Simultaneous Identification of Damping or Anti-damping Coefficient and Initial Value in PDEs from boundary measurement*

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Abstract

In this paper, the simultaneous identification of damping or anti-damping coefficient and initial value for some PDEs is considered. An identification algorithm is proposed based on the fact that the output of system happens to be decomposed into a product of an exponential function and a periodic function. The former contains information of the damping or anti-damping coefficient, while the latter does not. The convergence and error analysis are also developed. Three examples, namely, an anti-stable wave equation with boundary anti-damping, a Schrödinger equation with internal anti-damping, and two connected strings with middle joint anti-damping, are investigated and demonstrated by numerical simulations to show the effectiveness of the proposed algorithm.

Keywords: Identification, damping coefficient, anti-stable PDEs, anti-damping coefficient.

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1 Introduction

Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and inner product induced norm $\|\cdot\|$, and let $Y=\mathbb{R}$ (or $\mathbb{C}$). Consider the dynamic system in H,

$$\begin{cases} \dot{x}(t)=A(q)x(t),\ x(0)=x_0, \\ y(t)=Cx(t)+d(t), \end{cases} \tag{1.1}$$

where $A(q):D(A(q))\subset H\to H$ is the system operator depending on the coefficient q, which is assumed to be a generator of C_0-semigroup $\mathcal{T}_q=(T_q(t))_{t\in\mathbb{R}^+}$ on H, $C:H\to Y$ is the admissible observation operator for $\mathcal{T}_q$ ([**?**]), $x_0\in H$ is the initial value, and $d(t)$ is the external disturbance.

Various control systems which include damping can be formulated into system (**??**), where q is the damping coefficient. For a physical system, if the damping is produced by material itself that dissipates the energy stored in system, then the system keeps stable. Sometimes, however, the source of instability may arise from the negative damping. One example is the thermoacoustic instability in duct combustion dynamics and another is the stick-slip instability phenomenon in deep oil drilling, see, for instance, [**?**] and the references therein. In such cases, the negative damping will result in all the eigenvalues located in the right-half complex plane, and the open-loop plant is hence “anti-stable” (exponential stable in negative time) and the q in such kind of system is said to be the anti-damping coefficient.

A widely investigated problem in recent years is stabilization for anti-stable systems by imposing feedback controls. A breakthrough on stabilization for an anti-stable wave equation was first reached in [**?**], where a backstepping transformation is proposed to design the boundary state feedback control. By the backstepping method, [**?**] generalizes [**?**] to two connected anti-stable strings with joint anti-damping. Very recently, [**?**, **?**] investigate stabilization for anti-stable wave equation subject to external disturbance coming through the boundary input, where the sliding mode control and active disturbance rejection control technology are employed. It is worth pointing out that in all aforementioned works, the anti-damping coefficients are always supposed to be known.

On the other hand, a few stabilization results for anti-stable systems with unknown anti-damping coefficients are also available. In [**?**], a full state feedback adaptive control is designed for an anti-stable wave equation. By converting the wave equation into a cascade of two delay elements, an adaptive output feedback control and parameter estimator are designed in [**?**]. Unfortunately, no convergence of the parameter update law is provided in these works.

It can be seen in [**?**, **?**] that it is the uncertainty of the anti-damping coefficient which leads to complicated design for adaptive control and parameter update law. This comes naturally with the identification of unknown anti-damping coefficient. To the best of our knowledge, there are few studies on this regard. Our focus in the present paper is on simultaneous identification for both the anti-damping (or damping) coefficient and initial value for system (**??**), where the coefficient q is assumed to be in a prior parameter set $Q=[\underline{q},\overline{q}]$ ($\underline{q}$ or $\overline{q}$ may be infinity) and the initial value is supposed to be nonzero.

We proceed as follows. In Section **??**, we propose an algorithm to identify simultaneously the coefficient and initial value through the measured observation. The system may not suffer from the disturbance or it may suffer from the periodic or general bounded disturbance. In Section **??**, a wave equation with anti-damping term in the boundary is discussed. A Schrödinger equation with internal anti-damping term is investigated in Section **??**. Section **??** is devoted to coupled strings with middle joint anti-damping. In all these sections, numerical simulations are presented to verify the performance of the proposed algorithms. Some concluding remarks are presented in Section **??**.

2 Identification algorithm

Before giving the main results, we introduce the following well known Ingham's theorem [**?**, **?**, **?**] in Lemma **??** below.

Lemma 2.1. *Assume that the strictly increasing sequence $\{\omega_k\}_{k\in\mathbb{Z}}$ of real numbers satisfies the gap condtion*

$$\omega_{k+1}-\omega_k \geq \gamma \quad \text{for all} \quad k\in\mathbb{Z}, \tag{2.1}$$

for some $\gamma>0$. Then, for all $T>2\pi/\gamma$, there exist two positive constants C_1, C_2 depending only on γ and T such that

$$C_1\sum_{k\in\mathbb{Z}}|a_k|^2 \leq \int_0^T \left|\sum_{k\in\mathbb{Z}} a_k e^{it\omega_k}\right|^2 dt \leq C_2\sum_{k\in\mathbb{Z}}|a_k|^2, \tag{2.2}$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}}\in\ell^2$, where

$$C_1=\frac{2T}{\pi}\left(1-\frac{4\pi^2}{T^2\gamma^2}\right)>0,\;\; C_2=\frac{8T}{\pi}\left(1+\frac{4\pi^2}{T^2\gamma^2}\right)>0. \tag{2.3}$$

To begin with, we suppose that there is no external disturbance in system (**??**), that is,

$$\begin{cases} \dot{x}(t)=A(q)x(t),\; x(0)=x_0, \\ y(t)=Cx(t). \end{cases} \tag{2.4}$$

The following Theorem **??** indicates that the identification of the coefficient q and initial value x_0 can be achieved exactly simultaneously without error.

Theorem 2.1. *Let $A(q)$ in system (**??**) generate a C_0-semigroup $\mathcal{T}_q=(T_q(t))_{t\in\mathbb{R}^+}$ and suppose that $A(q)$ and C satisfy the following conditions:*

(i). $A(q)$ has a compact resolvent and all its eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ (or $\{\lambda_n\}_{n\in\mathbb{Z}}$) admit the following expansion:

$$\lambda_n=f(q)+i\mu_n, \quad \cdots<\mu_n<\mu_{n+1}<\cdots, \tag{2.5}$$

where $f:\,Q\to\mathbb{R}$ is invertible, μ_n is independent of q, and there exists an $L>0$ such that

$$\frac{\mu_n L}{2\pi}\in\mathbb{Z},\; \text{for all } n\in\mathbb{N}. \tag{2.6}$$

(ii). The corresponding eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ form a Riesz basis for H.

(iii). There exist two positive numbers κ and K such that $\kappa \le |\kappa_n| \le K$ for all $n \in \mathbb{N}$, where

$$\kappa_n := C\phi_n, \quad n \in \mathbb{N}. \tag{2.7}$$

Then both coefficient q and initial value x_0 can be uniquely determined by the output $y(t), t \in [0,T]$, where $T > 2L$. Precisely

$$q = f^{-1}\left(\frac{1}{L}\ln\frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}}\right), \quad \textit{for any } L < T_1 < T_2 - L, \tag{2.8}$$

and

$$x_0 = \frac{1}{L}\sum_{n\in\mathbb{N}}\frac{1}{\kappa_n}\left(\int_0^L y(t)e^{-\lambda_n t}dt\right)\phi_n. \tag{2.9}$$

Proof. Since $\{\phi_n\}_{n\in\mathbb{N}}$ forms a Riesz basis for H, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of eigenvectors of $A(q)^*$, which is bi-orthogonal to $\{\phi_n\}_{n\in\mathbb{N}}$, that is, $\langle\phi_n,\psi_m\rangle = \delta_{nm}$. In this way, we can express the initial value $x_0 \in H$ as

$$x_0 = \sum_{n\in\mathbb{N}}\langle x_0,\psi_n\rangle\phi_n,$$

and the solution to system (**??**) as

$$x(t) = T_q(t)x_0 = \sum_{n\in\mathbb{N}} e^{\lambda_n t}\langle x_0,\psi_n\rangle\phi_n. \tag{2.10}$$

