}$ has the property that for all $x, y \in \mathcal{M}$,

$$\overline{\tau}(x e_{\mathcal{N}} y) = \tau(xy). \tag{1.6.1}$$

(Weights are normally only defined on positive elements, but in addition $\overline{\tau}$ is defined on $\mathcal{M}_{e_\mathcal{N}}$.)

Furthermore, $\mathcal{M}_{e_\mathcal{N}}$, when viewed as a set of vectors of the form $t + K_{\overline{\tau}}$, is dense in the semicylic representation $\mathfrak{H}_{\overline{\tau}}$ in the $\|\cdot\|_{\overline{\tau}}$-norm. (Note that $K_{\overline{\tau}} = \{0\}$ since $\overline{\tau}$ is faithful.)

1.7 Projections of Finite Lifted Trace

In this section we state and prove a result which we use in Lemma 3.0.3. Our identification of $\langle \mathcal{M}, e_\mathcal{N} \rangle$ with $\mathcal{N}'_{\text{right}}$ (Remark 1.6.3) allows for the following proof.

Proposition 1.7.1. *([AET11] Lemma 3.4) We continue from Proposition 1.6.2. Let V be a closed subspace of $\mathfrak{H}_\tau$ with orthogonal projection $P_V : \mathfrak{H}_\tau \to V$. Then V is a closed right-$\mathcal{N}$-submodule, if and only if $P_V \in \langle \mathcal{M}, e_\mathcal{N} \rangle$.*

Proof. We use Definition 1.2.7. We note that V is a closed subspace of $\mathfrak{H}_\tau$. The following chain of bi-implications then hold:

$$\begin{aligned} & V \text{ is a right-}\mathcal{N}\text{-submodule} \\ \Leftrightarrow\ & V\mathcal{N}_{\text{right}} \subseteq V \\ \Leftrightarrow\ & Vn_r \subseteq V \qquad \forall\, n_r \in \mathcal{N}_{\text{right}} \\ \Leftrightarrow\ & n_r P_V = P_V n_r \text{ for all } n_r \in \mathcal{N}_{\text{right}} \qquad \text{[Zhu93] Corollary 18.3} \\ \Leftrightarrow\ & P_V \in \mathcal{N}'_{\text{right}} = \langle \mathcal{M}, e_\mathcal{N} \rangle. \end{aligned}$$

□

Remark 1.7.2. We express the proof of Proposition 1.7.1 in terms of operators acting on the left, in order to clarify the module language.

Using the notation of Proposition 1.7.1,

$$V \text{ is a closed right-}\mathcal{N}\text{-submodule of } \mathfrak{H}_\tau \iff J\mathcal{N}JV \subseteq V,$$

from Remark 1.4.4. Note that $a \in J\mathcal{N}J \iff a^* \in J\mathcal{N}J$. Thus,

$$\begin{aligned} & J\mathcal{N}JV \subseteq V \\ \Leftrightarrow\ & P_V Jn^*J = Jn^*JP_V \quad \text{for all } n \in \mathcal{N} \qquad \text{[Zhu93] Corollary 18.3} \\ \Leftrightarrow\ & P_V \in (J\mathcal{N}J)' = \langle \mathcal{M}, e_\mathcal{N} \rangle \qquad \text{Proposition 1.6.2 (c).} \end{aligned}$$

□

Definition 1.7.3. Suppose $V \subseteq \mathfrak{H}_\tau$ is a closed right-$\mathcal{N}$-submodule with orthogonal projection $P_V : \mathfrak{H}_\tau \to V$. Let $\overline{\tau}$ be the lifted trace of $\langle \mathcal{M}, e_\mathcal{N} \rangle$. We shall say that P_V has **finite lifted trace** if $\overline{\tau}(P_V) < \infty$. □

Chapter 2

W*-Dynamical Systems

Here we define the dynamical systems on von Neumann algebras that we are going to study, and also consider their Hilbert space representation.

2.1 Basic Definitions

Our ultimate goal in this dissertation is to prove a structure theorem for the following type of system.

Definition 2.1.1. Suppose $\mathcal{M}$ is a von Neumann algebra with a distinguished faithful normal trace τ. Let $\alpha : \mathcal{M} \to \mathcal{M}$ be a $*$-automorphism (an algebra automorphism that preserves the adjoint $(\alpha(a^*) = \alpha(a)^*)$ such that τ is α-preserving $(\tau(\alpha(a)) = \tau(a)$ for all $a \in \mathcal{M})$. We call the triple $(\mathcal{M}, \tau, \alpha)$ a ***W**-dynamical system.** □

In studying the structure of W^*-dynamical systems, the following type of system naturally appears, as we shall see in §2.4.

Definition 2.1.2. Suppose $\mathcal{R}$ is a von Neumann algebra with normal semifinite faithful tracial weight ρ. Let $\alpha : \mathcal{R} \to \mathcal{R}$ be a $*$-automorphism such that ρ satisfies

$$\rho \circ \alpha = \rho \tag{2.1.1}$$

(on the set of all positive elements $a \in \mathcal{R}$). We shall refer to the triple $(\mathcal{R}, \rho, \alpha)$ as a **semifinite *W**-dynamical system**.

2.2 Representation of W^*-dynamics

Recall from Chapter 1 that $\mathcal{M}$ is a von Neumann algebra with faithful normal trace τ and $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with trace $\tau|_{\mathcal{N}}$ which satisfies $\mathbb{1}_{\mathcal{M}} = \mathbb{1}_{\mathcal{N}}$.

Let $\alpha : \mathcal{M} \to \mathcal{M}$ be a $*$-automorphism, such that $\alpha|_{\mathcal{N}}$ is an $*$-automorphism on $\mathcal{N}$ and such that $(\mathcal{M}, \tau, \alpha)$ is a W^*-dynamical system. We shall examine the unitary operator, denoted U_α, that arises from α acting on $\mathcal{M}$. Afterwards, we shall see how U_α can be used to extend the dynamics of α to $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$.

As before, we assume that $\mathcal{M}$ acts on its GNS Hilbert space $\mathfrak{H}_\tau$. However, in cases where we believe confusion may arise, we will not hesitate to explicitly use the notation for the GNS- and semicyclic representation (§§1.1 and 1.3). The $*$-automorphism $\alpha : \mathcal{M} \mapsto \mathcal{M}$ induces a linear operator $U_\alpha : \mathfrak{H}_\tau \mapsto \mathfrak{H}_\tau$ defined (first on the dense subspace $\mathcal{M}\Omega$) by

$$U_\alpha(a\Omega) = \alpha(a)\Omega \qquad \text{for every } a \in \mathcal{M}. \tag{2.2.1}$$

As Ω is separating, U_α is well-defined.

Note that we have the following chain of equalities:

$$\begin{aligned} \|U_\alpha(x\Omega)\|^2 = \|\alpha(x)\Omega\|^2 &= \langle \alpha(x)\Omega, \alpha(x)\Omega\rangle \\ &= \tau(\alpha(x)^*\alpha(x)) = \tau(\alpha(x^*)\alpha(x)) \\ &= \tau(\alpha(x^*x)) = \tau(x^*x) = \langle x\Omega, x\Omega\rangle \\ &= \|x\Omega\|^2 . \end{aligned} \tag{2.2.2}$$

Thus, U_α is continuous on $\mathcal{M}\Omega$ and therefore may be uniquely extended to a linear operator ([Kre78] Theorem 2.7-11 or [KR97a] Corollary 1.2.3) with domain $\mathfrak{H}_\tau \equiv [\mathcal{M}\Omega]$ into $[\mathcal{M}\Omega]$, which we still denote by U_α.

Proposition 2.2.1. *The map $U_\alpha : \mathfrak{H}_\tau \to \mathfrak{H}_\tau$ is a unitary operator.*

Proof. It is sufficient, from [Kre78] Theorem 3.10-6 (f), to show that U_α is isometric and surjective onto $\mathfrak{H}_\tau$. Let $\xi \in \mathfrak{H}_\tau$ be arbitrary and let $(x_n\Omega)$ be a sequence in $\mathcal{M}\Omega$ approximating ξ in the norm of $\mathfrak{H}$. Using (2.2.2) and the continuity of U_α we have,

$$\begin{aligned} |\,\|\xi\| - \|U_\alpha(\xi)\|\,| &\leq |\,\|\xi\| - \|x_n\Omega\|\,| + |\,\|x_n\Omega\| - \|U_\alpha(\xi)\|\,| \\ &\leq \|\xi - x_n\Omega\| + |\,\|x_n\Omega\| - \|U_\alpha(\xi)\|\,| \\ &= \|\xi - x_n\Omega\| + |\,\|U_\alpha(x_n\Omega)\| - \|U_\alpha(\xi)\|\,| \\ &\leq \|\xi - x_n\Omega\| + \|U_\alpha(x_n\Omega) - U_\alpha(\xi)\| \to 0. \end{aligned}$$

Hence $\|\xi\| = \|U_\alpha(\xi)\|$, that is, U_α is isometric.

For surjectivity, let $\xi \in \mathfrak{H}_\tau$ be arbitrary and let $(x_n\Omega)$ be a sequence approximating ξ in the norm of $\mathfrak{H}_\tau$. Then for every $n \in \mathbb{N}$, using (2.2.1),

$$U_\alpha(\alpha^{-1}(x_n)\Omega) = x_n\Omega. \tag{2.2.3}$$

As U_α is isometric, $(\alpha^{-1}(x_n)\Omega)$ is a Cauchy sequence in $\mathfrak{H}_\tau$ and therefore has a limit, say $\zeta \in \mathfrak{H}_\tau$. Let us show that $U_\alpha(\zeta) = \xi$. Indeed, we have,

$$\begin{aligned} \|U_\alpha(\zeta) - \xi\| &\le \left\|U_\alpha(\zeta) - U_\alpha(\alpha^{-1}(x_n)\Omega)\right\| + \left\|U_\alpha(\alpha^{-1}(x_n)\Omega) - \xi\right\| \\ &\le \|U_\alpha\| \left\|\zeta - \alpha^{-1}(x_n)\Omega)\right\| + \|x_n\Omega - \xi\| \qquad \text{from (2.2.3)} \\ &\le \left\|\zeta - \alpha^{-1}(x_n)\Omega)\right\| + \|x_n\Omega - \xi\| \to 0. \end{aligned}$$

□

Let us discuss some properties of U_α.

We note that $U_\alpha^* = U_\alpha^{-1}$ behaves in the following manner:

$$\forall y \in \mathcal{M} \qquad U_\alpha^{-1}(y\Omega) = \alpha^{-1}(y)\Omega. \tag{2.2.4}$$

Note that our assumption that $\alpha(\mathcal{N}) = \mathcal{N}$ leads us to conclude that $[\mathcal{N}\Omega]$ is a reducing subspace for U_α i.e. $U_\alpha[\mathcal{N}\Omega] \subseteq [\mathcal{N}\Omega]$ and $U_\alpha^*[\mathcal{N}\Omega] \subseteq [\mathcal{N}\Omega]$. Hence, from [Zhu93] Corollary 18.3, we have

$$U_\alpha e_\mathcal{N} = e_\mathcal{N} U_\alpha. \tag{2.2.5}$$

We also have,

$$JU_\alpha = U_\alpha J. \tag{2.2.6}$$

(we use this in the Lemma 5.2.2). This is true, because,

$$JU_\alpha(x\Omega) = J\alpha(x)\Omega = \alpha(x^*)\Omega = U_\alpha(x^*\Omega) = U_\alpha J(x\Omega).$$

Remark 2.2.2. We can express α in terms of U_α. For every $a, b \in \mathcal{M}$,

$$U_\alpha a U_\alpha^*(b\Omega) = U_\alpha a\alpha^{-1}(b)\Omega = \alpha(a\alpha^{-1}(b))\Omega = \alpha(a)b\Omega.$$

Thus,

$$\alpha(a) = U_\alpha a U_\alpha^* \qquad \text{for all } a \in \mathcal{M}, \tag{2.2.7}$$

since $\mathcal{M}\Omega$ is dense in $\mathfrak{H}_\tau$. We note an obvious generalisation to (2.2.7):

$$\alpha^n(a) = U_\alpha^n(a)U_\alpha^{-n} \qquad \text{for all } n \in \mathbb{N}. \tag{2.2.8}$$

□

2.3 The Dynamics on $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$

Here we consider an important semifinite W^*-dynamical system that arises from a W^*-dynamical system.

We shall now extend α to the basic construction $\langle \mathcal{M}, e_{\mathcal{N}} \rangle \subseteq B(\mathfrak{H}_\tau)$, which we will denote by $\overline{\alpha}$ in the sequel. For every $x \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$, (cf. with (2.2.7))

$$\overline{\alpha}(x) := U_\alpha x U_\alpha^*. \tag{2.3.1}$$

In the special case where $x = e_{\mathcal{N}}$, using (2.2.5), (2.3.1) becomes

$$\overline{\alpha}(e_{\mathcal{N}}) = e_{\mathcal{N}}. \tag{2.3.2}$$

We show that $(\langle \mathcal{M}, e_{\mathcal{N}} \rangle, \overline{\tau}, \overline{\alpha})$ is a semifinite W^*-dynamical system. We first show,

Proposition 2.3.1. *The map $\overline{\alpha}$ defined by* (2.3.1) *has codomain $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$.*

Proof. As a first step, for all $a, b \in \mathcal{M}$, note that

$$\begin{aligned} \overline{\alpha}(a e_{\mathcal{N}} b) &= U_\alpha a e_{\mathcal{N}} b {U_\alpha}^{-1} = U_\alpha a {U_\alpha}^{-1} U_\alpha e_{\mathcal{N}} {U_\alpha}^{-1} U_\alpha b {U_\alpha}^{-1} \\ &= \alpha(a) U_\alpha e_{\mathcal{N}} {U_\alpha}^{-1} \alpha(b) = \alpha(a) e_{\mathcal{N}} \alpha(b), \end{aligned} \tag{2.3.3}$$

using (2.3.2). Thus, $\overline{\alpha}(a e_{\mathcal{N}} b) \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$. Next, recall from Proposition 1.6.2 (b), that $\mathcal{M}_{e_{\mathcal{N}}}$ is strong operator dense in $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$. So for any $x \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$, there is a net (x_λ) in $\mathcal{M}_{e_{\mathcal{N}}}$ such that for every $\xi \in \mathfrak{H}_\tau$,

$$x_\lambda \xi \to x \xi.$$

Thus, for every $\xi \in \mathfrak{H}_\tau$,

$$\begin{aligned} \|(\overline{\alpha}(x_\lambda) - \overline{\alpha}(x))\xi\| &= \|(U_\alpha x_\lambda U_\alpha^* - U_\alpha x U_\alpha^*)\xi\| \\ &\le \|U_\alpha\| \, \|(x_\lambda - x) U_\alpha^* \xi\| \to 0. \end{aligned} \tag{2.3.4}$$

In other words, $\overline{\alpha}(x)$ is the strong operator limit of a net whose elements are from $\mathcal{M}_{e_{\mathcal{N}}}$. Therefore, $\overline{\alpha}(x)$ belongs to the strong operator closure of $\mathcal{M}_{e_{\mathcal{N}}}$, that is, $\overline{\alpha}(x) \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$. ☐

Remark 2.3.2. We explicitly take note of an obvious generalisation of (2.3.3), which we use in Lemma 3.0.3. For every $n \in \mathbb{N}$ we have,

$$U_\alpha^n (a e_{\mathcal{N}} b) U_\alpha^{-n} = \alpha^n(a) e_{\mathcal{N}} \alpha^n(b). \tag{2.3.5}$$

☐

Proposition 2.3.3. *The map $\overline{\alpha} : \langle \mathcal{M}, e_{\mathcal{N}} \rangle \to \langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is a $*$-automorphism.*

Proof. With the use of Proposition 2.3.1 it is routine to check that $\overline{\alpha}$ is linear. Consider the map $\beta(x) := U_\alpha^* x U_\alpha$ defined for each $x \in \langle \mathcal{M}, e_\mathcal{N} \rangle$. Clearly, using similar arguments to those in Proposition 2.3.1, β is a map from $\langle \mathcal{M}, e_\mathcal{N} \rangle$ into $\langle \mathcal{M}, e_\mathcal{N} \rangle$. Moreover,

$$\overline{\alpha} \circ \beta(x) = U_\alpha (U_\alpha^* x U_\alpha) U_\alpha^* = x,$$

and similarly,

$$\beta \circ \overline{\alpha}(x) = x.$$

Hence, $\overline{\alpha}^{-1} = \beta$, so α is bijective.

The following calculation shows that $\overline{\alpha}$ is $*$-preserving:

$$\overline{\alpha}(x)^* = (U_\alpha x U_\alpha^*)^* = (x U_\alpha^*)^* (U_\alpha)^* = U_\alpha x^* U_\alpha^* = \overline{\alpha}(x^*).$$

Lastly, $\overline{\alpha}$ is multiplicative, because for every $x, y \in \langle \mathcal{M}, e_\mathcal{N} \rangle$

$$\overline{\alpha}(xy) = U_\alpha xy U_\alpha^* = U_\alpha x U_\alpha^* U_\alpha y U_\alpha^* = \overline{\alpha}(x)\overline{\alpha}(y).$$

□

Unlike the previous steps, to prove $\overline{\tau} \circ \overline{\alpha} = \overline{\tau}$ is not elementary. The main tool used in proving that $\overline{\tau}$ is $\overline{\alpha}$-preserving is the following.

Theorem 2.3.4 (Part of [SS08] Theorem 4.3.11)**.** *Let φ be a weight on $\langle \mathcal{M}, e_\mathcal{N} \rangle$ with $\varphi = \overline{\tau}$ on $\mathcal{M}_{e_\mathcal{N}}^+ := \{x \in \mathcal{M}_{e_\mathcal{N}} \mid \exists\, y \in \mathcal{M}_{e_\mathcal{N}} : x = y^* y\}$. If φ is normal, then $\varphi = \overline{\tau}$.* □

We will apply the above result to $\varphi = \overline{\tau} \circ \overline{\alpha}$. We have already remarked that $\overline{\tau}$ is normal (see the paragraph before (1.6.1)). As $\overline{\alpha}$ is a *-automorphism on a von Neumann algebra it is normal ([Con00] Proposition 46.6), hence, so is $\overline{\tau} \circ \overline{\alpha}$. So we just need to check equality of $\overline{\tau} \circ \overline{\alpha}$ and $\overline{\alpha}$ on $\mathcal{M}_{e_\mathcal{N}}^+$.

Corollary 2.3.5. *The $*$-automorphism $\overline{\alpha} : \langle \mathcal{M}, e_\mathcal{N} \rangle \to \langle \mathcal{M}, e_\mathcal{N} \rangle$ satisfies $\overline{\tau} \circ \overline{\alpha} = \overline{\alpha}$ i.e. $(\langle \mathcal{M}, e_\mathcal{N} \rangle, \overline{\tau}, \overline{\alpha})$ is a semifinite W^*-dynamical system.*

Proof. As remarked above, it is sufficient to show $\overline{\tau} \circ \overline{\alpha} = \overline{\tau}$ on $\mathcal{M}_{e_\mathcal{N}}^+$. We show something stronger by showing agreement on the dense subspace $\mathcal{M}_{e_\mathcal{N}}$. For all $a, b \in \mathcal{M}$,

$$\begin{aligned} &\overline{\tau} \circ \overline{\alpha}(a e_\mathcal{N} b) \\ &= \overline{\tau}(\alpha(a) e_\mathcal{N} \alpha(b)) && \text{using (2.3.3)} \\ &= \tau(\alpha(ab)) \\ &= \tau(ab) = \overline{\tau}(a e_\mathcal{N} b) && \text{using (1.6.1).} \end{aligned}$$

As $\overline{\tau}$ and $\overline{\alpha}$ are linear, $\overline{\tau} \circ \overline{\alpha} = \overline{\tau}$ on $\mathcal{M}_{e_\mathcal{N}}$. □

2.4 Representation of Semifinite W^*-dynamics

Let $(\mathcal{R}, \rho, \sigma)$ be a semifinite W^*-dynamical system. Just below, in a procedure similar to the one in §2.2, we can express, at least partially, the action of σ on $\mathcal{R}$ as a unitary U_σ acting on $\mathfrak{H}_\rho$.

We use the notation in §1.3.

For every $x \in \mathcal{K}_\rho$ ((1.3.1)) define

$$U_\sigma : \mathcal{K}_\rho/K_\rho \ni x + K_\rho \mapsto \sigma(x) + K_\rho \in \mathcal{K}_\rho/K_\rho.$$

We note that U_σ is well-defined, because using the fact that σ is a ρ-preserving $*$-automorphism on $\mathcal{R}$, we have $\rho(\sigma(x)^*\sigma(x)) = \rho(\sigma(x^*x)) = \rho(x^*x) < \infty$. This shows not only that $\sigma(x) \in \mathcal{K}_\rho$, but also that U_ρ is isometric on $\mathcal{K}_\rho/K_\rho$, that is, $\|x + K_\rho\|_\rho = \|\sigma(x) + K_\rho\|_\rho$ for all $x \in \mathcal{K}_\rho$. It follows that σ is bijective from $\mathcal{K}_\rho$ onto itself and hence $\{\sigma(x) + K_\rho : x \in K_\rho\} = \mathcal{K}_\rho/K_\rho$. Consequently, U_σ can be extended to a surjective isometric linear mapping from $\mathfrak{H}_\rho$ onto itself, which we still denote by U_σ. Thus, U_σ is a unitary operator on $\mathfrak{H}_\rho$ ([Kre78] Theorem 3.10-6 (h)).

Viewing $\mathcal{R}$ in its semicyclic representation $\pi_\rho(\mathcal{R})$ on $\mathfrak{H}_\rho$ allows us to express the action of σ on $\mathcal{R}$ in the following manner.

Proposition 2.4.1. *Define $\sigma_\rho : \pi_\rho(\mathcal{R}) \to \pi_\rho(\mathcal{R})$ by the prescription:*

$$\sigma_\rho(\pi_\rho(a)) := U_\sigma \pi_\rho(a) U_\sigma^* \qquad \textit{for all } a \in \mathcal{R}.$$

Then,

$$\sigma_\rho = \pi_\rho \circ \sigma \circ \pi_\rho^{-1}. \tag{2.4.1}$$

Proof. Note that for any $x \in \mathcal{R}$ and $y \in \mathcal{K}_\rho$,

$$\begin{aligned}
\sigma_\rho(\pi_\rho(x))(y + K_\rho) &= U_\sigma \pi_\rho(x) U_\rho^*(y + K_\rho) \\
&= U_\sigma \pi_\rho(x)(\sigma^{-1}(y) + K_\rho) \\
&= U_\sigma(x\sigma^{-1}(y) + K_\rho) \\
&= \sigma(x\sigma^{-1}(y)) + K_\rho \\
&= \sigma(x)\sigma(\sigma^{-1}(y)) + K_\rho \\
&= \sigma(x)y + K_\rho \\
&= \pi_\rho(\sigma(x))(y + K_\rho) \qquad\qquad \text{from (1.3.3)}
\end{aligned}$$

Thus, $\sigma_\rho(\pi_\rho(a)) = \pi_\rho(\sigma(a)) = \pi_\rho \circ \sigma \circ \pi_\rho^{-1}(\pi_\rho(a))$, for all $a \in \mathcal{R}$. □

Remark 2.4.2. Of interest to us, of course, is the special case where we consider the semifinite W^*-dynamical system, $(\langle \mathcal{M}, e_\mathcal{N} \rangle, \overline{\tau}, \overline{\alpha})$. However, the slightly more abstract approach clarifies the analogous structures and procedures that arise in comparison with finite W^*-dynamical systems. □

Chapter 3

Relative Weak Mixing

In this chapter, we prove a major lemma (Lemma 3.0.3) required to prove the main theorem of this dissertation. We state this essential result momentarily after the following definition:

Definition 3.0.1 (relative weak mixing)**.**
Let $(\mathcal{M}, \alpha, \tau)$ be a W^*-dynamical system and $\mathcal{N} \subseteq \mathcal{M}$ a von Neumann subalgebra such that $\alpha|_{\mathcal{N}}$ is a $*$-automorphism on $\mathcal{N}$ and $\mathbb{1}_{\mathcal{N}} = \mathbb{1}_{\mathcal{M}}$. Define $\mathcal{M} \cap \mathcal{N}^{\perp} := \{x \in \mathcal{M} : E_{\mathcal{N}}(x) = 0\}$. We say that α is **weakly mixing relative to $\boldsymbol{\mathcal{N}}$** if for any $a \in \mathcal{M} \cap \mathcal{N}^{\perp}$ we have

$$\frac{1}{n}\sum_{j=1}^{n} \left\| E_{\mathcal{N}}(a^*\alpha^j(a))\Omega \right\|_\tau^2 \to 0 \qquad \text{as } n \to \infty.$$

□

Remark 3.0.2. A remark concerning the use of the notation $\mathcal{M} \cap \mathcal{N}^{\perp}$. Recall, from §1.5, that the requirements that $\alpha|_{\mathcal{N}}$ is a $*$-automorphism on $\mathcal{N}$ and $\mathbb{1}_{\mathcal{N}} = \mathbb{1}_{\mathcal{M}}$ allow us to identify $[\mathcal{N}\Omega]$ with the Hilbert space obtained from the GNS construction on $(\mathcal{N}, \tau|_{\mathcal{N}})$. Thus, we observe that when $\xi \in [\mathcal{N}\Omega]^{\perp}$ is of the form $\xi = a\Omega$, where $a \in \mathcal{M}$, then

$$0 = e_{\mathcal{N}}(a\Omega) = E_{\mathcal{N}}(a)\Omega \iff E_{\mathcal{N}}(a) = 0 \iff a \in \mathcal{M} \cap \mathcal{N}^{\perp},$$

as Ω is separating for $\mathcal{M}$.

□

Lemma 3.0.3 (lack of weak mixing implies finite trace submodule ([AET11] Proposition 3.8))**.** *Let $(\mathcal{M}, \tau, \alpha)$ be a W^*-dynamical system and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$, such that $\alpha|_{\mathcal{N}}$ is a $*$-automorphism on $\mathcal{N}$ and $\mathbb{1}_{\mathcal{N}} = \mathbb{1}_{\mathcal{M}}$. If α is not weakly mixing relative to $\mathcal{N}$, then there is a U_α-invariant right-$\mathcal{N}$-submodule $V \subseteq [\mathcal{N}\Omega]^{\perp}$ such that P_V, the projection onto V, has finite lifted trace.* □

Roughly speaking, the proof of Lemma 3.0.3 can be outlined as follows. Working with the Hilbert space obtained from the semicylic representation and the extended dynamics (§2.4), we use von Neumann's ergodic theorem to obtain a vector v, say. This v will be associated with an operator X_v, which belongs to a noncommutative analogue of an L^2-space. Using a particular version of the spectral theorem allows us to express X_v as an "integral" of a collection of projections. It will be one of these projections, when represented as an operator on $\mathfrak{H}_\tau$, which will be selected to be our P_V.

3.1 Spectral Measures

Definition 3.1.1. Let Σ be a σ-algebra on a non-empty set $\mathcal{X}$. By a **spectral measure** on a Hilbert space $\mathfrak{H}$ we mean a mapping E from Σ to $\mathcal{P}(\mathfrak{H})$, the set of all projections in $B(\mathfrak{H})$ satisfying

(a) $E(\mathcal{X}) = I$ and

(b) Countable additivity: If (M_i) is a sequence of mutually disjoint elements from Σ, then $E(\cup_{i=1}^{\infty} M_i)$ is the strong operator limit of the sequence $(\sum_{i=1}^{k} E(M_i))$, that is, for all $x \in \mathfrak{H}$,

$$\lim_{k\to\infty} \sum_{i=1}^{k} E(M_i)x = E(\cup_{i=1}^{\infty} M_i)x.$$

□

Given a spectral measure E, as above, for every $x, y \in \mathfrak{H}$, let $E_{x,y}(\cdot) := \langle E(\cdot)x, y\rangle$. This mapping from Σ into $\mathbb{C}$ is a measure (see [Sch12] Lemma 4.4 and the discussion afterwards). In the special case of $x = y$, we shall write E_x instead of $E_{x,x}$. The measure E_x is positive and finite ([Sch12] Lemma 4.4).

3.2 Spectral Integrals and a Spectral Theorem for Unbounded Self-Adjoint Operators

We refer the reader to [Kre78] Chapter 10 for an introduction to unbounded linear operators.

Definition 3.2.1. Let $\mathcal{X}$ be a non-empty set, Σ a σ-algebra on $\mathcal{X}$, and E a spectral measure on a Hilbert space $\mathfrak{H}$. Denote by $\mathcal{S} = \mathcal{S}(\mathcal{X}, \Sigma, E)$ the set of all Σ-measurable functions $g : \mathcal{X} \to \mathbb{C} \cup \{\infty\}$ which are E-a.e. finite, that is, $E(\{t \in \mathcal{X} \mid g(t) = \infty\}) = 0$. □

Given a spectral measure E and a function $f \in \mathcal{S}$, we shall describe what we mean by a spectral integral $\mathbb{I}(f)$ (Theorem 3.2.5). A little later we will show how, in a special case, the process can be reversed; to each unbounded self-adjoint operator A we can obtain a unique spectral measure E_A, defined on the Borel sets of $\mathbb{R}$, such that A is associated with the identity function on $\mathbb{R}$.

In order to define the spectral integral $\mathbb{I}(f)$ for any $f \in \mathcal{S}$, we first need to describe $\mathbb{I}(f)$ in the case when f is bounded.

Definition 3.2.2. Let $\mathcal{X}$ be a non-empty set; Σ, a σ-algebra on $\mathcal{X}$; and $E : \Sigma \to \mathcal{P}(\mathfrak{H})$ a spectral measure on a Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathcal{X}, \Sigma)$, the space of all bounded Σ-measurable $\mathbb{C}$-valued functions equipped with the norm

$$\|f\|_{\mathcal{X}} = \sup\{|f(t)| \mid t \in \mathcal{X}\}.$$

□

When $f \in \mathcal{B}(\mathcal{X}, \Sigma)$, then $\mathbb{I}(f)$ may be described in terms of simple functions ([Sch12] §4.3.1- also see the top of [Sch12] p.78). The approach is similar to the development of the classical theory of Lebesgue integration. Recall that a simple function $c : \mathcal{X} \to \mathbb{C}$ is a finite linear combination of indicator functions. More precisely, $c = c(t)$ is defined as

$$c(t) := \sum_{i=1}^{n} r_i \chi_{K_i}(t),$$

where the r_i are complex numbers and the K_i belong to Σ and are pairwise disjoint. The integral of c in this case is $\mathbb{I}(c) = \sum_{i=1}^{n} r_i E(K_i)$. For more general bounded functions f, we approximate $\mathbb{I}(f)$, in the operator norm with a sequence of integrals of simple functions. We record this last result.

Proposition 3.2.3. *([Sch12] p.78) Let $(\mathcal{X}, \Sigma)$ be a measurable space and $E : \Sigma \to \mathcal{P}(\mathfrak{H})$ a spectral measure on a Hilbert space $\mathfrak{H}$. Let $f \in \mathcal{B}(\mathcal{X}, \Sigma)$. Then there is a sequence of simple functions (f_n), with each $f_n \in \mathcal{B}(\mathcal{X}, \Sigma)$ satisfying*

$$\|\mathbb{I}(f) - \mathbb{I}(f_n)\| \to 0 \qquad \textit{as } n \to \infty.$$

□

If $f \in \mathcal{B}(\mathcal{X}, \Sigma)$, we shall refer to $\mathbb{I}(f)$ as a **bounded spectral integral**, not only due to the bounded function from which it arises, but also from the fact that $\mathbb{I}(f)$ is a bounded linear operator ([Sch12] p.74).

