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Convergence of sequences of semi-continuous functions

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ABSTRACT

In this paper we investigate how three well-known modes of convergence for (real-valued) functions are related to one another. In particular, we consider order convergence, pointwise convergence and continuous convergence of sequences of nearly finite normal lower semi-continuous functions. There is a natural comparison to be made between the results we obtain for convergence of sequences of semi-continuous functions, and classic results on the convergence of sequences of measurable functions.

1. Introduction

Order convergence arises as a natural mode of convergence of sequences in a partially ordered set (poset). Indeed, order convergence is a generalization of convergence of sequences of real numbers, with respect to the usual metric topology on $\mathbb{R}$. Recall that a sequence (u_n) in a poset P order converges to u in P if

$$\begin{aligned}
&\exists(\lambda_n),\ (\mu_n)\subseteq P:\\
&\quad 1)\ \lambda_n \leqslant \lambda_{n+1} \leqslant u_{n+1} \leqslant \mu_{n+1} \leqslant \mu_n, \quad n\in\mathbb{N},\\
&\quad 2)\ \sup\{\lambda_n\colon n\in\mathbb{N}\} = u = \inf\{\mu_n\colon n\in\mathbb{N}\}.
\end{aligned} \tag{1}$$

It is clear that (1) is equivalent to the convergence with respect to the metric topology on $\mathbb{R}$, if $P=\mathbb{R}$. Order convergence also appears frequently in analysis, in particular in Riesz space (vector lattice) theory. In this regard, we may recall that σ-order continuous operators and integral operators acting between two Riesz space are characterized in terms of order convergence [11,21]. This includes, as a particular case, the L_p-spaces of p-summable functions on a measure space.

Another important application of order convergence is to entropy solutions of nonlinear conservation laws. A basic result in this regard is the following, see for instance [10, Section 11.4.2, Theorem 2].

Theorem 1. *Consider the Cauchy problem*

$$u_t + \big(F(u)\big)_x = 0, \qquad u(x,0) = u_0(x), \tag{2}$$

where $u_0 \in L_\infty(\mathbb{R})$ and $F:\mathbb{R}\to\mathbb{R}$ is continuously differentiable. For $\epsilon>0$, denote by u^ϵ the unique smooth solution of the viscous problem

$$u_t^\epsilon + \big(F(u^\epsilon)\big)_x = \epsilon u_{xx}^\epsilon, \qquad u^\epsilon(x,0) = u_0(x).$$

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Let (ϵ_n) be a sequence of positive numbers, converging to 0 in $\mathbb{R}$, such that the sequence (u^{ϵ_n}) is norm bounded in $L_\infty(\mathbb{R}\times[0,\infty))$, and converges almost everywhere to $u\in L_\infty(\mathbb{R}\times[0,\infty))$. Then u is the unique entropy solution of (2).

If $L_\infty(\Omega)$, for some σ-finite measure space (Ω,Σ,μ), is considered with the usual pointwise almost everywhere order, given by

$$u\leqslant v \quad\Leftrightarrow\quad \begin{pmatrix} \exists E\subset\Omega,\ \text{measure}(E)=0\text{:} \\ u(x)\leqslant v(x),\ x\in\Omega\setminus E \end{pmatrix}, \tag{3}$$

then $L_\infty(\Omega)$ is Dedekind complete Archimedean Riesz space. Order convergence in this Riesz space is precisely the bounded almost everywhere convergence used in Theorem 1, see [11, p. 484].

A recent application of order convergence of sequences to nonlinear partial differential equations (PDEs) is given in [16–18]. Spaces of generalized functions are constructed as completions of suitable uniform convergence spaces, the elements of which are nearly finite normal lower semi-continuous functions, as defined in Section 2, see also [16]. The spaces of generalized function so obtained contain generalized solutions of large classes of systems of nonlinear PDEs. The mentioned function spaces are equipped with the structure of a Riesz space in a natural way, and the sequences that converge with respect to the uniform convergence structures on these spaces are precisely the sequences that order converge.

