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Fixed Points of ψ-Contractive Mappings in Modular Spaces

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Abstract. The aim of this paper is to present fixed point results of ψ_ρ-contractive mappings in the framework of modular spaces. We also introduce (G, ψ_ρ)-contractive mappings and obtain fixed point results for such mappings. Some examples are presented to support the results proved herein.

1. Introduction

The concept of modular space was initiated by Nakano [24] and was redefined and generalized the notion of modular space by Musielak and Orlicz [23]. In addition, the most important development of these theory is due to Mazur and Orlicz, Luxemburg and Turpin (see, [19], [18], [31]). The study of fixed point theory in the context of modular function spaces was initiated by Khamsi [12] (see also [13]-[5]). Kuaket and Kumam [14] and Mongkolkeha and Kumam [20–22], considered and proved some fixed point and common fixed point results for generalized contraction mappings in modular spaces. Also, Kumam [15] obtained some fixed point theorems for nonexpansive mappings in arbitrary modular spaces. Recently, Kutbi and Latif [16] studied fixed points of multivalued mappings in modular function spaces. Recently, Chen [9] defined ψ-contractive mappings in complete metric spaces. Later, Nasine et al. [25] exteding the notion of ψ-contractive mappings obtained common fixed point results in ordered metric spaces. Also, Chandok and Dinu [8] obtained common fixed point results for weak ψ-contractive mappings in ordered metric spaces.

In 2008, Jachymski [11] investigated a new approach in metric fixed point theory by replacing an order structure with graph structure on a metric space. In this way, the results proved in ordered metric spaces are generalized (see for detail [11] and the reference therein). Recently, Aghanians and Nourouzi [3] proved the existence and uniqueness of fixed points for Banach and Kannan contractions defined on modular space endowed with a graph. Also, Öztürk et al. ([26]) obtained some fixed point results for mappings satisfying contractive condition of integral type in modular spaces endowed with a graph. For further work in this direction, we refer to (see, e.g., [1, 2, 27]).

The purpose of this paper is to introduce ψ-contractive mappings and to obtain fixed point results for such mappings in the setup of modular spaces. As an application of the results proved herein, we obtain fixed point results in the framework of modular space endowed with a graph. The existence of fixed points of mappings satisfying contractive condition of integral type is also obtained in such spaces. Some examples are provided to validate the results presented herein.

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2. Preliminaries

Let (X,d) be a metric space and $T : X \to X$ a mapping. A point x in X is said to be a fixed point of T if $Tx = x$ and denote the set of fixed points of T by $F(T)$. A self mapping T on X is called a Picard operator (PO) if $F(T) = \{z\}$ and for any initial approximation x in X, we have $\lim_{n\to\infty} T^n x = z$.

In the sequel the letters $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integer numbers, respectively.

Let Ψ be the collection of all mappings $\psi : {\mathbb{R}_+}^5 \to \mathbb{R}_+$ which satisfy the following conditions:

(C1) ψ is continuous;

(C2) ψ is strictly increasing in all the arguments;

(C3) For $t \in \mathbb{R}_+$ with $t > 0$, we have

$$\begin{aligned} \psi(t,t,t,0,2t) &< t, \ \ \psi(t,t,t,2t,0) < t, \ \ \psi(0,0,t,t,0) < t, \\ \psi(0,t,0,0,t) &< t, \ \ \psi(t,0,0,t,t) < t. \end{aligned}$$

Note that a mapping $\psi : {\mathbb{R}_+}^5 \to \mathbb{R}_+$ defined by

$$\psi(t_1,t_2,t_3,t_4,t_5) = k \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}, \text{ for } k \in (0,1)$$

satisfies conditions $(C1) - (C3)$. For more examples of such functions, we refer to [9].

Consistent with [23], some basic facts and notations needed in this paper are recalled as follows.

Definition 2.1. Let X be an arbitrary vector space. A functional $\rho : X \to [0,\infty]$ is called modular if for any x, y in X, the following conditions hold:

(m_1) $\rho(x) = 0$ if and only if $x = 0$;

(m_2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;

(m_3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ whenever $\alpha + \beta = 1$, and $\alpha, \beta \ge 0$.

If (m_3) is replaced with $\rho(\alpha x + \beta y) \le \alpha^s \rho(x) + \beta^s \rho(y)$ whenever $\alpha^s + \beta^s = 1$ and $\alpha, \beta \ge 0$,where $s \in (0,1]$, then ρ is called $s-$ convex modular. If $s = 1$, then we say that ρ is convex modular.

A trivial example of a modular functional is $\rho(x) = \sqrt{|x|}$, where $\rho : \mathbb{R} \to [0,\infty]$.