The following concept will allow us to define the unbounded spectral integrals in the sequel, using the bounded spectral integrals above.

Definition 3.2.4. Let $\mathcal{F} \subseteq \mathcal{S}(\mathcal{X}, \Sigma, E)$. By a **bounding sequence for $\mathcal{F}$** we mean a sequence of sets $(M_n) \subseteq \Sigma$ such that

(i) For all $f \in \mathcal{F}$, f is bounded on each M_n.

(ii) For each $n \in \mathbb{N}$, $M_n \subseteq M_{n+1}$.

(iii) $E(\cup_{i=1}^{\infty} M_i) = I$.

□

Theorem 3.2.5. *(Part of [Sch12] Theorem 4.13)*
Let E be a spectral measure on a Hilbert space $\mathfrak{H}$ and $f \in \mathcal{S}(\mathcal{X}, \Sigma, E)$. Then there exists an operator $\mathbb{I}(f)$, defined by (ii) below, with domain

$$\mathcal{D}(\mathbb{I}(f)) := \left\{ x \in \mathfrak{H} \,\middle|\, \int_{\mathcal{X}} |f(t)|^2 \, \mathrm{d}E_x < \infty \right\}.$$

Let (M_n) be a bounding sequence for f. Then we have:

(i) A vector $x \in \mathfrak{H}$ is in $\mathcal{D}(\mathbb{I}(f))$ if and only if the sequence $(\mathbb{I}(f\chi_{M_n})x)$ converges in $\mathfrak{H}$, or equivalently, if $\sup_{n \in \mathbb{N}}\{\|\mathbb{I}(f\chi_{M_n})x\|\} < \infty$.

(ii) For $x \in \mathcal{D}(\mathbb{I}(f))$, the limit of the sequence $(\mathbb{I}(f\chi_{M_n})x)$ does not depend on the bounding sequence (M_n). There is a linear operator $\mathbb{I}(f)$ on $\mathcal{D}(\mathbb{I}(f))$ defined by $\mathbb{I}(f)x = \lim_{n\to\infty} \mathbb{I}(f\chi_{M_n})x$.

(iii) $E(M_n)\mathbb{I}(f) \subseteq \mathbb{I}(f)E(M_n) = \mathbb{I}(f\chi_{M_n})$.

□

In accordance with p. 92 of [Sch12], we shall also use the notation $\int_{\mathcal{X}} f \, \mathrm{d}E$ for $\mathbb{I}(f)$. We now consider another "sequential description" of the spectral integral.

Let $\mathfrak{B}(\mathbb{R})$ denote the collection of all Borel sets of $\mathbb{R}$ and consider $f \in \mathcal{B}(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ and $[a, b] \subseteq \mathbb{R}$. Furthermore, assume f is continuous on $\mathcal{J} := [a, b]$. We extend f so that it is continuous on $[a-1, b]$. By a **partition** $\mathcal{P}$ we mean a finite sequence of numbers $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ such that

$$a - 1 < \lambda_0 < a < \lambda_1 < \cdots < \lambda_n = b.$$

Put $|\mathcal{P}| := \max_k\{|\lambda_k - \lambda_{k-1}|\}$. Now choose, for each $k \in \{1, 2, \ldots, n\}$, a $\zeta_k \in [\lambda_{k-1}, \lambda_k]$. We define the Riemann sum:

$$S_E(f, \{\lambda_i\}_{i=1}^n, \{\zeta_i\}_{i=1}^n, \mathcal{P}) := \sum_{k=1}^{n} f(\zeta_k) E((\lambda_{k-1}, \lambda_k]). \tag{3.2.1}$$

If there is no confusion, we shall use the symbols $S_E(f, \mathcal{P})$ and $S_E(\mathcal{P})$ instead of $S_E(f, \{\lambda_i\}_{i=1}^n, \{\zeta_i\}_{i=1}^n, \mathcal{P})$.

Proposition 3.2.6 ([Sch12] Proposition 4.1 and [Sch12] p. 66)**.** *Consider* $\mathbb{R}$ *with the Borel* σ*-algebra* $\mathfrak{B}(\mathbb{R})$*, and let* $E : \mathfrak{B}(\mathbb{R}) \to \mathcal{P}(\mathfrak{H})$ *be a spectral measure on a Hilbert space* $\mathfrak{H}$*. Let* $f \in \mathcal{B}(\mathbb{R}, \mathfrak{B}(\mathbb{R})))$ *be continuous on a closed interval* $\mathcal{J} := [a, b] \subseteq \mathbb{R}$*, with* $a < b$*. Then,* $\mathbb{I}(f\chi_{\mathcal{J}})$ *has the following property: for every* $\epsilon > 0$*, there exists a* $\delta = \delta(\epsilon) > 0$ *such that for every partition* $\mathcal{P}$ *satisfying* $|\mathcal{P}| < \delta$*, we have*

$$\|\mathbb{I}(f\chi_{\mathcal{J}}) - S_E(f\chi_{\mathcal{J}}, \mathcal{P})\| \leq \epsilon.$$

□

In order to work with unbounded self-adjoint linear operators, we will need to rely on the following version of the spectral theorem:

Theorem 3.2.7 ([Sch12] Theorem 5.7)**.** *Let* A *be a self-adjoint operator on a Hilbert space* $\mathfrak{H}$*. Then there exists a unique spectral measure* $E_A : \mathfrak{B}(\mathbb{R}) \to \mathcal{P}(\mathfrak{H})$ *such that*

$$A = \int_{\mathbb{R}} \lambda \, \mathrm{d}E_A(\lambda) := \int_{\mathbb{R}} \mathrm{id} \, \mathrm{d}E_A, \tag{3.2.2}$$

where, $\mathrm{id} : \mathbb{R} \to \mathbb{R}$*, is the identity function* $x \mapsto x$*.*

We shall refer to the right-hand side of Equation 3.2.2 as the ***spectral integral form of A****. For all Borel sets* B *in* $\mathbb{R}$*, we shall refer to the projections* $E_A(B)$ *as the* ***spectral projections of A****.* □

Remark 3.2.8. As in the spectral theorem for bounded normal linear operators (detailed, for instance, in [Mur90] §2.5), we will be able to restrict the domain of integration, $\mathbb{R}$, to the spectrum of the self-adjoint operator $A = \int_{\mathbb{R}} f \, \mathrm{d}E_A$. This is accomplished by examining all Borel measurable sets $\mathcal{Y} \subseteq \mathbb{R}$, such that $E_A(\mathcal{Y}) = 0$. The sets $\mathcal{Y}$ can be thought of as the parts of $\mathbb{R}$ whose corresponding spectral projections $E_A(\mathcal{Y})$ do not contribute to the integral $\int_{\mathbb{R}} f \, \mathrm{d}E$. We now make this more precise using results in [Sch12]. We let $\mathcal{O}$ denote all the open sets O of $\mathfrak{B}(\mathbb{R})$ for which $E_A(O) = 0$. The support of E_A, denoted $\mathrm{supp}(E_A)$, is then defined as

$$\mathrm{supp}(E_A) := \mathcal{X} \backslash \left(\bigcup_{O \in \mathcal{O}} O \right).$$

Then, for any Borel subset $B \subseteq \mathrm{supp}(E_A)$, $E_A(B) \neq 0$. Furthermore, the support of E_A is equal to the spectrum of A ([Sch12] Proposition 5.10). In particular, if $B = \int_{\mathbb{R}} f \, \mathrm{d}E_B$ is a positive self-adjoint operator, the spectrum of B is contained in $[0, \infty)$ ([SZ79] Corollary 9.26) and therefore $E_B(\mathcal{Y}) = 0$ for all measurable sets $\mathcal{Y} \subseteq (-\infty, 0)$. To remind us of this distinction, we use the notation $B = \int_0^\infty f \, \mathrm{d}E_B$ for positive self-adjoint operators. □

Given a self-adjoint operator A, Theorems 3.2.7 and 3.2.5 allow us to define a **functional calculus**

$$\mathcal{S}(\mathbb{R}, \mathcal{B}(\mathbb{R}), E_A)) \ni g \mapsto g(A) := \mathbb{I}(g), \tag{3.2.3}$$

associating complex-valued functions in $\mathcal{S}(\mathbb{R}, \mathcal{B}(\mathbb{R}), E_A))$ with operators defined through the use of Theorem 3.2.5. We record some useful properties of the functional calculus.

Theorem 3.2.9 ([Sch12] Theorem 5.9)**.** *Let A be a self-adjoint operator defined in a Hilbert space $\mathfrak{H}$, with spectral integral $A = \int_{\mathbb{R}} \lambda \, \mathrm{d}E$, where $E = E_A$, obtained from Theorem 3.2.7. Let $f, g \in \mathcal{S}(\mathbb{R}, \mathcal{B}(\mathbb{R}), E_A)$; α, $\beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(f(A))$. Then we have:*

(a) $\langle f(A)x, y\rangle = \int f(\lambda) \, \mathrm{d}E_{x,y}$.

(b) $\|f(A)x\|^2 = \int |f(\lambda)|^2 \, \mathrm{d}E_x$. *In particular, if $f(t) = g(t)$ E -a.e. on $\mathbb{R}$, then $f(A) = g(A)$.*

(c) $f(A)$ is bounded if and only if $f \in L^\infty(\mathbb{R}, E)$. In this case, $\|f(A)\| = \|f\|_\infty$.

(d) $\bar{f}(A) = f(A)^*$; in particular, $f(A)$ is self-adjoint if f is real-valued E-a.e. on $\mathbb{R}$.

(e) $(\alpha f + \beta g)(A) = \overline{\alpha f(A) + \beta g(A)}$.

(f) $(fg)(A) = f(A)g(A)$.

(g) $p(A) = \sum_n \alpha_n A^n$ for any polynomial $p(t) = \sum_n \alpha_n t^n \in \mathbb{C}[t]$.

(h) $\chi_M(A) = E(M)$ for every Borel set M on $\mathbb{R}$.

(i) If $f(t) \neq 0$ E-a.e. on $\mathbb{R}$, then $f(A)$ is invertible, and $f(A)^{-1} = \left(\frac{1}{f}\right)(A)$.

(j) If $f(t) \geq 0$ E-a.e. on $\mathbb{R}$, then $f(A) \geq 0$.

□

We mention a connection between the spectral measures and von Neumann algebras using Theorem 3.2.9 (h). First, however, we need to describe operators affiliated to a von Neumann algebra.

Let S and T be linear operators on a Hilbert space $\mathfrak{H}$. Recall (for instance, from [Kre78] p.526) that the notation $S \subset T$ means that

$$\mathcal{D}(S) \subseteq \mathcal{D}(T) \qquad \text{and} \qquad S = T|_{\mathcal{D}(S)}.$$

Proposition 3.2.10 ([Wes90] Proposition 7.1)**.** *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. Let S be an unbounded linear operator on $\mathfrak{H}$. The following are equivalent:*

(a) For all $M' \in \mathcal{A}'$*,* $M'S \subset SM'$*;*

(b) For all unitaries $U' \in \mathcal{A}'$*,* $U'S \subset SU'$*;*

(c) For all unitaries $U' \in \mathcal{A}'$*,* $U'S = SU'$*;*

(d) For all unitaries $U' \in \mathcal{A}'$*,* $(U')^*SU'$*.*

□

Definition 3.2.11. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. An operator S on $\mathfrak{H}$ satisfying the equivalent conditions in the Proposition 3.2.10 above is said to be **affiliated to $\mathcal{A}$** and is denoted by $S\,\eta\,\mathcal{A}$. □

Proposition 3.2.12 (§9.9 in [SZ79])**.** *Suppose we have a positive self-adjoint operator* A *affiliated to a von Neumann algebra* $\mathcal{A}$ *and* f *is a Borel measurable complex-valued function defined on* $[0, \infty)$*, bounded on compact sets. Then* $f(A)$ *is affiliated to* $\mathcal{A}$*. In particular, because* $E_A(G) = \chi_G(A)$ *for every Borel set* $G \subseteq [0, \infty)$*, we have that every spectral projection* $E_A(G)$ *of* A *is contained in* $\mathcal{A}$*.* □

3.3 Measurable Operators and Non-Commutative L^2-Spaces

There is an operator-theoretic version of the classical L^2-space, obtained from any semifinite von Neumann algebra $\mathcal{R}$, acting on a Hilbert space $\mathfrak{H}$, with faithful normal semifinite tracial weight ρ.

The following is presented in [Ter81]. Let $\overline{\mathcal{R}}$ denote the set of all closed densely defined operators affiliated with $\mathcal{R}$ and let $\mathcal{P}(\mathcal{R})$ denote the set of all projections in $\mathcal{R}$, that is, the set of all elements $P \in \mathcal{R}$ such that $P^2 = P$ and $P^* = P$. Let $\widetilde{\mathcal{R}}$ denote the set of all **ρ-measurable operators** (or simply measurable operators, if ρ is understood) in $\overline{\mathcal{R}}$, that is,

$$\widetilde{\mathcal{R}} := \{T \in \overline{\mathcal{R}} \mid \forall \delta > 0\, \exists P \in \mathcal{P}(\mathcal{R}) : P\mathfrak{H} \subseteq \mathcal{D}(T) \text{ and } \rho(\mathbb{1}_{\mathcal{R}} - P) \leq \delta\}. \tag{3.3.1}$$

The set $\widetilde{\mathcal{R}}$ is a complete Hausdorff topological $*$-algebra with respect to the strong sum and strong product ([Ter81] Chapter 1 Theorem 28, [Ter81] Chapter 1 Proposition 24 part 2)). The topology on $\widetilde{\mathcal{R}}$ is given by a base of neighbourhoods of 0 ([Ter81] Chapter 1 Proposition 27); each neighbourhood is given by,

$$N(\epsilon, \delta) := \{A \in \widetilde{\mathcal{R}} \mid \exists P \in \mathcal{P}(\mathcal{R}) \ : \|AP\| \leq \epsilon \text{ and } \rho(\mathbb{1}_{\mathcal{R}} - P) \leq \delta)\} \text{ for all } \epsilon, \delta > 0.$$

We shall refer to this topology as the **measure topology on $\widetilde{\mathcal{R}}$**. The term **topology of convergence in measure** is also used. The space $\widetilde{\mathcal{R}}$ has a countable basis for the neighbourhoods at 0, for instance, $\{N(1/n, 1/m) \mid n, m \in \mathbb{N}\}$. Therefore, there is a translation invariant metric d on $\widetilde{\mathcal{R}}$ which is compatible with the topology given by the neighbourhoods $N(\epsilon, \delta)$ ([Rud73] Theorem 1.24).

If (R_n) is a sequence of ρ-measurable operators converging in the measure topology to some $R \in \widetilde{\mathcal{R}}$, we shall use the notation

$$R_n \xrightarrow{\text{measure}} R.$$

A completion procedure can be performed with $\mathfrak{H}$. For all $\epsilon, \delta > 0$, let

$$O(\epsilon, \delta) := \{x \in \mathfrak{H} \mid \exists P \in \mathcal{P}(\mathcal{R}) \ : \|Px\| \leq \epsilon \text{ and } \rho(\mathbb{1}_R - P) \leq \delta\}. \tag{3.3.2}$$

The completion of $\mathfrak{H}$ with respect to the above topology is denoted by $\widetilde{\mathfrak{H}}$ ([Nel74] p. 107; see also [Wes90] 8.31). We refer to the topology on $\widetilde{\mathfrak{H}}$ arising from (3.3.2) as the **measure topology on $\widetilde{\mathfrak{H}}$**, or alternatively as the **topology of convergence in measure on $\widetilde{\mathfrak{H}}$**.

Remark 3.3.1. In an alternative approach, [Nel74] uses [Ter81]'s base of neighbourhoods of 0, but restricted to $\mathcal{R}$. The completion of $\mathcal{R}$ with respect to [Nel74]'s neighbourhood base is topologically $*$-isomorphic to $\widetilde{\mathcal{R}}$ ([Wes90] Theorem 8.37).

□

For every positive self-adjoint operator A affiliated to $\mathcal{R}$, put ([Ter81] p. 23)

$$\rho(A) := \sup_{n \in \mathbb{N}} \rho \left(\int_{(0,n]} \lambda \, \mathrm{d}E_A \right), \tag{3.3.3}$$

where $A = \int_0^\infty \lambda \, \mathrm{d}E_A$ is represented as a spectral integral.

Define **the noncommutative L^2-space**,

$$L^2(\rho) := \{A \in \widetilde{\mathcal{R}} \mid \rho(A^*A) < \infty\}, \tag{3.3.4}$$

with norm $\|A\|_{2,\rho} = \sqrt{\rho(A^*A)}$.

Remark 3.3.2. For our purposes, we have focussed on the special case of defining $L^2(\rho)$, however one can define noncommutative L^p-spaces for $p \in [1, \infty)$. □

We have that $L^2(\rho)$ is a Banach space ([Tak03b] Chapter IX Theorem 2.13 (ii)). By the arguments appearing just before Theorem 5 of [Nel74] (see also the comment in [Ter81] p. 23 "L^p spaces with respect to a trace"), there is a natural continuous mapping from $L^2(\rho)$ into $\widetilde{\mathcal{R}}$. Thus, if any sequence (R_n) of elements of $L^2(\rho)$ converges to some $R \in L^2(\rho)$ we have

$$R_n \xrightarrow{measure} R. \tag{3.3.5}$$

We record this for later.

Proposition 3.3.3. *Suppose $(R_n) \subseteq L^2(\rho)$ converges to some element $R \in L^2(\rho)$ in the norm of $L^2(\rho)$. Then (R_n) converges to R in the measure topology on $\widetilde{\mathcal{R}}$.* □

The space $L^2(\rho)$ can be identified with $\mathfrak{H}_\rho$. More precisely,

Proposition 3.3.4. *There exists a linear isometric surjection $\gamma : L^2(\rho) \to \mathfrak{H}_\rho$, such that $\gamma|_{\mathcal{K}_\rho}$ is the quotient map $g_\rho : \mathcal{K}_\rho \to \mathcal{K}_\rho / K_\rho$ from §1.3. The Banach space $L^2(\tau)$ is a Hilbert space when endowed with an inner product given by*

$$\langle A, B \rangle := \langle \gamma(A), \gamma(B) \rangle, \tag{3.3.6}$$

for all $A, B \in L^2(\tau)$, (with γ then unitary). The inner product in (3.3.6) *is compatible with the norm on $L^2(\rho)$ in the sense that for all $A \in L^2(\rho)$,*

$$\langle A, A \rangle = \|A\|_{2,\rho}^2 . \tag{3.3.7}$$

Proof. Consider the map

$$g : \mathcal{K}_\rho \to \mathfrak{H}_\rho,$$

defined by $g(R) = R + K_\rho$, for all $R \in \mathcal{K}_\rho$. This is just the quotient map $\mathcal{K}_\rho \to \mathcal{K}_\rho / K_\rho$, from §1.3, instead now with codomain $\mathfrak{H}_\rho$. From Equation (3.3.4),

$$L^2(\rho) \cap \mathcal{R} = \{R \in \mathcal{R} \mid \rho(R^*R) < \infty\} \stackrel{\text{def}}{=} \mathcal{K}_\rho. \tag{3.3.8}$$

Now, $L^2(\rho) \cap \mathcal{R}$ is dense in $L^2(\rho)$, by [Tak03b] Chapter IX Theorem 2.13 (ii), and g is an isometry, because

$$\|A\|_{2,\rho}^2 = \rho(A^*A) = \|A + K_\rho\|_\rho^2 , \text{ for all } A \in \mathcal{K}_\rho.$$

Therefore we may apply the bounded linear extension theorem ([Kre78] Theorem 2.7-11) to extend g to a bounded linear operator $\gamma : L^2(\rho) \to \mathfrak{H}_\rho$. In particular,

$$\gamma|_{\mathcal{K}_\rho} = g. \tag{3.3.9}$$

We shall show that γ is an isometry and is surjective onto $\mathfrak{H}_\rho$ (this is sufficient to show that γ is a unitary, by [Kre78] Theorem 3.10-6 (f), once we have established that $L^2(\tau)$ is a Hilbert space, below). The map γ is an isometry. Indeed, for any $T \in L^2(\rho)$, again using the density of $L^2(\rho) \cap \mathcal{R}$ in $L^2(\rho)$, there is a sequence (T_n) contained in $\mathcal{K}_\rho$ such that $\|T - T_n\|_{2,\rho} \to 0$, as $n \to \infty$. Thus, using Equation (3.3.9),

$$\begin{aligned} |\, \|\gamma(T)\|_\rho - \|T\|_{2,\rho} \,| &\leq |\, \|\gamma(T)\|_\rho - \|T_n\|_{2,\rho} \,| + |\, \|T_n\|_{2,\rho} - \|T\|_{2,\rho} \,| \\ &= |\, \|\gamma(T)\|_\rho - \|\gamma(T_n)\|_\rho \,| + |\, \|T_n\|_{2,\rho} - \|T\|_{2,\rho} \,| \\ &\leq \|\gamma(T) - \gamma(T_n)\|_\rho + \|T_n - T\|_{2,\rho} \\ &\leq \|\gamma\| \, \|T - T_n\|_{2,\rho} + \|T_n - T\|_{2,\rho} \,. \end{aligned}$$

So, for every $\epsilon > 0$ we can find n large enough, so that $|\, \|\gamma(T)\|_\rho - \|T\|_{2,\rho} \,| \leq \|\gamma(T) - \gamma(T_n)\|_2 + \|T_n - T\|_{2,\rho} < \epsilon$, thus, $\|\gamma(T)\|_\rho = \|T\|_{2,\rho}$.

To show surjectivity of γ, it is sufficient to show that $\gamma(L^2(\rho))$ is closed in $\mathfrak{H}_\rho$. To see why this is sufficient, note that we have the chain of inclusions,

$$g(L^2(\rho) \cap \mathcal{R})) \subseteq \gamma(L^2(\rho)) \subseteq \mathfrak{H}_\rho. \tag{3.3.10}$$

Therefore, because $g(L^2(\rho) \cap \mathcal{R})) \equiv \mathcal{K}_\rho / K_\rho$ is dense in $\mathfrak{H}_\rho$, so too is $\gamma(L^2(\rho))$.

Let us therefore show that $\gamma(L^2(\rho))$ is closed in $\mathfrak{H}_\rho$. Suppose that (t_n) is a sequence of elements in $\gamma(L^2(\rho))$ converging to some $t \in \mathfrak{H}_\rho$ in the norm of $\mathfrak{H}_\rho$. Let (T_n) denote the sequence of elements in $\mathcal{K}_\rho$ such that $\gamma(T_n) \equiv g(T_n) = t_n$. As γ is an isometry, (T_n) is a Cauchy sequence and therefore has a limit, say $T \in L^2(\rho)$. Thus, in the norm on $\mathfrak{H}_\rho$,

$$\begin{aligned} \|\gamma(T) - t\| &\leq \|\gamma(T) - \gamma(T_n)\| + \|\gamma(T_n) - t\| \\ &\leq \|T - T_n\|_{2,\rho} + \|\gamma(T_n) - t\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

the last inequality arise from the fact that γ is a linear isometry. Hence, $t \in \gamma(L^2(\rho))$ and therefore, $\gamma(L^2(\rho)))$ is closed in $\mathfrak{H}_\rho$. One can easily verify that (3.3.6) is an inner product for $L^2(\rho)$. Now, using the fact that γ is an isometry, (3.3.7) follows:

$$\langle A, A \rangle = \langle \gamma(A), \gamma(A) \rangle = \|\gamma(A)\|^2 = \|A\|_{2,\rho}^2 \,.$$

□

Remark 3.3.5. An alternative approach to establishing the inner product in (3.3.6) is via the trace on the non-commutative L^1-space (see [Tak03b] Chapter IX Lemma 2.12). □

We now show that the $L^2(\rho)$-limit of positive operators in $L^2(\rho) \cap \mathcal{R}$ is a positive operator. Note that we use $\mathcal{R}$ in its semicylic representation, discussed in §1.3.

Proposition 3.3.6. *Let $\mathcal{R}$ be a semifinite von Neumann algebra with normal faithful semifinite trace ρ. Represent $\mathcal{R}$ as operators on $\mathfrak{H}_\rho$ using the semicylic representation from §1.3. Let $(R_n) \subseteq L^2(\rho) \cap \mathcal{R}$ be a sequence of positive operators converging in the L^2-norm to some operator $R \in L^2(\rho)$. Furthermore, suppose that there is a $c > 0$ such that for all $n \in \mathbb{N}$, $\|R_n\| \leq c$. Then, R is a positive self-adjoint linear operator.*

Proof. It will be sufficient to show that for all $x \in \mathcal{D}(R)$,

$$\langle Rx, x\rangle \geq 0. \tag{3.3.11}$$

For if (3.3.11) is true, we have that R is symmetric and therefore, as R is a measurable operator, it will be self-adjoint ([Ter81] Chapter 1 Corollary 15 part 2)). As the sequence (R_n) converges to R in $L^2(\rho)$, (R_n) converges to R in $\widetilde{\mathcal{R}}$ (Proposition 3.3.3). The continuity of the map $\widetilde{\mathcal{R}} \times \widetilde{\mathfrak{H}_\rho} \to \widetilde{\mathfrak{H}_\rho}$ ([Nel74] Theorem 1), ensures that $R_n x \to Rx$ with respect to the measure topology on $\widetilde{\mathfrak{H}_\rho}$ for every $x \in \mathcal{D}(R)$. Hence, for every $\epsilon, \delta > 0$ there exists an $m \in \mathbb{N}$, such that for all natural numbers $n > m$ there is a projection $P \in \mathcal{R}$ such that

$$\|P(R_n x - Rx)\| \leq \epsilon \text{ and } \rho(\mathbb{1} - P) \leq \delta,$$

where $\mathbb{1}$ denotes the multiplicative unit in $\mathcal{R}$.

Thus, for an arbitrary, but fixed $x \in \mathcal{D}(R)$, there is a subsequence (R_{m_l}) of (R_n) and a projection $P_l \in \mathcal{R}$ such that

$$\|P_l(R_{m_l}x - Rx)\| \leq \frac{1}{l} \text{ and } \rho(\mathbb{1} - P_l) \leq \frac{1}{l}. \tag{3.3.12}$$

In order to simplify the notation, denote the subsequence (R_{m_l}) by (T_n).

We will now establish the convergence of $(\langle T_n x, y\rangle)$ to $\langle Rx, y\rangle$ for every $y \in \mathfrak{H}_\rho$ ((3.3.16) below). This will require establishing the convergence results shown in (3.3.14) and (3.3.15) below. We first prove why (3.3.14) is true. We first examine elements in the dense subspace $\mathcal{K}_\rho/K_\rho$ of $\mathfrak{H}_\rho$. Consider any $C \in L^2(\rho) \cap \mathcal{R}$. Then, using $\gamma : L^2(\rho) \to \mathfrak{H}_\rho$ obtained from Proposition 3.3.4, and the tracial property of ρ we expand and rearrange the expression $\|(\mathbb{1} - P_n)\gamma(C))\|^2$:

$$\begin{aligned}
&\langle(\mathbb{1} - P_n)\gamma(C)), (\mathbb{1} - P_n)\gamma(C))\rangle \\
=&\langle\gamma((\mathbb{1} - P_n)C), \gamma((\mathbb{1} - P_n)C)\rangle = \rho([(\mathbb{1} - P_n)C]^*(\mathbb{1} - P_n)C) \\
=&\rho((\mathbb{1} - P_n)C[(\mathbb{1} - P_n)C]^*) \\
=&\rho((\mathbb{1} - P_n)CC^*(\mathbb{1} - P_n)) \leq \|C\|^2 \rho(\mathbb{1} - P_n) \qquad \text{from [KR97b] p. 487.}
\end{aligned}$$

Taking into consideration the second inequality in (3.3.12), we have therefore shown for every $c \in \mathcal{K}_\rho/K_\rho$ that

$$\|(\mathbb{1} - P_n)c\| \xrightarrow[\infty]{n} 0. \tag{3.3.13}$$

To handle the more general case, let $y \in \mathfrak{H}_\rho$ be arbitrary and choose a $z \in \mathcal{K}_\rho/K_\rho$ such that $\|z - y\| < \frac{\epsilon}{3}$. Then,

$$\begin{aligned}
\|P_n y - y\| &\leq \|P_n y - P_n z\| + \|P_n z - z\| + \|z - y\| \\
&\leq \|y - z\| + \|P_n z - z\| + \|z - y\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3},
\end{aligned} \tag{3.3.14}$$

for any n large enough so that $\|P_n y - y\| < \epsilon$.

Also, using the first inequality in (3.3.12), note that

$$|\langle P_n y, T_n x - Rx\rangle| \leq \|y\| \, \|P_n(T_n x - Rx)\| \xrightarrow[\infty]{n} 0. \tag{3.3.15}$$

Now,

$$\begin{aligned} |\langle T_n x - Rx, y\rangle| &= |\langle T_n x - Rx, y - P_n y + P_n y\rangle| \\ &\leq |\langle T_n x - Rx, y - P_n y\rangle| + |\langle T_n x - Rx, P_n y\rangle| \\ &\leq |\langle T_n x, y - P_n y\rangle| + |\langle Rx, y - P_n y\rangle| + |\langle T_n x - Rx, P_n y\rangle| \\ &\leq \|y - P_n y\| \, \|T_n\| \, \|x\| + \\ &\qquad \|y - P_n y\| \, \|Rx\| + |\langle P_n y, T_n x - Rx\rangle| \xrightarrow[\infty]{n} 0, \end{aligned} \tag{3.3.16}$$

using our assumption that $(\|R_n\|)$ is bounded and Formulas (3.3.14) and (3.3.15). So we have shown that for every $x \in \mathcal{D}(R)$ and for every $y \in \mathfrak{H}_\rho$, $\langle T_n x, y\rangle \to \langle Rx, y\rangle$, as $n \to \infty$. However, for all $x \in \mathcal{D}(R)$, $\langle T_n x, x\rangle \geq 0$; so $\langle Rx, x\rangle \geq 0$. □

3.4 Extracting a Projection with Finite Trace from a Self-Adjoint Operator

The spectral theorem presented in §3.2 will be essential in the work that follows. The proposition below relies on the following intuitive idea: If positive operators A, B, C satisfy $A \leq B \leq C$, where $\leq$ is the usual ordering on positive operators, then the tracial weight ρ ought to respect such an ordering: $\rho(A) \leq \rho(B) \leq \rho(C)$.

Note that the proof of Proposition 3.4.1 does not rely on the representation of $\mathcal{R}$.