In this paper we investigate order convergence of sequences in the space $\mathcal{NL}(X)$ of nearly finite normal lower semi-continuous functions, see Section 2. Two characterizations of order convergence in this space are obtained, one in terms of pointwise convergence, and the other in terms of continuous convergence. Both characterizations have close analogies in the context of measurable functions. Indeed, we may recall [11, Theorem 71.3] that a sequence (u_n) in the space $\mathcal{M}(\Omega)$ of almost everywhere finite, real measurable functions on Ω order converges to $u\in\mathcal{M}(\Omega)$, with respect to the order (3), if and only if $(u_n(x))$ converges to $u(x)$ for almost all $x\in\Omega$. Egoroff's Theorem [9,14] states that, for a finite measure space, almost everywhere convergence in $\mathcal{M}(\Omega)$ is equivalent to almost uniform convergence.

Theorem 2. *Let (Ω,Σ,μ) be a finite measure space, and (u_n) a sequence in $\mathcal{M}(\Omega)$. If $(u_n(x))$ converges to $u(x)$ for almost all $x\in\Omega$, then the following statement holds*:

$$\begin{aligned}&\forall\epsilon>0\text{:}\\ &\exists A_\epsilon\subset\Omega,\quad \mu(A_\epsilon)<\epsilon\text{:}\\ &\quad (u_n)\ \textit{converges uniformly to } u \textit{ on } \Omega\setminus A_\epsilon.\end{aligned}$$

It should also be mentioned that, due to certain representation theorems for Archimedean Riesz spaces [11, Chapter 7], the characterizations of order convergence of sequences obtained for the space $\mathcal{NL}(X)$ may be applied to a large class of such Riesz spaces. Indeed, for an Archimedean Riesz space E, one may construct a topological space Θ such that

$$\mathcal{C}^\infty(\Theta)=\left\{u\in\mathcal{C}(\Theta,\overline{\mathbb{R}})\text{: } u^{-1}(\mathbb{R}) \text{ is dense in } \Theta\right\} \tag{4}$$

contains E as a sublattice. Here $\overline{\mathbb{R}}$ denotes the extended real line, equipped with the topology generated by all open rays in $\mathbb{R}$. The space $\mathcal{C}^\infty(\Theta)$ is, in turn, a Riesz subspace of the space $\mathcal{NL}(\Theta)$ of nearly finite normal lower semi-continuous functions. The notation $\mathcal{C}^\infty(\Theta)$ is unfortunate, since it may lead to confusion with spaces of smooth functions. However, it is established in the literature, and we will not try to change it here.

The paper is organized as follows. In Section 2 we obtain characterizations of order convergence of sequences of nearly finite normal lower semi-continuous functions in terms of pointwise convergence and continuous convergence. The implications of the results obtained in Section 2 for order convergence of sequences of continuous functions are discussed in Section 3, while Section 4 concerns the application of our results to large classes of Riesz spaces through the mentioned representation theorems.

2. Order convergence in $\mathcal{NL}(X)$

Throughout, X will denote a topological space. In this section we characterize order convergence of sequences of nearly finite normal lower semi-continuous functions on X. Before we formulate and prove these characterizations, a few basic concepts and results are recalled.

In this regard, see [2,8,19], an extended real valued function $u: X\to\overline{\mathbb{R}}$ is called normal lower semi-continuous whenever

$$I\big(S(u)\big)(x)=u(x),\quad x\in X. \tag{5}$$

Here I and S denote the lower and upper Baire operators [4], respectively, which are defined as

$$I(u)(x)=\sup\big\{\inf\big\{u(y)\text{: } y\in V\big\}\text{: } V\in\mathcal{V}(x)\big\},\quad x\in X \tag{6}$$

and

$$S(u)(x) = \inf\bigl\{\sup\bigl\{u(y):\ y \in V\bigr\}:\ V \in \mathcal{V}(x)\bigr\},\quad x \in X. \tag{7}$$

Here $\mathcal{V}(x)$ denotes the neighborhood filter at $x \in X$. The Baire operators, as well as their composition $I \circ S$, are idempotent and monotone with respect to the pointwise order on extended real valued functions,

$$u \leqslant v \quad \Leftrightarrow \quad \left(\begin{array}{l} \forall x \in X: \\ \quad u(x) \leqslant v(x) \end{array}\right). \tag{8}$$

That is, the identities

$$I\bigl(I(u)\bigr) = I(u),\qquad S\bigl(S(u)\bigr) = S(u),\qquad I\bigl(S\bigl(I\bigl(S(u)\bigr)\bigr)\bigr) = I\bigl(S(u)\bigr) \tag{9}$$

and the inequalities

$$I(u) \leqslant I(v),\qquad S(u) \leqslant S(v),\qquad I\bigl(S(u)\bigr) \leqslant I\bigl(S(v)\bigr) \tag{10}$$

hold for all extended real valued functions $u \leqslant v$ on X.