The modular functional $\rho : X \to [0,\infty]$ defines the following set

$$X_\rho = \{x \in X;\ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

called a modular space. The modular ρ is not sub-additive in general and therefore does not behave as a norm. One can associate to a modular an $F-$ norm.

Modular space X_ρ can be equipped with an $F-$norm defined by

$$\|x\|_\rho = \inf\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \le \alpha\}.$$

If ρ is convex modular, then

$$\|x\|_\rho = \inf\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \le 1\}$$

defines a norm on the modular space X_ρ, and is called the Luxemburg norm.

Define the $\rho-$ball, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x,r) = \{h \in X\rho : \rho(x-h) \le r\}.$$

Remark 2.2. [7] Followings are some consequences of condition (m_3):

(r_1) For $a, b \in \mathbb{R}$ with $|a| < |b|$ we have $\rho(ax) < \rho(bx)$ for all $x \in X$;

(r_2) For $a_1, ..., a_n \in \mathbb{R}_+$ with $\sum\limits_{i=1}^{n} a_i = 1$, we have

$$\rho\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} \rho(x_i), \text{ for any } x_1, ..., x_n \in X.$$

A function modular is said to satisfy (a) Δ_2-type condition if there exists $K > 0$ such that for any $x \in X_\rho$, we have $\rho(2x) \leq K\rho(x)$ (b) Δ_2-condition if $\rho(2x_n) \to 0$ as $n \to \infty$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 2.3. A sequence $\{x_n\}$ in modular space X_ρ is said to be:

(t_1) $\rho-$ convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$

(t_2) $\rho-$ Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

X_ρ is called $\rho-$ complete if any ρ-Cauchy sequence is ρ-convergent. Note that $\rho-$ convergent sequence is not necessarily a ρ-Cauchy sequence as ρ does not satisfy the triangle inequality. In fact, one can show that this will happen if and only if ρ satisfies the Δ_2-condition.

We know that [6] the norm and modular convergence are also the same when we deal with the Δ_2-type condition. Throughout this paper, we assume that modular function ρ is convex and satisfies the Δ_2-type condition.

Proposition 2.4. [20] Let X_ρ be a modular space. If $a, b \in \mathbb{R}_+$ with $b \geq a$, then $\rho(ax) \leq \rho(bx)$.

Proposition 2.5. [20] Suppose that X_ρ is a modular space, ρ satisfies the Δ_2-condition and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X_ρ. If $\rho(c(x_n - x_{n-1})) \to 0$, then

$\rho(\alpha l(x_n - x_{n-1})) \to 0$, as $n \to \infty$, where $c, l, \alpha \in \mathbb{R}_+$ with $c > l$ and $\frac{l}{c} + \frac{1}{\alpha} = 1$.

We define the notion of ψ_ρ-contractive mappings in the framework of modular space as follows:

Definition 2.6. Let X_ρ be a modular space. A self mapping T on X_ρ is said to be ψ_ρ-contractive mapping if there exist nonnegative numbers l, c with $l < c$ such that

$$\rho(c(Tx - Ty)) \leq \psi\left(\rho(l(x - y)), \rho(l(x - Tx)), \rho(l(y - Ty)), \rho\left(\frac{l}{2}(y - Tx)\right), \rho\left(\frac{l}{2}(x - Ty)\right)\right) \tag{1}$$

holds for any $x, y \in X_\rho$, where $\psi \in \Psi$.

We also state the following definitions and results from graph theory.

Let $\Delta = \{(x, x) : x \in X\}$ denotes the diagonal of $X \times X$, where X is any nonempty set equipped with a metric d. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph such that $\Delta \subseteq E(G)$. Further assume that G has no parallel edge and G is a weighted graph in the sense that each edge (x, y) is assigned the weight $d(x, y)$. Since d is a metric on X, the weight assigned to each vertex x to vertex y need not be zero and whenever a zero weight is assigned to some edge (x, y), it reduces to a (x, x) having weight 0. The graph G is identified by the pair $(V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ of vertices such that $x = x_0, ..., x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, ..., k\}$.

Recall that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Let G_x be the component of G consisting of all the edges and vertices which are contained in some path in G beginning at x. In $V(G)$, we define the relation R in the following way:

For $x, y \in V(G)$, we have xRy if and only if, there is a path in G from x to y. If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the equivalence class $[x]_G$ in $V(G)$ defined by the relation R is $V(G_x)$.