By (**??**), there exists an increasing sequence $\{K_n\} \subset \mathbb{Z}$ such that

$$\mu_n = \frac{2\pi K_n}{L},\ n \in \mathbb{N}, \tag{2.11}$$

which implies that $\{\mu_n\}$ satisfies the following gap condition

$$\mu_{n+1} - \mu_n = \frac{2\pi(K_{n+1}-K_n)}{L} \ge \frac{2\pi}{L} \triangleq \gamma. \tag{2.12}$$

In addition, by the assumption that $\{\phi_n\}$ forms a Riesz basis for H and $|C\phi_n|$ is uniformly bounded with respect to n, it follows from Proposition 2 of [**?**] or Theorem 2 of [**?**] that C is admissible for $\mathcal{T}_q$. So generally, we have

$$\begin{aligned} y(t) &= Cx(t) = \sum_{n\in\mathbb{N}} e^{\lambda_n t}\langle x_0,\psi_n\rangle C\phi_n \\ &= e^{f(q)t}\sum_{n\in\mathbb{N}} e^{i\mu_n t}\langle x_0,\psi_n\rangle C\phi_n \triangleq e^{f(q)t}P_L(t), \end{aligned} \tag{2.13}$$

where $P_L(t) = \sum_{n\in\mathbb{N}} e^{i\mu_n t}\langle x_0,\psi_n\rangle C\phi_n$ is well-defined and it follows from (**??**) that $P_L(t)$ is a Y-valued function with period L. For any $T_2 - L > T_1 > L$,

$$\begin{aligned} \int_{T_1}^{T_2} |y(t)|^2\,dt &= \int_{T_1-L}^{T_2-L} |y(t+L)|^2\,dt = \int_{T_1-L}^{T_2-L}\left|e^{f(q)(t+L)}P_L(t)\right|^2 dt \\ &= e^{2f(q)L}\int_{T_1-L}^{T_2-L}|y(t)|^2\,dt, \end{aligned} \tag{2.14}$$

that is

$$\|y\|_{L^2(T_1,T_2)} = e^{f(q)L}\|y\|_{L^2(T_1-L,T_2-L)}. \tag{2.15}$$

To obtain (**??**), we need to show that $\|y\|_{L^2(T_1,T_2)} \neq 0$ for $T_2 - T_1 > L$. Actually, it follows from (**??**) that

$$\begin{aligned} \|y\|^2_{L^2(T_1,T_2)} &= \int_{T_1}^{T_2} \left|e^{f(q)t}P_L(t)\right|^2 dt \geq C_3 \int_{T_1}^{T_2} |P_L(t)|^2 dt \\ &= C_3 \int_{T_1}^{T_2} \left|\sum_{n\in\mathbb{N}} e^{i\mu_n t}\langle x_0, \psi_n\rangle C\phi_n\right|^2 dt, \end{aligned} \tag{2.16}$$

where $C_3 = \min\left\{e^{2T_1 f(q)}, e^{2T_2 f(q)}\right\} > 0$. By Lemma **??** and the gap condition (**??**), it follows that for $T_2 - T_1 > \frac{2\pi}{\gamma} = L$,

$$\int_{T_1}^{T_2} \left|\sum_{n\in\mathbb{N}} e^{i\mu_n t}\langle x_0, \psi_n\rangle C\phi_n\right|^2 dt \geq C_1 \sum_{n\in\mathbb{N}} |\langle x_0, \psi_n\rangle C\phi_n|^2 \geq C_1\kappa^2 \sum_{n\in\mathbb{N}} |\langle x_0, \psi_n\rangle|^2, \tag{2.17}$$

where

$$C_1 = \frac{2(T_2 - T_1)}{\pi}\left(1 - \frac{L^2}{(T_2 - T_1)^2}\right) > 0 \text{ for } T_2 - T_1 > L. \tag{2.18}$$

The inequality (**??**) together with (**??**) gives

$$\|y\|^2_{L^2(T_1,T_2)} \geq C_1 C_3 \kappa^2 \sum_{n\in\mathbb{N}} |\langle x_0, \psi_n\rangle|^2. \tag{2.19}$$

Notice that $\{\phi_n\}_{n\in\mathbb{N}}$ forms a Riesz basis for H and so does $\{\psi_n\}_{n\in\mathbb{N}}$ for H, there are two positive numbers M_1, M_2 such that

$$M_1 \sum_{n\in\mathbb{N}} |\langle x_0, \psi_n\rangle|^2 \leq \|x_0\|^2 \leq M_2 \sum_{n\in\mathbb{N}} |\langle x_0, \psi_n\rangle|^2. \tag{2.20}$$

Combining (**??**) with (**??**) yields

$$\|y\|_{L^2(T_1,T_2)} \geq C\|x_0\| > 0, \tag{2.21}$$

where $C = \kappa\sqrt{\frac{C_1 C_3}{M_2}} > 0$. The identity (**??**) then follows from (**??**).

The inequality (**??**) means that system (**??**) is exactly observable for $T_2 - T_1 > L$. So the initial value x_0 can be uniquely determined by the output $y(t), t \in [T_1, T_2]$. We show next how to reconstruct the initial value from the output.

Actually, it follows from (**??**) that

$$\int_0^L e^{i(\mu_m - \mu_n)t} dt = \begin{cases} L, & n = m, \\ 0, & n \neq m. \end{cases} \tag{2.22}$$

Hence,

$$\int_0^L y(t)e^{-\lambda_n t} dt = \int_0^L \left(\sum_{m\in\mathbb{N}} e^{i(\mu_m - \mu_n)t}\langle x_0, \psi_m\rangle C\phi_m\right) dt = \kappa_n L \cdot \langle x_0, \psi_n\rangle, \tag{2.23}$$

Therefore the initial value x_0 can be reconstructed by

$$x_0 = \sum_{n\in\mathbb{N}} \langle x_0, \psi_n\rangle \phi_n = \frac{1}{L}\sum_{n\in\mathbb{N}} \frac{1}{\kappa_n}\left(\int_0^L y(t)e^{-\lambda_n t}dt\right)\phi_n. \tag{2.24}$$

This completes the proof of the theorem. □

Remark 2.1. Clearly, (**??**) and (**??**) provide an algorithm to reconstruct q and x_0 from the output. It may seem that the first condition of Theorem **??** is a bit restrictive but this is satisfied by some physical systems with constant damping discussed in Sections **??**-**??**. For the third condition, $|C\phi_n| \le K$ implies that C is admissible for $\mathcal{T}_q$ which ensures that the output belongs to $L^2_{loc}(0,\infty;Y)$, and $|C\phi_n| \ge \kappa$ implies that system (**??**) is exactly observable which ensures the unique determination of the initial value. It is easily seen from (**??**) that the coefficient q can always be identified as long as $\|y\|_{L^2(T_1,T_2)} \neq 0$ for some time interval $[T_1, T_2]$, which shows that the identifiability of coefficient q does not rely on the exact observability.

Remark 2.2. The condition $T_2 - T_1 > L$ in Theorem **??** is only used in the application of the Ingham inequality in (**??**) to ensure that $\|y\|_{L^2(T_1,T_2)} \neq 0$. In practical applications, however, this condition is not always necessary. Actually, any $L < T_1 < T_2$ is applicable in (**??**) as long as $\|y\|_{L^2(T_1,T_2)} \neq 0$. Similar remarks also apply for Corollary **??** and Theorem **??** below.

Now we come to the system with external disturbance which is inevitable in many situations. Suppose first that system (**??**) is corrupted by an unknown periodic disturbance $d(t) \neq 0$ in observation. The Corollary **??** below shows that we can construct not only the coefficient q and initial value x_0, but the disturbance $d(t)$ as well provided that $d(t)$ is a periodic function with some known period.