Remark 3. In the same way as the concept of normal lower semi-continuity is introduced above, we may consider the concept of a normal upper semi-continuous function: A function $u : X \to \overline{\mathbb{R}}$ is called normal upper semi-continuous if

$$S\bigl(I(u)\bigr) = u.$$

The concept of normal upper semi-continuity is dual to that of normal lower semi-continuity. In particular, the space of normal upper semi-continuous functions is lattice isomorphic to the space of normal lower semi-continuous functions. Thus the results we obtain here for normal lower semi-continuous functions apply also to normal upper semi-continuous functions.

A normal lower semi-continuous function u is called *nearly finite* if the set

$$\bigl\{x \in X:\ u(x) \in \mathbb{R}\bigr\} \tag{11}$$

is dense in X. In fact, if the set (11) is dense in X for some normal lower semi-continuous function, then it contains an open and dense set. That is, every nearly finite normal lower semi-continuous function is real valued on an open and dense set. We denote the set of such nearly finite normal lower semi-continuous functions by $\mathcal{NL}(X)$.

In view of (5) and the definition of the operators I and S given in (6) and (7), respectively, it is clear that the restriction of any $u \in \mathcal{NL}(X)$ to any open subset V of X is nearly finite and normal lower semi-continuous. That is,

$$\begin{array}{l} \forall u \in \mathcal{NL}(X),\ V \subseteq X \text{ open:} \\ \quad u_{|V} \in \mathcal{NL}(V), \end{array} \tag{12}$$

where $u_{|V}$ denotes the restriction of u to V.

With respect to the pointwise order (8), the space $\mathcal{NL}(X)$ is a Dedekind complete Archimedean Riesz space [16,20]. It should be noted that, although the algebraic operations are not given by the usual pointwise operations on real functions, they are in fact determined by those pointwise operations. The supremum and infimum of a set $A \subset \mathcal{NL}(X)$ are also defined in terms of the pointwise supremum and infimum, respectively [16]. In particular, if

$$\varphi_A(x) = \sup\bigl\{u(x):\ u \in A\bigr\},\quad x \in X$$

and

$$\psi_A(x) = \inf\bigl\{u(x):\ u \in A\bigr\},\quad x \in X,$$

then

$$\sup A = I\bigl(S(\varphi_A)\bigr),\qquad \inf A = I\bigl(S(\psi_A)\bigr) \tag{13}$$

provided the supremum and infimum, respectively, exist in $\mathcal{NL}(X)$.

While functions in $\mathcal{NL}(X)$ are typically not continuous, and may in fact fail to be continuous at any point of their domain of definition, some important properties of continuous functions remain valid. In particular, for $u, v \in \mathcal{NL}(X)$ and $D \subseteq X$ a dense set, it is true that

$$\left(\begin{array}{l} \forall x \in D: \\ \quad u(x) \leqslant v(x) \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{l} \forall x \in X: \\ \quad u(x) \leqslant v(x) \end{array}\right). \tag{14}$$

Regarding the continuity of normal lower semi-continuous functions, we may mention the following: For every $u \in \mathcal{NL}(X)$ and every $\epsilon > 0$ there exists an open and dense set $D_\epsilon(u) \subseteq X$ with the following property:

$$\begin{aligned} &\forall x \in D_\epsilon(u): \\ &\quad \omega(u, x) < \epsilon \end{aligned} \tag{15}$$

where $\omega(u, x)$ denotes the modulus of continuity of u at x, given by

$$\omega(u, x) = \inf\bigl\{\sup\bigl\{\bigl|u(x) - u(y)\bigr|: \; y \in V\bigr\}: \; V \in \mathcal{V}(x)\bigr\}.$$

It then follows from (15) that if X is a Baire space, then a function $u \in \mathcal{NL}(X)$ is real valued and continuous on a residual set.

Let us now proceed to establish the mentioned characterizations of order convergence of sequences in $\mathcal{NL}(X)$. These results are in part based on the following basic lemma.