Proposition 3.4.1. *Let $\mathcal{R}$ be a semifinite von Neumann algebra with normal faithful semifinite tracial weight ρ, acting on a Hilbert space $\mathfrak{H}$. Let $R \in L^2(\rho)$ be a non-zero positive self-adjoint operator. Apply Theorem 3.2.7 to R to obtain a unique spectral measure $E = E_R$ such that $R = \mathbb{I}(f) = \int_0^\infty \lambda \, \mathrm{d}E_R$, where $f : \mathbb{R} \ni \lambda \mapsto \lambda \in \mathbb{R}$ is the identity on $\mathbb{R}$. There is a non-zero spectral projection P_s of R, belonging to $\mathcal{R}$, with $\rho(P_s) < \infty$.*

Proof. For each $n \in \mathbb{N}$, let $M_n = [0, n]$. Then (M_n) is a bounding sequence for f using Definition 3.2.4 and the end of Remark 3.2.8. We are able to see from Theorem 3.2.5 that $\mathbb{I}(f)$ is approximable pointwise by bounded linear operators $\mathbb{I}(f\chi_{M_n})$. Such operators $\mathbb{I}(f\chi_{M_n})$ are, in turn, using Proposition 3.2.6, approximable in the operator norm by Riemann sums of the form $S(f\chi_{M_n}) = \sum_{k=1}^{t} f\chi_{M_n}(\xi_k)E((\lambda_{k-1}, \lambda_k])$. As $E((\lambda_{k-1}, \lambda_k]) \in \mathcal{R}$, from Proposition 3.2.12, every $\mathbb{I}(f\chi_{M_n})$ is norm-approximable by linear combinations of projections in $\mathcal{R}$ and therefore ([Zhu93] Theorem 20.3) belongs to $\mathcal{R}$. Thus, $\mathbb{I}(f\chi_{M_n})^*\mathbb{I}(f\chi_{M_n}) = \mathbb{I}(\overline{f}\chi_{M_n} f\chi_{M_n}) = \mathbb{I}(f^2\chi_{M_n}) \in \mathcal{R}$. Using parts (d) and (f) of Theorem 3.2.9, we have

$$R^*R = \mathbb{I}(f)^*\mathbb{I}(f) = f(R)^*f(R) = \bar{f}(R)f(R) = (\bar{f}f)(R) = f^2(R) = \mathbb{I}(f^2).$$

This, together with our assumption that $R \in L^2(\rho)$, implies that $\rho(\mathbb{I}(f^2)) < \infty$. Additionally, we also have that $\mathbb{I}(f^2) - \mathbb{I}(f^2\chi_{M_n}) \equiv f^2(R) - (f^2\chi_{M_n})(R) \in \widetilde{\mathcal{R}}$ (as $\widetilde{\mathcal{R}}$ is a $*$-algebra) and therefore we may view this last expression as the closed linear operator $\overline{f^2(R) - (f^2\chi_{M_n})(R)}$. Thus, part (e) of Theorem 3.2.9 implies that

$$f^2(R) - (f^2\chi_{M_n})(R) = (f^2 - f^2\chi_{M_n})(R).$$

Due to Theorem 3.2.9 part (j), we have that both $(f^2 - f^2\chi_{M_n})(R)$ and $f^2\chi_{M_n}(R)$ are positive operators, and therefore it follows that

$$0 \leq f^2\chi_{M_n}(R) \leq f^2(R).$$

Thus, we may express $\rho(f^2(R))$ as $\rho(f^2(R)) = \rho(f^2(R) - (f^2\chi_{M_n})(R)) + \rho((f^2\chi_{M_n})(R))$ and therefore obtain,

$$\rho((f^2\chi_{M_n})(R)) \leq \rho(f^2(R)).$$

Hence, $\rho(f^2\chi_{M_n}(R)) < \infty$.

The function $f^2\chi_{M_n}$ may be approximated below by a sequence of simple functions, (s_k) say, by taking the interval M_n, dividing it up into a number of subintervals of equal length $\frac{n}{k}$ and choosing the left most endpoint of each subinterval to form the coefficients of the indicator functions:

$$s_k = 0\chi_{\left[0, n\frac{1}{k}\right]} + \sum_{j=1}^{k-1} \left(n\frac{j}{k}\right)^2 \chi_{N_j}.$$

Here, $N_j := \left(n\frac{j}{k}, n\frac{(j+1)}{k}\right]$. Note that the N_j depend not only on the indices j, but also on the numbers k. As $\left(n\frac{j}{k}\right)^2 \chi_{N_j} \leq s_k \leq f^2\chi_{M_n}$, we have, using Theorem 3.2.9 parts (e) and (h),

$$\mathbb{I}\left(\left(n\frac{j}{k}\right)^2 \chi_{N_j}\right) \leq \mathbb{I}(s_k) \leq \mathbb{I}(f^2\chi_{M_n}).$$

Now fix a natural number $k \geq 2$ and then fix any $j \in \mathbb{N}$ so that $1 < j \leq k$. Using the Archimedean property of the natural numbers, there exists an $n \in \mathbb{N}$ such that $1 < \left(n\frac{j}{k}\right)^2$, so that

$$\mathbb{I}(\chi_{N_j}) \leq \mathbb{I}\left(\left(n\frac{j}{k}\right)^2 \chi_{N_j}\right).$$

Hence, $\rho(\mathbb{I}(\chi_{N_j})) \leq \rho(\mathbb{I}(s_k)) \leq \rho(\mathbb{I}(f^2\chi_{M_n})) < \infty$. As j was chosen to be strictly greater than 1, we have $N_j \subsetneq (\frac{nj}{k}, \infty)$, with $\frac{nj}{k} > 0$. We put $P_s := E(N_j)$. Finally, by Proposition 3.2.12, P_s belongs to $\mathcal{R}$. □

3.5 Invariance of our Projection P_s

We extend the dynamics to the measurable operators obtaining a result similar to Proposition 2.4.1. Notice in Proposition 3.5.1, below, we identify $\mathcal{R}$ with its semicyclic representation $\pi_\rho(\mathcal{R})$. As a consequence, $\sigma = \sigma_\rho$, where $\sigma_\rho(T) = U_\rho T U_\rho^*$ for all $T \in \mathcal{R}$, using Proposition 2.4.1. Hence, in this case, the W^*-dynamical system $(\mathcal{R}, \sigma, \rho)$ is identified with the system $(\pi_\rho(\mathcal{R}), \sigma_\rho, \rho)$.

Proposition 3.5.1. *Let $(\mathcal{R}, \rho, \sigma)$ be a semifinite W^*-dynamical system with faithful normal semifinite tracial weight ρ. Represent $\mathcal{R}$, using its semicyclic representation, as operators on $\mathfrak{H}_\rho$. Let $U_\rho : \mathfrak{H}_\rho \to \mathfrak{H}_\rho$ be the unitary satisfying $U_\rho(T + K_\rho) = \sigma(T) + K_\rho$, for all $T \in \mathcal{K}_\rho$ from §2.4. View $(\mathcal{R}, \sigma, \rho)$ as the W^*-dynamical system $(\pi_\rho(\mathcal{R}), \sigma_\rho, \rho)$. Then the map $\widetilde{\sigma} : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}$ given by $\widetilde{\sigma}(X) = U_\rho X U_\rho^*$ is well-defined and continuous in the topology of convergence in measure on $\widetilde{\mathcal{R}}$.*

Proof. Denote the unit in $\mathcal{R}$ by $\mathbb{1}$. Let $X \in \widetilde{\mathcal{R}}$. Then, using Equation (3.3.1), for any $\epsilon > 0$, there is a $Q \in \mathcal{P}(\mathcal{R})$ such that

$$Q\mathfrak{H}_\rho \subseteq \mathcal{D}(X) \quad \text{and} \quad \rho(\mathbb{1} - Q) \leq \epsilon. \tag{3.5.1}$$

We show that $U_\rho X U_\rho^* \in \widetilde{\mathcal{R}}$. First, observe that $U_\rho Q U_\rho^*$ is a projection. Second, $QU_\rho^*\mathfrak{H}_\rho = Q\mathfrak{H}_\rho \subseteq \mathcal{D}(X)$, so for $x \in U_\rho Q U_\rho^* \mathfrak{H}_\rho$ we have $U_\rho^* x \in \mathcal{D}(X)$, hence $x \in \mathcal{D}(XU_\rho^*) = \mathcal{D}(U_\rho X U_\rho^*)$, that is, $U_\rho Q U_\rho^* \mathfrak{H}_\rho \subseteq \mathcal{D}(U_\rho X U_\rho^*)$. Lastly, using Proposition 2.4.1,

$$\rho(\mathbb{1} - U_\rho Q U_\rho^*) = \rho(U_\rho(\mathbb{1} - Q)U_\rho^*) = \rho(\sigma(\mathbb{1} - Q)) = \rho(\mathbb{1} - Q) \leq \epsilon.$$

We continue proving the continuity result in the statement of the proposition. From §3.3, $\widetilde{\mathcal{R}}$ is metrizable. So we are able to show continuity by using sequential arguments. Thus, the density of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ ([Ter81] Chapter 1 Theorem 28) allows us to obtain a sequence (X_n) in $\mathcal{R}$ such that

$$X_n \to X \text{ in the measure topology on } \tilde{\mathcal{R}}.$$

Fix $\epsilon, \delta > 0$. There exists an $N \in \mathbb{N}$ such that

$$n > N \implies X_n - X \in N(\epsilon, \delta/2).$$

Thus, there exist $V, W \in \mathcal{P}(\mathcal{R})$ such that

$$V\mathfrak{H}_\rho \subseteq \mathcal{D}(X_n - X) \text{ and } \rho(\mathbb{1} - V) < \frac{\delta}{2},$$

as $X_n - X \in \widetilde{\mathcal{R}}$, and

$$\|(X_n - X)W\| \leq \epsilon \text{ and } \rho(I - W) \leq \frac{\delta}{2},$$

as $X_n - X \in N(\epsilon, \delta/2)$. If we put $P = V \wedge W$, then

(a) $P\mathfrak{H}_\rho \subseteq \mathcal{D}(X_n - X)$,
because $P\mathfrak{H}_\rho \subseteq V\mathfrak{H}_\rho$;

(b) $\rho(\mathbb{1} - P) \leq \delta$,
because

$$\begin{aligned}\rho(\mathbb{1} - (V \wedge W)) &= \rho((\mathbb{1} - V) \vee (\mathbb{1} - W)) \\ &\leq \rho(\mathbb{1} - V) + \rho(\mathbb{1} - W) \qquad \text{p.7 [Ter81]} \\ &\leq \delta/2 + \delta/2; \text{ and}\end{aligned}$$

(c) $\|(X_n - X)P\| \leq \|(X_n - X)W\| \leq \epsilon$,
because $(X_n - X)(V \wedge W)\mathfrak{H}_\rho \subseteq (X_n - X)W\mathfrak{H}_\rho$.

Given ϵ, δ, N, n and P as above, we wish to show, using (a), (b) and (c), that

(i) $U_\rho P U_\rho^* \mathfrak{H}_\rho \subseteq \mathcal{D}(U_\rho(X_n - X)U_\rho^*)$,

(ii) $\left\|U_\rho(X_n - X)U_\rho^* U_\rho P U_\rho^*\right\| \leq \epsilon$, and

(iii) $\rho(\mathbb{1} - U_\rho P U_\rho^*) \leq \delta$.

Parts (i) and (iii) use the same arguments appearing below (3.5.1), which we give for the sake of completeness.
For (i), we just observe that if $x \in \mathcal{D}(X_n - X)$ then,

$$U_\rho x \in \mathcal{D}((X_n - X)U_\rho^*) = \mathcal{D}(U_\rho(X_n - X)U_\rho^*),$$

and the inclusion $P\mathfrak{H}_\rho \subseteq \mathcal{D}(X_n - X)$, from (a) above, gives us (i).
For (ii), because $(X_n - X)P$ is a bounded linear operator (from (b) above),

$$\begin{aligned}\left\|U_\rho(X_n - X)U_\rho^* U_\rho P U_\rho^*\right\| &= \left\|U_\rho(X_n - X)P U_\rho^*\right\| \\ &\leq \|U_\rho\| \, \|(X_n - X)P\| \, \left\|U_\rho^*\right\| \leq \epsilon.\end{aligned}$$

For (iii):

$$\begin{aligned}\rho(\mathbb{1} - U_\rho P U_\rho^*) &= \rho(U_\rho(\mathbb{1} - P)U_\rho^*) \\ &= \rho(\sigma_\rho(\mathbb{1} - P)) \qquad \text{using Proposition 2.4.1} \\ &= \rho(\mathbb{1} - P) \leq \delta \qquad \text{because } \rho \text{ is } \sigma_\rho\text{-preserving.}\end{aligned}$$

Hence, for every $\epsilon, \delta > 0$, there exists an $N \in \mathbb{N}$ such that $X_n - X \in N(\epsilon, \delta/2)$ for all $n > N$, implies that $U_\rho(X_n - X)U_\rho^* \in N(\epsilon, \delta)$ for all $n > N$. Hence, we have shown that the map $\tilde{\sigma}$ is continuous.

□

Corollary 3.5.2. *Let $(\mathcal{R}, \rho, \sigma)$ be a semifinite W^*-dynamical system with faithful normal semifinite tracial weight ρ. Let $U_\rho : \mathfrak{H}_\rho \to \mathfrak{H}_\rho$ be the unitary satisfying $U_\rho(T+K_\rho) = \sigma(T) + K_\rho$, for all $T \in \mathcal{K}_\rho$ from §2.4. Represent $\mathcal{R}$ as operators on $\mathfrak{H}_\rho$ using the semicyclic representation and view $\widetilde{\mathcal{R}}$ as operators on $\mathfrak{H}_\rho$. Let $\gamma : L^2(\rho) \to \mathfrak{H}_\rho$ be the Hilbert space isomorphism obtained in Proposition 3.3.4. Suppose that $X \in L^2(\rho)$ satisfies $U_\rho x = x$, where $x := \gamma(X) \in \mathfrak{H}_\rho$. Then, using the map $\widetilde{\sigma}$ in Proposition 3.5.1, we have $\widetilde{\sigma}(X) = X$.*

Proof. In $\mathfrak{H}_\rho$, approximate x with a sequence of elements $(\gamma(X_n)) \subseteq \mathcal{K}_\rho/K_\rho$, where $X_n \in \mathcal{K}_\rho$. Then the continuity of U_ρ gives us, in $\mathfrak{H}_\rho$,

$$U_\rho x = \lim_n U_\rho \gamma(X_n) = \lim_n (\sigma(X_n) + K_\rho) = \lim_n \gamma(\sigma(X_n)),$$

because γ is the quotient map $\mathcal{K}_\rho \to \mathcal{K}_\rho/K_\rho$ when restricted to $\mathcal{K}_\rho$. Thus, from Proposition 3.3.4, we have in $L^2(\rho)$,

$$\sigma(X_n) \longrightarrow \gamma^{-1}(U_\rho \gamma(X).$$

Hence,

$$\sigma(X_n) \xrightarrow{measure} \gamma^{-1}(U_\rho \gamma(X)),$$

using Proposition 3.3.3. On the other hand,

$$\begin{aligned} \gamma(X_n) \to \gamma(X) &\Rightarrow X_n \to X \text{ in } L^2(\rho) && \text{Proposition 3.3.4} \\ &\Rightarrow X_n \xrightarrow{measure} X && \text{Proposition 3.3.3} \\ &\Rightarrow \widetilde{\sigma}(X_n) \to \widetilde{\sigma}(X) && \text{Proposition 3.5.1.} \end{aligned}$$

As remarked before the statement of Proposition 3.5.1, due to the identification of $\mathcal{R}$ with $\pi_\rho(\mathcal{R})$, we have $\sigma = \sigma_\rho$, with $\sigma_\rho(T) = U_\rho T U_\rho^*$ for all $T \in \mathcal{R}$. In particular, $\sigma(X_n) = U_\rho X_n U_\rho^*$ for all $n \in \mathbb{N}$. Hence, $(\sigma(X_n))$ converges to both $\widetilde{\sigma}(X)$ and to $\gamma^{-1}(U_\rho\gamma(X)) = \gamma^{-1}(U_\rho x) = \gamma^{-1}(x) = \gamma^{-1}(\gamma(X)) = X$, using our assumption that $U_\rho x = x$. As $\widetilde{\mathcal{R}}$ is a Hausdorff space under the topology of convergence in measure ([Ter81] Chapter 1 Theorem 28), we have the required equality. □

We will use the Spectral Theorem to "pass" the relationship "$U_\rho X U_\rho^* = X$" to the projection P_s obtained in Proposition 3.4.1. There $\mathcal{R}$ was not expressed in its semicylic representation, but we do use the form in the following proposition.

Proposition 3.5.3. *Let $(\mathcal{R}, \rho, \sigma)$ be a semifinite W^*-dynamical system with faithful normal semifinite tracial weight ρ. Let $\mathcal{R}$ be represented as operators on $\mathfrak{H}_\rho$ using the semicyclic representation. Let $R \in L^2(\rho)$ be a non-zero positive self-adjoint operator. Apply Theorem 3.2.7 to R to obtain a unique spectral measure $E = E_R$ such that $R = \mathbb{I}(f) = \int_0^\infty \lambda \, dE_R$, where $f : \mathbb{R} \ni \lambda \mapsto \lambda \in \mathbb{R}$ is the identity on $\mathbb{R}$. Under the additional assumption that the operator R satisfies $U_\rho R U_\rho^* = R$, the projection P_s, obtained in Proposition 3.4.1, satisfies $U_\rho P_s U_\rho^* = P_s$.*

Proof. Using Definition 3.1.1, it is clear that F, defined by $F(B) = U_\rho E(B)U_\rho^*$ for all Borel sets B, is a spectral measure. Thus, it makes sense to consider the integral $\mathbb{K}(f) := \int_0^\infty f(\lambda)\,\mathrm{d}F$. Similar to $\mathbb{I}(f)$ in Proposition 3.4.1, we first approximate $\mathbb{K}(f)$ point-wise on $\mathcal{D}(\mathbb{K}(f))$ using Theorem 3.2.5:

$$\mathbb{K}(f)x = \lim_n \mathbb{K}(f\chi_{M_n})x, \text{ for all } x \in \mathcal{D}(\mathbb{K}(f)),$$

where $M_n := [0, n]$, for all $n \in \mathbb{N}$. Just as we approximated $\mathbb{I}(f\chi_{M_n})$, for each $n \in \mathcal{N}$, in Proposition 3.4.1, with Riemann sums of the form

$$S_E(f\chi_{M_n}) = \sum_{k=1}^{t} f\chi_{M_n}(\xi_k)(E((\lambda_{k-1}, \lambda_k]),$$

we approximate each $\mathbb{K}(f\chi_{M_n})$ with Riemann sums

$$S_F(\{\xi_k\}_{k=1}^t, \{\lambda_k\}_{k=1}^t) = S_F(f\chi_{M_n}) = \sum_{k=1}^{t} f\chi_{M_n}(\xi_k)F((\lambda_{k-1}, \lambda_k]).$$

The overall goal is to show that $F = E$. We do this by showing that $\mathbb{K}(f) = \mathbb{I}(f)$. As a first step, we now show that the $\mathbb{K}(f\chi_{M_n})$ can be replaced by the $U_\rho\mathbb{I}(f\chi_{M_n})U_\rho^*$, thereby approximating both $\mathbb{K}(f)$ and $U_\rho\mathbb{I}(f)U_\rho^*$ point-wise.

More precisely, we wish to show that for every $n \in \mathbb{N}$, we can make the right most side of the following inequality as small as we wish, by making the partitions $\mathcal{Z}$ as fine as we wish:

$$\begin{aligned} &\left\|\mathbb{K}(f\chi_{M_n}) - U_\rho\mathbb{I}(f\chi_{M_n})U_\rho^*\right\| \leq \\ &\left\|\mathbb{K}(f\chi_{M_n}) - U_\rho[S_E(f\chi_{M_n}, \mathcal{Z})]U_\rho^*\right\| + \left\|U_\rho S_E(f\chi_{M_n}, \mathcal{Z}) - \mathbb{I}(f\chi_{M_n})U_\rho^*\right\|. \end{aligned} \tag{3.5.2}$$

Observe that for any Riemann sum $S_E(f\chi_{M_n})$ with partition $\mathcal{Z} = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$,

$$\begin{aligned} U_\rho S_E(f\chi_{M_n})U_\rho^* &= U_\rho \left(\sum_{k=1}^{t} f\chi_{M_n} E((\lambda_{k-1}, \lambda_k])\right) U_\rho^* \\ &= \sum_{k=1}^{t} f\chi_{M_n} U_\rho E((\lambda_{k-1}, \lambda_k])U_\rho^* = S_F(f\chi_{M_n}). \end{aligned}$$

Thus, continuing with (3.5.2),

$$\begin{aligned} &\left\|\mathbb{K}(f\chi_{M_n}) - U_\rho\mathbb{I}(f\chi_{M_n})U_\rho^*\right\| \\ &\leq \|\mathbb{K}(f\chi_{M_n}) - S_F(f\chi_{M_n}, \mathcal{Z})\| + \|U_\rho\| \, \|S_E(f\chi_{M_n}, \mathcal{Z}) - \mathbb{I}(f\chi_{M_n})\| \left\|U_\rho^*\right\|, \end{aligned}$$

and applying Proposition 3.2.6 to both $\mathbb{K}(f\chi_{M_n})$ and $\mathbb{I}(f\chi_{M_n})$, we can make the right hand side of (3.5.2) as small as we wish, as claimed. Therefore, $\mathbb{K}(f\chi_{M_n}) = U_\rho\mathbb{I}(f\chi_{M_n})U_\rho^*$, for all $n \in \mathbb{N}$.

So, $\mathbb{K}(f)x = \lim_n U_\rho\mathbb{I}(f\chi_{M_n})U_\rho^* x$, for all $x \in \mathcal{D}(\mathbb{K}(f))$.
We now show that $\mathcal{D}(\mathbb{K}(f)) = \mathcal{D}(\mathbb{I}(f))$. From Theorem 3.2.5, we have,

$$\mathcal{D}(\mathbb{I}(f)) = \left\{ x \in \mathfrak{H}_\rho \mid \int_0^\infty f(t)^2 \, \mathrm{d}E_x < \infty \right\} \text{ and}$$

$$\mathcal{D}(\mathbb{K}(f)) = \left\{ x \in \mathfrak{H}_\rho \mid \int_0^\infty f(t)^2 \, \mathrm{d}F_x < \infty \right\}.$$

With the above in mind,

$$\begin{aligned} x \in \mathcal{D}(\mathbb{K}(f)) &\iff \int_0^\infty f(t)^2 \, \mathrm{d}F_x < \infty \iff \int_0^\infty f(t)^2 \, \mathrm{d}E_{U_\rho^* x} < \infty \\ &\iff U_\rho^* x \in \mathcal{D}(\mathbb{I}(f)) \iff x \in U_\rho \mathcal{D}(\mathbb{I}(f)). \end{aligned}$$

It follows that,

$$\begin{aligned} &\mathcal{D}(\mathbb{K}(f)) = U_\rho \mathcal{D}(\mathbb{I}(f)) = \mathcal{D}(\mathbb{I}(f)U_\rho^*) \\ =&\mathcal{D}(U_\rho \mathbb{I}(f) U_\rho^*) = \mathcal{D}(U_\rho R U_\rho^*) \\ =&\mathcal{D}(R) \qquad \text{(from our assumption that } U_\rho R U_\rho^* = R) \\ =&\mathcal{D}(\mathbb{I}(f)). \end{aligned}$$

Thus, for all $x \in \mathcal{D}(\mathbb{K}(f)) \equiv \mathcal{D}(\mathbb{I}(f))$,

$$\mathbb{K}(f)x = \lim_n U_\rho \mathbb{I}(f\chi_{M_n})U_\rho^* x = U_\rho \lim_n \mathbb{I}(f\chi_{M_n})U_\rho^* x = U\mathbb{I}(f)U_\rho^* x = \mathbb{I}(f)x.$$

From the uniqueness of the spectral measure (Theorem 3.2.7) it follows that $F = E$. In particular, because P_s is a spectral projection, $UP_sU^* = P_s$. □

3.6 Proof of Lemma 3.0.3

The following result will be used in the beginning of Proposition 3.6.2's proof.

Theorem 3.6.1 (part of [Tak03b] Chapter IX Theorem 2.13 (ii))**.** *Let $\mathcal{R}$ be a semifinite von Neumann algebra with a normal faithful semifinite tracial weight ρ. Then the space $L^2(\rho)$ is invariant under multiplication from $\mathcal{R}$ from both the left and right. Furthermore, for all $x \in L^2(\rho)$ and for all $a \in \mathcal{R}$,*

(3.6.1) $$\|xa\|_{2,\rho} \leq \|a\| \, \|x\|_{2,\rho} \qquad \|ax\|_{2,\rho} \leq \|a\| \, \|x\|_{2,\rho}.$$

□

Proposition 3.6.2. *Let $\mathcal{R}$ be a semifinite von Neumann algebra with normal faithful semifinite tracial weight ρ, represented on $\mathfrak{H}_\rho$ using the semicyclic representation. Let $R \in L^2(\rho)$ be a non-zero positive self-adjoint operator. Let $(R_n) \subseteq L^2(\rho)$ converge to R in the norm on $L^2(\rho)$. Apply Theorem 3.2.7 to R to obtain a unique spectral measure $E = E_R$ such that $R = \mathbb{I}(f) = \int_0^\infty \lambda \, \mathrm{d}E_R$, where $f : \mathbb{R} \ni \lambda \mapsto \lambda \in \mathbb{R}$ is the identity on $\mathbb{R}$. Suppose further that we have a projection $e \in \mathcal{R}$ satisfying $R_n e = 0$ for all $n \in \mathbb{N}$. Then the projection P_s obtained in Proposition 3.4.1 satisfies $P_s e = 0$.*

Proof. For each $n \in \mathbb{N}$, let $M_n := [0, n]$. The overall plan is to show the following chain of implications:

$$R_n e = 0 \implies Re = 0 \implies \mathbb{I}(f\chi_{M_n})e = 0 \implies E((0,\infty))e = 0 \implies P_s e = 0.$$

Both $R_n e$ and Re belong $L^2(\rho)$, using Theorem 3.6.1. Thus, $R_n e - Re \in L^2(\rho)$. Using (3.6.1),

$$\|R_n e - Re\|_{2,\rho} = \|(R_n - R)e\|_{2,\rho} \leq \|e\| \, \|R_n - R\|_{2,\rho} \xrightarrow[\infty]{n} 0.$$

Thus, $Re = 0$.

As, $\mathcal{D}(E(M_n)\mathbb{I}(f)) = \mathcal{D}(\mathbb{I}(f))$, for every $n \in \mathbb{N}$, we have, using part (iii) of Theorem 3.2.5,

$$E(M_n)\mathbb{I}(f)z = \mathbb{I}(f\chi_{M_n})z \qquad \text{for all } z \in \mathcal{D}(\mathbb{I}(f)). \tag{3.6.2}$$

Notice now that,

$$x \in \mathcal{D}(\mathbb{I}(f)e) \implies ex \in \mathcal{D}(\mathbb{I}(f)).$$

So, as a special case of (3.6.2),

$$E(M_n)Rex = \mathbb{I}(f\chi_{M_n})ex \qquad \text{for all } x \in \mathcal{D}(\mathbb{I}(f)e).$$

From this, $\mathbb{I}(f\chi_{M_n})e = 0$, since $Re = 0$. Thus, employing property (a) of Theorem 3.2.9, for all $n \in \mathbb{N}$ and for all $x \in \mathfrak{H}_\rho$,

$$0 = \langle \mathbb{I}(f\chi_{M_n})ex, ex \rangle = \int_{\mathbb{R}} f\chi_{M_n}(t) \, \mathrm{d}E_{ex}.$$

Since, $f\chi_{M_n}(t) > 0$ on $(0, n]$ for each $n \in \mathbb{N}$, it follows that $E_{ex}((0,n]) := \langle E((0,n])ex, ex \rangle = 0$ for all $x \in \mathfrak{H}_\rho$, and therefore $E_{ex}((0,\infty)) = 0$.

Indeed, if $E_{ex}((0,n]) > 0$ for some $n \in \mathbb{N}$, then, for every natural number $k > 1$

$$E_{ex}\left(\left[\frac{1}{k}, n\right]\right) > 0,$$

because $\lim_k E_{ex}\left(\left[\frac{1}{k}, n\right]\right) = E_{ex}((0,n])$ (using [Sch12] Lemma 4.5 with $N_k = [1/k, n]$). However,

$$\int_{\mathbb{R}} f\chi_{M_n}(t) \, \mathrm{d}E_{ex} \geq f\left(\frac{1}{k}\right) E_{ex}\left(\left[\frac{1}{k}, n\right]\right) > 0,$$

because, $f(1/k) = 1/k > 0$, contradicting the assumption that $\int_{\mathbb{R}} f\chi_{M_n}\,\mathrm{d}E_{ex} = 0$.

Thus, for all $x \in \mathfrak{H}_\rho$, we have, $0 = \langle E((0,\infty))ex, ex\rangle = \langle eE((0,\infty))ex, x\rangle$. Thus, $0 = eE((0,\infty))e = (E((0,\infty))e)^*E((0,\infty))e$. Therefore,

$$\|E((0,\infty))e\|^2 = \|(E((0,\infty))e)^*E((0,\infty))e\|^2 = 0,$$

and thus, we have $E((0,\infty))e = 0$. Since, $P_s \leq E((0,\infty))$,

$$P_s e = P_s E((0,\infty))e = 0.$$

□

To obtain the limit b_1 in Lemma 3.0.3 below, we need to use von Neumann's mean ergodic theorem which we state for the reader's convenience.

Theorem 3.6.3 (von Neumann's mean ergodic theorem ([Pet89] Chapter 2 Theorem 1.2))**.** *Suppose $\mathfrak{H}$ is a Hilbert space and suppose that $U : \mathfrak{H} \to \mathfrak{H}$ is a contractive linear operator, that is, U is a bounded linear operator satisfying*

$$\forall x \in \mathfrak{H} \;\; \|Ux\| \leq \|x\|\,.$$

Let $\mathfrak{F} := \{x \in \mathfrak{H} : Ux = x\}$, the fixed point space of U. Let Q be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{F}$. Then, for every $x \in \mathfrak{H}$,

$$\frac{1}{n}\sum_{j=0}^{n} U^j x \to Qx$$

in $\mathfrak{H}$, as $n \to \infty$. □

As an aside, we note that if U is as above and belongs to a von Neumann algebra $\mathcal{A}$, then the projection Q, being a strong operator limit of elements in $\mathcal{A}$, also belongs to $\mathcal{A}$.

We are ready to give a proof of Lemma 3.0.3.

Proof of Lemma 3.0.3. Suppose that there is an element $a \in \mathcal{M} \cap \mathcal{N}^\perp$ such that

$$\frac{1}{n}\sum_{j=1}^{n} \left\|E_{\mathcal{N}}(a^*\alpha^j(a))\Omega\right\|_\tau^2 \nrightarrow 0 \qquad \text{as } n \to \infty.$$

Define $b := ae_{\mathcal{N}}a^* \in \langle \mathcal{M}, e_{\mathcal{N}}\rangle$. Note that b is self-adjoint; hence $\bar{\alpha}^j(b)$ is self-adjoint. As $\overline{\tau}(b^*b) = \overline{\tau}(ae_{\mathcal{N}}a^*ae_{\mathcal{N}}a^*) = \overline{\tau}(aE_{\mathcal{N}}(a^*a)e_{\mathcal{N}}a^*) = \tau(aE_{\mathcal{N}}(a^*a)a^*) < \infty$ (using Lemma 1.5.3) and $\overline{\tau}(\bar{\alpha}^j(b)^*\bar{\alpha}^j(b)) = \overline{\tau}(\bar{\alpha}^j(b^*b)) = \overline{\tau}(b^*b)$ (using Corollary 2.3.5), the operators b and $\bar{\alpha}^j(b)$, for each $j \in \mathbb{N}$, belong to $\mathcal{K}_{\overline{\tau}}$. Thus, according to the paragraph appearing before (1.3.2) (with $\overline{\tau}$ replacing ρ), $\overline{\tau}(\bar{\alpha}^j(b)^*b)$ is well-defined.