Corollary 2.1. *Suppose that all the conditions in Theorem* **??** *are satisfied. If the disturbance $d(t)$ in system (**??**) is a periodic function with period l, then the coefficient q can be uniquely determined by the output $y(t), t \in [0,T]$ with $T > l + L$ as*

$$q = f^{-1}\left(\frac{1}{L}\ln\frac{\|\bar{y}\|_{L^2(T_1,T_2)}}{\|\bar{y}\|_{L^2(T_1-L,T_2-L)}}\right), \quad \textit{for any } L < T_1 < T_2 - L, \tag{2.25}$$

where

$$\bar{y}(t) = y(t+l) - y(t),\ t > 0. \tag{2.26}$$

In addition, if $l/L = n_1/n_2$ is rational, where $n_1, n_2 \in \mathbb{N}$, then the disturbance $d(t)$ can be reconstructed from $y(t), t \in [0,(n_2+1)l]$ by

$$d(t) = \frac{e^{n_1 L f(q)}y(t) - y(t+n_1 L)}{e^{n_1 L f(q)} - 1},\ 0 < t < l, \tag{2.27}$$

and the initial value x_0 can be reconstructed by

$$x_0 = \frac{1}{L}\sum_{n\in\mathbb{N}} \frac{1}{\kappa_n}\left(\int_0^L [y(t) - d(t)]\, e^{-\lambda_n t}dt\right)\phi_n. \tag{2.28}$$

Proof. With the same analysis as that in the proof of Theorem **??**, we can show that

$$y(t) = e^{f(q)t}P_L(t) + d(t), \tag{2.29}$$

where $P_L(t)$ is defined in (**??**). Since the disturbance $d(t)$ is with the period l, we have

$$\bar{y}(t) = y(t+l) - y(t) = e^{f(q)t}\left[e^{f(q)l}P_L(t+l) - P_L(t)\right] \triangleq e^{f(q)t}\bar{P}_L(t),\ t>0. \tag{2.30}$$

Clearly, $\bar{P}_L(t) = e^{f(q)l}P_L(t+l) - P_L(t)$ is also a Y-valued periodic function with period L. As a result, (**??**) follows from the similar arguments in the proof of Theorem **??**.

Now suppose that $l/L = n_1/n_2$ for some $n_1, n_2 \in \mathbb{N}$. Then $n_1L = n_2l$, and hence

$$\begin{aligned} y(t+n_1L) - y(t) &= e^{f(q)(t+n_1L)}P_L(t+n_1L) + d(t+n_1L) - e^{f(q)t}P_L(t) - d(t) \\ &= e^{n_1Lf(q)+f(q)t}P_L(t) - e^{f(q)t}P_L(t) + d(t+n_2l) - d(t) \\ &= \left[e^{n_1Lf(q)} - 1\right]e^{f(q)t}P_L(t). \end{aligned} \tag{2.31}$$

Therefore,

$$d(t) = y(t) - e^{f(q)t}P_L(t) = \frac{e^{n_1Lf(q)}y(t) - y(t+n_1L)}{e^{n_1Lf(q)} - 1},\ 0<t<l. \tag{2.32}$$

After having recovered the coefficient q and disturbance $d(t)$, the reconstruction of initial value x_0 from (**??**) is a direct consequence of Theorem **??**. This completes the proof. □

For general bounded disturbance $d(t)$ in observation, it should be noted that system (**??**) is supposed to be anti-stable in Theorem **??** below whereas in Theorem **??** and Corollary **??**, there is no constraint on the stability of system.

Theorem 2.2. *Suppose that system (**??**) is anti-stable and all the conditions in Theorem **??** are satisfied. If the inverse of $f(q)$ is continuous and the disturbance $d(t)$ is bounded, i.e. $|d(t)| \le M$ for some $M>0$ and all $t \ge 0$, then for any $T_2 - L > T_1 > L$,*

$$\lim_{T_1\to+\infty} q_{T_1} = q,\quad \lim_{T_1\to+\infty} \|\hat{x}_{0T_1} - x_0\| = 0, \tag{2.33}$$

where

$$q_{T_1} = f^{-1}\left(\frac{1}{L}\ln\frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}}\right),\qquad L<T_1<T_2-L, \tag{2.34}$$

and

$$\hat{x}_{0T_1} = \frac{1}{L}\sum_{n\in\mathbb{N}}\frac{1}{\kappa_n}\left(\int_{T_1}^{T_1+L} y(t)e^{-\lambda_n t}dt\right)\phi_n,\quad T_1 \ge 0. \tag{2.35}$$

Moreover, for sufficiently large T_1, the errors $|f(q_{T_1}) - f(q)|$ and $\|\hat{x}_{0T_1} - x_0\|$ satisfy

$$|f(q_{T_1}) - f(q)| < \frac{4}{L}\frac{M\sqrt{T_2-T_1}}{\|y\|_{L^2(T_1-L,T_2-L)} - M\sqrt{T_2-T_1}}, \tag{2.36}$$

and

$$\|\hat{x}_{0T_1} - x_0\| \le \frac{CM}{\kappa\sqrt{L}}e^{-f(q)T_1},\ \text{for some } C>0. \tag{2.37}$$

Proof. Introduce

$$y_e(t) = CT_q(t)x_0 = y(t) - d(t) = e^{f(q)t}P_L(t), \tag{2.38}$$

where $P_L(t)$ is defined in (??). We first show that

$$\lim_{T_1\to+\infty} \|y_e\|_{L^2(T_1,T_2)} = +\infty. \tag{2.39}$$

Since system (??) is anti-stable, the real part of the eigenvalues $f(q) > 0$. It then follows from (??) that

$$\|y_e\|^2_{L^2(T_1,T_2)} = \int_{T_1}^{T_2} \left|e^{f(q)t}P_L(t)\right|^2 dt \geq e^{2f(q)T_1} \int_{T_1}^{T_2} \left|\sum_{n\in\mathbb{N}} e^{i\mu_n t}\langle x_0, \psi_n\rangle C\phi_n\right|^2 dt. \tag{2.40}$$

Using the same arguments as (??)-(??) in the proof of Theorem ??, we have

$$\|y_e\|_{L^2(T_1,T_2)} \geq Ce^{f(q)T_1}\|x_0\|, \tag{2.41}$$

where $C = \kappa\sqrt{\frac{C_1}{M_2}} > 0$. Since $f(q) > 0$, $x_0 \neq 0$, (??) holds. Therefore, for sufficiently large T_1,

$$\frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}} = \frac{\|y_e + d\|_{L^2(T_1,T_2)}}{\|y_e + d\|_{L^2(T_1-L,T_2-L)}} \leq \frac{\|y_e\|_{L^2(T_1,T_2)} + \|d\|_{L^2(T_1,T_2)}}{\|y_e\|_{L^2(T_1-L,T_2-L)} - \|d\|_{L^2(T_1-L,T_2-L)}}. \tag{2.42}$$

Since $|d(t)| \leq M$, for any finite time interval I,

$$\|d\|_{L^2(I)} = \left(\int_I |d(t)|^2 dt\right)^{\frac{1}{2}} \leq M\sqrt{|I|}, \tag{2.43}$$

where $|I|$ represents the length of the time interval I. Hence,

$$\frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}} \leq \frac{\|y_e\|_{L^2(T_1,T_2)} + M\sqrt{T_2 - T_1}}{\|y_e\|_{L^2(T_1-L,T_2-L)} - M\sqrt{T_2 - T_1}} = \frac{e^{Lf(q)} + \varepsilon(T_1,T_2)}{1 - \varepsilon(T_1,T_2)}, \tag{2.44}$$

where

$$\varepsilon(T_1,T_2) = \frac{M\sqrt{T_2 - T_1}}{\|y_e\|_{L^2(T_1-L,T_2-L)}}. \tag{2.45}$$

Similarly,

$$\frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}} \geq \frac{\|y_e\|_{L^2(T_1,T_2)} - \|d\|_{L^2(T_1,T_2)}}{\|y_e\|_{L^2(T_1-L,T_2-L)} + \|d\|_{L^2(T_1-L,T_2-L)}} \geq \frac{e^{Lf(q)} - \varepsilon(T_1,T_2)}{1 + \varepsilon(T_1,T_2)}. \tag{2.46}$$

It is clear from (??) and (??) that $\lim_{T_1\to+\infty} \varepsilon(T_1,T_2) = 0$. This together with (??) and (??) gives

$$\lim_{T_1\to+\infty} \frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}} = e^{Lf(q)}. \tag{2.47}$$

Since $f^{-1}(q)$ is continuous,

$$\lim_{T_1\to+\infty} q_{T_1} = f^{-1}\left(\frac{1}{L}\ln \lim_{T_1\to+\infty} \frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}}\right) = q. \tag{2.48}$$