Definition 2.7. ([11, Definition 2.1]) A mapping $f : X \to X$ is called a Banach $G-$contraction if and only if:

(a) There is an edge between x and y implies that there is an edge between $f(x)$ and $f(y)$ for any $x, y \in X$, that is, f preserves edges of G;

(b) There exists α in $[0, 1)$ such that there is an edge between x and y implies

$$d(T(x), T(y)) \leq \alpha d(x, y).$$

for all $x, y \in X$. That is, T decreases weights of edges of G.

Let X_ρ be a modular space endowed with a graph G and $T : X_\rho \to X_\rho$ a mapping. Denote

$$X_T = \{x \in X : (x, Tx) \in E(G)\}.$$

For any $x, y \in V' \subseteq V(G), (x, y) \in E' \subseteq E(G)$, then (V', E') is called a subgraph of G.

Öztürk et al. ([26]) defined the notions of C_ρ-graph and orbitally G_ρ-continuity as follows:

Definition 2.8. Let $\{T^n x\}$ be a sequence, there exists $C > 0$ such that $\rho\left(C\left(T^n x - x^*\right)\right) \to 0$ for $x^* \in X_\rho$ and $\left(T^n x, T^{n+1} x\right) \in E(G)$ for all $n \in \mathbb{N}$. Then a graph G is called a C_ρ-graph if there exists a subsequence $\{T^{n_p} x\}$ of $\{T^n x\}$ such that $(T^{n_p} x, x^*) \in E(G)$ for $p \in \mathbb{N}$.

Definition 2.9. A mapping $T : X_\rho \to X_\rho$ is called orbitally G_ρ-continuous if for all $x, y \in X_\rho$ and any sequence $(n_p)_{p\in\mathbb{N}}$ of positive integers, and also there exists $C > 0$ such that

$$\rho\left(C\left(T^{n_p} x - y\right)\right) \to 0, \quad \left(T^{n_p} x, T^{n_p+1} x\right) \in E(G) \quad \text{imply} \quad \rho\left(C\left(T\left(T^{n_p} x\right) - T(y)\right)\right) \to 0$$

as $p \to \infty$.

3. Fixed Points of ψ_ρ-Contractive Mappings

In this section, we obtain several fixed point results in the setup of modular space. We start with the following result.

Theorem 3.1. Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ a ψ_ρ-contractive mapping. Then T has a unique fixed point.

Proof. Let x be a given point in X. We construct a sequence $\{x_n\}$ such that $x_n = T^n x$, $n = 1, 2, 3, \ldots$. Now we show that the following statement hold:

$$\rho\left(c\left(T^n x - T^{n+1} x\right)\right) < \rho\left(c\left(T^{n-1} x - T^n x\right)\right) \tag{2}$$

for all $n \geq 1$. Suppose that $\rho\left(c\left(T^{n-1} x - T^n x\right)\right) \leq \rho\left(c\left(T^n x - T^{n+1} x\right)\right)$ for some $n \in \mathbb{N}$. If follows from (1) and Remark 2.2 that

$$\begin{aligned}
\rho\left(c\left(T^n x - T^{n+1} x\right)\right) \leq \psi\Big(&\rho\left(l\left(T^{n-1} x - T^n x\right)\right), \rho\left(l\left(T^{n-1} x - T^n x\right)\right), \rho\left(l\left(T^n x - T^{n+1} x\right)\right), \\
&\rho\left(\tfrac{l}{2}\left(T^{n-1} x - T^{n+1} x\right)\right), \rho\left(\tfrac{l}{2}\left(T^n x - T^n x\right)\right)\Big) \\
\leq \psi\Big(&\rho\left(l\left(T^{n-1} x - T^n x\right)\right), \rho\left(l\left(T^{n-1} x - T^n x\right)\right), \rho\left(l\left(T^n x - T^{n+1} x\right)\right), \\
&\rho\left(l\left(T^{n-1} x - T^n x\right)\right) + \rho\left(l\left(T^n x - T^{n+1} x\right)\right), 0\Big) \\
\leq \psi\Big(&\rho\left(c\left(T^{n-1} x - T^n x\right)\right), \rho\left(c\left(T^{n-1} x - T^n x\right)\right), \rho\left(c\left(T^n x - T^{n+1} x\right)\right), \\
&\rho\left(c\left(T^{n-1} x - T^n x\right)\right) + \rho\left(c\left(T^n x - T^{n+1} x\right)\right), 0\Big)
\end{aligned} \tag{3}$$

$$\leq \psi\left(\rho\left(c\left(T^n x - T^{n+1}x\right)\right), \rho\left(c\left(T^n x - T^{n+1}x\right)\right), \rho\left(c\left(T^n x - T^{n+1}x\right)\right),\right.$$