Let $\gamma : L^2(\overline{\tau}) \to \mathfrak{H}_{\overline{\tau}}$ be the Hilbert space isomorphism from Proposition 3.3.4.

Observe, using the tracial property of $\overline{\tau}$ and the equalities $e_{\mathcal{N}} m e_{\mathcal{N}} = E_{\mathcal{N}}(m) e_{\mathcal{N}} = e_{\mathcal{N}} E_{\mathcal{N}}(m)$ (from Lemma 1.5.3), that for all $m \in \mathcal{M}$,

$$\begin{aligned}
\langle U^j_{\bar{\alpha}}(\gamma(b)), \gamma(b)\rangle_{\overline{\tau}} = \langle \gamma(\bar{\alpha}^j(b)), \gamma(b)\rangle_{\overline{\tau}} &= \overline{\tau}(b(U^j_\alpha b U^{-j}_\alpha)) \\
&= \overline{\tau}(a e_{\mathcal{N}} a^* U^j_\alpha (a e_{\mathcal{N}} a^*) U^{-j}_\alpha) \\
&= \overline{\tau}(a e_{\mathcal{N}} a^* \alpha^j(a) e_{\mathcal{N}} \alpha^j(a)^*) && \text{from (2.3.5)} \\
&= \overline{\tau}(e^2_{\mathcal{N}} E_{\mathcal{N}}(a^* \alpha^j(a)) \alpha^j(a)^* a) \\
&= \overline{\tau}(e_{\mathcal{N}} E_{\mathcal{N}}(a^* \alpha^j(a)) \alpha^j(a)^* a e_{\mathcal{N}}) \\
&= \overline{\tau}(E_{\mathcal{N}}(a^* \alpha^j(a)) e_{\mathcal{N}} \alpha^j(a)^* a e_{\mathcal{N}}) \\
&= \overline{\tau}(E_{\mathcal{N}}(a^* \alpha^j(a)) e_{\mathcal{N}} E_{\mathcal{N}}(a^* \alpha^j(a))^*) \\
&= \left\| E_{\mathcal{N}}(a^* \alpha^j(a)) \Omega \right\|^2_\tau && \text{Equation (1.6.1).}
\end{aligned}$$

Thus, for any $n \in \mathbb{N}$,

$$\frac{1}{n} \sum_{j=1}^{n} \left\| E_{\mathcal{N}}(a^* \alpha^j(a)) \Omega \right\|^2_\tau = \langle n^{-1} \sum_{j=1}^{n} U^j_{\bar{\alpha}}(\gamma(b)), \gamma(b)\rangle_{\overline{\tau}} \xrightarrow[\infty]{n} \langle b_1, \gamma(b)\rangle_{\overline{\tau}} \neq 0,$$

where $b_1 = Q(\gamma(b)) \in L^2(\overline{\tau})$ and Q is the orthogonal projection onto the fixed point space of $U_{\bar{\alpha}}$, obtained from von Neumann's mean ergodic theorem.

As $b = (e_{\mathcal{N}} a^*)^*(e_{\mathcal{N}} a^*)$ is positive, $(1/n) \sum_{j=1}^{n} \bar{\alpha}^j(b)$ is positive for every $n \in \mathbb{N}$. Recall from Proposition 3.3.4 that the Hilbert space isomorphism $\gamma : L^2(\overline{\tau}) \to \mathfrak{H}_{\overline{\tau}}$ when restricted to $\mathcal{K}_{\overline{\tau}}$ is the quotient map $g : \mathcal{K}_{\overline{\tau}} \to \mathcal{K}_{\overline{\tau}}/K_{\overline{\tau}}$. Thus, because $\gamma((1/n) \sum_{j=1}^{n} \bar{\alpha}^j(b)) = n^{-1} \sum_{j=1}^{n} U^j_{\bar{\alpha}}(g(b)) \to b_1$, in $\mathfrak{H}_{\overline{\tau}}$, we have

$$\left(\frac{1}{n}\right) \sum_{j=1}^{n} \bar{\alpha}^j\,(b) = \gamma^{-1}\left(\gamma((1/n) \sum_{j=1}^{n} \bar{\alpha}^j(b))\right) \to \gamma^{-1}(b_1),$$

in $L^2(\overline{\tau})$. Furthermore,

$$\begin{aligned}
\left\| \frac{1}{n} \sum_{j=1}^{n} \bar{\alpha}^j(b) \right\| &\leq \frac{1}{n} \sum_{j=1}^{n} \left\| \bar{\alpha}^j \right\| \|b\| \\
&\leq \frac{1}{n} \sum_{j=1}^{n} \|b\| = \|b\| < \infty.
\end{aligned}$$

So $B_1 := \gamma^{-1}(b_1)$ is a possibly unbounded positive self-adjoint linear operator in $\mathfrak{H}_{\overline{\tau}}$ by Proposition 3.3.6. Applying Proposition 3.4.1, with $R = B_1$, we obtain a non-zero projection P_s of finite lifted trace. Though P_s is an operator on $\mathfrak{H}_{\bar{\tau}}$, we shall later identify the operator with our required operator P_V on $\mathfrak{H}_\tau$.

We now wish to apply Proposition 3.6.2. We note that for all $t \in \mathcal{N}$, $0 = \langle a\Omega, t\Omega\rangle_\tau = \tau(t^*a) = \tau(a^{**}t^*) = \langle t^*\Omega, a^*\Omega\rangle_\tau$, therefore $a^*\Omega \in \mathcal{M}\Omega \cap [\mathcal{N}\Omega]^\perp$. Thus,

$$E_{\mathcal{N}}(a^*)\Omega = 0. \tag{3.6.3}$$

Using Equations (2.2.5) and (3.6.3),

$$e_{\mathcal{N}}\alpha^j(a^*)\Omega = e_{\mathcal{N}}U_\alpha^j(a^*\Omega) = U_\alpha^j e_{\mathcal{N}}(a^*\Omega) = U_\alpha^j E_{\mathcal{N}}(a^*)\Omega = 0.$$

Thus, for all $c \in \mathcal{N}$, using Proposition 1.6.2 (c)

$$\bar{\alpha}^j(b)Jc^*J\Omega = Jc^*J\bar{\alpha}^j(b)\Omega = Jc^*J\alpha^j(a)e_{\mathcal{N}}\alpha^j(a^*)\Omega = 0.$$

By linearity, $\frac{1}{n}\sum_{j=1}^n \bar{\alpha}^j(b)Jc^*J\Omega = 0$. Hence, as $[J\mathcal{N}J\Omega] = [\mathcal{N}\Omega]$ (because $Jt^*J\Omega = Jt^*\Omega = t\Omega$ for every $t \in \mathcal{N}$), we have that $\left(\frac{1}{n}\sum_{j=1}^n \bar{\alpha}^j(b)\right) e_{\mathcal{N}} = 0$. Now let $\bar{\pi} : \langle \mathcal{M}, e_{\mathcal{N}}\rangle \to B(\mathfrak{H}_{\bar{\tau}})$ denote the semicyclic representation of $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$. Then we have that

$$\bar{\pi}\left(\frac{1}{n}\sum_{j=1}^n \bar{\alpha}^j(b)\right)\bar{\pi}(e_{\mathcal{N}}) = \bar{\pi}\left(\left(\frac{1}{n}\sum_{j=1}^n \bar{\alpha}^j(b)\right)e_{\mathcal{N}}\right) = 0.$$

Thus, we have verified the "$R_n e = 0$" part in the statement of Proposition 3.6.2 with the roles of R_n and e played by $\bar{\pi}(\frac{1}{n}\sum_{j=1}^n \bar{\alpha}^j(b))$ and $\bar{\pi}(e_{\mathcal{N}})$, respectively.

Hence,

$$P_s\bar{\pi}(e_{\mathcal{N}}) = 0. \tag{3.6.4}$$

Next, due to the equality $U_{\bar{\alpha}}b_1 = b_1$, obtained using von Neumann's ergodic theorem, by Corollary 3.5.2, we have $U_{\bar{\alpha}}B_1U_{\bar{\alpha}}^* = B_1$. Thus,

$$U_{\bar{\alpha}}P_sU_{\bar{\alpha}}^* = P_s, \tag{3.6.5}$$

from Proposition 3.5.3. Additionally, P_s belongs to $\bar{\pi}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$ (Proposition 3.4.1). We therefore put $P_V := \bar{\pi}^{-1}(P_s) \in \langle \mathcal{M}, e_{\mathcal{N}}\rangle$, where V is the range of P_V in $\mathfrak{H}_\tau$. So Equation (3.6.5) becomes,

$$\begin{aligned}
& \bar{\pi}\circ\bar{\alpha}\circ\bar{\pi}^{-1}(\bar{\pi}(P_V)) = \bar{\pi}(P_V) && \text{from Proposition 2.4.1}\\
\implies\ & \bar{\pi}\bar{\alpha}(P_V) = \bar{\pi}(P_V) && \\
\implies\ & \bar{\alpha}(P_V) = P_V && \text{due to the faithfulness of } \bar{\pi}\\
\implies\ & U_\alpha P_V U_\alpha^* = P_V && \text{from Equation (2.2.7).}
\end{aligned}$$

Thus, P_V is U_α-invariant.

Using (3.6.4),

$$\bar{\pi}^{-1}(P_s\bar{\pi}(e_{\mathcal{N}})) = 0 \implies \bar{\pi}^{-1}(P_s)\bar{\pi}^{-1}(\bar{\pi}(e_{\mathcal{N}})) = P_V e_{\mathcal{N}} = 0.$$

Thus, $V \perp [\mathcal{N}\Omega]$.

Lastly, Proposition 1.7.1 implies that V is a right-$\mathcal{N}$-submodule. □

Chapter 4

Finite Rank Approximation

In this chapter we study a finite rank approximation to the U_α-invariant right-$\mathcal{N}$-submodule V obtained in Chapter 3. We first present the following definition before stating the main result of this chapter, Lemma 4.0.2.

Definition 4.0.1 (finite-rank modules)**.** (Definition 3.5 [AET11]) Suppose that $\mathcal{A}$ is a von Neumann algebra and $\mathfrak{H}$ is an $\mathcal{A}$-$\mathcal{A}$-module. A left- (respectively right-) $\mathcal{A}$-submodule W of $\mathfrak{H}$ has **finite rank** if there are some $\xi_1, \xi_2, \ldots \xi_r \in W$ such that $W = \overline{\sum_{i=1}^{r} \mathcal{A}\xi_i}$ (respectively, $W = \overline{\sum_{i=1}^{r} \xi_i \mathcal{A}}$). We say, in either case, that $\xi_1, \xi_2, \ldots \xi_r$ **spans** W. The numerical value of its **rank** is the least $r \geq 1$ for which this is possible. □

Lemma 4.0.2. *([AET11] Lemma 4.1) Suppose that $(\mathcal{M}, \tau, \alpha)$ is a finite von Neumann dynamical system, where the GNS space $\mathfrak{H}_\tau$ is separable, $\mathcal{N} \subseteq \mathcal{M} \cap \mathcal{M}'$ is an α-invariant von Neumann subalgebra and $V \subseteq \mathfrak{H}_\tau$ is a U_α-invariant right-$\mathcal{N}$-submodule of finite lifted trace. Then, for any $\epsilon > 0$ there is a further U_α-invariant right-$\mathcal{N}$-submodule $V_1 \subseteq V$ such that*

- $\overline{\tau}(P_V - P_{V_1}) < \epsilon$;
- *V_1 has finite rank, say at most $r \geq 1$;*
- *there is a set of mutually orthogonal vectors $\{\xi_1, \xi_2, \ldots, \xi_r\}$ such that $V_1 = \overline{\sum_{i=1}^{r} \xi_i \mathcal{N}}$; and*
- *there exists a unitary matrix of contractive operators $U = (u_{ji})_{1 \leq i,j \leq r} \in \mathcal{U}_{r \times r}(\mathcal{N})$ such that*

$$U_\alpha(\xi_i) = \sum_{j=1}^{r} \xi_j u_{ji} \text{ for all } i = 1, 2, \ldots, r. \tag{4.0.1}$$

We call such a V_1 a ***rank r approximation*** *to V. Furthermore, for every $x \in V$, there exists a sequence (x_n) converging to x with x_n belonging to a rank r_n approximation to V, where $r_n \leq n$ for every $n = 1, 2, 3, \ldots$.* □

We note that our assumptions about $\mathcal{N}$, in the Lemma above, have been strengthened from the previous chapters. Namely, that we require that $\mathcal{N}$ lies in the centre of $\mathcal{M}$ in order to ensure that P_V can be identified as a decomposable operator (Definition 4.1.6).

We also note that we added to the statement and deviated slightly from the proof of Lemma 4.0.2 as it appears in [AET11]. In particular, we have found it necessary to add the final part discussing the approximation of vectors in V in order to prove Proposition 5.2.2 and we have introduced simpler arguments to obtain the u_{ji}.

In §§4.1-4.4 we work under more general conditions than what is described in Lemma 4.0.2 in that we deal with a state instead of a trace.

Sections 4.1-4.3 contain the background on direct integral theory, though we begin proving a small part of Lemma 4.0.2 in §4.2. We focus on the proof from §4.4 onwards.

4.1 Direct Integrals of Hilbert Spaces, Decomposable Operators and Decomposable von Neumann Algebras

The concept of a direct integral of Hilbert spaces extends the idea of the direct sum of Hilbert spaces. The definition we present here is from [KR97b] Chapter 14.

Definition 4.1.1. Let (X, Σ, μ) be a measure space. Suppose further that X is a σ-compact locally compact Borel measure space (by σ-compact, we mean that X is a countable union of compact sets from Σ) with μ the completion of a Borel measure on X. Let $\{\mathfrak{H}_p\}$ be a family of separable Hilbert spaces (referred to as **fibres**) indexed by $p \in X$. A separable Hilbert space $\mathfrak{H}_\oplus$ is called the **direct integral of $\{\mathfrak{H}_p\}$ over (X, μ)** denoted $\mathfrak{H}_\oplus = \int_X^\oplus \mathfrak{H}_p \, \mathrm{d}\mu(p)$ if for every $x \in \mathfrak{H}$ there is a map $p \mapsto x(p)$ such that

1. $x(p) \in \mathfrak{H}_p$ for each p.
2. $p \mapsto \langle x(p), y(p) \rangle$ is μ-integrable, for every $x, y \in \mathfrak{H}_\oplus$ and
$$\langle x, y \rangle = \int_X \langle x(p), y(p) \rangle \, \mathrm{d}\mu(p).$$
3. For every $p \in X$, if $u_p \in \mathfrak{H}_p$ and $p \mapsto \langle u_p, y(p) \rangle$ is μ-integrable for every $y \in \mathfrak{H}_\oplus$, then there is a $u \in \mathfrak{H}_\oplus$ such that $u(p) = u_p$ for almost every p.

The map $p \mapsto x(p)$ is referred to as a **measurable cross section of Hilbert spaces**. We shall refer to (X, μ) as the **domain of integration** for $\mathfrak{H}_\oplus$. □

Remark 4.1.2. When considering a subspace of $\mathfrak{H}_\oplus$ which is itself to be viewed as a direct integral (see Proposition 4.6.1), we weaken condition 1 in Definition 4.1.1 to almost all p, rather than all p. □

Remark 4.1.3. Alternative terminology exists for some of the objects just introduced. The map $p \mapsto x(p)$ can also be called a decomposition of x. When the set of Hilbert space fibres $\{\mathfrak{H}_p\}$ is understood, $p \mapsto x(p)$ is just referred to as a "measurable section". □

Remark 4.1.4. Let $x, y \in \mathfrak{H}_\oplus$ and $c \in \mathbb{C}$. Then the measurable cross section $p \mapsto z(p)$ corresponding to $ax + y$ will agree with $p \mapsto ax(p) + y(p)$ for almost all p. Conversely, if two measurable cross sections $p \mapsto x(p)$ and $p \mapsto y(p)$ are equal almost everywhere, then $x = y$. Thus, we will sometimes not distinguish between the vector x and the its corresponding mapping $p \mapsto x(p)$. For details, see [KR97b] Remark 14.1.2. □

Remark 4.1.5. For convenience, we shall identify Hilbert space fibres $\mathfrak{H}_p$ of the same Hilbert dimension $\dim(\mathfrak{H}_p)$ (with the convention that the infinite dimensional Hilbert space has dimension "∞"). This identification can be done rigorously- see [Nie80]. Moreover, we remark that for all $n = 1, 2, 3, \ldots, \infty$ the set $W_n = \{p \in X \mid \dim(\mathfrak{H}_p) = n\}$ is measurable ([KR97b] Remark 14.1.5). The assumption of separability of each Hilbert space fibre ensures that X is a partition of the W_n's. □

The broad initial idea of proving Lemma 4.0.2 is to represent all the mathematical objects that we have been working with, such as the basic construction, on a direct integral of Hilbert spaces.

Having described a Hilbert space divided into Hilbert space fibres, we would like to know which linear operators on $\mathfrak{H}_\oplus$ can be represented "fibre-wise".

Definition 4.1.6. Let $\mathfrak{H}_\oplus$ be a direct integral of Hilbert spaces $\{\mathfrak{H}_p\}$ over a measure space (X, Σ, μ). An operator T on $\mathfrak{H}_\oplus$ is called **decomposable** (with respect to this direct integral decomposition) if

(a) there is a map $p \mapsto T(p)$ defined on X such that $T(p) \in B(\mathfrak{H}_p)$ for every $p \in X$;

(b) for every $x \in \mathfrak{H}_\oplus$, $T(p)x(p) = (Tx)(p)$ for almost all p.

We refer to the mapping $p \mapsto T(p)$ as a **measurable section (of operators)**. Denote the identity operator on $\mathfrak{H}_p$ by I_p. We call T **diagonalizable** if $T(p) = f(p)I_p$ for some $f \in L^\infty(\mu)$. □

Remark 4.1.7. Suppose T is a diagonalisable operator. We use the symbol M_f to denote such an operator T. If the diagonalisable operator T is a projection, then it can be shown that $T = M_{\chi_R}$ for some measurable set $R \subseteq X$ ([KR97b] Remark 14.1.7, [Knu11] Corollary 2.19). For a measurable set X_0, $M_{\chi_{X_0}}$ is referred to as the **diagonalisable projection corresponding to X_0.** □

Theorem 4.1.8 ([KR97b] Theorem 14.1.10)**.** *If $\mathfrak{H}_\oplus$ is the direct integral of $\{\mathfrak{H}_p\}$, then the set of all decomposable operators is a von Neumann algebra and its commutant coincides with the von Neumann algebra of all diagonalisable operators. So, by the double commutant theorem ([KR97b] Theorem 5.3.1), the von Neumann algebra of decomposable operators is the commutant of the von Neumann algebra of diagonalisable operators.* □

Definition 4.1.9. Let $\mathfrak{H}_\oplus$ be a direct integral of $\{\mathfrak{H}_p\}$ over (X, μ) and let $\mathcal{A}$ be a C^*-algebra. A representation ϕ of $\mathcal{A}$ on $\mathfrak{H}_\oplus$ is called **decomposable** with **decomposition** $p \mapsto \phi_p$ if

(a) $\phi(\mathcal{A})$ consists of decomposable operators;

(b) there are representations $\phi_p : \mathcal{A} \to B(\mathfrak{H}_p)$ such that $\phi(a)(p) = \phi_p(a)$ for almost all $p \in X$.

□

The following definition is taken from [KR97b] Definition 14.1.14, but we have added some notation and terminology for later use.

Definition 4.1.10. Let $\mathfrak{H}_\oplus$ be a direct integral of $\{\mathfrak{H}_p\}$ over (X, μ). Let $\mathcal{B}$ be a von Neumann algebra on $\mathfrak{H}_\oplus$, and suppose that $\mathcal{A} \subseteq \mathcal{B}$ is a norm separable, strong operator dense C^*-subalgebra. Then $\mathcal{B}$ is said to be **decomposable** with decomposition $p \mapsto \mathcal{B}_p$, if

(a) the identity representation ι of $\mathcal{A}$, $p \mapsto \iota_p$, is decomposable;

(b) $\iota_p(\mathcal{A})$ is strong operator dense in the von Neumann algebra $\mathcal{B}_p$ almost everywhere.

We write $\mathcal{B} = \int_X^\oplus \mathcal{B}_p \, \mathrm{d}\mu$ and say that $\mathcal{B}$ is the **direct integral** of $\{\mathcal{B}_p\}$. We refer to the mapping $p \mapsto \mathcal{B}_p$ as a **measurable section of von Neumann algebras**. The image of $\iota_p(\mathcal{A})$ is usually denoted by $\mathcal{A}_p$. □

A link between decomposable operators and decomposable von Neumann algebras is as expected.

Proposition 4.1.11 ([KR97b] Theorem 14.1.16)**.** *Let $\mathfrak{H}_\oplus$ be the direct integral of Hilbert spaces $\{\mathfrak{H}_p\}$ over (X, Σ, μ) and $\mathcal{B}$ a von Neumann subalgebra of the algebra of decomposable operators. Then $\mathcal{B}$ is decomposable with unique decomposition $\mathcal{B} = \int_X^\oplus \mathcal{B}_p \,\mathrm{d}\mu$.* □

Remark 4.1.12. We will need, in Proposition 4.5.4 a result from [Dix81] and in Proposition 4.7.11 a result from [Nie80]. As a result, we will assume without proof that the approaches used in those books are compatible to the one presented in [KR97a]. □

We are now in the position to start transferring our mathematical objects to a direct integral framework.

4.2 The Cyclic Vector

Let $\mathcal{R}$ be a von Neumann algebra with faithful normal state ρ, and $\mathcal{S}$ be an abelian von Neumann subalgebra of $\mathcal{R}$ with faithful normal state $\rho|_\mathcal{S}$. Assume that $\mathcal{R}$ is in its cyclic representation acting on the separable GNS Hilbert space $\mathfrak{H}_\rho$ and that Ω_ρ is the corresponding cyclic vector.

In this section we use $\mathcal{S}$ to obtain a direct integral $\mathfrak{H}_\oplus^\mathcal{S}$ over a measure space. To do this, we use a result in [KR97b] (see Theorem 4.2.1 below), but we will have need to modify the fibres given by that result in order to ensure that we have a probability space as our domain of integration for $\mathfrak{H}_\oplus^\mathcal{S}$ (Proposition 4.2.6). In doing so, we will modify our cyclic vector Ω_ρ so that fibre-wise its components are almost everywhere unit vectors (Definition 4.2.10 and the remarks afterwards).

The following theorem ensures that given an abelian von Neumann algebra $\mathcal{B}$, there exists a direct integral $\mathfrak{H}_\oplus$ of Hilbert spaces making $\mathcal{B}$ unitarily equivalent to the von Neumann algebra of all diagonalisable operators on $\mathfrak{H}_\oplus$.

Theorem 4.2.1 ([KR97b] Theorem 14.2.1)**.** *Suppose $\mathcal{B}$ is an abelian von Neumann algebra on a separable Hilbert space $\mathfrak{H}$. There exists a locally compact complete separable metric measure space (X, Σ, μ), and a direct integral $\mathfrak{H}_\oplus = \int_X^\oplus \mathfrak{H}_p \,\mathrm{d}\mu$ over a family of Hilbert spaces $\{\mathfrak{H}_p\}$ such that there is a unitary operator $D : \mathfrak{H} \to \mathfrak{H}_\oplus$ with $D\mathfrak{H}D^* = \mathfrak{H}_\oplus$. Furthermore, $D\mathcal{B}D^*$ is the von Neumann algebra (Theorem 4.1.8) of all diagonalisable operators relative to $\{\mathfrak{H}_p\}$.* □

Let us apply the above result to our $\mathcal{S}$, which we place in a remark for ease of reference.

Remark 4.2.2. Applying Theorem 4.2.1 to $\mathcal{S}$ above, denote by $\mathfrak{H}_\oplus^\mathcal{S} = \int_Y^\oplus \mathfrak{H}_p^\mathcal{S} \,\mathrm{d}\mu$ the direct integral decomposition over the measure space (Y, Ψ, μ) such that Φ is a unitary $\Phi : \mathfrak{H}_\rho \to \mathfrak{H}_\oplus^\mathcal{S}$ making $\Phi\mathcal{S}\Phi^{-1}$ the von Neumann algebra of all diagonalisable operators.

We shall denote by φ the map $\varphi : B(\mathfrak{H}_\rho) \ni A \mapsto \Phi A \Phi^{-1} \in B(\mathfrak{H}_\oplus^{\mathcal{S}})$. In particular, $\varphi(\mathcal{S})$ is the von Neumann algebra of all diagonalisable operators acting on $\mathfrak{H}_\oplus^{\mathcal{S}}$. □

Proposition 4.2.3. *Suppose $(\mathcal{R}, \rho)$ is a von Neumann algebra with finite faithful state ρ acting on its GNS Hilbert space $\mathfrak{H}_\rho$. Let $\mathcal{S}$ be an abelian von Neumann subalgebra of $\mathcal{R}$ and let Φ be the unitary in Remark 4.2.2 which turns $\varphi(\mathcal{S}) = \Phi\mathcal{S}\Phi^{-1}$ into the set of all diagonalisable operators on the direct integral $\mathfrak{H}_\oplus^{\mathcal{S}} = \int_Y^\oplus \mathfrak{H}_p^{\mathcal{S}}\, \mathrm{d}\mu$. Let $\Omega_\rho \in \mathfrak{H}_\rho$ be the distinguished cyclic and separating vector of $\mathcal{R}$. Then $\Phi(\Omega_\rho)(p) \neq 0$ for almost all $p \in Y$.*

Proof. Note that for all $T \in \mathcal{S}$,

$$\Phi(T\Omega_\rho) = \Phi T \Phi^{-1}\Phi(\Omega_\rho) = \varphi(T)\Phi(\Omega_\rho). \tag{4.2.1}$$

So, if $T \in \mathcal{S}$ is positive and non-zero, as Φ is a unitary and ρ is faithful,

$$\langle \varphi(T)\Phi(\Omega_\rho), \Phi(\Omega_\rho)\rangle = \langle \Phi(T\Omega_\rho), \Phi(\Omega_\rho)\rangle = \langle T\Omega_\rho, \Omega_\rho\rangle = \rho(T) > 0. \tag{4.2.2}$$

Write Γ for $\Phi(\Omega_\rho)$. Suppose on the contrary that $\|\Gamma(p)\| = 0$ for some measurable set K of non-zero μ-measure. Consider the diagonalisable projection M_{χ_K} corresponding to K :

$$M_{\chi_K}(p) = \begin{cases} I_p & \text{if } p \in K \\ 0 & \text{otherwise.} \end{cases}$$

Then, as M_{χ_K} is not the zero operator, using (4.2.2),

$$\begin{aligned} 0 < \rho(\varphi^{-1}(M_{\chi_K})) &= \langle M_{\chi_K}\Gamma, \Gamma\rangle = \int_Y \langle \chi_K(p)\Gamma(p), \Gamma(p)\rangle\, \mathrm{d}\mu \\ &= \int_Y \chi_K(p)\langle \Gamma(p), \Gamma(p)\rangle\, \mathrm{d}\mu = \int_K \langle \Gamma(p), \Gamma(p)\rangle\, \mathrm{d}\mu = \int_K \|\Gamma(p)\|^2\, \mathrm{d}\mu, \end{aligned}$$

thus contradicting our assumption about K. □

Due to Proposition 4.2.3, we will feel free to divide through by $\|\Phi(\Omega_\rho)(p)\|$ almost everywhere.

We would like to examine in §4.5 vector states of the form $\langle T(p)\xi(p), \xi(p)\rangle$, where T is a decomposable operator in a decomposable von Neumann algebra and $\xi(p)$ is a unit vector almost everywhere. Unfortunately, attempting to define such a unit vector Γ' as

$$\Gamma'(p) = \frac{1}{\|\Phi(\Omega_\rho)(p)\|}\Phi(\Omega_\rho)(p) \qquad p\text{ -almost everywhere}$$

creates a technical impasse. For instance, it is not clear that replacing $\varphi(\Omega_\rho)$ with Γ' in (4.2.2) will yield the same state ρ. To overcome this, we will have to redefine the inner product on each $\mathfrak{H}_p^{\mathcal{S}}$ for almost each p. It is surprising that this does not affect the inner product on $\mathfrak{H}_\oplus^{\mathcal{S}}$ (Proposition 4.2.6).

It is easily checked that the following definition does indeed give an inner product on almost every Hilbert space fibre $\mathfrak{H}_p^{\mathcal{S}}$.

Definition 4.2.4. We use the assumptions and notation of Remark 4.2.2 and Proposition 4.2.3. For every $p \in Y$ such that $\|\Phi(\Omega_\rho)(p)\| \neq 0$ denote by $\mathfrak{H}'_p$ the Hilbert space $\mathfrak{H}_p^{\mathcal{S}}$ endowed with the following inner product:

$$\langle s, t\rangle' := \frac{1}{\|\Phi(\Omega_\rho)(p)\|^2}\langle s, t\rangle \qquad s, t \in \mathfrak{H}_p^{\mathcal{S}}.$$

□

We explicitly note that for $\Phi(\Omega_\rho)(p) \neq 0$, that under the new inner product we have

$$(\|\Phi(\Omega_\rho)(p)\|')^2 = \langle \Phi(\Omega_\rho)(p), \Phi(\Omega_\rho)(p)\rangle' = \frac{1}{\|\Omega_\rho(p)\|^2}\langle \Phi(\Omega_\rho)(p), \Phi(\Omega_\rho)(p)\rangle = 1. \tag{4.2.3}$$

Definition 4.2.5. Consider the measure space (Y, Ψ, μ) from Remark 4.2.2. Define the following set map $\nu : \Psi \to [0, \infty]$4.2.3

$$\nu(K) = \int_K \|\Phi(\Omega_\rho)(p)\|^2 \,\mathrm{d}\mu.$$

□

Proposition 4.2.6. *The set map ν defined in Definition 4.2.5 is a probability measure on the measurable space (Y, Ψ) and satisfies*

$$\langle x, y\rangle = \int_Y \langle x(p), y(p)\rangle' \,\mathrm{d}\nu.$$

Proof. Due to the fact that $\|\Phi(\Omega_\rho)(p)\| \geq 0$, [Rud87] Theorem 1.29 not only ensures that ν is indeed a measure, but also that for all $x, y \in \mathfrak{H}_\oplus^{\mathcal{S}}$,

$$\begin{aligned}
\langle x, y\rangle &= \int_Y \langle x(p), y(p)\rangle \,\mathrm{d}\mu \\
&= \int_Y \frac{1}{\|\Omega_\rho(p)\|^2}\langle x(p), y(p)\rangle \,\|\Omega_\rho(p)\|^2 \,\mathrm{d}\mu \\
&= \int_Y \langle x(p), y(p)\rangle' \,\mathrm{d}\nu.
\end{aligned}$$

Moreover, $\nu(Y) = \int_Y \|\Phi(\Omega_\rho)(p)\|^2 \,\mathrm{d}\mu = \|\Phi(\Omega_\rho)\|^2 = \|\Omega_\rho\|^2 = 1.$ □

Proposition 4.2.7. *Consider the measure ν from Proposition 4.2.6 (and Definition 4.2.5). We have that the measure μ is equivalent to ν, i.e. for all $K \in \Psi$, $\mu(K) = 0 \iff \nu(K) = 0$.*

Proof. If $K \in \Psi$ satisfies $\mu(K) = 0$, then clearly $\nu(K) = 0$. Conversely, suppose $\mu(K) \neq 0$. Then $\mu(K) > 0$. As $\|\Phi(\Omega_\rho)(p)\|^2 > 0$ for almost all $p \in Y$ (Proposition 4.2.3), $\nu(K) = \int_K \|\Phi(\Omega_\rho)(p)\|^2 \, \mathrm{d}\mu \neq 0$. □

Remark 4.2.8. We mention two consequences of Proposition 4.2.7 which we use later:

(i) $L^\infty(\mu) = L^\infty(\nu)$ (this is straight forward to prove; we use this in Proposition 4.4.2).