Lemma 4. *Let X be a Baire space. Assume that $\mathcal{A} \subset \mathcal{NL}(X)$ satisfies*

$$B_{\mathcal{A}} = \left\{ x \in X \;\middle|\; \begin{array}{l} \forall M \in \mathbb{R}: \\ \exists u \in \mathcal{A}: \\ \quad u(x) > M \end{array} \right\} \quad \text{is of first Baire category}. \tag{16}$$

Then there exists $u_0 \in \mathcal{NL}(X)$ such that

$$u \leqslant u_0, \quad u \in \mathcal{A}. \tag{17}$$

The corresponding statement for sets bounded from below is also true.

Proof. Assume that $\mathcal{A} \subset \mathcal{NL}(X)$ satisfies (16). Then the function

$$\psi : X \ni x \mapsto \sup\bigl\{u(x): \; u \in \mathcal{A}\bigr\} \tag{18}$$

is real valued on $R = X \setminus B_{\mathcal{A}}$. Since we clearly have

$$u \leqslant \psi, \quad u \in \mathcal{A}$$

it follows by (10) and the normal lower semi-continuity of each $u \in \mathcal{A}$ that

$$u \leqslant v = I\bigl(S(\psi)\bigr), \quad u \in \mathcal{A}.$$

Furthermore, (5) and (9) imply that v is normal lower semi-continuous. We claim that v is nearly finite. In this regard, suppose that

$$\begin{aligned} &\exists U \subseteq X \text{ a nonempty open set:} \\ &\forall x \in U: \\ &\quad v(x) = \infty. \end{aligned} \tag{19}$$

Note that $I(S(\psi)) \leqslant S(\psi)$. Therefore (19) implies

$$S(\psi)(x) = \infty, \quad x \in U. \tag{20}$$

In view of (7) it follows from (20) that

$$\begin{aligned} &\forall x \in U, \; M > 0, \; W \in \mathcal{V}(x): \\ &\exists y \in W \cap U: \\ &\quad \psi(y) > M. \end{aligned} \tag{21}$$

That is, for every $M > 0$, the set $U_M = \{x \in U: \; \psi(x) > M\}$ is dense in U. Since ψ is lower semi-continuous, being the pointwise supremum of lower semi-continuous functions, it follows that U_M is also open. Therefore the set

$$P = \left\{ x \in U \;\middle|\; \begin{array}{l} \exists M \in \mathbb{R}: \\ \forall u \in \mathcal{A}: \\ \quad u(x) \leqslant M \end{array} \right\} = \bigl\{x \in U: \; \psi(x) < \infty\bigr\} = \bigcup_{M \in \mathbb{N}} (U \setminus U_M)$$

is of first Baire category in U, being the union of countably many closed and nowhere dense subsets of U. Furthermore,

$$U = P \cup (U \cap B_{\mathcal{A}}). \tag{22}$$

Since $B_{\mathcal{A}}$ is of first Baire category in X and U is open in X, the set $U \cap B_{\mathcal{A}}$ is of first Baire category in U. Therefore (22) implies that U is not a Baire space, being the union of two sets of first Baire category, hence of countably many closed nowhere dense sets. However, since U is an open subset of a Baire space, U is also a Baire space, see [6, Exercise 46.9], contrary to the conclusion that U is not a Baire space. Therefore (19) cannot hold. Consequently, the set of points $\{x \in X\colon |v(x)| < \infty\}$ is dense in X, which proves that v is nearly finite.

The statement for sets bounded from below follows in the same way. □

Theorem 5. *A sequence (u_n) in $\mathcal{NL}(X)$ order converges to $u \in \mathcal{NL}(X)$ in $\mathcal{NL}(X)$ if and only if*

$$\begin{aligned}
&\forall \epsilon > 0\colon\\
&\exists A_\epsilon \subseteq X \text{ an open and dense set}\colon\\
&\forall x \in A_\epsilon\colon\\
&\exists N_\epsilon(x) \in \mathbb{N},\ V_\epsilon(x) \in \mathcal{V}(x)\colon\\
&\forall n \in \mathbb{N},\ n \geqslant N_\epsilon(x),\ y \in V_\epsilon(x)\colon\\
&\quad \big|u_n(y) - u(y)\big| \leqslant \epsilon.
\end{aligned} \tag{23}$$