$$\left.\rho\left(c\left(T^n x - T^{n+1}x\right)\right) + \rho\left(c\left(T^n x - T^{n+1}x\right)\right), 0\right) < \rho\left(c\left(T^n x - T^{n+1}x\right)\right)$$

a contradiction and hence we have $\rho\left(c\left(T^n x - T^{n+1}x\right)\right) < \rho\left(c\left(T^{n-1}x - T^n x\right)\right)$ for all $n \geq 1$. Set, $L_n = \rho\left(c\left(T^n x - T^{n-1}x\right)\right)$. Thus, from (2), we have that $\{L_n\}$ is a nonincreasing sequence and bounded from below. So convergent to some $L \geq 0$. Note that

$$L_n \leq \psi\left(L_n, L_n, L_n, 2L_n, 0\right).$$

On taking limit as $n \to \infty$ on both sides of above inequality, we have $L \leq \psi\left(L, L, L, 2L, 0\right)$. Now $L > 0$ further implies that $L \leq \psi\left(L, L, L, 2L, 0\right) < L$ a contradiction. Hence,

$$\rho\left(c\left(T^n x - T^{n+1}x\right)\right) \to 0, \quad \text{as} \quad n \to \infty. \tag{4}$$

Next, we show that $\{lT^n x\}_{n\in\mathbb{N}}$ is a ρ-Cauchy sequence. If not, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$ with $m_k > n_k \geq k$ such that the following hold:

$$\rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right) \geq \varepsilon, \text{ and } \rho\left(c\left(T^{n_k-1}x - T^{m_k}x\right)\right) < \varepsilon. \tag{5}$$

Now, choose $\alpha \in \mathbb{R}_+$ such that $\frac{l}{c} + \frac{1}{\alpha} = 1$, then we have

$$\begin{aligned}
&\rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right) \leq \rho\left(c\left(T^{n_k}x - T^{m_k}x\right)\right)\\
&\leq \psi\left(\rho\left(l\left(T^{n_k-1}x - T^{m_k-1}x\right)\right), \rho\left(l\left(T^{n_k-1}x - T^{n_k}x\right)\right), \rho\left(l\left(T^{m_{k-1}}x - T^{m_k}x\right)\right),\right.\\
&\left.\rho\left(\tfrac{l}{2}\left(T^{n_k-1}x - T^{m_k}x\right)\right), \rho\left(\tfrac{l}{2}\left(T^{m_k-1}x - T^{n_k}x\right)\right)\right).
\end{aligned} \tag{6}$$

Also, we have

$$\begin{aligned}
\rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right) &= \rho\left(l\left(T^{n_k}x - T^{n_k-1}x + T^{n_k-1}x - T^{m_k}x\right)\right)\\
&= \rho\left(\tfrac{1}{\alpha}\alpha l\left(T^{n_k}x - T^{n_k-1}x\right) + \tfrac{l}{c}c\left(T^{n_k-1}x - T^{m_k}x\right)\right)\\
&\leq \rho\left(\alpha l\left(T^{n_k}x - T^{n_k-1}x\right)\right) + \rho\left(c\left(T^{n_k-1}x - T^{m_k}x\right)\right)\\
&\leq \rho\left(\alpha l\left(T^{n_k}x - T^{n_k-1}x\right)\right) + \varepsilon.
\end{aligned} \tag{7}$$

Now by (4), (5), (7) and Proposition 2.5, we have

$$\lim_{k\to\infty} \rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right) = \varepsilon. \tag{8}$$

From inequality (6), we have

$$\begin{aligned}
\rho\left(l\left(T^{n_k-1}x - T^{m_k-1}x\right)\right) &= \rho\left(l\left(T^{n_k-1}x - T^{m_k}x + T^{m_k}x - T^{m_k-1}x\right)\right)\\
&= \rho\left(\tfrac{l}{c}c\left(T^{n_k-1}x - T^{m_k}x\right) + \tfrac{1}{\alpha}\alpha l\left(T^{m_k}x - T^{m_k-1}x\right)\right)\\
&\leq \rho\left(c\left(T^{n_k-1}x - T^{m_k}x\right)\right) + \rho\left(\alpha l\left(T^{m_k}x - T^{m_k-1}x\right)\right)
\end{aligned} \tag{9}$$

$$\leq \varepsilon + \rho\left(\alpha l\left(T^{m_k}x - T^{m_k-1}x\right)\right).$$