(ii) a statement P is true ν-a.e if and only if P is true μ-a.e (we use this in Corollary 4.2.9).

□

As a consequence of Propositions 4.2.6, we have

Corollary 4.2.9. *Consider the collection of Hilbert spaces $\{\mathfrak{H}'_p\}$ defined in Definition 4.2.4 and the measure space (Y, Ψ, ν) in Proposition 4.2.6. The Hilbert space $\mathfrak{H}^{\mathcal{S}}_\oplus = \int_Y^\oplus \mathfrak{H}^{\mathcal{S}}_p \, \mathrm{d}\nu$ from Remark 4.2.2, is also given by $\mathfrak{H}^{\mathcal{S}}_\oplus = \int_Y^\oplus \mathfrak{H}'_p \, \mathrm{d}\nu$.*

Proof. We verify Definition 4.1.1.

Let $x \in \mathfrak{H}^{\mathcal{S}}_\oplus$. Then we can form a corresponding map $p \mapsto x'(p)$ algebraically as follows. For each $p \in Y$, treating $\mathfrak{H}_p$ as a vector space, we have that $\mathfrak{H}_p = \mathfrak{H}'_p$. So we set $x'(p) = x(p)$ for all $p \in Y$.

The map $p \mapsto \langle x'(p), y'(p)\rangle$ is equal almost everywhere to

$$p \mapsto \frac{1}{\|\Phi(\Omega_\rho)(p)\|^2} \langle x(p), y(p)\rangle.$$

The map $p \mapsto \frac{1}{\|\Phi(\Omega_\rho)(p)\|^2}$ is μ-measurable and thus ν-measurable (as both measures are defined on the same σ-algebra Ψ). Thus, $p \mapsto \frac{1}{\|\Phi(\Omega_\rho)(p)\|^2} \langle x(p), y(p)\rangle$ is ν-measurable. Proposition 4.2.6 shows that $\langle x, y\rangle = \int_Y \langle x(p), y(p)\rangle' \, \mathrm{d}\nu$.

Lastly, suppose that for each $p \in Y$ there is a $x_p \in \mathfrak{H}'_p$ such that for all $y \in \mathfrak{H}^{\mathcal{S}}_\oplus$, $p \mapsto \langle x_p, y(p)\rangle'$ is ν-measurable. Then, because $\mathfrak{H}^{\mathcal{S}}_\oplus = \int_Y \mathfrak{H}^{\mathcal{S}}_p \, \mathrm{d}\mu$, there is exists a map $p \mapsto x(p)$ with $x(p) = x_p$ almost everywhere, with $x(p) \in \mathfrak{H}_p$. In a similar manner to the beginning of the proof, by viewing $\mathfrak{H}^{\mathcal{S}}_p$ and $\mathfrak{H}'_p$ as the same vector space, we have that the map $p \mapsto x'(p)$ with $x'(p) = x(p)$ satisfies $x'(p) = x_p$ almost everywhere. □

Definition 4.2.10. In $\mathfrak{H}_{\oplus}^{\mathcal{S}} = \int_Y^{\oplus} \mathfrak{H}'_p \, \mathrm{d}\nu$, we define the vector Γ as

$$\Gamma = \Phi\Omega_{\rho}.$$

□

Remark 4.2.11. From (4.2.3) we have $\|\Gamma(p)\| = 1$ almost everywhere and therefore, $\|\Gamma\|^2 = \int_Y \|\Gamma(p)\|^2 \, \mathrm{d}\nu = 1$. □

4.3 Decomposing von Neumann Algebras and their States

We require a few results (Propositions 4.3.1 and 4.3.2), in order to establish that $\varphi(\mathcal{R}) := \Phi\mathcal{R}\Phi^{-1}$ is a decomposable von Neumann algebra (Proposition 4.3.3). This is followed by a sequence of propositions that build the theory for the corresponding decomposition of the state on $\varphi(\mathcal{R})$. Strictly speaking, all of this is presented in [KR97b], but not in the form that we require.

Proposition 4.3.1. *We use the notation and assumptions in Remark 4.2.2 and Proposition 4.2.3. In particular, we consider the map* $\varphi : B(\mathfrak{H}_{\rho}) \to B(\mathfrak{H}_{\oplus}^{\mathcal{S}})$*, defined by* $\varphi(a) = \Phi a \Phi^{-1}$ *and recall that* $\mathcal{R}$ *is a von Neumann algebra with faithful normal trace* ρ *acting on its GNS space* $\mathfrak{H}_{\rho}$*. We have that* $\varphi(\mathcal{R}) := \Phi\mathcal{R}\Phi^{-1}$ *is a von Neumann algebra.*

Proof. We note that φ sends the identity operator $\mathbb{1}_{\mathcal{R}} \in \mathcal{R}$ to the identity operator in $\mathcal{B}(\mathfrak{H}_{\oplus}^{\mathcal{S}})$.

It is routine to verify that $\varphi(\mathcal{R})$ is a $*$-algebra.

Let us show that $\varphi(\mathcal{R})$ is closed in the strong operator topology. Let (r_{λ}) be a net in $\varphi(\mathcal{R})$ converging in the strong operator topology to some operator $r \in B(\mathfrak{H}_{\oplus}^{\mathcal{S}})$. For each λ, there is an $a_{\lambda} \in \mathcal{R}$ such that $r_{\lambda} = \Phi a_{\lambda} \Phi^*$. Then (a_{λ}) converges to $\Phi^* r \Phi$ in the strong operator topology. Indeed, for every $x \in \mathfrak{H}_{\oplus}^{\mathcal{S}}$, because $(\Phi a_{\lambda} \Phi^*)$ converges to r,

$$\|a_{\lambda}x - \Phi^* r \Phi x\| = \|\Phi^*(\Phi a_{\lambda}\Phi^* - r)\Phi x\| \leq \|\Phi^*\| \, \|(\Phi a_{\lambda}\Phi^* - r)\Phi x\| \to 0.$$

Thus, as $\mathcal{R}$ is a von Neumann algebra, we have that $\Phi^* r \Phi \in \mathcal{R}$. Hence, there is an $a \in \mathcal{R}$ such that $a = \Phi^* r \Phi$. So, $r \in \varphi(\mathcal{R})$.

□

We note that φ on $\mathcal{R}$ is normal (ultra-weakly continuous) ([SZ79] Corollary 5.13). The map φ on $\mathcal{R}$ also preserves the commutant of $\mathcal{S}$.

Proposition 4.3.2. *With the notation and assumptions of Proposition 4.2.3, we have* $\varphi(\mathcal{S})' = \varphi(\mathcal{S}')$.

Proof. Let $f \in \varphi(\mathcal{S}')$. Then, for some $a' \in \mathcal{S}'$, $f = \Phi a' \Phi^*$. We want to show that $f \in \varphi(\mathcal{S})'$. So let $g = \Phi a \Phi^* \in \varphi(\mathcal{S})$ for some $a \in \mathcal{S}$. Then,

$$fg = \Phi a' \Phi^* \Phi a \Phi^* = \Phi a' a \Phi^* = \Phi a a' \Phi^* = \Phi a \Phi^* \Phi a' \Phi^* = gf.$$

Conversely, if $f \in \varphi(\mathcal{S})'$, then $f \Phi a \Phi^* = \Phi a \Phi^* f$ for every $a \in \mathcal{S}$. Put $a' := \Phi^* f \Phi$. Then, $a' : \mathfrak{H}_\rho \to \mathfrak{H}_\rho$ and for every $a \in \mathcal{S}$,

$$a'a = \Phi^* f \Phi a = \Phi^* f \Phi a \Phi^* \Phi = \Phi^* \Phi a \Phi^* f \Phi = a \Phi^* f \Phi = aa'.$$

Thus, $a' \in \mathcal{S}'$. Hence, $f = \Phi a' \Phi^* \in \varphi(\mathcal{S}')$. □

The next result summarises many of the main points made so far in this chapter.

Proposition 4.3.3. *With the notation and assumptions of Proposition 4.2.3, we further assume that $\mathcal{S}$ is **central** in $\mathcal{R}$ i.e. $\mathcal{S} \subseteq \mathcal{R} \cap \mathcal{R}'$. Then $\varphi(\mathcal{R}) := \Phi \mathcal{R} \Phi^{-1}$ is a decomposable von Neumann algebra with respect to the direct integral Hilbert space $\mathfrak{H}_\oplus^{\mathcal{S}} = \int_Y \mathfrak{H}'_p \, \mathrm{d}\nu$, where ν is a probability measure, with decomposition denoted by (see Definition 4.1.10)*

$$\varphi(\mathcal{R}) = \int_Y^\oplus \varphi(\mathcal{R})_p \, \mathrm{d}\nu.$$

The von Neumann algebra $\varphi(\mathcal{R})$ consists solely of decomposable operators and $\varphi(\mathcal{S})$ is the von Neumann algebra of diagonalisable operators.

Proof. We can take ν to be a probability measure by Corollary 4.2.9. As $\mathcal{S} \subseteq \mathcal{R}'$, we have $\mathcal{R} \subseteq \mathcal{S}'$. Therefore, as Φ is unitary, $\varphi(\mathcal{R}) \subseteq \varphi(\mathcal{S}') = \varphi(\mathcal{S})'$, from Proposition 4.3.2. Recall that $\varphi(\mathcal{S})'$ is the von Neumann algebra of all decomposable operators (from Remark 4.2.2 and Theorem 4.1.8). Thus, $\varphi(\mathcal{R})$ is a von Neumann algebra consisting solely of decomposable operators, and is therefore a decomposable von Neumann algebra (Proposition 4.1.11). From, Proposition 4.2.3, $\varphi(\mathcal{S})$ is the von Neumann algebra of all diagonalisable operators. □

We turn our attention to decomposing the associated state ρ of $\mathcal{R}$.

Proposition 4.3.4. *Consider a von Neumann algebra $\mathcal{B}$ acting on a Hilbert space $\mathfrak{H}$. Let $s : \mathcal{B} \to \mathbb{C}$ be the linear functional*

(4.3.1) $$s(a) = \langle a\xi, \xi \rangle \qquad a \in \mathcal{B},$$

where $\xi \in \mathfrak{H}$ is a unit vector. We have the following properties:

(i) $s(\mathbb{1}_{\mathcal{B}}) = 1$.

(ii) *s is positive.*

(iii) s is normal.

(iv) Suppose that $a \in \mathcal{B}^+$ and ξ is separating for $\mathcal{B}$. If $s(a) = 0$, then $a = 0$.

Proof.

(i)
$s(\mathbb{1}_{\mathcal{B}}) = \|\xi\|^2 = 1.$

(ii)
If $a \in \mathcal{B}^+$, then there is a $b \in \mathcal{B}$ such that $a = b^*b$. Thus,

$$s(a) = \|b\xi\|^2 \geq 0.$$

(iii)
We use the definition of normality in Definition 1.1.2. Let (b_λ) be an increasing net in $\mathcal{B}^+$ with supremum $b = \sup\{b_\lambda\}$. Recall that $b = \sup\{b_\lambda\}$ is the strong operator limit of the net (b_λ) ([Zhu93] Theorem 17.1). In particular,

$$b\xi = \lim_\lambda b_\lambda \xi.$$

Therefore, we have $\lim_\lambda s(b_\lambda) = s(b)$:

$$\begin{aligned} |s(b_\lambda) - s(b)| &= |\langle (b_\lambda \xi - b\xi, \xi\rangle| \\ &\leq \|b_\lambda \xi - b\xi\| \, \|\xi\| = \|b_\lambda \xi - b\xi\| \to 0. \end{aligned}$$

Now, as s is positive, we have that $(s(b_\lambda))$ is a bounded monotone increasing net with limit $\sup_\lambda\{s(b_\lambda)\}$. Thus, $\sup_\lambda\{s(b_\lambda)\} = \lim_\lambda s(b_\lambda) = s(b)$.

(iv)
Suppose that $a \in \mathcal{B}^+$ and write $a = b^*b$ for some $b \in \mathcal{B}$. If $s(a) = 0$, then using the separating property of ξ,

$$0 = s(a) = \|b\xi\|^2 \implies b\xi = 0 \implies b = 0.$$

So $a = 0$.

□

Proposition 4.3.5. *Continuing from Proposition 4.3.3, we have that Γ (Definition 4.2.10) is a separating vector for $\varphi(\mathcal{R})$, that is, if $r \in \mathcal{R}$ satisfies $\varphi(r)\Gamma = 0$, then $\varphi(r) = 0$. Furthermore, Γ is cyclic for $\varphi(\mathcal{R})$, that is, $[\varphi(\mathcal{R})\Gamma] = \mathfrak{H}^{\mathcal{S}}_{\oplus} = \int_Y^{\oplus} \mathfrak{H}'_p \, \mathrm{d}\nu$.*

Proof. Let $r \in \mathcal{R}$. Then,

$$\begin{aligned}
\varphi(r)\Gamma &= 0 \\
\Phi r \Phi^{-1}\Phi\Omega_\rho &= 0 \\
\Phi(r\Omega_\rho) &= 0 \\
r\Omega_\rho &= 0 && \text{as } \Phi \text{ is an isometry} \\
r &= 0 && \Omega_\rho \text{ is separating.}
\end{aligned}$$

To prove the cyclic property, we just need to show that

$$\mathfrak{H}_\oplus^\mathcal{S} = \Phi([\mathcal{R}\Omega_\rho]) \subseteq [\varphi(\mathcal{R})\Gamma].$$

If $x \in \Phi([\mathcal{R}\Omega_\rho]) = \mathfrak{H}_\oplus^\mathcal{S}$, there is a sequence of elements $(r_n\Omega_\rho)$ such that $\lim_n r_n\Omega_\rho = \Phi^{-1}x$. As Φ is continuous,

$$\lim_n \Phi(r_n\Omega_\rho) = \Phi(\lim_n r_n\Omega_\rho) = \Phi\Phi^{-1}x = x.$$

Now

$$\Phi(\mathcal{R}\Omega_\rho) = \Phi\mathcal{R}\Phi^{-1}\Phi(\Omega_\rho) = \varphi(\mathcal{R})\Gamma. \tag{4.3.2}$$

So, from (4.3.2), we have that $\Phi(r_n\Omega_\rho) = \varphi(r_n)\Gamma$ and therefore $x \in [\varphi(\mathcal{R})\Gamma]$. □

Proposition 4.3.6. *We use the notation and assumptions of Proposition 4.3.3 and Definition 4.2.10. Define on $\varphi(\mathcal{R})$, $\omega := \rho \circ \varphi^{-1}$. Then, ω is a normal faithful state such that for all $a \in \varphi(\mathcal{R})$*

(i) $\omega(a) = \langle a\Gamma, \Gamma\rangle$.

(ii) There are linear functionals

$$\omega_p(a(p)) := \langle a(p)\Gamma(p), \Gamma(p)\rangle \tag{4.3.3}$$

such that for almost all p, ω_p is a normal faithful state on $\varphi(\mathcal{R})_p$ and $\omega(a) = \int_Y \omega_p(a(p))\,\mathrm{d}\nu$.

Proof. For all $a \in \varphi(\mathcal{R})$

$$\begin{aligned}
\omega(a) &= \rho \circ \varphi^{-1}(a) = \langle \varphi^{-1}(a)\Omega_\rho, \Omega_\rho\rangle \\
&= \langle \Phi\varphi^{-1}(a)\Omega_\rho, \Phi\Omega_\rho\rangle && \text{as } \Phi \text{ is preserves the inner product} \\
&= \langle a\Gamma, \Gamma\rangle.
\end{aligned}$$

From Proposition 4.3.4, ω is a normal state. As Γ is separating (Proposition 4.3.5), we have that ω is faithful (Proposition 4.3.4).

We now have

$$\omega(a) = \langle a\Gamma, \Gamma\rangle = \int_Y \langle a(p)\Gamma(p), \Gamma(p)\rangle \, \mathrm{d}\nu.$$

Except on a set E_1 of measure zero, $\|\Gamma(p)\| = 1$ (Remark 4.2.11). So, from Proposition 4.3.4, for all $p \in E_1$ ω_p is a normal state. As a special case of the proof in [KR97b] Lemma 14.1.19, we have that ω_p is faithful for almost p, except in a null set E_2 . Thus, for almost all $p \in Y\backslash(E_1 \cup E_2)$, ω_p is a faithful normal state on $\varphi(\mathcal{R})_p$. □

Remark 4.3.7. We refer to the notation ω_p in Proposition 4.3.6. As ω_p is faithful for almost all p we have from (4.3.3) that $\Gamma(p)$ is separating for $\varphi(\mathcal{R})_p$ for almost all $p \in Y$. □

In the next result, we use [KR97b] Lemma 14.1.3. We note in that Lemma that when [KR97b] refers to a set $\{x_a\}$ of vectors "spanning" a Hilbert space $\mathfrak{H}$, they mean that the set of all finite linear combinations of elements from $\{x_a\}$ is dense in $\mathfrak{H}$.

Proposition 4.3.8. *Using the notation and assumptions of Proposition 4.3.3 and Definition 4.2.10, we have that for almost all p, $\Gamma(p)$ is cyclic for $\varphi(\mathcal{R})_p$, that is, $[\varphi(\mathcal{R})_p\Gamma(p)] = \mathfrak{H}'_p$, for almost all $p \in Y$.*

Proof. We have $\varphi(\mathcal{R})\Gamma$ dense in $\mathfrak{H}^{\mathcal{S}}_{\oplus}$ (Proposition 4.3.5). So $\mathcal{A}\Gamma$ is dense in $\mathfrak{H}^{\mathcal{S}}_{\oplus}$ in terms of Definition 4.1.10 (with $\mathcal{B}$ replaced with $\varphi(\mathcal{R})$), since $\mathcal{A}$ is strongly dense in $\varphi(\mathcal{R})$. Thus, using [KR97b] Lemma 14.1.3, for almost all p,

$$\overline{\{(a\Gamma)(p) : a \in \mathcal{A}\}} = \mathfrak{H}'_p. \tag{4.3.4}$$

On the other hand, $\iota_p(\mathcal{A})$ is strongly dense in $\varphi(\mathcal{R})_p$ (Definition 4.1.10), where $\iota(a) = a(p)$ a.e. (Definition 4.1.9) since $\iota = \mathrm{id}_{\mathcal{A}}$, for all $a \in \mathcal{A}$. As a result, $\{a(p) \mid a \in \mathcal{A}\}$ is strongly dense in $\varphi(\mathcal{R})_p$ a.e. In particular, for almost all p

$$\overline{\{a(p)\Gamma(p) \mid a \in \mathcal{A}} = \overline{\varphi(\mathcal{R})_p\Gamma(p)}.$$

However, since $(a\Gamma)(p) = a(p)\Gamma(p)$ a.e (Definition 4.1.6), we have from (4.3.4), for almost all p,

$$\overline{\varphi(\mathcal{R})_p\Gamma(p)} = \mathfrak{H}'_p.$$

□

4.4 Decomposing the Jones Projection

Recall that the Jones projection, $e_{\mathcal{S}}$ is the projection from $\mathfrak{H}_\rho \equiv [\mathcal{R}\Omega_\rho]$ onto $[\mathcal{S}\Omega_\rho]$. We do not directly decompose $\Phi e_{\mathcal{S}}\Phi^{-1}$. Rather, we first define a map Q (Definition 4.4.1) and show that it projects onto $[\varphi(\mathcal{S})\Gamma]$.

Definition 4.4.1. We continue using the notation and assumptions from Proposition 4.3.3. Recall that $\mathfrak{H}_\oplus^{\mathcal{S}} = \int_Y^\oplus \mathfrak{H}'_p \, \mathrm{d}\nu$ is the Hilbert space obtained from Corollary 4.2.9 and that Γ is the vector defined in Definition 4.2.10. Define an operator Q on $\mathfrak{H}_\oplus^{\mathcal{S}}$: for all $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$,

$$(Qx)(p) = \langle x(p), \Gamma(p)\rangle \Gamma(p) \qquad \text{for almost all } p \in Y.$$

□

We shall show shortly that Q is well-defined. We proceed in steps:

1. Show that $Qx \in \mathfrak{H}_\oplus^{\mathcal{S}}$ for every $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$.
2. Show that $Qx \in [\varphi(\mathcal{S})\Gamma]$ for every $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$.
3. Show that Q is a projection.
4. Show that $\Phi e_{\mathcal{S}}\Phi^{-1}$ projects onto $\Phi([\mathcal{S}\Omega_\rho]) = [\varphi(\mathcal{S})\Gamma]$.

We take a moment to establish some technical results before we carry out the aforementioned steps.

Proposition 4.4.2. *We use the notation and assumptions in Proposition 4.3.3 and Definition 4.2.10. Recall, from the beginning of §4.2, that $\Omega_\rho \in \mathfrak{H}_\rho$ is the distinguished cyclic vector of $\mathcal{R}$. Then, we have the equality*

$$\Phi(\mathcal{S}\Omega_\rho) = \{p \mapsto f(p)\Gamma(p) \mid f \in L^\infty(\nu)\}.$$

Proof. From Remark 4.2.2 and Corollary 4.2.9, $\varphi(\mathcal{S})$ is the von Neumann algebra of all diagonalisable operators on $\mathfrak{H}_\oplus^{\mathcal{S}} = \int_Y^\oplus \mathfrak{H}'_p \, \mathrm{d}\nu$. However, as noted in Remark 4.2.8, $L^\infty(\mu) = L^\infty(\nu)$. Thus, $n \in \mathcal{S}$ if and only if there exists an $f \in L^\infty(\nu)$ such that $M_f = \varphi(n)$. In terms of this,

$$\Phi(n\Omega_\rho)(p) = (\Phi n \Phi^{-1}\Phi\Omega_\rho)(p) = (\varphi(n)\Gamma)(p) = f(p)\Gamma(p). \tag{4.4.1}$$

for almost all p, from which the results follows. □

The next proposition uses standard results and convergence arguments in complete spaces.

Proposition 4.4.3. *We use the notation and assumptions in Proposition 4.3.3, and Definition 4.2.10. There exists a unitary* $\theta : L^2(\nu) \to [\varphi(\mathcal{S})\Gamma]$ *such that for all* $f \in L^\infty(\nu)$,

$$\theta(f) = M_f\Gamma.$$

Proof. The map

$$\mathcal{S} \ni n \mapsto \varphi(n) \in B(\mathfrak{H}_\oplus^{\mathcal{S}}),$$

is a $*$-isomophism onto $\varphi(\mathcal{S})$. So we may define

$$\theta' : L^\infty(\nu) \ni f \mapsto M_f\Gamma \in [\varphi(\mathcal{S})\Gamma] \subseteq \mathfrak{H}_\oplus^{\mathcal{S}}.$$

Viewing $L^\infty(\nu)$ as a subset of $L^2(\nu)$, the inner product is preserved: for every $f, g \in L^\infty(\nu)$,

$$\begin{aligned}
\langle f, g\rangle_{L^2(\nu)} &= \int_Y f(p)\overline{g(p)}\,\mathrm{d}\nu \\
&= \int_Y f(p)\overline{g(p)}\langle \Gamma(p), \Gamma(p)\rangle\,\mathrm{d}\nu \\
&= \int_Y \langle f(p)\Gamma(p), g(p)\Gamma(p)\rangle\,\mathrm{d}\nu = \langle \theta'(f), \theta'(g)\rangle.
\end{aligned}$$

Thus, θ' is continuous on $L^\infty(\nu)$, and therefore can be extended to a linear operator

$$\theta : L^2(\nu) \to [\varphi(\mathcal{S})\Gamma].$$

In particular, θ is an isometry and therefore injective. Let $x \in [\varphi(\mathcal{S})\Gamma]$. Then there is a sequence $(\varphi(n_i)\Gamma)$ converging to x. So there is a sequence $(f_i) \subseteq L^\infty(\nu)$ such that for each $i \in \mathbb{N}$, $\varphi(n_i) = \theta(f_i)$. As θ is an isometry and $(\varphi(n_i)\Gamma)$ is a Cauchy sequence, (f_i) is a Cauchy sequence in $L^2(\nu)$. Thus there is an $f \in L^2(\nu)$ such that $\lim_i f_i = f$. Hence, using the continuity of θ,

$$x = \lim_i \varphi(n_i)\Gamma = \lim_i \theta(f_i) = \theta(\lim f_i) = \theta(f). \tag{4.4.2}$$

This shows that θ is bijective mapping preserving the inner product and hence a Hilbert space isomorphism. □

Proposition 4.4.4. *Continuing from Proposition 4.4.3,* $[\varphi(\mathcal{S})\Gamma]$ *is equal to the space*

$$L^2(\nu)\Gamma := \{p \mapsto g(p)\Gamma(p) \mid g \in L^2(\nu)\}$$

and $\theta(g) = g\Gamma$ *for all* $g \in L^2(\nu)$, *where* $g\Gamma$ *is the map* $p \mapsto g(p)\Gamma(p)$.

Proof. If $f \in L^2(\nu)$, there is a sequence of elements (f_i) in $L^\infty(\nu)$, such that $\lim_i f_i = f$ in $L^2(\nu)$. Thus, using the continuity of θ,

$$\theta(f) = \theta(\lim_i f_i) = \lim_i \theta(f_i) = \lim_i f_i \Gamma. \tag{4.4.3}$$

$f\Gamma \in \mathfrak{H}_\oplus^{\mathcal{S}}$: for any $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$,

$$p \mapsto \langle f(p)\Gamma(p), x(p)\rangle = f(p)\langle \Gamma(p), x(p)\rangle$$

is integrable, since it is measurable (both f and $p \mapsto \langle \Gamma(p), x(p)\rangle$ are measurable) and

$$\int_Y |f(p)\langle \Gamma(p), x(p)\rangle| \,\mathrm{d}\nu \leq \left[\int_Y |f(p)|^2 \,\mathrm{d}\nu\right]^{\frac{1}{2}} \left[\int_Y \|x(p)\|^2 \,\mathrm{d}\nu\right]^{\frac{1}{2}} < \infty. \tag{4.4.4}$$

So there is a vector $u \in \mathfrak{H}_\oplus^{\mathcal{S}}$ such that $u(p) = f(p)\langle \Gamma(p), x(p)\rangle$ for almost all p (Definition 4.1.1). Therefore, $f\Gamma \in \mathfrak{H}_\oplus^{\mathcal{S}}$.

We show that $\lim_i f_i\Gamma = f\Gamma$. Note that

$$\begin{aligned} \|f_i\Gamma - f\Gamma\|^2 &= \int_Y \|f_i(p)\Gamma(p) - f(y)\Gamma(p)\|^2 \,\mathrm{d}\nu \\ &= \int_Y |f_i(p) - f(p)|^2 \|\Gamma(p)\|^2 \,\mathrm{d}\nu \\ &= \int_Y |f_i(p) - f(p)|^2 \,\mathrm{d}\nu \to 0, \end{aligned} \tag{4.4.5}$$

because $f_i \to f$ in $L^2(\nu)$. Comparing (4.4.3) and (4.4.5), from the uniqueness of limits in Hilbert spaces, we have that $\theta(f) = f\Gamma$. Hence, $\theta(L^2(\nu)) = L^2(\nu)\Gamma$.

However, from Theorem 4.4.3, θ is surjective onto $[\varphi(\mathcal{S})\Gamma]$. Hence, $[\varphi(\mathcal{S})\Gamma] = \theta(L^2(\nu)) = L^2(\nu)\Gamma$.

□

We begin with the steps outlined after Definition 4.4.1.

Proposition 4.4.5. *Let Q be the operator defined in Definition 4.4.1. Then, $Qx \in \mathfrak{H}_\oplus^{\mathcal{S}}$ for every $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$.*

Proof. We use Definition 4.1.1. Let $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$. Clearly, for almost all $p \in Y$, $\langle x(p), \Gamma(p)\rangle\Gamma(p)$ is a scalar multiple of $\Gamma(p)$ and therefore belongs to $\mathfrak{H}'_p$. Next, for all $y \in \mathfrak{H}_\oplus^{\mathcal{S}}$, the equality

$$\langle\langle x(p), \Gamma(p)\rangle\Gamma(p), y(p)\rangle = \langle x(p), \Gamma(p)\rangle\langle \Gamma(p), y(p)\rangle,$$

shows that $p \mapsto \langle\langle x(p), \Gamma(p)\rangle\Gamma(p), y(p)\rangle$ is measurable being the pointwise product of measurable functions. Hence, as $\|\Gamma(p)\|^2 = 1$ for almost all $p \in Y$ (from (4.2.3)) we have

$$\begin{aligned}
&\int_Y |\langle\langle x(p), \Gamma(p)\rangle\Gamma(p), y(p)\rangle| \,\mathrm{d}\nu \\
&\leq \int_y \|x(p)\| \, \|\Gamma(p)\| \, \|\Gamma(p)\| \, \|y(p)\| \,\mathrm{d}\nu = \int_Y \|x(p)\| \, \|y(p)\| \,\mathrm{d}\nu \\
&\leq \left[\int_Y \|x(p)\|^2 \,\mathrm{d}\nu\right]^{\frac{1}{2}} \left[\int_Y \|y(p)\|^2 \,\mathrm{d}\nu\right]^{\frac{1}{2}} \\
&< \infty.
\end{aligned} \tag{4.4.6}$$

So, $Qx \in \mathfrak{H}_\oplus^{\mathcal{S}}$ for all $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$. □

Proposition 4.4.6. *We use the assumptions and notation of Proposition 4.3.3. Let Q be defined as in Definition 4.4.1. Then, $Qx \in [\varphi(\mathcal{S})\Gamma]$ for every $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$.*

Proof. As a special case of (4.4.6) (when $x = y$),

$$\begin{aligned}
\int_Y |\langle x(p), \Gamma(p)\rangle|^2 \,\mathrm{d}\nu &= \int_Y \langle x(p), \Gamma(p)\rangle\langle\Gamma(p), x(p)\rangle \,\mathrm{d}\nu \\
\leq \int_Y \|x(p)\|^2 \,\mathrm{d}\nu &= \|x\|^2 < \infty.
\end{aligned} \tag{4.4.7}$$

The calculation above thus shows that $p \mapsto \langle x(p), \Gamma(p)\rangle$ belongs to $L^2(\nu)$ and therefore, as $L^2(\nu)\Gamma = [\varphi(\mathcal{S})\Gamma]$ (Proposition 4.4.4), $Qx \in [\varphi(\mathcal{S})\Gamma]$. □

Proposition 4.4.7. *We continue from Proposition 4.4.6. The operator Q is a projection.*

Proof. We check the following:

(i) For all $x \in [\varphi(\mathcal{S})\Gamma]$, $Qx = x$.

(ii) For all $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$, $Q^2x = x$.

(iii) For all $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$, $Q^*x = Qx$.