We next show the convergence of the initial value. Similar to the arguments (**??**)-(**??**) in the proof of Theorem **??**, we have

$$x_0 = \frac{1}{L}\sum_{n\in\mathbb{N}} \frac{1}{\kappa_n}\left(\int_{T_1}^{T_1+L} y_e(t)e^{-\lambda_n t}dt\right)\phi_n, \quad \forall\, T_1 \geq 0. \tag{2.49}$$

It then follows from (**??**) that

$$\hat{x}_{0T_1} - x_0 = \frac{1}{L}\sum_{n\in\mathbb{N}} \frac{1}{\kappa_n}\left(\int_{T_1}^{T_1+L} d(t)e^{-\lambda_n t}dt\right)\phi_n, \quad \forall\, T_1 \geq 0. \tag{2.50}$$

In view of the Riesz basis property of $\{\phi_n\}$, it follows that

$$\begin{aligned} \|\hat{x}_{0T_1} - x_0\|^2 &= \left\|\frac{1}{L}\sum_{n\in\mathbb{N}} \frac{1}{\kappa_n}\left(\int_{T_1}^{T_1+L} d(t)e^{-\lambda_n t}dt\right)\phi_n\right\|^2 \\ &\leq \frac{M_2}{L^2}\sum_{n\in\mathbb{N}}\left|\frac{1}{\kappa_n}\int_{T_1}^{T_1+L} d(t)e^{-\lambda_n t}dt\right|^2 \\ &\leq \frac{M_2}{L^2\kappa^2}\sum_{n\in\mathbb{N}}\left|e^{-\lambda_n T_1}\int_0^L d(t+T_1)e^{-\lambda_n t}dt\right|^2 \\ &= \frac{M_2}{L^2\kappa^2}e^{-2f(q)T_1}\sum_{n\in\mathbb{N}}\left|\int_0^L \left(d(t+T_1)e^{-f(q)t}\right)e^{-i\mu_n t}dt\right|^2, \end{aligned} \tag{2.51}$$

where $M_2 > 0$ is introduced in (**??**). To estimate the last series in (**??**), we need the Riesz basis (sequence) property of the exponential system $\Lambda := \left\{f_n = e^{i\mu_n t}\right\}_{n\in\mathbb{N}}$. There are two cases according to the relation between the sets $\{K_n\}_{n\in\mathbb{N}}$ introduced in (**??**) and integers $\mathbb{Z}$:

Case 1: $\{K_n\}_{n\in\mathbb{N}} = \mathbb{Z}$, *that is,* $\Lambda = \left\{e^{i\frac{2n\pi}{L}t}\right\}_{n\in\mathbb{Z}}$. In this case, since $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$, Λ forms a Riesz basis for $L^2[-\frac{L}{2},\frac{L}{2}]$.

Case 2: $\{K_n\}_{n\in\mathbb{N}} \subsetneq \mathbb{Z}$. In this case, it is noted that the exponential system $\left\{e^{i\mu_n t}\right\}_{n\in\mathbb{N}}$ forms a Riesz sequence in $L^2[-\frac{L}{2},\frac{L}{2}]$.

In each case above, by the propositions of Riesz basis and Riesz sequence (see, e.g., [**?**, p. 32-35, p.154]), there exists a positive constant $C_4 > 0$ such that

$$\sum_{n\in\mathbb{N}} |(g,f_n)|^2 \leq C_4\|g\|^2_{L^2[-\frac{L}{2},\frac{L}{2}]}, \tag{2.52}$$

for all $g \in L^2[-\frac{L}{2},\frac{L}{2}]$, where $(\cdot,\cdot)$ here denotes the inner product in $L^2[-\frac{L}{2},\frac{L}{2}]$.

We return to the estimation of $\|\hat{x}_{0T_1} - x_0\|$. By variable substitution of $t = \frac{L}{2} - s$ in (**??**), together with (**??**), gives

$$\begin{aligned} \|\hat{x}_{0T_1} - x_0\|^2 &\leq \frac{M_2}{L^2\kappa^2}e^{-2f(q)T_1}\sum_{n\in\mathbb{N}}\left|\int_{-\frac{L}{2}}^{\frac{L}{2}}\left[d\left(T_1+\frac{L}{2}-s\right)e^{f(q)(s-\frac{L}{2})}e^{-i\mu_n\frac{L}{2}}\right]e^{i\mu_n s}ds\right|^2 \\ &\leq \frac{M_2C_4}{L^2\kappa^2}e^{-2f(q)T_1}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left|d\left(T_1+\frac{L}{2}-s\right)e^{f(q)(s-\frac{L}{2})}e^{-i\mu_n\frac{L}{2}}\right|^2 ds \\ &\leq \frac{M^2M_2C_4}{L\kappa^2}e^{-2f(q)T_1}. \end{aligned} \tag{2.53}$$

Therefore,

$$\|\hat{x}_{0T_1} - x_0\| \leq \sqrt{\frac{M_2C_4}{L}}\frac{M}{\kappa}e^{-f(q)T_1}, \tag{2.54}$$

which implies that $\|\hat{x}_{0T_1} - x_0\|$ will tend to zero as $T_1 \to +\infty$ for $f(q) > 0$. The inequality (**??**) with the positive number $C = \sqrt{M_2C_4}$ is also concluded.

Finally, we estimate $|f(q_{T_1}) - f(q)|$. Setting T_1 large enough so that $\varepsilon(T_1, T_2) < 1$, it follows from (**??**) and (**??**) that

$$\begin{aligned} Lf(q_{T_1}) &= \ln \frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}} \leq \ln \frac{e^{Lf(q)} + \varepsilon(T_1,T_2)}{1-\varepsilon(T_1,T_2)} \\ &= Lf(q) + \ln\left[1 + \frac{(1+e^{-Lf(q)})\varepsilon(T_1,T_2)}{1-\varepsilon(T_1,T_2)}\right] \\ &< Lf(q) + \frac{2\varepsilon(T_1,T_2)}{1-\varepsilon(T_1,T_2)}. \end{aligned} \tag{2.55}$$

Similarly, for T_1 large enough so that $\varepsilon(T_1, T_2) \leq \frac{1}{4}$, it follows from (**??**) and (**??**) that

$$\begin{aligned} Lf(q_{T_1}) &= \ln \frac{\|y\|_{L^2(T_1,T_2)}}{\|y\|_{L^2(T_1-L,T_2-L)}} \geq \ln \frac{e^{Lf(q)} - \varepsilon(T_1,T_2)}{1+\varepsilon(T_1,T_2)} \\ &= Lf(q) + \ln\left[1 - \frac{(1+e^{-Lf(q)})\varepsilon(T_1,T_2)}{1+\varepsilon(T_1,T_2)}\right] \\ &> Lf(q) - \frac{4\varepsilon(T_1,T_2)}{1+\varepsilon(T_1,T_2)}, \end{aligned} \tag{2.56}$$

where the last inequality comes from the fact that $\ln(1-x) \geq -2x$, for $x \in [0, \frac{1}{2}]$.

Combining (**??**) and (**??**), and setting T_1 large enough so that $\varepsilon(T_1, T_2) \leq \frac{1}{4}$, we have

$$|f(q_{T_1}) - f(q)| < \max\left\{\frac{2}{L}\left(\frac{\varepsilon(T_1,T_2)}{1-\varepsilon(T_1,T_2)}\right), \frac{4}{L}\left(\frac{\varepsilon(T_1,T_2)}{1+\varepsilon(T_1,T_2)}\right)\right\} < \frac{4\varepsilon(T_1,T_2)}{L}. \tag{2.57}$$

The error estimation (**??**) comes from the fact that

$$\varepsilon(T_1,T_2) = \frac{M\sqrt{T_2-T_1}}{\|y-d\|_{L^2(T_1-L,T_2-L)}} \leq \frac{M\sqrt{T_2-T_1}}{\|y\|_{L^2(T_1-L,T_2-L)} - M\sqrt{T_2-T_1}}. \tag{2.58}$$

We thus complete the proof of the theorem. □

Remark 2.3. Theorem **??** shows that when system (**??**) is anti-stable, then q_{T_1} defined in (**??**) can be regarded as an approximation of the coefficient q when T_1 is sufficiently large. Roughly speaking, the $\varepsilon(T_1, T_2)$ defined in (**??**) reflects the ratio of the energy, in L^2 norm, of the disturbance $d(t)$ which is an unwanted signal, with the energy of the real output signal $y_e(t)$. We may regard $1/\varepsilon(T_1, T_2)$ as signal-to-noise ratio (SNR) which is well known in signal analysis. Theorem **??** indicates that q_{T_1} defined in (**??**) is an approximation of the coefficient q when SNR is large enough. However, if system (**??**) is stable, *i.e.* $f(q) < 0$, similar analysis shows that the output will decay exponentially, which implies that the unknown disturbance will account for a large proportion in observation and the SNR can not be too large. In this case, it is difficult to extract enough useful information from the corrupted observation as that with large SNR.