Note that

$$\begin{aligned}
\rho\left(\tfrac{l}{2}\left(T^{n_k-1}x - T^{m_k}x\right)\right) &= \rho\left(\tfrac{l}{2}\left(T^{n_k-1}x - T^{n_k}x + T^{n_k}x - T^{m_k}\right)\right) \\
&= \rho\left(\tfrac{1}{2}l\left(T^{n_k-1}x - T^{n_k}x\right) + \tfrac{1}{2}l\left(T^{n_k}x - T^{m_k}\right)\right) \\
&\leq \rho\left(l\left(T^{n_k-1}x - T^{n_k}x\right)\right) + \rho\left(l\left(T^{n_k}x - T^{m_k}\right)\right).
\end{aligned} \tag{10}$$

Also,

$$\begin{aligned}
\rho\left(\tfrac{l}{2}\left(T^{m_k-1}x - T^{n_k}x\right)\right) &= \rho\left(\tfrac{l}{2}\left(T^{m_k-1}x - T^{m_k}x + T^{m_k}x - T^{n_k}\right)\right) \\
&= \rho\left(\tfrac{1}{2}l\left(T^{m_k-1}x - T^{m_k}x\right) + \tfrac{1}{2}l\left(T^{m_k}x - T^{n_k}\right)\right) \\
&\leq \rho\left(l\left(T^{m_k-1}x - T^{m_k}x\right)\right) + \rho\left(l\left(T^{n_k}x - T^{m_k}\right)\right).
\end{aligned} \tag{11}$$

Using (8), (9), (10) and (11), and arranging the (6), we obtain that

$$\begin{aligned}
&\rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right) \leq \rho\left(c\left(T^{n_k}x - T^{m_k}x\right)\right) \\
&\leq \psi\Big(\rho\left(l\left(T^{n_k-1}x - T^{m_k-1}x\right)\right), \rho\left(l\left(T^{n_k-1}x - T^{n_k}x\right)\right), \rho\left(l\left(T^{m_{k-1}}x - T^{m_k}x\right)\right), \\
&\rho\left(\tfrac{l}{2}\left(T^{n_k-1}x - T^{m_k}x\right)\right), \rho\left(\tfrac{l}{2}\left(T^{m_k-1}x - T^{n_k}x\right)\right)\Big) \\
&\leq \psi\Big(\varepsilon + \rho\left(\alpha l\left(T^{m_k}x - T^{m_k-1}x\right)\right), \rho\left(l\left(T^{n_k-1}x - T^{n_k}x\right)\right), \\
&\rho\left(l\left(T^{m_k-1}x - T^{m_k}x\right)\right), \rho\left(l\left(T^{n_k-1}x - T^{n_k}x\right)\right) + \rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right), \\
&\rho\left(l\left(T^{m_k-1}x - T^{m_k}x\right)\right) + \rho\left(l\left(T^{n_k}x - T^{m_k}x\right)\right)\Big).
\end{aligned} \tag{12}$$

On taking limit as $k \to \infty$, (4), (8) and Proposition 2.5 give $\varepsilon \leq \psi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \varepsilon$, a contradiction. Thus $\{lT^n x\}_{n\in\mathbb{N}}$ is ρ-Cauchy sequence. Since X_ρ is a ρ-complete, there exists a point z in X_ρ such that $\rho\left(l\left(T^n x - z\right)\right) \to 0$ as $n \to \infty$.
Suppose that $Tz \neq z$, that is, $\rho\left(c\left(z - Tz\right)\right) > 0$. Thus

$$\begin{aligned}
&\rho\left(c\left(T^n x - Tz\right)\right) \\
&\leq \psi\Big(\rho\left(l\left(T^{n-1}x - z\right)\right), \rho\left(l\left(T^{n-1}x - T^n x\right)\right), \rho\left(l\left(z - Tz\right)\right), \\
&\rho\left(\tfrac{l}{2}\left(T^{n-1}x - Tz\right)\right), \rho\left(\tfrac{l}{2}\left(z - T^n x\right)\right)\Big) \\
&\leq \psi\Big(\rho\left(l\left(T^{n-1}x - z\right)\right), \rho\left(l\left(T^{n-1}x - T^n x\right)\right), \rho\left(l\left(z - Tz\right)\right), \\
&\rho\left(l\left(T^{n-1}x - T^n x\right)\right) + \rho\left(l\left(T^n x - Tz\right)\right), \rho\left(\tfrac{l}{2}\left(z - T^n x\right)\right)\Big)
\end{aligned} \tag{13}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, we get that

$$\rho\left(c\left(z - Tz\right)\right) \leq \psi\left(0, 0, \rho\left(l\left(z - Tz\right)\right), \rho\left(l\left(z - Tz\right)\right), 0\right) < \rho\left(l\left(z - Tz\right)\right) < \rho\left(c\left(z - Tz\right)\right) \tag{14}$$

a contradiction. Hence, $Tz = z$.
Uniqueness can be obtained easily, so we omit it.