Let $x \in [\varphi(\mathcal{S})\Gamma]$. There is an $f \in L^2(\nu)$ such that $x = f\Gamma$ using Proposition 4.4.4. Then, for almost all $p \in Y$,

$$(Qx)(p) = \langle x(p), \Gamma(p)\rangle\Gamma(p) = \langle f(p)\Gamma(p), \Gamma(p)\rangle\Gamma(p) = f(p) \|\Gamma(p)\|^2 \Gamma(p) = f(p)\Gamma(p).$$

Thus $Qx = x$, because $\|Qx - x\|^2 = \int_Y \|f(p)\Gamma(p) - f(p)\Gamma(p)\|^2 \,\mathrm{d}\nu = 0$. Hence, combining this with Proposition 4.4.6, the range of Q is $[\varphi(\mathcal{S})\Gamma]$.

If $x \in \mathfrak{H}_{\oplus}^{\mathcal{S}}$, then

$$\begin{aligned}(Q^2x)(p) = (QQx)(p) &= \langle (Qx)(p), \Gamma(p)\rangle\Gamma(p) = \langle\langle x(p), \Gamma(p)\rangle\Gamma(p), \Gamma(p)\rangle\Gamma(p)\\ &= \langle x(p), \Gamma(p)\rangle \|\Gamma(p)\|^2 \Gamma(p)\\ &= (Qx)(p).\end{aligned}$$

Lastly, from [KR97b] Proposition 14.1.8, Q has its adjoint given by $(Q^*x)(p) = Q(p)^*x(p)$ a.e. So, for almost all $p \in Y$, for all $s, t \in \mathfrak{H}_p$,

$$\begin{aligned}\langle (Q^*)(p)s, t\rangle &= \langle s, Q(p)t\rangle = \langle s, \langle t, \Gamma(p)\rangle\Gamma(p)\rangle\\ &= \overline{\langle t, \Gamma(p)\rangle}\langle s, \Gamma(p)\rangle\\ &= \langle s, \Gamma(p)\rangle\langle\Gamma(p), t\rangle = \langle\langle s, \Gamma(p)\rangle\Gamma(p), t\rangle\\ &= \langle Q(p)s, t\rangle.\end{aligned}$$

So, for all $x, y \in \mathfrak{H}_{\oplus}^{\mathcal{S}}$,

$$\langle Qx, y\rangle = \int_Y \langle Q(p)x(p), y(p)\rangle \,\mathrm{d}\nu = \int_Y \langle Q^*(p)x(p), y(p)\rangle \,\mathrm{d}\nu = \langle Q^*x, y\rangle.$$

□

Proposition 4.4.8. *We assume the notation of Proposition 4.3.3 and recall that $e_{\mathcal{S}}$ is the Jones projection (see the paragraph before Definition 4.4.1). Then, the operator $\varphi(e_{\mathcal{S}}) := \Phi e_{\mathcal{S}}\Phi^{-1}$ is a projection onto $[\varphi(\mathcal{S})\Gamma]$. Hence, Q, defined in Definition 4.4.1, satisfies $Q = \varphi(e_{\mathcal{S}})$, that is, for all $x \in \mathfrak{H}_{\oplus}^{\mathcal{S}}$,*

$$(\varphi(e_{\mathcal{S}})x)(p) = \langle x(p), \Gamma(p)\rangle\Gamma(p) \qquad \text{for almost all } p \in Y. \tag{4.4.8}$$

Proof. From a similar calculation to (4.2.1), $\Phi(\mathcal{S}\Omega) = \varphi(\mathcal{S})\Gamma$. In a similar manner to the proof of Proposition 4.3.5, we have $\Phi([\mathcal{S}\Omega]) = [\Phi(\mathcal{S}\Omega)]$ and therefore,

$$\Phi([\mathcal{S}\Omega]) = [\Phi(\mathcal{S}\Omega)] = [\varphi(\mathcal{S})\Gamma].$$

□

We now study the effect the decomposition of the Jones projection has on the elements in $\varphi(\mathcal{R})$. The subsequent results Corollary 4.4.9 and Proposition 4.4.10 will be used in the decomposition of the basic construction in §4.5.

Corollary 4.4.9. *We use the assumptions and notation in Proposition 4.3.3 and recall that $e_{\mathcal{S}}$ is the Jones projection (see the paragraph before Definition 4.4.1). Then, for every $a, b \in \mathcal{R}$, $\varphi(ae_{\mathcal{S}}b) := \Phi ae_{\mathcal{S}}b\Phi^{-1}$ is a decomposable operator. Furthermore, for almost every p and every $x \in \mathfrak{H}'_p$,*

$$[\varphi(ae_{\mathcal{S}}b)(p)]x = \langle\varphi(b)(p)x, \Gamma(p)\rangle\varphi(a)(p)\Gamma(p).$$

Proof. The operators $\varphi(a)$ and $\varphi(b)$ are decomposable using Proposition 4.3.3. We see that $\varphi(e_{\mathcal{S}})$ is a decomposable operator by defining, for almost all p, $\varphi(e_{\mathcal{S}})(p) \in B(\mathfrak{H}'_p)$ using the right hand side of (4.4.8): for every $x \in \mathfrak{H}'_p$,

$$\varphi(e_{\mathcal{S}})(p)x := \langle x, \Gamma(p)\rangle\Gamma(p).$$

Thus, writing

$$\varphi(ae_{\mathcal{S}}b) = \Phi a\Phi^{-1}\Phi e_{\mathcal{S}}\Phi^{-1}\Phi b\Phi^{-1} = \varphi(a)\varphi(e_{\mathcal{S}})\varphi(b),$$

we see that $\varphi(ae_{\mathcal{S}}b)$ is decomposable ([KR97b] Proposition 14.1.8). For all $b \in \mathcal{R}$, for almost all $p \in Y$ and for all $x \in \mathfrak{H}'_p$, using the expression for $\varphi(e_{\mathcal{S}})$ in Proposition 4.4.8, since $[\varphi(e_{\mathcal{S}})\varphi(b)](p) = \varphi(e_{\mathcal{S}})\varphi(b)(p)$ a.e. ([KR97b] Proposition 14.1.8 (ii)), we have

$$[\varphi(e_{\mathcal{S}})\varphi(b)](p)x = \varphi(e_{\mathcal{S}})(p)\varphi(b)(p)x = \langle\varphi(b)(p)x, \Gamma(p)\rangle\Gamma(p).$$

Thus,

$$[\varphi(ae_{\mathcal{S}}b)(p)]x = \varphi(a)(p)\langle\varphi(b)(p)x, \Gamma(p)\rangle\Gamma(p) = \langle\varphi(b)(p)x, \Gamma(p)\rangle\varphi(a)(p)\Gamma(p).$$

□

We make an observation that will be useful in the following result. Let $\{e_i\}$ be an orthonormal basis for a finite k-dimensional Hilbert space $\mathfrak{H}$. Let v, $w \in \mathfrak{H}$.

$$\begin{aligned} \sum_{i=1}^{k}\langle v, e_i\rangle\langle e_i, w\rangle &= \sum_{i=1}^{k}\langle\langle v, e_i\rangle e_i, w\rangle \\ &= \left\langle \sum_{i=1}^{k}\langle v, e_i\rangle e_i, w \right\rangle. \end{aligned} \tag{4.4.9}$$

However, $v = \sum_{i=1}^{k}\langle v, e_i\rangle e_i$, so

$$\sum_{i=1}^{k}\langle v, e_i\rangle\langle e_i, w\rangle = \langle v, w\rangle. \tag{4.4.10}$$

Similarly, if $\mathfrak{H}$ is separable and infinite dimensional, let (x_i) be a total orthonormal sequence for $\mathfrak{H}$. As before, let v and w belong to $\mathfrak{H}$. From the proof of Theorem 3.6-3

[Kre78], write $v = \sum_{i=1}^{\infty} \langle v, e_i \rangle e_i$. Hence,

$$
\begin{aligned}
\sum_{i=1}^{\infty} \langle v, e_i \rangle \langle e_i, w \rangle &= \lim_{k \to \infty} \sum_{i=1}^{k} \langle v, e_i \rangle \langle e_i, w \rangle \\
&= \lim_{k \to \infty} \left\langle \sum_{i=1}^{k} \langle v, e_i \rangle e_i, w \right\rangle \\
&= \left\langle \lim_{k \to \infty} \sum_{i=1}^{k} \langle v, e_i \rangle e_i, w \right\rangle \\
&= \left\langle \sum_{i=1}^{\infty} \langle v, e_i \rangle e_i, w \right\rangle \\
&= \langle v, w \rangle.
\end{aligned} \tag{4.4.11}
$$

Proposition 4.4.10. *With the assumptions of Proposition 4.3.3, for any $p \in Y$, consider $\mathfrak{H}'_p$, a Hilbert space fibre in the direct integral $\mathfrak{H}_{\oplus}^{\mathcal{S}} = \int_Y^{\oplus} \mathfrak{H}'_p \, \mathrm{d}\nu$. Let tr denote the canonical trace on $B(\mathfrak{H}'_p)$ (see for instance [Mur90] §2.4). Then, for all $a, b \in \mathcal{R}$ and for almost all $p \in Y$,*

$$
\mathrm{tr}(\varphi(ae_{\mathcal{S}}b))(p)) = \langle \varphi(a)(p)\Gamma(p), \varphi(b^*)(p)\Gamma(p) \rangle.
$$

Proof. Let (x_n) be a total orthonormal set in $\mathfrak{H}'_p$, if $\dim(\mathfrak{H}'_p) = \infty$, or a sequence consisting of an orthonormal basis followed by 0's if $\dim(\mathfrak{H}'_p) < \infty$. From [Mur90] §2.4 and Corollary 4.4.9,

$$
\begin{aligned}
\mathrm{tr}(\varphi(ae_{\mathcal{S}}b))(p)) &= \sum_{n=1}^{\infty} \langle \varphi(ae_{\mathcal{S}}b))(p)x_n, x_n \rangle \\
&= \sum_{n=1}^{\infty} \langle \langle \varphi(b)(p)x_n, \Gamma(p) \rangle \varphi(a)(p)\Gamma(p), x_n \rangle \\
&= \sum_{n=1}^{\infty} \langle \varphi(b)(p)x_n, \Gamma(p) \rangle \langle \varphi(a)(p)\Gamma(p), x_n \rangle \\
&= \sum_{n=1}^{\infty} \langle \varphi(a)(p)\Gamma(p), x_n \rangle \langle x_n, \varphi(b)^*(p)\Gamma(p) \rangle \\
&= \langle \varphi(a)(p)\Gamma(p), \varphi(b)^*(p)\Gamma(p) \rangle = \langle \varphi(a)(p)\Gamma(p), \varphi(b^*)(p)\Gamma(p) \rangle
\end{aligned}
$$

using (4.4.10) or (4.4.11), depending on whether $\mathfrak{H}'_p$ has finite or infinite dimension. □

4.5 Decomposing the Basic Construction and the Finite Lifted Trace

We now strengthen our assumptions: we require a trace τ in place of a state ρ. This is in order to ensure that the modular conjugation operator J from §1.4 is continuous.

The following result lays the foundation for us to show that the basic construction $\langle\mathcal{M}, e_{\mathcal{N}}\rangle$ (§1.6) is decomposable.

Proposition 4.5.1. *Recall that $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal trace τ represented on its GNS space $\mathfrak{H}_\tau$ and that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with trace $\tau|_{\mathcal{N}}$. Thus, the results of the previous three sections apply with $\mathcal{R}$ replaced with $\mathcal{M}$ and $\mathcal{S}$ replaced with $\mathcal{N}$, respectively. Suppose further that $\mathcal{N}$ is central. Then, for all $n \in \mathcal{N}$, $Jn^*J = n$, and $\mathcal{N} = J\mathcal{N}J$.*

Proof. Let $n \in \mathcal{N}$ and $m \in \mathcal{M}$. Then,

$$Jn^*Jm\Omega = Jn^*m^*\Omega = J(mn)^*\Omega = (mn)\Omega = nm\Omega,$$

as $n \in \mathcal{M}'$. Hence, $Jn^*J = n$. Thus, $J\mathcal{N}J = \mathcal{N}$. □

Proposition 4.5.2. *We use the assumptions and notation in Proposition 4.5.1. From Proposition 4.3.3, with $\mathcal{N}$ replacing $\mathcal{S}$, let Φ be the unitary such that $\varphi(\mathcal{N}) := \Phi\mathcal{N}\Phi^{-1}$ is the von Neumann algebra of all diagonalisable operators on the direct integral $\mathfrak{H}_\oplus^{\mathcal{N}} = \int_Y^\oplus \mathfrak{H}_p' \,\mathrm{d}\nu$ with domain of integration (Y, Σ, ν). Recall that $\langle\mathcal{M}, e_{\mathcal{N}}\rangle$ denotes the basic construction (§1.6). Then $\varphi(\langle\mathcal{M}, e_{\mathcal{N}}\rangle) := \Phi\langle\mathcal{M}, e_{\mathcal{N}}\rangle\Phi^{-1}$ is a decomposable von Neumann algebra and $\varphi(\langle\mathcal{M}, e_{\mathcal{N}}\rangle) = \varphi(\mathcal{N})'$ is the von Neumann algebra of all decomposable operators.*

Proof. As $\varphi(\mathcal{N})$ is the von Neumann algebra of all diagonalisable operators, from Proposition 4.2.2 (with the role of $\mathcal{S}$ replaced here by $\mathcal{N}$), $\varphi(\mathcal{N})'$ is the von Neumann algebra of all decomposable operators (Theorem 4.1.8). From Proposition 1.6.2 (c), we have $(J\mathcal{N}J)' = \langle\mathcal{M}, e_{\mathcal{N}}\rangle$. Using Proposition 4.5.1, we have $\mathcal{N}' = \langle\mathcal{M}, e_{\mathcal{N}}\rangle$. Hence, from Proposition 4.3.2:

$$\varphi(\mathcal{N})' = \varphi(\mathcal{N}') = \varphi(\langle\mathcal{M}, e_{\mathcal{N}}\rangle). \tag{4.5.1}$$

As $\varphi(\langle\mathcal{M}, e_{\mathcal{N}}\rangle)$ is a von Neumann algebra of decomposable operators, it is a decomposable von Neumann algebra (Proposition 4.1.11). □

Proposition 4.5.3. *We use the notation and assumptions from Proposition 4.5.2. Recall that $\overline{\tau}$ is the finite lifted trace of the basic construction $\langle\mathcal{M}, e_{\mathcal{N}}\rangle$ (after Remark 1.6.3). Then, using Proposition 4.4.10, we have for all $a, b \in \mathcal{M}$,*

$$\overline{\tau}(ae_{\mathcal{N}}b) = \int_Y \mathrm{tr}(\varphi(ae_{\mathcal{N}}b)(p)) \,\mathrm{d}\nu.$$

Proof. We have from (1.6.1) and the tracial property of τ, $\overline{\tau}(ae_{\mathcal{N}}b) = \tau(ab) = \tau(ba) = \tau(b^{**}a)$. So,

$$\begin{aligned}\overline{\tau}(ae_{\mathcal{N}}b) &= \langle a\Omega, b^*\Omega\rangle = \langle \varphi(a)\Gamma, \varphi(b^*)\Gamma\rangle \\ &= \int_Y \langle \varphi(a)(p)\Gamma(p), \varphi(b^*)(p)\Gamma(p)\rangle \, \mathrm{d}\nu \\ &= \int_Y \mathrm{tr}(\varphi(ae_{\mathcal{S}}b))(p)) \, \mathrm{d}\nu.\end{aligned}$$

We used Proposition 4.4.10 for the last equality. □

Proposition 4.5.4. *We use the notation and assumptions in Propositions 4.5.1, 4.5.2 and 4.5.3. For all* $a \in \langle \mathcal{M}, e_{\mathcal{N}}\rangle^+$,

$$\overline{\tau}(a) = \int_Y \mathrm{tr}(\varphi(a)(p)) \, \mathrm{d}\nu.$$

Proof. Consider the following function $\Xi : \langle \mathcal{M}, e_{\mathcal{N}}\rangle^+ \to [0, \infty]$:

$$\Xi(a) := \int_Y \mathrm{tr}(\varphi(a)(p)) \, \mathrm{d}\nu. \tag{4.5.2}$$

We shall see shortly, that Ξ is well-defined. Note that we can extend the domain of Ξ to also include $\mathcal{M}_{e_{\mathcal{N}}}$, the set of all linear combinations of $ae_{\mathcal{N}}b$ (see also Proposition 4.5.3). We show that Ξ is a weight on $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$ such that $\Xi = \overline{\tau}$ on $\mathcal{M}_{e_{\mathcal{N}}}$. To show that Ξ is normal we show that it is the (possibly infinite) sum of positive normal linear functionals on $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$ ([SS08] pp. 56-57). It will then follow from Theorem 2.3.4 that $\Xi = \overline{\tau}$ on $\langle \mathcal{M}, e_{\mathcal{N}}\rangle^+$.

From [Dix81] Proposition 1 Ch 1 Part II, there exists a countable set of vectors $\{x_n\}$ contained in $\mathfrak{H}_{\oplus}^{\mathcal{N}}$, such that for almost all $p \in Y$, the non-zero elements of $\{x_n(p)\}$ form an orthonormal basis for $\mathfrak{H}'_p$. This allows us to express the function $p \mapsto \mathrm{tr}(\varphi(a)(p))$ almost everywhere as

$$p \mapsto \sum_{n=1}^{\infty} \langle \varphi(a)(p)x_n(p), x_n(p)\rangle. \tag{4.5.3}$$

As $\langle \varphi(a)(p)x_n, x_n\rangle \geq 0$, for each $t \in \mathbb{N}$, $p \mapsto \sum_{n=1}^{t} \langle \varphi(a)(p)x_n(p), x_n(p)\rangle$ is an element in a monotone increasing sequence (in t). Moreover $p \mapsto \langle \varphi(a)(p)x_n(p), x_n(p)\rangle$ is measurable for every a and x_n and therefore (4.5.3) is a measurable function, being the pointwise limit of a sequence of increasing measurable functions ([Rud87] Theorem

1.14). In particular, Ξ is well-defined. Thus, the monotone convergence theorem yields

$$\begin{aligned}\Xi(a) &= \int_Y \sum_{n=1}^{\infty} \langle \varphi(a)(p)x_n(p), x_n(p)\rangle \,\mathrm{d}\nu = \sum_{n=1}^{\infty} \int_Y \langle \varphi(a)(p)x_n(p), x_n(p)\rangle \,\mathrm{d}\nu \\ &= \sum_{n=1}^{\infty} \langle \varphi(a)x_n, x_n\rangle.\end{aligned}$$

Using Proposition 4.3.4, $\langle \varphi(a)x_n, x_n\rangle$ is normal for each $n \in \mathbb{N}$. Thus Ξ is the sum of positive normal linear functionals.

It is easy to check that Ξ is a weight. We use [KR97b] Theorem 14.1.8. For every $u, w \in \varphi(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$, with the convention that $\infty + \infty = \infty$:

$$\begin{aligned}\Xi(u+w) &= \int_Y \mathrm{tr}(u(p)+w(p)) \,\mathrm{d}\nu = \int_Y \mathrm{tr}(u(p)) + \mathrm{tr}(w(p)) \,\mathrm{d}\nu \\ &= \int_Y \mathrm{tr}(u(p)) \,\mathrm{d}\nu + \int_Y \mathrm{tr}(w(p)) \,\mathrm{d}\nu = \Xi(u) + \Xi(w).\end{aligned}$$

and for $k > 0$,

$$\Xi(ku) = \int_Y \mathrm{tr}(ku(p)) \,\mathrm{d}\nu = \int_Y k\mathrm{tr}(u(p)) \,\mathrm{d}\nu = k\int_Y \mathrm{tr}(u(p)) \,\mathrm{d}\nu = k\Xi(u).$$

Lastly, we recall, from Proposition 4.5.3, that for all $a, b \in \mathcal{M}$,

$$\Xi(ae_{\mathcal{N}}b) = \overline{\tau}(ae_{\mathcal{N}}b),$$

so $\Xi|_{\mathcal{M}_{e_{\mathcal{N}}}} = \overline{\tau}|_{\mathcal{M}_{e_{\mathcal{N}}}}$. From our remarks after (4.5.2), we have shown that $\overline{\tau}(a) = \int_Y \mathrm{tr}(a(p)) \,\mathrm{d}\nu$ for all positive $a \in \langle \mathcal{M}, e_{\mathcal{N}}\rangle$. ☐

4.6 Decomposing the Range of Finite Lifted Projections

We first show sufficient conditions for a closed subspace of $\mathfrak{H}_\oplus$ to be a direct integral of Hilbert spaces.

Proposition 4.6.1. *Let $\mathfrak{G}$ be a closed subspace of a direct integral $\mathfrak{H}_\oplus = \int_X \mathfrak{H}_p \,\mathrm{d}\mu$ over (X, Σ, μ). Suppose there is a decomposable projection P on $\mathfrak{H}_\oplus$ with range $\mathfrak{G}$. Then $\mathfrak{G} = \int_Y^\oplus P(p)(\mathfrak{H}_p) \,\mathrm{d}\mu$.*

Proof. We use Definition 4.1.1 and Remark 4.1.2. The assumption of P being decomposable means that for almost all $p \in Y$, $P(p)$ is a projection ([KR97b] Lemma 14.1.20).

So, for almost all p, $\mathfrak{G}_p := P(p)\mathfrak{H}_p$ is a closed subspace. For convenience, for those values of p for which $P(p)$ is not a projection, set $\mathfrak{G}_p = \{0\}$. We also note that for every $x \in \mathfrak{G}$, there is an $w \in \mathfrak{H}_\oplus$ such that $Pw = x$ and therefore $x(p) \in P(p)\mathfrak{H}_p$, a.e.

The measurability and integrability conditions on $p \mapsto \langle x(p), y(p)\rangle$ for every $y \in \mathfrak{G}$ are trivially satisfied because $\mathfrak{G} \subseteq \mathfrak{H}_\oplus$.

For every $p \in X$, $u_p \in \mathfrak{G}_p$, suppose that

$$p \mapsto \langle u_p, y(p)\rangle \tag{4.6.1}$$

is μ-integrable for every $y \in \mathfrak{G}$. Consider, any $x \in \mathfrak{H}_\oplus^{\mathcal{S}}$. Then, for almost all p,

$$\begin{aligned}\langle u_p, x(p)\rangle &= \langle P(p)u_p, x(p)\rangle = \langle u_p, P(p)x(p)\rangle \\ &= \langle u_p, (Px)(p)\rangle,\end{aligned}$$

which is μ-integrable by (4.6.1). Hence, from Definition 4.1.1, there is a $u \in \mathfrak{H}_\oplus^{\mathcal{S}}$ such that

$$u(p) = u_p,$$

for almost all p. So,

$$\begin{aligned}&(Pu)(p) = P(p)u(p) = P(p)u_p = u_p = u(p) \qquad \text{almost everywhere} \\ \implies &\|Pu - u\|^2 = \int_Y \|(Pu)(p) - u(p)\|^2 \,\mathrm{d}\mu = 0 \\ \implies &u = Pu \in \mathfrak{G}.\end{aligned}$$

□

Proposition 4.6.2. *We use the notation and assumptions in Propositions 4.5.1, 4.5.2 and 4.5.3. Suppose that* $P \in \langle \mathcal{M}, e_{\mathcal{N}}\rangle$*, so that* $\varphi(P)$ *is a decomposable operator in* $\varphi(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$ *(Proposition 4.5.2). Then*

$$\overline{\tau}(P) = \int_Y \dim(\varphi(P)(p)\mathfrak{H}'_p)\,\mathrm{d}\nu,$$

where $\dim(G)$ *is the Hilbert space dimension of the Hilbert space* G*.*

Proof. Theorem 4.5.4 yields

$$\overline{\tau}(P) = \int_Y \mathrm{tr}(\varphi(P)(p))\,\mathrm{d}\nu.$$

As remarked in the proof of Proposition 4.6.1, we have that for almost every p, $\varphi(P)(p)$ is a projection. Hence, from [Dix81] Theorem 5 Chapter 6 Part I, $\mathrm{tr}(\varphi(P)(p)) = \dim(\varphi(P)(p)\mathfrak{H}'_p)$ almost everywhere. □

Let us apply the two results above to the P_V mentioned in the statement of Lemma 4.0.2. As $P_V \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$, $\varphi(P_V)$ is decomposable (Proposition 4.5.2). Applying Proposition 4.6.1 to ΦV, we have

$$\mathfrak{L} := \Phi V = \int_Y^{\oplus} \mathfrak{L}_p \, \mathrm{d}\nu, \tag{4.6.2}$$

where $\mathfrak{L}_p = \varphi(P_V)(p)(\mathfrak{H}'_p)$. We have from Proposition 4.6.2,

$$\overline{\tau}(P_V) = \int_Y \dim(\mathfrak{L}_p) \, \mathrm{d}\nu. \tag{4.6.3}$$

Corollary 4.6.3. *Continuing with the notation and assumptions of Proposition 4.6.2, P_V is of finite lifted trace if and only if the map $p \mapsto \dim(\mathfrak{L}_p)$ in (4.6.3) is ν-integrable.* □

Definition 4.6.4. Let $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Consider the map $p \mapsto \dim(\mathfrak{L}_p)$ from (4.6.3). Let

$$X_k := \{p \in Y \mid \dim(\mathfrak{L}_p) = k\},$$

and

$$X_{\leq k} := \{p \in Y \mid \dim(\mathfrak{L}_p) \leq k\}.$$

Analogously, define $X_{\leq k}, X_{\geq k}$ etc.

We remark that X_k (and therefore $X_{\leq k}$) are measurable sets, using similar arguments to [KR97b] Remark 14.1.5. □

Proposition 4.6.5. *We use the notation from Corollary 4.6.3 and Definition 4.6.4. For every $\epsilon > 0$, there exists a sufficiently large $r \in \mathbb{N}$ such that*

$$\int_{X_{>r}} \dim(\mathfrak{L}_p) \, \mathrm{d}\nu < \epsilon.$$

Proof. For convenience denote by $f : Y \mapsto \mathbb{N} \cup \{\infty\}$ the map $p \mapsto \dim(\mathfrak{L}_p)$. For each $n \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, let

$$f_n := \chi_{X_n} f \qquad .$$

Now consider $X := \cup_{n=0}^{\infty} X_n$ (that is, the union of all the X_n's except X_∞) and the function

$$g := \chi_X f.$$

As $\overline{\tau}(P_V) = \int_Y f \, \mathrm{d}\nu < \infty$, we have that $\nu(Y \backslash X) = 0$. Thus, $\int_Y g \, \mathrm{d}\nu = \int_Y f \, \mathrm{d}\nu$.

Note that (f_n) is an monotonically increasing sequence with limit function g. Thus, the monotone convergence theorem implies that

$$\lim_n \int f_n \, \mathrm{d}\nu = \int_Y g \, \mathrm{d}\nu.$$

In other words, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $N < r < \infty$ then

$$\epsilon > \int_Y f \, \mathrm{d}\nu - \int_Y f_r \, \mathrm{d}\nu = \int_{Y \setminus X_r} f \, \mathrm{d}\nu = \int_{X_{>r}} \dim(\mathfrak{L}_p) \, \mathrm{d}\nu.$$

□

4.7 Decomposing the Dynamics

We now wish to represent the dynamics $U_\alpha : \mathfrak{H}_\tau \to \mathfrak{H}_\tau$ (see Chapter 2 §2.2) on the Hilbert space $\mathfrak{H}_\oplus^\mathcal{N}$.

Roughly speaking, given a vector $(p \mapsto x(p)) \in \mathfrak{H}_\oplus^\mathcal{N}$, the dynamics may permute values p whose corresponding Hilbert space fibres are of the same dimension, before modifying each value $x(p)$. This is where Remark 4.1.5 is used in order to ensure that $\Phi U_\alpha \Phi^{-1}(p)$ is well-defined on each fibre $\mathfrak{H}'_p$.

We wish to set up a $*$-isomorphism between $\mathcal{N}$ and $L^\infty(\nu)$. Note that the map

$$\vartheta : L^\infty(\nu) \ni f \mapsto M_f \in \varphi(\mathcal{N}) \tag{4.7.1}$$

is a $*$-isomorphism. Recall that we have assumed that the $*$-automorphism α on $\mathcal{M}$ when restricted to $\mathcal{N}$ is a $*$-automorphism of $\mathcal{N}$. Define $A : L^\infty(\nu) \to L^\infty(\nu)$ by the prescription

$$A(f) = \vartheta^{-1} \circ \varphi \circ \alpha \circ \varphi^{-1} \circ \vartheta(f) \qquad f \in L^\infty(\nu). \tag{4.7.2}$$

In order to describe A as accurately as possible, we shall need the following definition. Note that we have specialised the definition to $L^\infty(\nu)$.

Definition 4.7.1 ([MS93] Example 2.1.10)**.** Let (X, Σ, ν) be a measure space and $Z \subseteq X$ a measurable subset. Let $T : Z \to X$ be a measurable transformation. Define, for each $f \in L^\infty(\mu)$,

$$C_T(f)(x) := \begin{cases} f \circ T(x) & \text{if } x \in Z \\ 0 & \text{otherwise.} \end{cases} \tag{4.7.3}$$

If C_T is continuous on $L^\infty(\mu)$, we say that C_T is a **generalised composition operator**. When $Z = X$, then we just speak of C_T being a **composition operator**. □

The following proposition is the consequence of results from descriptive set theory ([MS93] Corollary 2.1.14).

Proposition 4.7.2. *The linear operator A in* (4.7.2) *is a composition operator C_S for some ν-preserving measurable map $S : Y \to Y$.*

Proof. Note that, because ϑ, ϕ, and α have inverses, that A is a $*$-isomorphism and therefore, in particular, $A(fg) = A(f)A(g)$ for $f, g \in L^\infty(\nu)$. As Y is a completely separable metrizable space (i.e Y is a **Polish space**) (Theorem 4.2.1) we have that A is a generalised composition operator C_S for some measurable mapping $S : \mathcal{D}(S) \subseteq Y \to Y$ ([MS93] Corollary 2.1.14). Furthermore, A is injective and $A\{\chi_K \mid K \in \Psi\} \subseteq \{\chi_K \mid K \in \Psi\}$. It follows from the remarks after [MS93] Corollary 2.1.14 that A is a composition operator.

Moreover, S preserves the measure ν. Indeed, from Equation 4.7.2, we have for every $f \in L^\infty(\nu)$,

$$\begin{aligned} A(f) &= \vartheta^{-1}\varphi\alpha\varphi^{-1}\vartheta(f) \\ f \circ S &= \vartheta^{-1}\varphi\alpha\varphi^{-1}\vartheta(f) \\ \varphi^{-1}\vartheta(f \circ S) &= \alpha\varphi^{-1}\vartheta(f) \end{aligned} \tag{4.7.4}$$

Recall that $\tau \circ \alpha = \tau$, and so using ω from Proposition 4.3.6, as well as (4.7.4), we compute,

$$\omega(\vartheta(f \circ S)) = \tau \circ \varphi^{-1}\vartheta(f \circ S) = \tau\alpha\varphi^{-1}\vartheta(f) = \tau \circ \varphi^{-1}\vartheta(f) = \omega(\vartheta(f)). \tag{4.7.5}$$

Now, Proposition 4.3.6 gives $\omega(M_f) = \langle M_f\Gamma, \Gamma\rangle$ for all $f \in L^\infty(\nu)$, and therefore,

$$\begin{aligned} \omega(\vartheta(f)) &= \omega(M_f) = \langle M_f\Gamma, \Gamma\rangle \\ &= \int_Y \langle f(p)\Gamma(p), \Gamma(p)\rangle \,\mathrm{d}\nu = \int_Y f(p) \left\|\Gamma(p)\right\|^2 \,\mathrm{d}\nu \\ &= \int_Y f(p)\,\mathrm{d}\nu. \end{aligned}$$

Therefore, for any $K \in \Psi$, using (4.7.5) (with f replaced by χ_K),

$$\int \chi_K \circ S \,\mathrm{d}\nu = \int \chi_K \,\mathrm{d}\nu,$$

and so, it follows that $\nu(S^{-1}(K)) = \nu(K)$, i.e. S is ν-preserving.