Remark 2.4. The anti-stability assumption in Theorem **??** is almost necessary since otherwise, we may have the case of $y(t) = Cx(t) + d(t) = 0$ for which we cannot obtain anything for identification.

Remark 2.5. It is well known that inverse problems are usually ill-posed in the sense of Hadamard, that is, arbitrarily small error in the measurement data may lead to indefinitely large error in the solution. Theorem **??** shows that if system (**??**) is anti-stable, our algorithm is robust against bounded unknown disturbance in measurement data. Actually, similar to the analysis in Theorem **??**, it can be shown that when system (**??**) is not anti-stable, the algorithm in Theorem **??** is also numerically stable in the presence of small perturbations in the measurement data, as long as the perturbation is relatively small in comparison to the output. Some numerical simulations are presented subsequently in Example **??** in Section **??**.

3 Application to wave equation

In this section, we apply the algorithm proposed in the previous section to identification of the anti-damping coefficient and initial values for a one-dimensional vibrating string equation described by

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1,\ t > 0, \\ u(0,t) = 0,\ \ u_x(1,t) = qu_t(1,t), & t \geq 0, \\ y(t) = u_x(0,t) + d(t), & t \geq 0, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & 0 \leq x \leq 1, \end{cases} \tag{3.1}$$

where x denotes the position, t the time, $0 < q \neq 1$ the unknown anti-damping coefficient, $u_0(x)$ and $u_1(x)$ the unknown initial displacement and initial velocity, respectively, and $y(t)$ is the boundary measured output corrupted by the disturbance $d(t)$.

Let $\mathcal{H} = H_E^1(0,1) \times L^2(0,1)$, where $H_E^1(0,1) = \{f \in H^1(0,1) | f(0) = 0\}$, equipped with the inner product $\langle \cdot, \cdot \rangle$ and the inner product induced norm

$$\|(f,g)\|^2 = \int_0^1 \left[|f'(x)|^2 + |g(x)|^2 \right] dx.$$

Define the system operator $\mathcal{A} : D(\mathcal{A}) (\subset \mathcal{H}) \to \mathcal{H}$ as

$$\begin{cases} \mathcal{A}(f,g) = (g, f''), \\ D(\mathcal{A}) = \left\{ (f,g) \in H^2(0,1) \times H_E^1(0,1) \mid f(0) = 0, f'(1) = qg(1) \right\}, \end{cases} \tag{3.2}$$

and the observation operator $\mathcal{C}$ from $\mathcal{H}$ to $\mathbb{C}$ as

$$\mathcal{C}(f,g) = f'(0), \quad (f,g) \in D(\mathcal{A}). \tag{3.3}$$

It is indicated in [**?**] that the operator $\mathcal{A}$ generates a C_0-group on $\mathcal{H}$.

Lemma 3.1. *[**?**] Let $\mathcal{A}$ be defined by (**??**) and let $q \neq 1$. Then the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues given by*

$$\lambda_n = \frac{1}{2} \ln \frac{1+q}{q-1} + in\pi, \quad n \in \mathbb{Z},\ \ if\ \ q > 1, \tag{3.4}$$

or

$$\lambda_n = \frac{1}{2}\ln\frac{1+q}{1-q} + i\frac{2n+1}{2}\pi, \quad n \in \mathbb{Z}, \ \ if \ \ 0 < q < 1, \tag{3.5}$$

and the corresponding eigenfunctions $\Phi_n(x)$ are given by

$$\Phi_n(x) = \left(\frac{\sinh \lambda_n x}{\lambda_n}, \sinh \lambda_n x\right), \quad \forall\, n \in \mathbb{Z}. \tag{3.6}$$

Moreover, $\{\Phi_n(x)\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $\mathcal{H}$.

Lemma 3.2. *[?] Let $\mathcal{A}$ be defined by (??) and let $q \neq 1$. Then the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is given by*

$$\begin{cases} \mathcal{A}^*(v,h) = -(h, v''), \\ D(\mathcal{A}^*) = \left\{(v,h) \in H^2(0,1)\times H^1(0,1) \mid v(0)=0, v'(1)+qh(1)=0\right\}, \end{cases} \tag{3.7}$$

and $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})$. The eigenvector $\Psi_n(x)$ of $\mathcal{A}^*$ corresponding to $\overline{\lambda}_n$ is given by

$$\Psi_n(x) = \left(\frac{\sinh \overline{\lambda}_n x}{\overline{\lambda}_n}, -\sinh \overline{\lambda}_n x\right), \ \forall\, n \in \mathbb{Z}. \tag{3.8}$$

It is easy to verify that for any $n, m \in \mathbb{Z}$, $\langle \Phi_n, \Psi_m\rangle = \delta_{nm}$. System (**??**) can be written as the following evolutionary equation in $\mathcal{H}$:

$$\frac{dX(t)}{dt} = \mathcal{A}X(t), \ t > 0, \quad X(0) = (u_0, u_1), \tag{3.9}$$

where $X(t) = (u(\cdot,t), u_t(\cdot,t))$, and the solution of (**??**) is given by

$$X(t) = \sum_{n\in\mathbb{Z}} e^{\lambda_n t}\langle X(0), \Psi_n\rangle \Phi_n. \tag{3.10}$$

Thus

$$y(t) = \mathcal{C}X(t) + d(t) = \sum_{n\in\mathbb{Z}} e^{\lambda_n t}\langle X(0), \Psi_n\rangle \mathcal{C}\Phi_n + d(t) = \sum_{n\in\mathbb{Z}} e^{\lambda_n t}\langle X(0), \Psi_n\rangle + d(t). \tag{3.11}$$

It can be seen from Lemma **??** that when $q = 1$, the real part of the eigenvalues is $+\infty$, while for $0 < q \neq 1$, the real part is finite positive. Hence, we suppose $1 \notin Q$ as usual (see, e.g., [**?**, **?**]), where $Q = [\underline{q}, \overline{q}]$ is the prior parameter set.

We take $q \in Q = (1, +\infty)$ as an example to illustrate how to apply the algorithms proposed in the previous section to the simultaneous identification for the anti-damping coefficient q and initial values. The following Corollaries **??**-**??** are the direct consequences of Theorem **??**, Corollary **??** and Theorem **??**, respectively, by noticing that for system (**??**), the relevant function and parameters now are

$$f(q) = \frac{1}{2}\ln\frac{q+1}{q-1}, \quad \mu_n = n\pi, \quad L = 2, \quad \kappa_n = 1. \tag{3.12}$$

Corollary 3.1. *Suppose that $d(t)=0$ in system (**??**), then both the coefficient q and initial values $u_0(x)$ and $u_1(x)$ can be uniquely determined by the output $y(t), t\in[0,T]$, where $T>4$. Specifically, q can be recovered exactly from*

$$q=\frac{\|y\|_{L^2(T_1,T_2)}+\|y\|_{L^2(T_1-2,T_2-2)}}{\|y\|_{L^2(T_1,T_2)}-\|y\|_{L^2(T_1-2,T_2-2)}},\quad 2\le T_1<T_2-2, \tag{3.13}$$

and the initial values $u_0(x)$ and $u_1(x)$ can be reconstructed from

$$u_0(x)=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\int_0^2 y(t)e^{-\lambda_n t}dt\right)\frac{\sinh\lambda_n x}{\lambda_n},\ u_1(x)=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\int_0^2 y(t)e^{-\lambda_n t}dt\right)\sinh\lambda_n x. \tag{3.14}$$