4. Fixed Points of $\left(G,\psi_\rho\right)$-Contractive Mappings

We define the $\left(G,\psi_\rho\right)$-contractive mappings as follows:

Definition 4.1. Let X_ρ be a modular space endowed with a graph G. A self mapping T on X_ρ is called $\left(G,\psi_\rho\right)$-contractive mapping if

i. T preserves edges of G;

ii. there exist nonnegative numbers l, c with $l < c$ such that

$$\rho\left(c\left(Tx-Ty\right)\right)\leq\psi\left(\rho\left(l\left(x-y\right)\right),\rho\left(l\left(x-Tx\right)\right),\rho\left(l\left(y-Ty\right)\right),\ \rho\left(\frac{l}{2}\left(x-Ty\right)\right),\rho\left(\frac{l}{2}\left(y-Tx\right)\right)\right)$$

holds for all $(x,y)\in E(G)$, where $\psi\in\Psi$.

Remark 4.2. Let X_ρ be a modular space endowed with a graph G and $T : X_\rho \to X_\rho$ a $\left(G,\psi_\rho\right)$-contractive mapping. If there exists $x_0 \in X_\rho$ such that $Tx_0 \in [x_0]_{\tilde{G}}$, then following statements hold:

i. T is both a $\left(G^{-1},\psi_\rho\right)$-contractive mapping and $\left(\tilde{G},\psi_\rho\right)$-contractive mapping,

ii. $[x_0]_{\tilde{G}}$ is $T-$invariant and $T_{|[x_0]_{\tilde{G}}}$ is a $\left(\tilde{G}_{x_0},\psi_\rho\right)$-contractive mapping.

Theorem 4.3. Let X_ρ be ρ-complete modular space endowed with a graph G and $T : X_\rho \to X_\rho$. If following statements hold:

i. G is weakly connected and $C_\rho-$graph;

ii. T is a $\left(\tilde{G},\psi_\rho\right)$-contractive mapping;

iii. X_T is nonempty.

Then T is a PO.

Proof. If $x \in X_T$, then $Tx \in [x]_{\tilde{G}}$ and $\left(T^n x, T^{n+1}x\right) \in E\left(G\right)$ for all $n \in \mathbb{N}$. Note that

$$\rho\left(c\left(T^n x-T^{n+1}x\right)\right)<\rho\left(c\left(T^{n-1}x-T^n x\right)\right). \tag{15}$$

holds all $n \in \mathbb{N}$. Indeed, from Definition 4.1, Remark 2.2 and (3), we have

$$\rho\left(c\left(T^n x-T^{n+1}x\right)\right)<\rho\left(c\left(T^{n-1}x-T^n x\right)\right). \tag{16}$$

Set $h_n = \rho\left(c\left(T^n x - T^{n+1}x\right)\right)$. Following arguments similar to those in the proof of Theorem 3.1, we obtain that

$$\rho\left(c\left(T^n x-T^{n+1}x\right)\right)\to 0,\quad n\to\ \infty. \tag{17}$$

and $\{lT^n x\}$ is ρ-Cauchy sequence. By ρ-completeness of X_ρ, there exists $x^* \in X_\rho$ such that $\rho\left(l\left(T^n x - x^*\right)\right) \to 0$ as $n \to \infty$ and $\left(T^n x, T^{n+1}x\right) \in E\left(G\right)$ for all $n \in \mathbb{N}$ and G is a C_ρ-graph, then there exists subsequence $\{T^{n_p}x\}$

such that $(T^{n_p}x, x^*) \in E(G)$ for each $p \in \mathbb{N}$. Also, $(T^{n_p}x, x^*) \in E(G)$ for each $p \in \mathbb{N}$. From Definition 4.1, and Remark 2.2, we have

$$\begin{aligned}
&\rho\left(c\left(T^{n_p+1}x - Tx^*\right)\right)\\
&\leq \psi\left(\rho\left(l\left(T^{n_p}x - x^*\right)\right), \rho\left(l\left(T^{n_p}x - T^{n_p+1}x\right)\right), \rho\left(l\left(x^* - Tx^*\right)\right),\right.\\
&\left.\rho\left(\tfrac{l}{2}\left(T^{n_p}x - Tx^*\right)\right), \rho\left(\tfrac{l}{2}\left(x^* - T^{n_p+1}x\right)\right)\right)\\
&\leq \psi\left(\rho\left(l\left(T^{n_p}x - x^*\right)\right), \rho\left(l\left(T^{n_p}x - T^{n_p+1}x\right)\right), \rho\left(l\left(x^* - Tx^*\right)\right),\right.\\
&\left.\rho\left(l\left(T^{n_p}x - x^*\right)\right) + \rho\left(l\left(x^* - Tx^*\right)\right), \rho\left(\tfrac{l}{2}\left(x^* - T^{n_p+1}x\right)\right)\right).
\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}
\rho\left(c\left(x^* - Tx^*\right)\right) &\leq \psi\left(0, 0, \rho\left(l\left(x^* - Tx^*\right)\right), \rho\left(l\left(x^* - Tx^*\right)\right), 0\right)\\
&\leq \psi\left(0, 0, \rho\left(c\left(x^* - Tx^*\right)\right), \rho\left(c\left(x^* - Tx^*\right)\right), 0\right) < \rho\left(c\left(x^* - Tx^*\right)\right),
\end{aligned}$$