□

The operator $C_S : L^\infty(\nu) \to L^\infty(\nu)$ is invertible, since A is. Define

$$\overline{C}_S : L^2(\nu) \to L^2(\nu) : f \mapsto f \circ S \tag{4.7.6}$$

Note, using the "four step process", that

$$\begin{aligned} \left\|\overline{C}_S f\right\|^2 &= \int_Y |f \circ S|^2 \,\mathrm{d}\nu = \int_Y |f|^2 \,\mathrm{d}(\nu \circ S^{-1}) \\ &= \int_Y |f|^2 \,\mathrm{d}\nu = \|f\|^2 . \end{aligned} \tag{4.7.7}$$

Furthermore, $\overline{C}_S$ is surjective. Indeed, for any $g \in L^2(\nu)$, let (g_n) be a sequence in $L^\infty(\nu)$ such that $g = \lim_n g_n$. Thus, there is a sequence (f_n) in $L^\infty(\nu)$ such that for all $n \in \mathbb{N}$,

$$C_S(f_n) = g_n.$$

Since C_S is isometric, and (g_n) is Cauchy, it follows that (f_n) is Cauchy in $L^2(\nu)$. Let $f = \lim_n f_n$. Therefore,

$$\overline{C}_S(f) = \lim_n C_S(f_n) = \lim_n g_n = g.$$

So, $\overline{C}_S$ is bijective. From [MS93] Theorem 2.2.14, S is invertible. We now ensure that S preserves the sets Y_n from Definition 4.7.5.

We single out a special case of Proposition 4.6.1.

Definition 4.7.3. We use the notation in Proposition 4.5.2. Recall that (Y, Ψ, ν) is the probability measure space serving as the domain of integration for the direct integral Hilbert space $\mathfrak{H}_\oplus^\mathcal{N} = \int_Y \mathfrak{H}'_p \, \mathrm{d}\nu$. Let $K \in \Psi$ and consider $M_K := M_{\chi_K}$ the diagonalisable projection corresponding to K. Define, using Proposition 4.6.1 (with $P = M_K$),

$$\int_K^\oplus \mathfrak{H}'_p \, \mathrm{d}\nu := M_K(\mathfrak{H}_\oplus^\mathcal{N}) = \int_Y M_K(p)\mathfrak{H}'_p \, \mathrm{d}\nu.$$

□

Remark 4.7.4. We use the notation of Definition 4.7.3 above.

Note that $\int_K^\oplus \mathfrak{H}'_p \, \mathrm{d}\nu$ is a $\varphi(\mathcal{N})$-submodule. Indeed, for every element in $\varphi(\mathcal{N})$ there is a corresponding diagonalisable operator M_f for some $f \in L^\infty(\nu)$. Proposition 14.1.8 [KR97b] ensures that $M_f M_K$ is decomposable and hence,

$$(M_f M_K x)(p) = \begin{cases} f(p)x(p) & \text{if } p \in K \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $M_f M_K x \in \int_K^\oplus \mathfrak{H}'_p \, \mathrm{d}\nu$. □

Definition 4.7.5. Let $k \in \mathbb{N}$. Consider the direct integral $\mathfrak{H}_\oplus^\mathcal{N} = \int_Y^\oplus \mathfrak{H}'{}_p \, \mathrm{d}\nu$ from Proposition 4.5.2. Let

$$Y_k := \{p \in Y \mid \dim(\mathfrak{H}'_p) = k\},$$

and

$$Y_{>k} := \{p \in Y \mid \dim(\mathfrak{H}'_p) > k\}.$$

□

In the proof of Propositions 4.7.6 and 4.7.8, we deviate from the idea of a proof outlined at the bottom of p. 33 [AET11].

Proposition 4.7.6. *We continue from Remark 4.7.4. Let S be the map obtained in Proposition 4.7.2. Then, for $n \in \mathbb{N} \cup \{\infty\}$, with Y_n defined in Definition 4.7.5, $SY_n = Y_n$, up to a set of measure zero.*

Proof. Let $n \in \mathbb{N} \cup \{\infty\}$ be arbitrary. Let $K \in \Psi$ satisfy $K \subseteq Y_n$. Put $M_K = M_{\chi_K} \in \varphi(\mathcal{N}) \subseteq \varphi(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$. Then M_K satisfies the hypotheses of Proposition 4.6.2. Hence,

$$\begin{aligned} \overline{\tau}(\varphi^{-1}(M_K)) &= \int_Y \dim(M_K(p)\mathfrak{H}'_p)\,\mathrm{d}\nu \\ &= \int_K \dim(M_K(p)\mathfrak{H}'_p)\,\mathrm{d}\nu + \int_{Y\setminus K} \dim(M_K(p)\mathfrak{H}'_p)\,\mathrm{d}\nu \\ &= \int_K \dim(\mathfrak{H}'_p)\,\mathrm{d}\nu + 0 \\ &= \int_K n\,\mathrm{d}\nu = n\nu(K). \end{aligned} \tag{4.7.8}$$

On the other hand, as $\overline{\tau}$ is $\overline{\alpha}$-invariant,

$$\begin{aligned} \overline{\tau}(\varphi^{-1}(M_K)) &= \overline{\tau}(\overline{\alpha}(\varphi^{-1}(M_K))) \\ &= \overline{\tau}(\alpha(\varphi^{-1}(M_K))) && \text{as } \overline{\alpha} = \alpha \text{ on } \mathcal{N} \\ &= \overline{\tau}(\varphi^{-1}(M_{\chi_K \circ S})) && \text{from (4.7.4)} \\ &= \int_{S^{-1}(K)} \dim(\mathfrak{H}'_p)\,\mathrm{d}\nu. \end{aligned} \tag{4.7.9}$$

In particular, for $K = Y_n$, the above equations (4.7.8) and (4.7.9) give us

$$n\nu(Y_n) = \int_{S^{-1}(Y_n)} \dim(\mathfrak{H}_p)\,\mathrm{d}\nu.$$

Consider $\nu(Y_n \Delta S^{-1}(Y_n))$, where for sets A and B, $A\Delta B$ is the symmetric difference $A\Delta B := (A\setminus B) \cup (B\setminus A)$. To prove Proposition 4.7.6 we will need to show that $\nu(Y_n \Delta S^{-1}(Y_n)) = 0$.

If $\nu(Y_n \Delta S^{-1}(Y_n)) > 0$, then as the Y_n's partition Y, there is some $j \neq n$, such that $\nu(Y_j \cap S^{-1}(Y_n)) > 0$. Consider, in place of K in (4.7.8) and (4.7.9),

$$K_j = S(Y_j \cap S^{-1}(Y_n)) \subseteq Y_n,$$

and note that $\nu(K_j) = \nu(S^{-1}(K_j)) = \nu(Y_j \cap S^{-1}(Y_n)) > 0$. We have

$$\overline{\tau}(\varphi^{-1}(M_{K_j})) = n\nu(K_j), \tag{4.7.10}$$

and, since $S^{-1}(K_j) \subseteq Y_j$,

$$\overline{\tau}(\varphi^{-1}(M_{K_j})) = \int_{S^{-1}(K_j)} \dim(\mathfrak{H}_p)\,\mathrm{d}\nu = j\nu(S^{-1}(K_j)) = j\nu(K_j), \tag{4.7.11}$$

due to S being ν-preserving (Theorem 4.7.2). Thus $n = j$, a contradiction. So, $\nu(Y_n \Delta S^{-1}(Y_n)) = 0$. □

Remark 4.7.7. We make some remarks that will be used in Proposition 4.7.8 below. We use the notation from Proposition 4.7.6 and its proof. Let $K \in \Psi$ and put $M = M_{\chi_K}$. Observe that

$$\varphi(P_V)M = M\varphi(P_V), \tag{4.7.12}$$

because $\varphi(\langle \mathcal{M}, e_{\mathcal{N}} \rangle) = \varphi(\mathcal{N})'$ (see (4.5.1)).

Now, using (4.7.12), note that $M\varphi(P_V)$ is a projection:

$$\begin{aligned}(M\varphi(P_V))^2 &= M\varphi(P_V)M\varphi(P_V) = M\varphi(P_V)\varphi(P_V)M \\ &= M\varphi(P_V)M = M^2\varphi(P_V) = M\varphi(P_V),\end{aligned}$$

and,

$$(M\varphi(P_V))^* = \varphi(P_V)^*M^* = \varphi(P_V)M = M\varphi(P_V).$$

□

We define some notation, similar to Definition 4.7.5, that will be useful in the next result (Proposition 4.7.8) and the rest of the chapter.

Proposition 4.7.8. *We continue from Remark 4.7.7. The map* $g : Y \ni p \mapsto \dim(\mathfrak{L}_p) \in \mathbb{N} \cup \{\infty\}$*, where the* $\mathfrak{L}_p$ *are obtained in* (4.6.2)*, is preserved by* S *i.e.* $g \circ S = g$ *a.e.*

Proof. We wish to employ the same technique as in the last proposition's proof, but require some modification as it is not necessarily true that $P_V \in \mathcal{N}$.

Take any set $K \in \Psi$. Put $M_K := M_{\chi_K} \in \varphi(\mathcal{N})$. We have that $\varphi(P_V)M_K = M_K\varphi(P_V)$ (Remark 4.7.7). On the other hand, in a similar manner to Definition 4.7.3, $M_K\varphi(P_V) = \int_K \mathfrak{L}_p \, \mathrm{d}\nu$. Hence, in a similar way to (4.6.3) ($\varphi(P_V) = P_{\mathfrak{L}}$ replaced by $M_K P_{\mathfrak{L}} = P_{\mathfrak{L}_K}$, where $\mathfrak{L}_K = \int_Y^{\oplus} \mathfrak{L}_p \, \mathrm{d}\nu$ as in Definition 4.7.3),

$$\overline{\tau}(\varphi^{-1}(M_K)P_V) = \int_K \dim(\mathfrak{L}_p) \, \mathrm{d}\nu.$$

The $\overline{\alpha}$-invariance of $\overline{\tau}$ enables us to write,

$$\overline{\tau}(P_V\varphi^{-1}(M_K)) = \overline{\tau}(\overline{\alpha}(P_V\varphi^{-1}(M_K))) = \overline{\tau}(P_V\varphi^{-1}(M_{S^{-1}(K)})) = \int_{S^{-1}(K)} \dim(\mathfrak{L}_p) \, \mathrm{d}\nu,$$

since $\overline{\alpha}(P_V) = P_V$ (as seen at the end of Lemma 3.0.3's proof at the end of §3.6). Define, for each $n \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, $X_n := \{p \in Y \mid \dim(\mathfrak{L}_p) = n\}$. The X_n's are measurable, as mentioned in Definition 4.6.4. In a similar manner to what was done before in Proposition 4.7.6, $\nu(X_n \Delta S^{-1}(X_n)) = 0$. So, S preserves the X_n (up to a set of measure zero) and we have, for almost all p,

$$g \circ S|_{X_n} = g|_{X_n} \circ S|_{X_n} = g|_{X_n}. \tag{4.7.13}$$

As the X_n's form a partition of Y, we have $g \circ S = g$ a.e. □

We now transfer U_α the action acting on $\mathfrak{H}_\tau$ onto $\mathfrak{H}_\oplus^{\mathcal{N}}$.

Definition 4.7.9. We continue from Proposition 4.7.8. Recall that $\mathfrak{H}_\oplus^{\mathcal{N}}$ is the Hilbert space, Φ is the unitary and φ the map all mentioned in Proposition 4.5.2. Consider U_α and $\overline{\alpha}$ from §2.4. Define the map $U_\beta : \mathfrak{H}_\oplus^{\mathcal{N}} \to \mathfrak{H}_\oplus^{\mathcal{N}}$ by the prescription

$$U_\beta := \Phi U_\alpha \Phi^{-1},$$

Define $\beta : \varphi(\langle \mathcal{M}, e_{\mathcal{N}} \rangle) \to \varphi(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, by the prescription

$$\beta := \varphi \circ \overline{\alpha} \circ \varphi^{-1}.$$

□

It is clear that U_β is a unitary operator and, comparing the definition of β with that of $\overline{\alpha}$ in (2.3.1) and the results thereafter, that β is a $*$-isomorphism.

Remark 4.7.10. The fact that Φ is a unitary allows us to transfer many properties of the dynamics to the direct integral framework. For instance, a closed subspace $W \subseteq \mathfrak{H}_\tau$ is U_α-invariant if and only if $\Phi(W)$ is U_β-invariant. Indeed, for all $x \in W$,

$$U_\alpha x \in W \iff \Phi U_\alpha \Phi^{-1} \Phi x \in \Phi(W) \iff U_\beta \Phi(x) \in \Phi(W).$$

□

We require in Proposition 4.7.11 that the Hilbert space fibres of the same dimension are identified as the same Hilbert space (Remark 4.1.5).

Proposition 4.7.11. *With the notation of Proposition 4.5.2, we continue from Definition 4.7.9. Let S be the function obtained in Proposition 4.7.2. There exists a measurable section of unitary operators $p \mapsto \Upsilon(p)$ such that for all $x \in \mathfrak{H}_\oplus^{\mathcal{N}}$,*

$$(U_\beta x)(p) = \Upsilon(p)(x(Sp)).$$

Proof. For every diagonalisable operator $M_f \in \varphi(\mathcal{N})$, recalling that $\overline{\alpha}$ agrees with α on $\mathcal{N}$ (compare (2.2.7) and (2.3.1)), we use (4.7.4) (with $\vartheta(f) = M_f$) to see that,

$$\begin{aligned}\beta(M_f) &= \Phi\overline{\alpha}(\varphi^{-1}(M_f))\Phi^{-1} = \Phi\alpha(\varphi^{-1}(M_f))\Phi^{-1}\\ &= \Phi\varphi^{-1}(M_{f\circ S})\Phi^{-1} = M_{f\circ S}.\end{aligned}$$

Hence, for every $x \in \mathfrak{H}_\oplus^\mathcal{N}$,

$$(\beta(M_f)x)(p) = (M_{f\circ S}x)(p) = (f\circ S)(p)x(p).$$

Therefore, replacing p with $S^{-1}p$ above:

$$(\beta(M_f)x)(S^{-1}p) = f(p)x(S^{-1}p) \in \mathfrak{H}_{S^{-1}p}. \tag{4.7.14}$$

Define $F : \mathfrak{H}_\oplus^\mathcal{N} \to \mathfrak{H}_\oplus^\mathcal{N}$, by the prescription $(Fx)(p) = x(Sp)$. For now assume that F is a well-defined decomposable unitary operator on $\mathfrak{H}_\oplus^\mathcal{N}$, with Hilbert adjoint F^* given by $(F^*x)(p) = x(S^{-1}p)$. Using (4.7.14),

$$(F^*\beta(M_f)x)(p) = (\beta(M_f)x)(S^{-1}p) = f(p)x(S^{-1}p) = f(p)(F^*x)(p) = (M_fF^*x)(p)$$

and we have $\beta(M_f) = FM_fF^*$. In other words, $M_f = F^*U_\beta M_f(F^*U_\beta)^*$. Let $\Upsilon := F^*U_\beta$. Then Υ is a decomposable operator (Theorem 4.1.8), since $\Upsilon M_f = M_f\Upsilon$. Thus,

$$(U_\beta x)(p) = (F\Upsilon x)(p) = \Upsilon(p)x(Sp).$$

We return to the assertions made about F and show that it is a decomposable unitary operator. We check that for every $x \in \mathfrak{H}_\oplus^\mathcal{N}$ there is a vector in $\mathfrak{H}_\oplus^\mathcal{N}$ corresponding to $p \mapsto x(Sp)$. First,

$$(Fx)(p) = x(Sp) \in \mathfrak{H}'_{Sp} = \mathfrak{H}_p,$$

by Proposition 4.7.6 and Remark 4.1.5. Next, we check that Fx has suitable measurability properties. We use [Nie80]'s formulation of direct integrals to do this. From [Nie80] p. 15, $p \mapsto x(p)$ is measurable in the usual sense as a $Y \to \mathfrak{H}_\oplus^\mathcal{N}$ function. Thus, by Proposition 4.7.2, and considering the composition of measurable maps,

$$p \mapsto Sp \mapsto x(Sp),$$

we see that the map $p \mapsto x(Sp)$ is a measurable map.

In a similar manner to (4.7.7),

$$\begin{aligned}\int_Y \|x(Sp)\|^2 \,\mathrm{d}\nu &= \int_Y \|x(p)\|^2 \,\mathrm{d}(\nu\circ S^{-1})\\ &= \int_Y \|x(p)\|^2 \,\mathrm{d}\nu < \infty.\end{aligned}$$

Thus, $p \mapsto \|x(Sp)\|$ is square-integrable. So there is a vector $v \in \mathfrak{H}_\oplus^\mathcal{N}$ such that $v(p) = x(Sp)$ for almost all p ([Nie80] p.15) and hence, F is decomposable.

To see that F is unitary, we check that F is an isometry and surjective. For the isometry part, note that because S is ν-preserving. the four step process yields,

$$\|x\|^2 = \int_Y \|x(p)\|^2 \, \mathrm{d}\nu = \int_Y \|x(S(p))\|^2 \, \mathrm{d}\nu = \|Fx\|^2 .$$

For the surjectivity, note that as above one can show that there is an operator G on $\mathfrak{H}^{-1}$ such that $(Gx)(p) = x(S^{-1}p)$. Thus, for any $x \in \mathfrak{H}_\oplus^\mathcal{N}$ we have a vector $Gx \in \mathfrak{H}_\oplus^\mathcal{N}$ such that $FGx(p) = x(p)$. Finally, we note that $F^* = G$ as $(GFx)(p) = x(p)$ for all $x \in \mathfrak{H}_\oplus^\mathcal{N}$. □

Remark 4.7.12. If S is the identity on Y, then $C_S = \mathrm{id}_{L^\infty(\nu)}$ and therefore, using (4.7.2), we have that β is the identity on $\varphi(\mathcal{N})$. Thus, $U_\beta \varphi(n) U_\beta^* = \varphi(n)$ for every $n \in \mathcal{N}$ and so $U_\beta \in \varphi(\mathcal{N})' = \varphi(\langle \mathcal{M}, e_\mathcal{N} \rangle)$. Thus, U_β is a decomposable operator (Proposition 4.5.2). Reversing our argument, we have that S is the identity on Y if U_β is a decomposable operator. □

4.8 Finite Rank Modules

We now define $\mathfrak{L}_{\leq r}$, which is the direct integral representation of our V_1 (a fact which we single out explicitly in Proposition 4.8.6 below). Note that Remark 4.7.7 applies here.

Definition 4.8.1. Consider the direct integral $\mathfrak{L} = \int_Y^\oplus \mathfrak{L}_p \, \mathrm{d}\nu$ from (4.6.2). Let $P_V \in \langle \mathcal{M}, e_\mathcal{N} \rangle$ be a projection of finite lifted trace. Let $M_{\leq r}$ be the diagonalisable projection corresponding to $X_{\leq r}$ (Definition 4.6.4) i.e. $M_{\leq r} := M_{\chi X_{\leq r}}$. Define

$$\mathfrak{L}_{\leq r} := M_{\leq r} \mathfrak{L}.$$

□

Remark 4.8.2. We now show that

$$\mathfrak{L}_{\leq r} = I := \int_{X_{\leq r}}^\oplus \mathfrak{L}_p \, \mathrm{d}\nu \oplus \int_{X_{>r}}^\oplus \{0\} \, \mathrm{d}\nu, \tag{4.8.1}$$

where we have used the notation from Definition 4.6.4.

This direct integral, I, is well-defined by Proposition 4.6.1, with projection $P := M_{\leq r} \varphi(P_V)$. The projection P is indeed decomposable, since $\varphi(P_V)$ is decomposable by Proposition 1.7.1 and Proposition 4.5.2, and so $P(p) = \chi_{X_{\leq r}}(p) \varphi(P_V)(p)$. Note that

$$P(p)\mathfrak{H}'_p = \begin{cases} \mathfrak{L}_p & \text{for } p \in X_{\leq r} \\ \{0\} & \text{for } p \in Y \backslash X_{\leq r} = X_{>r}, \end{cases} \tag{4.8.2}$$

and this explains the notation on the most right-hand side of (4.8.1). Furthermore, Proposition 4.6.1 gives

$$\begin{aligned} I &= P\mathfrak{H}_{\oplus}^{\mathcal{N}} = M_{\leq r}\varphi(P_V)\mathfrak{H}_{\oplus}^{\mathcal{N}} \\ &= M_{\leq r}\Phi P_V \Phi^{-1}\mathfrak{H}_{\oplus}^{\mathcal{N}} = M_{\leq r}\Phi P_V \mathfrak{H}_{\tau} = M_{\leq r}\Phi V = M_{\leq r}\mathfrak{L}, \end{aligned}$$

using (4.6.2). □

In order to construct the right-$\mathcal{N}$-basis of Lemma 4.0.2, we will have to modify the proof technique found in [Gla03] Lemmas 9.3 and 9.4. The aforementioned result requires some terminology, which we present now.

Definition 4.8.3. Let $(X, \mathcal{X}, \mu)$ be a probability measure space. We call a collection $\mathcal{H} \subseteq \mathcal{X}$ **hereditary** when for every $A \in \mathcal{H}$ and every subset $B \in \mathcal{X}$ of A we have that B also belongs to $\mathcal{H}$. We say that a hereditary collection $\mathcal{H}$ **saturates** $\mathcal{X}$ if for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $B \in \mathcal{H}$ with $B \subseteq A$ and $\mu(B) > 0$. □

Lemma 4.8.4 ([Gla03] Lemma 3.17)**.** *If $\mathcal{H}$ is a hereditary collection which saturates $\mathcal{X}$, then there exists a countable measurable partition $\{A_i : i \in \mathbb{N}\}$, of X, with $A_i \in \mathcal{H}$ for every i.* □

Proposition 4.8.5. *We continue to use the notation and assumptions from Definitions 4.5.1 and 4.5.2. Consider $\mathfrak{L}_{\leq r}$ from Definition 4.8.1. Then, there is a set of r vectors $\{\zeta_i\}$ such that $\mathfrak{L}_{\leq r} = \overline{\sum_{i=1}^{r} \zeta_i \varphi(\mathcal{N})}$ (so that $\mathfrak{L}_{\leq r}$ is a right-$\varphi(\mathcal{N})$-submodule of finite rank at most r). The ζ_i are mutually orthogonal. The non-zero elements of $\{\zeta_1(p), \zeta_2(p), \ldots, \zeta_r(p)\}$ form an orthonormal basis for $\mathfrak{L}_p$ for almost every $p \in X_{\leq r}$.*

Proof. Let $X = \{p \in X_{\leq r} \mid \dim(\mathfrak{L}_p) > 0\}$. Let $\mathcal{X} := \{X \cap T \mid T \in \Psi\}$. For every $x \in \mathfrak{L}_{\leq r}$, define $h_X(x) = \{p \in X \mid \|x(p)\| > 0\}$. Set $\mathcal{H}_{\mathfrak{L}_{\leq r}} := \{h_X(x) \mid x \in \mathfrak{L}_{\leq r}\}$. Then, $\mathcal{H}_{\mathfrak{L}_{\leq r}} \subseteq \mathcal{X}$. We note also that $\nu(h_X(x)) > 0$ if $x \neq 0$.

We check that $\mathcal{H}_{\mathfrak{L}_{\leq r}}$ is hereditary. Let $x \in \mathfrak{L}_{\leq r}$. The set

$$h_X(x) = X \cap \{p \in Y \mid \|x(p)\| > 0\},$$

is measurable; X is just the union of measurable sets and the set $\{p \in Y \mid \|x(p)\|^2 > 0\}$ is measurable from Definition 4.1.1. Now suppose that we have an arbitrary $x_0 \in \mathfrak{L}_{\leq r}$ and a B is a measurable set such that $B \subseteq h_X(x_0)$. Let us show that there is some $x_1 \in \mathfrak{L}_{\leq r}$ such that $B = h_X(x_1)$. Indeed,

$$B = \{p \in X \mid \chi_B(p)\, \|x_0(p)\| > 0\} = \{p \in X \mid \|M_{\chi_B}(p)x_0(p)\| > 0\}.$$

So, $B = h_X(M_{\chi_B}x_0)$. Next, we show that $\mathcal{H}_{\mathfrak{L}_{\leq r}}$ saturates $\mathcal{X}$. Let $A \in \mathcal{X}$ with $\nu(A) > 0$ and suppose on the contrary that there is no such set $B \in \mathcal{H}_{\mathfrak{L}_{\leq r}}$ in Definition 4.8.3. In other words, every $x \in \mathfrak{L}_{\leq r}$ satisfies

$$\|x(p)\| = 0 \qquad \text{for almost all } p \in A. \tag{4.8.3}$$

So, for those almost all $p \in A$, we have that $\mathfrak{L}_p = (\mathfrak{L}_{\leq r})_p = \{0\}$ using [KR97b] Lemma 14.1.3. (If $x_1, x_2, \dots$ spans $\mathfrak{L}_{\leq r}$, then $x_1(p), x_2(p), \dots$ spans $\mathfrak{L}_p$ for almost all $p \in X$, so $\mathfrak{L}_p = \{0\}$ for almost all p by (4.8.3). Note that separability is used here since we take a countable intersection of the subsets of A given by (4.8.3) for each $x = x_1, x_2, \dots$.) This contradicts the definition of X and $A \subseteq X$ with $\nu(A) > 0$.

From Lemma 4.8.4, we have that there exists a countable partition of X, say $\{K_i\}$, such that $K_i \in \mathcal{H}_{\mathfrak{L}_{\leq r}}$ for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let x_i denote the element in $\mathfrak{L}_{\leq r}$ such that $K_i = h(x_i)$. Put,

$$x_i'(p) = \begin{cases} \frac{1}{\|x_i(p)\|} x_i(p) & \text{if } p \in K_i \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$f_k := \sum_{j=1}^{k} x_j'.$$

We will now show that there is a $\zeta_1 \in \mathfrak{L}_{\leq r}$ such that ζ_1 is the limit of the partial sums (f_k). (In terms of the statement of Lemma 4.0.2, we shall have $\xi_1 := \Phi^{-1}(\zeta_1)$; see the proof of Proposition 4.8.6, below.) Obtaining this ζ_1 is the first step in a Gram-Schmidt process for modules that give $\zeta_2, \zeta_3, \dots, \zeta_r$ as well.

Let us show that (f_k) is a Cauchy sequence in the Hilbert space $\mathfrak{L}_{\leq r}$.

Assume that $n > m$. Then we have,

$$\begin{aligned}
\|f_n - f_m\|^2 &= \left\| \sum_{j=m+1}^{n} x_j' \right\|^2 = \int_Y \left\| \sum_{j=m+1}^{n} x_j'(p) \right\|^2 \mathrm{d}\nu \\
&\leq \int_Y \left(\sum_{j=m+1}^{n} \|x_j'(p)\| \right)^2 \mathrm{d}\nu \\
&= \sum_{j=m+1}^{n} \int_Y \|x_j'(p)\|^2 \mathrm{d}\nu + \sum_{j \neq k} \int_Y \|x_j'(p)\| \, \|x_k'(p)\| \, \mathrm{d}\nu \\
&= \sum_{j=m+1}^{n} \int_{K_i} 1 \, \mathrm{d}\nu + 0 \\
&= \sum_{j=m+1}^{n} \nu(K_j) \qquad (4.8.4)
\end{aligned}$$

(the cross terms yield zero, because the $K_i's$ form a partition of X). Now $(\sum_{j=1}^{t} \nu(K_j))$ is a Cauchy sequence, because, by the monotone convergence theorem,

$$\infty > \nu(X) = \int_X 1 \, \mathrm{d}\nu = \lim_t \int_{\cup_{i=1}^t K_i} 1 \, \mathrm{d}\nu = \lim_t \sum_{i=1}^{t} \nu(K_i).$$

So, from (4.8.4), (f_n) is a Cauchy sequence and hence there is an $\zeta_1 \in \mathfrak{L}_{\leq r}$ such that $\zeta_1 = \lim_n f_n$.

Note that, for each i, when restricted to K_i, $(f_n|_{K_i})$ converges to $\zeta_1|_{K_i}$. So

$$\|\zeta_1(p)\| = \|f_n(p)\| = 1,$$

for n large enough, for almost all $p \in X$.

Let us now construct ζ_2, using a Gram-Schmidt process for modules. First define

$$\mathfrak{L}_2 := \{x \in \mathfrak{L}_{\leq r} \mid \langle x(p), \zeta_1(p)\rangle = 0 \text{ a.e.}\}.$$

It is routine to show that $\mathfrak{L}_2$ is closed.

So, $\mathfrak{L}_2$ is a direct integral of Hilbert spaces $\{(\mathfrak{L}_2)_p\}$ (Proposition 4.6.1). We note that

$$\dim((\mathfrak{L}_2)_p) = \mathfrak{L}_p - 1 \tag{4.8.5}$$

for $p \in \{l \in Y \mid 0 < \dim(\mathfrak{L}_l) \leq r\}$, while for $p \in \{l \in Y \mid \dim(\mathfrak{L}_l) > r\}$, $\dim((\mathfrak{L}_2)_p) = 0$.

Now put $X_2 := \{p \in Y \mid \dim((\mathfrak{L}_2)_p) > 0\}$ and note that we have

$$\mathfrak{L}_2 = \int_{X_2} (\mathfrak{L}_2)_p \, \mathrm{d}\nu \oplus \int_{Y\setminus X_2} \{0\} \, \mathrm{d}\nu.$$

Performing the same procedure on $\mathfrak{L}_2$ that gave us ζ_1 (when performed on $\mathfrak{L}_{\leq r}$) we obtain a vector $\zeta_2 \in \mathfrak{L}_2$. From the definition of $\mathfrak{L}_2$, $\langle \zeta_2, \zeta_1 \rangle = 0$.

From (4.8.5), we may repeat the process until we obtain vectors $\zeta_1, \zeta_2, \ldots, \zeta_r$. The vectors ζ_i are mutually pointwise orthogonal due to the definition

$$\mathfrak{L}_{i+1} = \{x \in \mathfrak{L}_i \mid \langle x(p), \zeta_i(p)\rangle = 0 \text{ a.e.}\}.$$

With the vectors $\{\zeta_i\}$ constructed above we show that $\mathfrak{L}_{\leq r} = \overline{\sum_{i=1}^r \zeta_i \varphi(\mathcal{N})}$. By construction $\zeta_i \in \mathfrak{L}_{\leq r}$ for each i. As $\mathfrak{L}_{\leq r}$ is a right-$\varphi(\mathcal{N})$-submodule, $\sum_{i=1}^r \zeta \varphi(\mathcal{N}) \subseteq \mathfrak{L}_{\leq r}$ and as $\mathfrak{L}_{\leq r}$ is closed, $\overline{\sum_{i=1}^r \zeta_i \varphi(\mathcal{N})} \subseteq \mathfrak{L}_{\leq r}$. Conversely, let $x \in \mathfrak{L}_{\leq r}$. We must approximate x with elements contained in $\sum_{i=1}^r \zeta_i \varphi(\mathcal{N})$. Note that for almost all $p \in Y$,

$$x(p) = \sum_{j=1}^r \langle x(p), \zeta_j(p)\rangle \zeta(p), \tag{4.8.6}$$

since pointwise, the process that gave $\zeta_1, \zeta_2, \ldots, \zeta_r$ is the usual Gram-Schmidt process which ends only once a basis for $\mathfrak{L}_p$ (with $\dim(\mathfrak{L}_p) \leq r$) is obtained for almost every p.