Corollary 3.2. *Suppose that the disturbance $d(t)$ in system (**??**) is a periodic function with period l, then the coefficient q can be uniquely recovered by the output $y(t), t\in[0,T]$, where $T>2+l$ as*

$$q=\frac{\|\bar{y}\|_{L^2(T_1,T_2)}+\|\bar{y}\|_{L^2(T_1-2,T_2-2)}}{\|\bar{y}\|_{L^2(T_1,T_2)}-\|\bar{y}\|_{L^2(T_1-2,T_2-2)}},\quad 2\le T_1<T_2-2, \tag{3.15}$$

where

$$\bar{y}(t)=y(t+l)-y(t),\quad t\ge 0. \tag{3.16}$$

In particular, if the period l is rational, that is, there exist $n_1,n_2\in\mathbb{N}$ such that $l=n_1/n_2$, then the disturbance can be recovered as well from $y(t), t\in[0,2n_1+l]$ as

$$d(t)=\frac{(q+1)^{n_1}y(t)-(q-1)^{n_1}y(t+2n_1)}{(q+1)^{n_1}-(q-1)^{n_1}},\ 0\le t\le l, \tag{3.17}$$

and the initial values $u_0(x)$ and $u_1(x)$ can be reconstructed as

$$\begin{aligned} u_0(x)&=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\int_0^2 [y(t)-d(t)]\,e^{-\lambda_n t}dt\right)\frac{\sinh\lambda_n x}{\lambda_n},\\ u_1(x)&=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\int_0^2 [y(t)-d(t)]\,e^{-\lambda_n t}dt\right)\sinh\lambda_n x. \end{aligned} \tag{3.18}$$

Corollary 3.3. *Suppose that $q\in Q=(1,+\infty)$ in system (**??**) and the disturbance is bounded, i.e. $|d(t)|\le M$ for some $M>0$ and all $t\ge 0$. Then for any $T_2-2>T_1\ge 2$,*

$$\lim_{T_1\to+\infty} q_{T_1}=q,\quad \lim_{T_1\to+\infty}\|(\hat{u}_{0T_1},\hat{u}_{1T_1})-(u_0,u_1)\|=0, \tag{3.19}$$

where

$$q_{T_1}=\frac{\|y\|_{L^2(T_1,T_2)}+\|y\|_{L^2(T_1-2,T_2-2)}}{\|y\|_{L^2(T_1,T_2)}-\|y\|_{L^2(T_1-2,T_2-2)}},\quad 2\le T_1<T_2-2. \tag{3.20}$$

and

$$\begin{aligned} \hat{u}_{0T_1}(x)&=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\int_{T_1}^{T_1+2} y(t)e^{-\lambda_n t}dt\right)\frac{\sinh\lambda_n x}{\lambda_n},\\ \hat{u}_{1T_1}(x)&=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\int_{T_1}^{T_1+2} y(t)e^{-\lambda_n t}dt\right)\sinh\lambda_n x. \end{aligned} \tag{3.21}$$

To end this section, we present some numerical simulations for system (**??**) to illustrate the performance of the algorithm.

Example 3.1. *The observation with some random noises when system (??) is stable.*

Simple spectral analysis together with Theorem **??** shows that Corollary **??** is also valid for $q \in Q = (-\infty, -1)$. In this example, the damping coefficient q and initial values $u_0(x), u_1(x)$ are chosen as

$$q = -3, \quad u_0(x) = -3 \sin \pi x, \quad u_1(x) = \pi \cos \pi x. \tag{3.22}$$

In this case, the output can be obtained from (**??**) (with $d(t) = 0$), where the infinite series is approximated by a finite one, that is, $\{n \in \mathbb{Z}\}$ is replaced by $\{n \in \mathbb{Z} \mid -5000 \leq n \leq 5000\}$. Some random noises are added to the measurement data and we use these data to test the algorithm proposed in Corollary **??**.

Let $T_1 = 2, T_2 = 2.5$, then the damping coefficient q can be recovered from (**??**), and the initial values $u_0(x), u_1(x)$ can be reconstructed from (**??**). Table **??** indicates the numerical results for the damping coefficients (the second column in Table **??**) and Figure **??**-**??** for the initial values in various cases of noise levels. In Table **??**, the absolute errors of the real damping coefficient and the recovered ones and the L^2-norm of the differences between the exact initial values and the reconstructed ones are also shown.

It is worth pointing out that in reconstruction of the initial values from (**??**), the infinite series is approximated by a finite one once again, that is, $\{n \in \mathbb{Z}\}$ is replaced by $\{n \in \mathbb{Z} \mid |n| \leq 1000\}$, which accounts for the zero value of the reconstructed initial velocity at the left end. This is also the reason that the errors of the initial velocity (the last column in Table **??**) are relatively large even if there is no random noise in the measured data.

Figure 1: without random noise

Figure 2: with 1% random error

Example 3.2. *The observation with periodic disturbance when system (??) is anti-stable.*

The anti-damping coefficient q and initial values $u_0(x)$ and $u_1(x)$ are chosen as

$$q = 3, \quad u_0(x) = 3 \sin \pi x, \quad u_1(x) = \pi \cos \pi x, \tag{3.23}$$

Figure 3: with 3% random error

Figure 4: with 5% random error

Table 1: Absolute errors with different noise levels

Noise Level	Recovered q	Errors for q	Errors for $u_0(x)$	Errors for $u_1(x)$
0	-3.0000	9.3259E-15	1.1744E-08	2.2215E-01
1%	-2.9994	6.2498E-04	1.1618E-03	2.2748E-01
3%	-2.9979	2.0904E-03	3.5604E-03	2.6662E-01
5%	-3.0132	1.3224E-02	4.9227E-03	3.1340E-01

and the observation is corrupted by the disturbance

$$d(t) = 3\sin \pi t + 4\cos \pi t, \tag{3.24}$$

being with period $l = 2$. The relevant parameters in Corollary **??** are chosen as follows:

$$T = 4.5, \quad n_1 = n_2 = 1, \quad T_1 = 2, \quad T_2 = 2.5.$$

For this case, the output is also obtained from (**??**) by a finite series approximation. The anti-damping coefficient q is recovered from (**??**):

$$q = \frac{\|\bar{y}\|_{L^2(2,\,2.5)} + \|\bar{y}\|_{L^2(0,\,0.5)}}{\|\bar{y}\|_{L^2(2,\,2.5)} - \|\bar{y}\|_{L^2(0,\,0.5)}} = 3.0000. \tag{3.25}$$

The periodic disturbance can be estimated from (**??**):

$$d(t) = 2y(t) - y(t+2), \quad 0 \le t \le 2, \tag{3.26}$$

and the initial values $u_0(x), u_1(x)$ can be reconstructed from (**??**). Figure **??** depicts the output and disturbance and Figure **??** the initial values.

Figure 5: output $y(t)$ (upper) and periodic disturbance $d(t)$ (lower)

Figure 6: initial values u_0 (upper) and u_1 (lower)

Example 3.3. *The observation with general bounded disturbance when system (**??**) is anti-stable.*

The anti-damping coefficient and initial values are also chosen as that in (**??**), and the observation is corrupted by the bounded disturbance:

$$d(t) = 2\sin\frac{1}{1+t} + 3\cos 10t. \tag{3.27}$$

The observation is also obtained from (**??**) by a finite series approximation, see Figure **??** for the output with disturbance. The relevant parameters in Corollary **??** are chosen to be $T_2 = T_1 + 3$, and let T_1 be different values increasing from 2 to 10. The corresponding anti-damping coefficients q_{T_1} recovered from (**??**) are depicted in Figure **??**. It is seen that q_{T_1} converges to the real value $q = 3$ as T_1 increases. Setting $T_1 = 0,\ 3,\ 7$ in (**??**) and reconstructing the initial values produce results in Figure **??** from which we can see that the reconstructed initial values become closer to the real ones as T_1 increases.

4 Application to Schrödinger equation

In this section, we consider a quantum system described by the following Schrödinger equation:

$$\begin{cases} u_t = -iu_{xx} + qu, & 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \ \ u(1,t) = 0, & t \geq 0, \\ y(t) = u(0,t) + d(t), & t \geq 0, \\ u(x,0) = u_0(x), & 0 \leq x \leq 1. \end{cases} \tag{4.1}$$

where $u(x,t)$ is the complex-valued state, i is the imaginary unit, and the potential $q > 0$ and $u_0(x)$ are the unknown anti-damping coefficient and initial value, respectively.