which implies that $\rho(c(x^* - Tx^*)) = 0$, that is, $x^* = Tx^*$.
To prove the uniqueness, we proceed as follows: Let $y^* \in X_\rho - \{x^*\}$ be a fixed point of T. As G is a C_ρ-graph, there exists a subsequence $\{T^{n_p}x\}$ of $\{T^n x\}$ such that $(T^{n_p}x, x^*) \in E(G)$ and $(T^{n_p}x, y^*) \in E(G)$ for each $p \in \mathbb{N}$. Since G is weakly connected, so we have $(x^*, y^*) \in E\left(\tilde{G}\right)$. Thus,

$$\begin{aligned}
\rho\left(c\left(x^* - y^*\right)\right) = \rho\left(c\left(Tx^* - Ty^*\right)\right) &\leq \psi\left(\rho\left(l\left(x^* - y^*\right)\right), \rho\left(l\left(x^* - Tx^*\right)\right), \rho\left(l\left(y^* - Ty^*\right)\right),\right.\\
&\left.\rho\left(\tfrac{l}{2}\left(x^* - Ty^*\right)\right), \rho\left(\tfrac{l}{2}\left(y^* - Tx^*\right)\right)\right)\\
&\leq \psi\left(\rho\left(l\left(x^* - y^*\right)\right), \rho\left(l\left(x^* - Tx^*\right)\right), \rho\left(l\left(y^* - Ty^*\right)\right),\right.\\
&\left.\rho\left(l\left(x^* - y^*\right)\right) + \rho\left(l\left(y^* - Ty^*\right)\right), \rho\left(l\left(y^* - x^*\right)\right) + \rho\left(l\left(x^* - Tx^*\right)\right)\right),
\end{aligned}$$

so

$$\rho\left(c\left(x^* - y^*\right)\right) \leq \psi\left(\rho\left(l\left(x^* - y^*\right)\right), 0, 0, \rho\left(l\left(x^* - y^*\right)\right), \rho\left(l\left(x^* - y^*\right)\right)\right) < \rho\left(l\left(x^* - y^*\right)\right) < \rho\left(c\left(x^* - y^*\right)\right),$$

a contradiction and the result follows.
In Theorem 4.3, if we replace the condition that G is a C_ρ-graph with orbitally G_ρ-continuity of T, then we have the following theorem.

Theorem 4.4. Let X_ρ be a ρ-complete modular space endowed with a graph G, and $T : X_\rho \to X$ a $\left(\tilde{G}, \psi_\rho\right)$-contractive and orbitally G_ρ-continuous mapping. If X_T is nonempty and the graph G is weakly connected, then T is a PO.

Proof. Let $x \in X_T$. From Theorem 4.3, it follows that $\{lT^n x\}$ is a ρ-Cauchy sequence in X_ρ. By ρ-completeness of X_ρ, there exists $x^* \in X_\rho$ such that $\rho(l(T^n x - x^*)) \to 0$. As $\left(T^n x, T^{n+1}x\right) \in E(G)$ for all $n \in \mathbb{N}$ and T is orbitally G_ρ-continuous, so we have $\rho(l(T(T^n x) - T(x^*))) \to 0$, as $n \to \infty$. That is, $Tx^* = x^*$. Assume that y^* is another fixed point of T. Following arguments similar to those in the proof of Theorem 4.3, we obtain that $x^* = y^*$.

Define $\Phi = \{\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \varphi$ is a Lebesgue integral mapping which is summable, nonnegative and satisfies $\int_0^\epsilon \varphi(t)dt > 0$, for each $\epsilon > 0\}$.