Then, $x = \sum_{j=1}^r f_j \zeta_j$, where f_j denotes the map $p \mapsto \langle x(p), \zeta(p)\rangle I(p)$. We employ a similar approximation argument to the one seen in (4.4.5). Note that $f_j \in L^2(\nu)$, using the same argument appearing in (4.4.7) (replacing $\Gamma(p)$ with $\zeta_j(p)$). Let (s_n^j) be a sequence of $L^\infty(\nu)$-functions approximating f_j in $L^2(\nu)$. Then, using the orthonormality of $\zeta_1(p), \zeta_2(p), \ldots, \zeta_r(p)$,

$$\begin{aligned}\left\| x - \sum_{j=1}^{r} s_n^{(j)} \zeta_j \right\|^2 &= \int_Y \left\| \left[\sum_{j=1}^{r} (f_j - s_n^{(j)}) \right] \zeta_j \right\|^2 \mathrm{d}\nu \\ &\leq \int_Y \sum_{j=1}^{r} |f_j - s_n^{(j)}|^2 \, \mathrm{d}\nu \\ &= \sum_{j=1}^{r} \int_Y \left| f_j - s_n^{(j)} \right|^2 \mathrm{d}\nu \to 0 \qquad \text{as } n \to 0\end{aligned}$$

□

We now turn our attention to approximating any $x \in \Phi(V)$ in the statement of Lemma 4.0.2. Let $P_r := M_{\chi_{X_{\leq r}}}$, where $X_{\leq r}$ is defined in Definition 4.6.4 For each $n \in \mathbb{N}$, let $x_n := P_n x$. We note, by Definition 4.8.1, that x_n belongs to the finite rank approximate $\mathfrak{L}_{\leq n}$, where the rank r_n of $\mathfrak{L}_{\leq n}$ is at most n by Proposition 4.8.5. Let $P = M_{\chi_{X_{<\infty}}}$ and set $z = Px$.

We wish to show two things:

1. $z = x$.
2. $x_n \to z$.

For the first item,

$$\|x - z\|^2 = \int_Y \|x(p) - z(p)\|^2 \, \mathrm{d}\nu = \int_{X_\infty} \|x(p) - z(p)\|^2 \, \mathrm{d}\nu.$$

On the other hand,

$$\begin{aligned}\overline{\tau}(P_V) < \infty &\implies \int_Y \dim(\mathfrak{L}_p) \, \mathrm{d}\nu < \infty \\ &\implies \int_{X_\infty} \dim(\mathfrak{L}_p) \, \mathrm{d}\nu = 0 \\ &\implies \nu(X_\infty) = 0.\end{aligned}$$

Thus, $x = z$.

For the second item, put $f_n(p) = \|x_n(p)\|^2$ and $f(p) = \|x(p)\|^2$. Then (f_n) is an increasing sequence of functions converging pointwise to f, since $f_n(p) = \chi_{X_{\leq n}} f(p)$. Thus, we may apply the monotone convergence theorem to obtain,

$$\lim_n \|x_n\|^2 = \lim_n \int_Y f_n(p) \, \mathrm{d}\nu = \int_Y \lim_n f_n(p) \, \mathrm{d}\nu = \int_Y f(p) \, \mathrm{d}\nu = \|x\|^2 .$$

Now as $(x - x_n) \perp x_n$ we have

$$\|x\|^2 - \|x_n\|^2 = \|x - x_n\|^2 .$$

Thus, $\|x - x_n\|^2 \to 0$.

We transfer our constructions from the direct integral framework, relying on the fact that Φ is a unitary.

Proposition 4.8.6. *Consider $\mathfrak{L}_{\leq r}$ from Proposition 4.8.5. Define $V_1 := \Phi^{-1}\mathfrak{L}_{\leq r}$. Then V_1 is a U_α-invariant right-$\mathcal{N}$-submodule $\mathfrak{H}_\tau$ with finite rank at most r.*

Proof. For each $i \in \{1, 2, \dots, r\}$, using the vectors ζ_i in Proposition 4.8.5,

$$\xi_i = \Phi^{-1}(\zeta_i). \tag{4.8.7}$$

Let us show the right-$\mathcal{N}$-submodule property. Take a $v \in V_1$. Then there is a sequence in $\mathfrak{L}$, say (w_n) consisting of element of the form

$$w_j = \sum_{i=1}^{r} \zeta_i \varphi(n_j^{(i)}),$$

where $n_j^{(i)} \in \mathcal{N}$, such that $\lim_j w_j = \Phi(v) \in \mathfrak{L}_{\leq r}$.

Then, $\lim_j \Phi^{-1}(w_j) = \Phi^{-1}(\lim_j w_j) = \Phi^{-1}\Phi(v) = v$ and

$$\begin{aligned} \Phi^{-1}(w_j) &= \sum_{i=1}^{r} \Phi^{-1}\left(\zeta_i \varphi(n_j^{(i)})\right) = \sum_{i=1}^{r} \Phi^{-1}\left(\varphi(n_j^{(i)})\zeta_i\right) \\ &= \sum_{i=1}^{r} \Phi^{-1}\varphi(n_j^{(i)})\Phi\Phi^{-1}(\zeta_i) \\ &= \sum_{i=1}^{r} n_j^{(i)}\xi_i. \end{aligned} \tag{4.8.8}$$

We show that V_1 is U_α-invariant.

The space $\mathfrak{L}_{\leq r}$ is U_β-invariant: For all $x \in \mathfrak{L}$

$$\begin{aligned} (U_\beta M_{\leq r}x)(p) &= \Upsilon(p)(M_{\leq r}x)(Sp) && \text{Proposition 4.7.11} \\ &= \Upsilon(p)\chi_{X_{\leq r}}(Sp)x(Sp) \\ &= \Upsilon(p)\chi_{X_{\leq r}}(p)x(Sp) && \text{Proposition 4.7.8} \\ &= \chi_{X_{\leq r}}(p)\Upsilon(p)x(Sp) && \text{Proposition 4.1.8} \\ &= (M_{\leq r}U_\beta x)(p) && \text{Definition 4.8.1.} \end{aligned}$$

So, $U_\beta M_{\leq r}x \in M_{\leq r}\mathfrak{L} = \mathfrak{L}_{\leq r}$ since $\mathfrak{L}$ is U_β-invariant.

This shows that $U_\beta\mathfrak{L}_{\leq r} \subseteq \mathfrak{L}_{\leq r}$. Similarly, $U_\beta^{-1}\mathfrak{L}_{\leq r} = U_{\beta^{-1}}\mathfrak{L}_{\leq r} \subseteq \mathfrak{L}_{\leq r}$. Thus, $U_\beta\mathfrak{L}_{\leq r} = \mathfrak{L}_{\leq r}$. So, $U_\alpha V_1 = V_1$. □

Note that, for every $\epsilon > 0$, using Proposition 4.6.5, with the r given in the statement of Proposition 4.6.5, and V_1 by Proposition 4.8.6

$$\begin{aligned}\overline{\tau}(P_V - P_{V_1}) &= \overline{\tau}(P_V) - \overline{\tau}(P_{V_1}) \\ &= \int_Y \dim(\mathfrak{L}_p)\,\mathrm{d}\nu - \int_{X_{\leq r}} \dim(\mathfrak{L}_p)\,\mathrm{d}\nu \\ &= \int_{X_{>r}} \dim(\mathfrak{L}_p)\,\mathrm{d}\nu < \epsilon \qquad .\end{aligned}$$

We now study how the u_{ji}'s in the statement of Lemma 4.0.2 are obtained.

Define, using Definition 4.7.9 and Corollary 4.7.11

$$v_{ji}(p) := \langle (U_\beta \zeta_i)(p), \zeta_j(p)\rangle = \langle \Upsilon(p)\zeta_i(Sp), \zeta_j(p)\rangle. \tag{4.8.9}$$

Recall that $\Upsilon(p)$ is unitary for almost all p and that for each p, $\{\zeta_k(p)\}$ forms an orthonormal basis for $\mathfrak{H}'_p$. So, for almost all p, $v_{ji}(p)$ is a contraction:

$$|v_{ji}(p)| \leq \|\Upsilon(p)\|\,\|\zeta_i(p)\|\,\|\zeta_j(p)\| \leq 1. \tag{4.8.10}$$

We know that $p \mapsto \langle (U_\beta\zeta_i)(p), \zeta_j(p)\rangle$ is a measurable function. So, $M_{v_{ji}} \in \varphi(\mathcal{N})$.

We also have, for all $k \in \{1, 2, \ldots, r\}$ and for almost all p,

$$\sum_{j=1}^{r} v_{jk}(p)\zeta_j(p) = \sum_{j=1}^{r} \zeta_j(p)v_{jk}(p) = \sum_{j=1}^{r} \zeta_j(p)\langle (U_\beta\zeta_k)(p), \zeta_j(p)\rangle = (U_\beta\zeta_k)(p), \tag{4.8.11}$$

by Proposition 4.8.5. Hence, for all $k \in \{1, 2, \ldots, r\}$,

$$U_\beta\zeta_k = \sum_{j=1}^{r} \zeta_j v_{jk} = \sum_{j=1}^{r} v_{jk}\zeta_j. \tag{4.8.12}$$

Now define

$$u_{ji} := \varphi^{-1}(M_{v_{ji}}). \tag{4.8.13}$$

Using our right-$\mathcal{N}$-basis defined in (4.8.7), we have that for all $k \in \{1, 2, \ldots, r\}$, $U_\alpha\xi_k =$

$\sum_{j=1}^{r} \xi_j u_{jk}$:

(4.8.14)
$$\begin{aligned}
\sum_{j=1}^{r} \xi_j u_{jk} &= \sum_{j=1}^{r} \xi_j \varphi^{-1}(M_{v_{ji}}) \\
&= \sum_{j=1}^{r} \varphi^{-1}(M_{v_{ji}})\xi_j && \text{as } \mathcal{N} \text{ is central} \\
&= \sum_{j=1}^{r} \Phi^{-1} M_{v_{ji}} \Phi\Phi^{-1}(\zeta_j) \\
&= \sum_{j=1}^{r} \Phi^{-1} M_{v_{ji}} \zeta_j \\
&= \sum_{j=1}^{r} \Phi^{-1}(\zeta_j v_{jk}) && \text{by definition of } M_{v_{ji}} \\
&= \Phi^{-1} \sum_{j=1}^{r} (\zeta_j v_{jk}) \\
&= \Phi^{-1} U_\beta \zeta_k = \Phi^{-1}\Phi U_\alpha \Phi^{-1}\zeta_k = U_\alpha \xi_k.
\end{aligned}$$

We now show that $U = (u_{ji})$ is a unitary operator in $\mathcal{U}_{r\times r}(\mathcal{N})$. As φ^{-1} is a $*$-automorphism, we have that $\varphi^{-1}(M_1) = \mathbb{1}_{\mathcal{N}}$ and $\varphi^{-1}(0) = 0$. Thus, if $U_\oplus := (v_{ji})$ is a unitary matrix in $\mathcal{U}_{r\times r}(\mathcal{N})$, then, $U = (u_{ji}) = (\varphi^{-1}(M_{v_{ji}}))$ is a unitary matrix in $\mathcal{U}_{r\times r}(\mathcal{N})$.

Let $U_\oplus^*$ be the adjoint of $U_\oplus$, that is, $U_\oplus^* = (v_{ij}^*)$ (the (j,i) entry in $U_\oplus^*$ is v_{ij}^*, as opposed to v_{ji} in $U_\oplus$), where $v_{ij}^* = \overline{v_{ij}}$, the bar denoting complex conjugation (the involution of $L^\infty(\nu)$). Let us verify that $U_\oplus U_\oplus^* = I$ and $U_\oplus^* U_\oplus = I$, where I is the matrix consisting of constant one function on its diagonal and zeros everywhere else.

We calculate the entries $(U_\oplus^* U_\oplus)_{ji}$ of the matrix $U_\oplus^* U_\oplus$ at an arbitrary p :

(4.8.15)
$$\begin{aligned}
(U_\oplus^* U_\oplus)_{ji}(p) &= \sum_{k=1}^{r} v_{ik}^*(p) v_{kj}(p) = \sum_{k=1}^{r} \overline{\langle (U_\beta\zeta_j)(p), \zeta_k(p)\rangle} \langle (U_\beta\zeta_i)(p), \zeta_k(p)\rangle \\
&= \sum_{k=1}^{r} \langle (U_\beta\zeta_i)(p), \zeta_k(p)\rangle \langle \zeta_k(p), (U_\beta\zeta_j)(p)\rangle \\
&= \langle \Upsilon(p)\zeta_i(Sp), \Upsilon(p)\zeta_j(Sp)\rangle = \langle \zeta_i(Sp), \zeta_j(Sp)\rangle.
\end{aligned}$$

In the second last equality above we used (4.4.10) and Proposition 4.7.11. Similarly,

(4.8.16)
$$(U_\oplus U_\oplus^*)_{ji}(p) = \langle \zeta_i(Sp), \zeta_j(Sp)\rangle.$$

Thus, from (4.8.15),

$$(U_\oplus^* U_\oplus)_{ji}(p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, from (4.8.16)

$$(U_\oplus U_\oplus^*)_{ji}(p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 5

The Structure Theorem

In this chapter we use the two fundamental lemmas obtained in Chapters 3 and 4, respectively, along with the concept of central vectors (Definition 5.1.1), to obtain our goal, a structure theorem for asymptotically abelian systems (Theorem 5.2.3).

5.1 Central Vectors

In this section we treat Lemma 4.6 [AET11] (Proposition 5.1.4), but we adapt its proof in order to apply the same techniques used in Chapter 3.

Definition 5.1.1. Recall that $\mathcal{M}$ is a von Neumann algebra with faithful normal trace τ represented on its GNS Hilbert space $\mathfrak{H}_\tau$ and $\widetilde{M}$ is the $*$-algebra of τ-measurable operators (see (3.3.1)). Consider $L^2(\tau)$, the non-commutative L^2-space arising from the von Neumann algebra $(\mathcal{M}, \tau)$ (see (3.3.4)). A vector $\xi \in L^2(\tau)$ is called **central** if, using the multiplication in $\widetilde{M}$ (see the paragraph after (3.3.1)),

$$m\xi = \xi m \tag{5.1.1}$$

for all $m \in \mathcal{M}$. □

Remark 5.1.2. Note what Definition 5.1.1 says in the case where m in (5.1.1) is chosen to be any unitary $u \in \mathcal{M}$: in the $*$-algebra $\widetilde{\mathcal{M}}$,

$$x = \overline{u^* \overline{xu}} = \overline{u^* xu} = u^* xu, \tag{5.1.2}$$

since u is unitary. In other words, using Definition 3.2.11,

$$x \, \eta \, \mathcal{M}'. \tag{5.1.3}$$

However, $x \in L^2(\tau) \subseteq \widetilde{M}$, thus, in particular,

$$x \, \eta \, \mathcal{M}. \tag{5.1.4}$$

So, x is affiliated to $(\mathcal{M} \cap \mathcal{M}')$. Indeed, let P be a projection of $\mathfrak{H}_\tau \oplus \mathfrak{H}_\tau$ onto $\mathcal{G}_x := \{(\xi, x\xi) \mid \xi \in \mathcal{D}(x)\}$, the graph of x. Then, applying [SZ79] Lemma 9.6 to (5.1.3) and (5.1.4), yields

$$P \in M_2(\mathcal{M}) \cap M_2(\mathcal{M}') = M_2(\mathcal{M} \cap \mathcal{M}'), \tag{5.1.5}$$

where $M_2(\mathcal{A})$ is the set of all 2×2 matrices with entries in $\mathcal{A}$. Another application of [SZ79] Lemma 9.6, this time to (5.1.5), shows that

$$x \, \eta \, (\mathcal{M} \cap \mathcal{M}'). \tag{5.1.6}$$

□

We require a preliminary result in order to prove Proposition 5.1.4.

In the case of a *positive* self-adjoint operator R affiliated to a von Neumann $\mathcal{A}$, we know (Proposition 3.2.12) that T's spectral projections belong to $\mathcal{A}$. From this we derive the following:

Proposition 5.1.3. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. Suppose that T is a closed self-adjoint linear operator in $\mathfrak{H}$ affiliated to $\mathcal{A}$. There exists a sequence T_n of operators contained in $\mathcal{A}$ such that for all $x \in \mathcal{D}(T)$,*

$$T_n x \to Tx. \tag{5.1.7}$$

Proof. As T is closed, use the polar decomposition ([KR97b] Theorem 6.1.11) to write $T = U|T|$, where $U \in \mathcal{A}$ is a partial isometry and $|T| := \sqrt{T^*T}$ is affiliated to $\mathcal{A}$.

Now $|T|$ is self-adjoint and positive and therefore, from Theorem 3.2.7, there is a spectral measure $E = E_{|T|}$ such that

$$|T| = \int_0^\infty \mathrm{id} \, \mathrm{d}E. \tag{5.1.8}$$

Using Theorem 3.2.5, there is a sequence (S_n) of bounded linear operators with $S_n = (\chi_{M_n}\mathrm{id})(|T|) := \mathbb{I}(\chi_{M_n}\mathrm{id})$, such that for all $x \in \mathcal{D}(|T|)$,

$$S_n x \to |T|x. \tag{5.1.9}$$

Every S_n is affiliated to $\mathcal{A}$, using Proposition 3.2.12 and thus, because S_n is a bounded operator (Theorem 3.2.9 part (c)) we have $S_n \in \mathcal{A}$.

From the continuity of U, we have from (5.1.9), for every $x \in \mathcal{D}(|T|)$

$$T_n x \to Tx. \tag{5.1.10}$$

where $T_n := US_n$ for every $n \in \mathbb{N}$.

To complete the proof, note that $\mathcal{D}(|T|) = \mathcal{D}(T)$ ([Sch12] Lemma 7.1). □

We describe a representation of $L^2(\tau)$ acting on itself, which will be used in Proposition 5.1.4. It is a non-commutative generalisation of viewing $L^2(\tau)$ as multiplication operators on $L^2(\tau)$ itself.

Consider, for every $a \in \widetilde{\mathcal{M}}$, the operator $\pi(a)$ with domain

$$\mathcal{D}(\pi(a)) := \{x \in L^2(\tau) \mid ax \in L^2(\tau)\} \tag{5.1.11}$$

and defined by

$$\pi(a)x := ax \qquad \text{for all } x \in \mathcal{D}(\pi(a)). \tag{5.1.12}$$

Then $\pi(\widetilde{\mathcal{M}})$ represents $\widetilde{\mathcal{M}}$ concretely on $L^2(\tau)$ (in the terminology of [Tak03b] Chapter IX Definition 2.11; see also Definition 2.4 and Theorem 2.5).

In particular, for $a \in L^2(\tau) \subseteq \widetilde{\mathcal{M}}$, and $m \in \mathcal{M}$, we know from [Tak03b] Chapter IX Theorem 2.13 (ii), that $am \in L^2(\tau)$. So $\mathcal{M} \subseteq \mathcal{D}(\pi(a))$ for every $a \in L^2(\tau)$. In particular, for all $a \in L^2(\tau)$,

$$\mathbb{1} \in \mathcal{D}(\pi(a)). \tag{5.1.13}$$

Proposition 5.1.4 ([AET11] Lemma 4.6)**.** *With the notation in Definition 5.1.1, define the centre of the von Neumann algebra* $\mathcal{M}$

$$\mathcal{Z}(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}'. \tag{5.1.14}$$

Then the closure $\overline{\mathcal{Z}(\mathcal{M})}$ *in* $L^2(\tau)$ *is equal to the set of all central vectors in* $L^2(\tau)$*.*

Proof. Let $x \in L^2(\tau)$ be a central vector. Consider, the vectors $y, z \in L^2(\tau)$,

$$y := \frac{1}{2}(x + x^*) \tag{5.1.15}$$

and

$$z := \frac{1}{2i}(x - x^*), \tag{5.1.16}$$

where $i = \sqrt{-1} \in \mathbb{C}$. As we are working in the $*$-algebra $\widetilde{\mathcal{M}}$, both y and z are closed and self-adjoint. Furthermore, both y and z are affiliated to $\mathcal{Z}(\mathcal{M})$, since x is (using Remark 5.1.2).

From Proposition 5.1.3, with $\mathcal{A} = \pi(\mathcal{Z}(\mathcal{M}))$ (see (5.1.11)) acting on the Hilbert space $L^2(\tau)$ (Proposition 3.3.4), there exist two sequences (y_n) and (z_n) in $\mathcal{Z}(\mathcal{M})$ such that

$$y_n h = \pi(y_n)h \to \pi(y)h = yh \qquad \text{for all } h \in \mathcal{D}(\pi(y)) \subset L^2(\tau) \tag{5.1.17}$$

and similarly,

(5.1.18) $$z_n h \to zh \qquad \text{for all } h \in \mathcal{D}(\pi(z)) \subset L^2(\tau).$$

Now, from (5.1.13) $\mathbb{1}$ belongs to both $\mathcal{D}(\pi(y))$ and to $\mathcal{D}(\pi(z))$, and therefore using $h = \mathbb{1}$ in (5.1.17) and (5.1.18), in the norm of $L^2(\tau)$,

$$x = \lim_n (y_n + iz_n),$$

since $w\mathbb{1} = w$ for all $w \in L^2(\tau)$. So, $x \in \overline{\mathcal{Z}(\mathcal{M})}$. □

5.2 Asymptotically Abelian Systems

We now specialise to the case of asymptotically abelian W^*-dynamical system.

Definition 5.2.1 (Definition 1.10 [AET11]). Let $(\mathcal{M}, \tau, \alpha)$ be a W^*-dynamical system. Assume that $\mathcal{M}$ acts on its GNS space $\mathfrak{H}_\tau \equiv [\mathcal{M}\Omega]$. We say that $(\mathcal{M}, \tau, \alpha)$ is **asymptotically abelian** if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left\| (\alpha^k(a)b - b\alpha^k(a))\Omega \right\|_{\mathfrak{H}_\tau} = 0 \qquad \text{for all } a, b \in \mathcal{M}.$$

□

Recall that $L^2(\tau)$ is a Hilbert space (Proposition 3.3.4) and, from [Tak03b] Chapter IX Theorem 2.13 (ii), as τ is finite, $\mathcal{M} \cap L^2(\tau) = \mathcal{M}$ is dense in $L^2(\tau)$. Thus,

(5.2.1) $$(L^2(\tau), \mathrm{id}, \mathbb{1})$$

is a GNS triple for $(\mathcal{M}, \tau)$.

From (5.2.1) we can define a modular conjugation operator $J_\tau : L^2(\tau) \to L^2(\tau)$ given by

(5.2.2) $$J_\tau a = J_\tau(a\mathbb{1}) = a^*\mathbb{1} = a^* \qquad \text{for all } a \in L^2(\tau) \subseteq \widetilde{\mathcal{M}}.$$

Right multiplication in $L^2(\tau)$ behaves like the right multiplication of $\mathcal{M}$ on $\mathfrak{H}_\tau$ (see Remark 1.4.4, Proposition 1.4.3), because $\forall x, a \in L^2(\tau)$

(5.2.3) $$J_\tau a^* J_\tau x = J_\tau a^* x^* = J_\tau (xa)^* = xa.$$

Thus, using the unitary γ in Proposition 3.3.4, $\xi \in L^2(\tau)$ is a central vector if and only if for all $m \in \mathcal{M}$,

(5.2.4) $$m\zeta = Jm^*J\zeta,$$

where $\zeta = \gamma(\xi)$. Let us also call vectors $\zeta \in \mathfrak{H}_\tau$ **central** if they satisfy (5.2.4).

In the next proposition's proof, we closely follow what was done in [AET11]. However, we apply the statement we added to Lemma 4.0.2, since it is not clear how [AET11]'s version of this lemma is used in Proposition 5.2.2's proof. Also, the u_{ji} in Lemma 4.0.2 are claimed to be unitary in [AET11], but in general appears to only be contractions. This, however, is good enough for Proposition 5.2.2's proof.

Proposition 5.2.2 ([AET11] Proposition 4.7)**.** *Using the notation from Definition 5.2.1, with the additional assumption that $\mathfrak{H}_\tau$ is separable, suppose $\mathcal{N}$ is a α-invariant central von Neumann subalgebra of $\mathcal{M}$, and let $V \subseteq \mathfrak{H}_\tau$ be an α-invariant right-$\mathcal{N}$-submodule of $\mathcal{M}$ having finite lifted trace. Then all elements of V are central vectors.*

Proof. It is sufficient to prove this proposition when V is of finite rank. Indeed, suppose that every finite rank approximate to V consists solely of central vectors. Let $x \in V$ be arbitrary. Then, from Lemma 4.0.2, there is a sequence of vectors (x_n) converging to x with each x_n belonging to some finite rank approximate of V. By assumption each x_n will be central. For every $a \in \mathcal{M}$,

$$ax = \lim_n ax_n = \lim_n Ja^*Jx_n = Ja^*J \lim_n x_n = Ja^*Jx = xa.$$

Thus, all vectors in V are central.

So we may assume that V actually has finite rank. Since, for any $a\Omega \in \mathcal{M}\Omega$, $U_\alpha^j(a\Omega) = \alpha^j(a)\Omega$ and α is asymptotically abelian, we have for any $b \in \mathcal{M}$,

$$\frac{1}{n}\sum_{k=1}^{n} \left\| bU_\alpha^k(a\Omega) - U_\alpha^k(a\Omega)b \right\|_{\mathfrak{H}_\tau} = \frac{1}{n}\sum_{k=1}^{n} \left\| (b\alpha^k(a) - \alpha^k(a)b)\Omega \right\|_{\mathfrak{H}_\tau} \to 0.$$

Approximating an arbitrary $\xi \in \mathfrak{H}_\tau \equiv [\mathcal{M}\Omega]$ by elements of $\mathcal{M}\Omega$, it follows that for each fixed $b \in \mathcal{M}$ and $\xi \in \mathfrak{H}_\tau$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left\| bU_\alpha^k(\xi) - U_\alpha^k(\xi)b \right\|_{\mathfrak{H}_\tau} = 0. \tag{5.2.5}$$

On the other hand, with $\xi_1, \ldots, \xi_r$ given by Lemma 4.0.2, we know that

$$U_\alpha^{-k}(\xi_i) = \sum_{j=1}^{r} \xi_j u_{ji}^{-k} \quad \text{for all } i = 1, 2, \ldots, r,$$

writing $U^{-k} := (u_{ji}^{(-k)})_{1\leq i,j\leq r}$ (with $u_{ji}^{(-k)}$, by definition being given by Lemma 4.0.2, with U_α^{-k} instead of U_α). We have

$$U_\alpha^{-k}(\xi_i) = \sum_{j=1}^{r} \xi_j u_{ji}^{(-k)} \implies \xi_i = \sum_{j=1}^{r} U_\alpha^k(\xi_j)\alpha^k(u_{ji}^{(-k)}) \quad \text{for all } i = 1, 2, \ldots, r. \tag{5.2.6}$$

The latter implication (5.2.6) is true, because

$$
\begin{aligned}
&\sum_{j=1}^{r}[U_\alpha^k(\xi_j)]\alpha^k(u_{ji}^{(-k)}) \\
=&\sum_{j=1}^{r} J\alpha^k(u_{ji}^{(-k)})^* J[U_\alpha^k(\xi_j)] && \text{as } \alpha^k(u_{ji}^{(-k)}) \in \mathcal{N} \\
=&\sum_{j=1}^{r} JU_\alpha^k u_{ji}^{(-k)} U_\alpha^{-k} J[U_\alpha^k(\xi_j)] && \text{from (2.2.8)} \\
=&\sum_{j=1}^{r} U_\alpha^k J u_{ji}^{(-k)} J U_\alpha^{-k}[U_\alpha^k(\xi_j)] && \text{from (2.2.6)} \\
=&\sum_{j=1}^{r} U_\alpha^k(\xi_j u_{ji}^{(-k)}) && \text{as } u_{ji}^{(-k)}\xi_j = J(u_{ji}^{(-k)})^* J\xi_j = \xi_j u_{ji}^{(-k)} \\
=&U_\alpha^k \sum_{j=1}^{r}(\xi_j u_{ji}^{(-k)}) \\
=&U_\alpha^k(U_\alpha^{-k}(\xi_i)) && \text{hypothesis of (5.2.6)} \\
=&\xi_i.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|b\xi_i - \xi_i b\|_{\mathfrak{H}_\tau} \\
&= \left\| \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{j=1}^{r} bU_\alpha^k(\xi_j)\alpha^k(u_{ji}^{(-k)}) - \sum_{j=1}^{r} U_\alpha^k(\xi_j)\alpha^k(u_{ji}^{(-k)})b\right)\right\|_{\mathfrak{H}_\tau} \\
&= \left\| \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{j=1}^{r} bU_\alpha^k(\xi_j)\alpha^k(u_{ji}^{(-k)}) - \sum_{j=1}^{r} Jb^*JU_\alpha^k(\xi_j)\alpha^k(u_{ji}^{(-k)})\right)\right\|_{\mathfrak{H}_\tau} \\
&= \left\| \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{r}\left(J\alpha^k(u_{ji}^{(-k)})^*J\left[bU_\alpha^k(\xi_j) - Jb^*JU_\alpha^k(\xi_j)\right]\right)\right\| \qquad (5.2.7) \\
&\leq \sum_{j=1}^{r}\frac{1}{n}\sum_{k=1}^{n}\left\|J\alpha^k(u_{ji}^{(-k)})J\right\| \left\|bU_\alpha^k(\xi_j) - U_\alpha^k(\xi_j)b\right\|_{\mathfrak{H}_\tau} \\
&\leq \sum_{j=1}^{r}\frac{1}{n}\sum_{k=1}^{n}\left\|bU_\alpha^k(\xi_j) - U_\alpha^k(\xi_j)b\right\|_{\mathfrak{H}_\tau},
\end{aligned}
$$

because $J\alpha^k(u_{ji}^{(-k)})J$ is a contractive operator, by Lemma 4.0.2 (and $\|Jx\| = \|x\|$ for all $x \in \mathfrak{H}_\tau$). From (5.2.5), by making k large enough, we can make the right hand side of

(5.2.7) as small as we like, so that $b\xi_i = \xi_i b$ for every $i \in \{1, 2, \ldots, r\}$. By taking linear combinations with coefficients in $\mathcal{N}$ and then a completion, we have that all vectors in V are central, as required. □

Theorem 5.2.3 (structure theorem for asymptotically abelian systems;[AET11] Theorem 1.14)**.** *If $(\mathcal{M}, \tau, \alpha)$ is an asymptotically abelian W^*-dynamical system, then α is weakly mixing relative to the centre $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$.*

Proof. Suppose, for the sake of contradiction, that α were not weakly mixing relative to $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}$. Then, Lemma 3.0.3 gives a non-trivial right-$\mathcal{Z}(\mathcal{M})$-submodule $V \subseteq [\mathcal{Z}(\mathcal{M})\Omega]^\perp$ of finite lifted trace. Such a V must consist of central vectors by Proposition 5.2.2. However, Proposition 5.1.4 now gives $V \subseteq [\mathcal{Z}(\mathcal{M})\Omega]$, a contradiction. □

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