Let $\mathcal{H} = L^2(0,1)$ be equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and the inner product induced

Figure 7: output $y(t)$(upper) and coefficient q_r(lower)

Figure 8: initial values u_0(upper) and u_1(lower)

norm $\|\cdot\|$. Introduce the operator $\mathcal{A}$ defined by

$$\begin{cases} \mathcal{A}\phi = -i\phi'' + q\phi, \\ D(\mathcal{A}) = \left\{\phi \in H^2(0,1) \mid \phi'(0) = \phi(1) = 0\right\}. \end{cases} \tag{4.2}$$

A straightforward verification shows that such defined $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.

Lemma 4.1. *[?] Let $\mathcal{A}$ be defined by (??). Then the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues given by*

$$\lambda_n = q + i\left(n - \frac{1}{2}\right)^2 \pi^2, \quad n \in \mathbb{N}, \tag{4.3}$$

and the corresponding eigenfunctions $\phi_n(x)$ are given by

$$\phi_n(x) = \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi x, \quad n \in \mathbb{N}. \tag{4.4}$$

In addition, $\{\phi_n(x)\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $\mathcal{H}$.

System (**??**) can be rewritten as the following evolutionary equation in $\mathcal{H}$:

$$\frac{dX(t)}{dt} = \mathcal{A}X(t),\ t > 0, \quad X(0) = u_0, \tag{4.5}$$

and the solution of (**??**) is given by

$$X(t) = \sum_{n\in\mathbb{N}} e^{\lambda_n t}\langle X(0), \phi_n\rangle\phi_n. \tag{4.6}$$

Thus

$$y(t) = \sqrt{2}\sum_{n\in\mathbb{N}} e^{\lambda_n t}\langle u_0, \phi_n\rangle + d(t). \tag{4.7}$$

The relevant function and parameters in Theorems **??**-**??** for system (**??**) are

$$f(q) = q, \quad \mu_n = \left(n - \frac{1}{2}\right)^2 \pi^2, \quad L = \frac{8}{\pi}, \quad \kappa_n = \sqrt{2}. \tag{4.8}$$

Parallel to Section **??**, we have three corollaries corresponding to the exact observation, observation with periodic disturbance and observation with general bounded disturbance, respectively, for system (**??**). Here we only list the last one and the former two are omitted.

Corollary 4.1. *Suppose that* $q \in Q = (0, +\infty)$ *in system (**??**) and the disturbance is bounded, i.e.* $|d(t)| \le M$ *for some* $M > 0$ *and all* $t \ge 0$*. Then for any* $T_2 - \frac{8}{\pi} > T_1 > \frac{8}{\pi}$*,*

$$\lim_{T_1 \to +\infty} q_{T_1} = q, \quad \lim_{T_1 \to +\infty} \|\hat{u}_{0T_1} - u_0\| = 0, \tag{4.9}$$

where

$$q_{T_1} = \frac{\pi}{8} \ln \frac{\|y\|_{L^2(T_1, T_2)}}{\|y\|_{L^2(T_1 - \frac{8}{\pi}, T_2 - \frac{8}{\pi})}}, \quad \frac{8}{\pi} < T_1 < T_2 - \frac{8}{\pi}, \tag{4.10}$$

and

$$\hat{u}_{0T_1}(x) = \frac{\pi}{8} \sum_{n \in \mathbb{Z}} \left(\int_{T_1}^{T_1 + \frac{8}{\pi}} y(t) e^{-\lambda_n t} dt \right) \cos\left(n - \frac{1}{2}\right) \pi x. \tag{4.11}$$

We also give a numerical simulation to test the algorithm proposed in Corollary **??** for system (**??**), where the anti-damping coefficient q and initial value $u_0(x)$ are chosen as

$$q = 0.7, \quad u_0(x) = \sin \pi x + i \cos \pi x, \tag{4.12}$$

and the observation is corrupted by the disturbance

$$d(t) = 2 \sin \frac{t}{10 + t} + 3i \cos 20t. \tag{4.13}$$

The observation can be obtained from (**??**) by a finite series approximation, that is, $\{n \in \mathbb{N}\}$ is replaced by $\{n \in \mathbb{N} \mid n \le 5000\}$. Figure **??** depicts the real part and imaginary part of the output. The relevant parameters in Corollary **??** are chosen to be $T_2 = T_1 + 1$, and T_1 increasing from 2.55 to 10. The corresponding anti-damping coefficients q_{T_1} recovered from (**??**) are shown in Figure **??**. It is obvious that q_{T_1} is convergent to real value $q = 0.7$ as T_1 increases. Setting $T_1 = 0,\ 3,\ 7$, respectively, in (**??**), the reconstructed initial values are shown in Figure **??** from which it is seen that the errors between the reconstructed initial values and the real ones become smaller as T_1 increases.

5 Application to coupled strings equation

In this section, we consider the following two connected anti-stable strings with joint anti-damping described by

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), & x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1),\ t > 0, \\ u\left(\frac{1}{2}^-, t\right) = u\left(\frac{1}{2}^+, t\right), & t \ge 0, \\ u_x\left(\frac{1}{2}^-, t\right) - u_x\left(\frac{1}{2}^+, t\right) = q u_t(1,t), & t \ge 0, \\ u(0,t) = u_x(1,t) = 0, & t \ge 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & 0 \le x \le 1, \\ y(t) = u_x(0,t) + d(t), & t \ge 0, \end{cases} \tag{5.1}$$

Figure 9: real part (upper) and imaginary part (lower) of output $y(t)$ with disturbance

Figure 10: anti-damping coefficient q_r

Figure 11: real part (upper) and imaginary part (lower) of initial value $u_0(x)$

where $q > 0, q \neq 2$ is the unknown anti-damping constant. System (**??**) models two connected strings with joint vertical force anti-damping, see [**?**, **?**, **?**] for more details.

Let $\mathcal{H} = H_E^1(0,1) \times L^2(0,1)$ be equipped with the inner product $\langle \cdot, \cdot \rangle$ and its induced norm

$$\|(u,v)\|^2 = \int_0^1 \left[|u'(x)|^2 + |v(x)|^2 \right] dx,$$

where $H_E^1(0,1) = \left\{ u |\ u \in H^1(0,1), u(0) = 0 \right\}$. Then system (**??**) can be rewritten as an evolutionary equation in $\mathcal{H}$ as follows

$$\frac{d}{dt} X(t) = \mathcal{A} X(t), \tag{5.2}$$

where $X(t) = (u(\cdot,t), u_t(\cdot,t)) \in \mathcal{H}$ and $\mathcal{A}$ is defined by

$$\mathcal{A}(u,v) = (v(x), u''(x)), \tag{5.3}$$

with the domain

$$D(\mathcal{A}) = \left\{ (u,v) \in H^1(0,1) \times H^1_E(0,1) \left| \begin{array}{l} u(0) = u'(1) = 0, u|_{[0,\frac{1}{2}]} \in H^2(0,\frac{1}{2}), \\ u|_{[\frac{1}{2},1]} \in H^2(\frac{1}{2},1), u'(\frac{1}{2}^-) - u'(\frac{1}{2}^+) = qv(\frac{1}{2}) \end{array} \right. \right\}, \tag{5.4}$$

where $u|_{[a,b]}$ denotes the function $u(x)$ confined to $[a,b]$.

We assume without loss of generality that the prior parameter set for q is $Q = (2,+\infty)$ since the case for $Q = (0,2)$ is very similar.

Lemma 5.1. *[?] Let $\mathcal{A}$ be defined by (??)-(??) and $q \in Q = (2,+\infty)$. Then $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$ and eigenvalues of $\mathcal{A}$ are algebraically simple and separated, which are given by*

$$\lambda_n = \frac{1}{2}\ln\frac{q+2}{q-2} + in\pi, \quad n \in \mathbb{Z}. \tag{5.5}$$

The corresponding eigenfunctions $\Phi_n(x)$ are given by

$$\Phi_n(x) = (\phi_n(x), \lambda_n\phi_n(x)), \quad \forall\, n \in \mathbb{Z}, \tag{5.6}$$

where

$$\phi_n(x) = \begin{cases} \dfrac{\sqrt{2}}{\lambda_n}\cosh\dfrac{\lambda_n}{2}\sinh\lambda_n x, & 0 < x < \dfrac{1}{2}, \\ \dfrac{\sqrt{2}}{\lambda_n}\sinh\dfrac{\lambda_n}{2}\cosh\lambda_n(1-x), & \dfrac{1}{2} < x < 1. \end{cases} \tag{5.7}$$

and $\{\Phi_n(x)\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $\mathcal{H}$. In addition, $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.