Corollary 4.5. Let X_ρ be ρ-complete modular space endowed with a graph G and $T : X_\rho \to X_\rho$. Suppose that:

i. G is weakly connected and $C_\rho-$graph;

ii. there exist nonnegative numbers l, c with $l < c$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)\,dt \leq \int_0^{\psi\left(\rho(l(x-y)),\rho(l(x-Tx)),\rho(l(y-Ty)),\rho\left(\frac{l}{2}(x-Ty)\right),\rho\left(\frac{l}{2}(y-Tx)\right)\right)} \varphi(t)\,dt$$

for all $(x,y) \in E\left(\tilde{G}\right)$, $\psi \in \Psi$ and $\varphi \in \Phi$;

iii. X_T is nonempty.

Then T is a PO.

Corollary 4.6. Let X_ρ be ρ-complete modular space endowed with a graph G and $T : X_\rho \to X_\rho$. Suppose that:

i. G is weakly connected;

ii. there exist nonnegative numbers l, c with $l < c$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)\,dt \leq \int_0^{\psi\left(\rho(l(x-y)),\rho(l(x-Tx)),\rho(l(y-Ty)),\rho\left(\frac{l}{2}(x-Ty)\right),\rho\left(\frac{l}{2}(y-Tx)\right)\right)} \varphi(t)\,dt$$

for all $(x,y) \in E\left(\tilde{G}\right)$, $\psi \in \Psi$ and $\varphi \in \Phi$;

iii. X_T is nonempty;

iv. T is orbitally G_ρ-continuous.

Then T is a PO.

Corollary 4.7. Let X_ρ be ρ-complete modular space and $T : X_\rho \to X_\rho$. Suppose that there exist nonnegative numbers l, c with $l < c$ such that

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)\,dt \leq \int_0^{\psi\left(\rho(l(x-y)),\rho(l(x-Tx)),\rho(l(y-Ty)),\rho\left(\frac{l}{2}(x-Ty)\right),\rho\left(\frac{l}{2}(y-Tx)\right)\right)} \varphi(t)\,dt$$

holds for all $x, y \in X_\rho$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then T has a unique fixed point.

Example 4.8. Let $X_\rho = [0,1]$ and $\rho(x) = |x|$ for all $x \in X_\rho$. Consider

$$E(G) = \{(x,y) \ : \ x, y \in [0,1]\}.$$

Define the mapping $\psi : \mathbb{R}_+{}^5 \to \mathbb{R}_+$ by

$$\psi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{5}\max\left\{t_1,t_2,t_3,\frac{t_4}{2},\frac{t_5}{2}\right\},$$

Define $T : X_\rho \to X_\rho$ as $Tx = \frac{x}{4}$.

Note that G is weakly connected and C_ρ-graph, X_T is nonempty and T is both a $(\tilde{G},\psi_\rho)$-contractive mapping and a ψ_ρ-contractive mapping with $c = \frac{1}{2}$, $l = \frac{1}{3}$. In addition, T is orbitally G_ρ-continuous. Thus, all the conditions of Theorem 3.1, Theorem 4.3, and Theorem 4.4 are satisfied. Moreover $x = 0$ is a unique fixed point of T.

Example 4.9. Let $X_\rho = \{0, 1, 2, 3\}$ and $\rho(x) = |x|$ for all $x \in X_\rho$. Consider

$$E\left(\tilde{G}\right) = \{(0,0), (0,1), (1,1), (1,3), (2,2), (0,3), (2,3), (3,3)\}.$$

Define $T : X_\rho \to X_\rho$ by

$$Tx = \begin{cases} 0, & x \in \{0,1\} \\ 1, & x \in \{2,3\} \end{cases}.$$

A mapping $\psi : \mathbb{R}_+{}^5 \to \mathbb{R}_+$ is defined by

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{2}{3} \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}.$$

Clearly, G is weakly connected and C_ρ-graph, X_T is nonempty and T is orbitally G_ρ-continuous and $(\tilde{G}, \psi_\rho)$-contractive mapping with $c = 4$, $l = 3$. Note that T is not ψ_ρ-contractive mapping. Indeed, we have

$$\rho(c(T1 - T2)) \leq \psi\left(\rho(l(1-2)), \rho(l(1-T1)), \rho(l(2-T2)),\ \rho\left(\tfrac{l}{2}(1-T2)\right), \rho\left(\tfrac{l}{2}(2-T1)\right)\right).$$

$4 \leq 2$ is obtained, an absurd statement. Hence all the conditions of Theorem 4.3 and Theorems 4.4 are satisfied. However, Theorem 3.1 is not applicable in this case.

Remark 4.10. Recently, Paknazar et al. [28] gave an existence of solution of integral equations in Musielak-Orlicz modular spaces. Hajji and Hanebaly [10] also applied their fixed point result to obtain the solution of perturbed integral equations in modular function spaces (see also [30]). Our results can also be employed to solve such integral equations in the framework of modular function space endowed with a graph.

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