Lemma 5.2. *[?] Let $\mathcal{A}$ be defined by (??)-(??) and $q \in Q = (2,+\infty)$. Then the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is given by*

$$\mathcal{A}^*(u,v) = -(v, u''), \tag{5.8}$$

with the domain

$$D(\mathcal{A}^*) = \left\{ (u,v) \in H^1(0,1) \times H^1_E(0,1) \left| \begin{array}{l} u(0) = u'(1) = 0, u|_{[0,\frac{1}{2}]} \in H^2(0,\frac{1}{2}), \\ u|_{[\frac{1}{2},1]} \in H^2(\frac{1}{2},1), u'(\frac{1}{2}^-) - u'(\frac{1}{2}^+) = -qv(\frac{1}{2}) \end{array} \right. \right\}, \tag{5.9}$$

and $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})$. The eigenfunctions $\Psi_n(x)$ of $\mathcal{A}^*$ corresponding to $\overline{\lambda}_n$ are given by

$$\Psi_n(x) = \left(\overline{\phi}_n(x), -\overline{\lambda}_n\overline{\phi}_n(x)\right), \quad \forall\, n \in \mathbb{Z}. \tag{5.10}$$

We show that $\{\Psi_n(x)\}$ is biorthogonal to $\{\Phi_n(x)\}$. Actually, for any $n,m \in \mathbb{Z}$,

$$\begin{aligned} \langle \Phi_n, \Psi_m \rangle &= \int_0^1 \left[\phi_n'(x)\cdot\phi_m'(x) - \lambda_n\phi_n(x)\cdot\lambda_m\phi_m(x)\right]dx \\ &= 2\cosh\frac{\lambda_n}{2}\cosh\frac{\lambda_m}{2}\int_0^{\frac{1}{2}}\cosh(\lambda_n-\lambda_m)x\,dx \\ &\quad -2\sinh\frac{\lambda_n}{2}\sinh\frac{\lambda_m}{2}\int_{\frac{1}{2}}^1\cosh(\lambda_n-\lambda_m)(1-x)\,dx \\ &= 2\cosh\frac{\lambda_n-\lambda_m}{2}\int_0^{\frac{1}{2}}\cosh(\lambda_n-\lambda_m)x\,dx = \delta_{nm}. \end{aligned} \tag{5.11}$$

Hence, the solution of (??) can be expressed as

$$X(t)=\sum_{n\in\mathbb{Z}} e^{\lambda_n t}\langle X(0),\Psi_n\rangle\Phi_n. \tag{5.12}$$

Define the observation operator $\mathcal{C}$ from $\mathcal{H}$ to $\mathbb{C}$ to be

$$\mathcal{C}(f,g)=f'(0),\quad (f,g)\in D(\mathcal{A}). \tag{5.13}$$

Then

$$y(t)=\mathcal{C}X(t)+d(t)=\sum_{n\in\mathbb{Z}} e^{\lambda_n t}\langle X(0),\Psi_n\rangle\mathcal{C}\Phi_n+d(t) \tag{5.14}$$

The following Corollary **??** is the direct consequence of Theorem **??** by noticing that for system (??), the relevant function and parameters now are

$$f(q)=\frac{1}{2}\ln\frac{q+2}{q-2},\quad \mu_n=n\pi,\quad L=2,\quad \kappa_n=\sqrt{2}\cosh\frac{\lambda_n}{2}, \tag{5.15}$$

and simple calculation shows that

$$\frac{\sqrt{2}}{2}\left[\left(\frac{q+2}{q-2}\right)^{\frac{1}{4}}-\left(\frac{q-2}{q+2}\right)^{\frac{1}{4}}\right]\le|\kappa_n|\le\frac{\sqrt{2}}{2}\left[\left(\frac{q+2}{q-2}\right)^{\frac{1}{4}}+\left(\frac{q-2}{q+2}\right)^{\frac{1}{4}}\right],\ \ \forall\, n\in\mathbb{Z}. \tag{5.16}$$

The corollaries that correspond to Theorem **??** and Corollary **??** are also omitted here.

Corollary 5.1. *Suppose that $q\in Q=(2,+\infty)$ in system (??) and the disturbance is bounded, i.e. $|d(t)|\le M$ for some $M>0$ and all $t\ge 0$. Then for any $T_2-2>T_1\ge 2$,*

$$\lim_{T_1\to+\infty} q_{T_1}=q,\qquad \lim_{T_1\to+\infty}\|(\hat{u}_{0T_1},\hat{u}_{1T_1})-(u_0,u_1)\|=0, \tag{5.17}$$

where

$$q_{T_1}=\frac{2\left(\|y\|_{L^2(T_1,T_2)}+\|y\|_{L^2(T_1-2,T_2-2)}\right)}{\|y\|_{L^2(T_1,T_2)}-\|y\|_{L^2(T_1-2,T_2-2)}},\ 2\le T_1<T_2-2. \tag{5.18}$$

and

$$\begin{gathered}\hat{u}_{0T_1}(x)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{Z}}\left(\int_{T_1}^{T_1+2}y(t)e^{-\lambda_n t}dt\right)\frac{\sinh\lambda_n x}{\lambda_n}, & 0<x<\dfrac{1}{2},\\ \dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{Z}}\left(\int_{T_1}^{T_1+2}y(t)e^{-\lambda_n t}dt\right)\tanh\frac{\lambda_n}{2}\cdot\frac{\cosh\lambda_n(1-x)}{\lambda_n}, & \dfrac{1}{2}<x<1.\end{cases}\\ \hat{u}_{1T_1}(x)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{Z}}\left(\int_{T_1}^{T_1+2}y(t)e^{-\lambda_n t}dt\right)\sinh\lambda_n x, & 0<x<\dfrac{1}{2},\\ \dfrac{1}{2}\displaystyle\sum_{n\in\mathbb{Z}}\left(\int_{T_1}^{T_1+2}y(t)e^{-\lambda_n t}dt\right)\tanh\frac{\lambda_n}{2}\cdot\cosh\lambda_n(1-x), & \dfrac{1}{2}<x<1.\end{cases}\end{gathered} \tag{5.19}$$

As before, we present some numerical simulations for system (??) to showcase the effectiveness of the algorithm proposed in Corollary **??**. The anti-damping coefficient q and initial values $u_0(x)$ and $u_1(x)$ are chosen to be

$$q=3,\quad u_0(x)=\sin x,\quad u_1(x)=\cos x, \tag{5.20}$$

and the observation is corrupted by the disturbance

$$d(t) = \sin \frac{t^2}{10+t} + \cos 10t. \tag{5.21}$$

The observation is obtained from (**??**) by a finite series approximation, that is, $\{n \in \mathbb{N}\}$ is replaced by $\{n \in \mathbb{N} | \ |n| \leq 5000\}$. Figure **??** depicts the output. The relevant parameters in Corollary **??** are chosen to be $T_2 = T_1 + 1$, and T_1 increasing from 2 to 8. The corresponding anti-damping coefficients q_{T_1} recovered from (**??**) are plotted in Figure **??**. It can be seen that q_{T_1} converges to the real value $q = 3$ as T_1 increases. Let $T_1 = 0,\ 3,\ 7$ in (**??**), and the reconstructed initial values are shown in Figure **??**, from which it is seen that the reconstructed initial values become closer to the real ones as T_1 increases.

Figure 12: output $y(t)$ (upper) and coefficient q_r (lower)

Figure 13: initial values u_0 (upper) and u_1 (lower)

6 Concluding remarks

In this paper, we propose an algorithm to reconstruct simultaneously an anti-damping coefficient and an initial value for some anti-stable PDEs. When the measured output is exact or with periodic disturbance, the recovered values are exact whereas if the measured output suffers from bounded unknown disturbance, the approximated values of the anti-damping coefficient and initial value can also be obtained. Some numerical examples are carried out to validate the effectiveness of the algorithm. It is very promising to apply the algorithm presented here to stabilization of anti-stable systems with an unknown anti-damping coefficient.

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