Proof. Assume that $(u_n) \subset \mathcal{NL}(X)$ order converges to u in $\mathcal{NL}(X)$. Then

$$\begin{aligned}
&\exists (\lambda_n),\ (\mu_n) \subset \mathcal{NL}(X)\colon\\
&\quad 1)\ \lambda_n \leqslant \lambda_{n+1} \leqslant u_n \leqslant \mu_{n+1} \leqslant \mu_n,\\
&\quad 2)\ \sup\{\lambda_n\colon n \in \mathbb{N}\} = u = \inf\{\mu_n\colon n \in \mathbb{N}\}.
\end{aligned} \tag{24}$$

Let $D \subseteq X$ be the open and dense set $D = \{x \in X\colon |\lambda_1(x)|, |\mu_1(x)| < \infty\}$. It follows from (24) that

$$\begin{aligned}
&\forall n \in \mathbb{N},\ x \in D\colon\\
&\quad \big|u(x)\big|, \big|\lambda_n(x)\big|, \big|\mu_n(x)\big| < \infty
\end{aligned} \tag{25}$$

so that the pointwise algebraic operations involving the functions u and λ_n, μ_n for $n \in \mathbb{N}$ are well defined.

Now suppose that

$$\begin{aligned}
&\exists \epsilon > 0,\ U \subseteq D \text{ nonempty and open}\colon\\
&\forall x \in U,\ n \in \mathbb{N}\colon\\
&\quad u(x) - \lambda_n(x) > \epsilon
\end{aligned} \tag{26}$$

so that

$$\begin{aligned}
&\forall x \in U,\ n \in \mathbb{N}\colon\\
&\quad \lambda_n(x) < u(x) - \epsilon.
\end{aligned} \tag{27}$$

Consider the function

$$v(x) = \begin{cases} u(x) - \frac{\epsilon}{2} & \text{if } x \in U,\\ u(x) & \text{if } x \notin U. \end{cases}$$

By (27) it follows that

$$\lambda_n \leqslant v \leqslant u, \quad n \in \mathbb{N}. \tag{28}$$

Since the operators I and S are monotone and idempotent, see (9) and (10), it follows that

$$\lambda_n \leqslant I\big(S(v)\big) \leqslant u, \quad n \in \mathbb{N} \tag{29}$$

which implies that

$$\big|I\big(S(v)\big)(x)\big| < \infty, \quad x \in D. \tag{30}$$

Since $I(S(v))$ is normal lower semi-continuous and D is open and dense in X, it follows that $v \in \mathcal{NL}(X)$. Because $U \subseteq D$ is open, it follows by (6) and (7) that

$$I\big(S(v)\big)(x) = I\big(S(u)\big)(x) - \frac{\epsilon}{2}, \quad x \in U.$$

Since u is normal lower semi-continuous it now follows that

$$I\big(S(v)\big)(x) = u(x) - \frac{\epsilon}{2}, \quad x \in U.$$

This shows that $I(S(v)) \neq u$, so that, by (28), u cannot be the supremum of the increasing sequence (λ_n). This conclusion contradicts (24) so that (26) cannot hold. Therefore the set

$$D_\epsilon = \left\{ x \in D \;\middle|\; \begin{array}{l} \exists n(x) \in \mathbb{N}: \\ \quad 0 \leqslant u(x) - \lambda_{n(x)}(x) \leqslant \epsilon \end{array} \right\}$$

is dense in D for every $\epsilon > 0$. Consider any $x_0 \in D_\epsilon$. By definition of D_ϵ we have

$$\begin{aligned} &\exists n(x_0) \in \mathbb{N}: \\ &\quad 0 \leqslant u(x_0) - \lambda_{n(x_0)}(x_0) \leqslant \epsilon \end{aligned}$$

so that

$$\lambda_{n(x_0)}(x_0) \leqslant u(x_0) \leqslant \lambda_{n(x_0)}(x_0) + \epsilon. \tag{31}$$

Consider any open set $V \in \mathcal{V}(x_0)$, with $V \subseteq D$. Since V is open, u and $\lambda_{n(x_0)}$ are normal lower semi-continuous on V. If the set

$$\big\{x \in V\colon\ u(x) > \lambda_n(x) + \epsilon\big\} \tag{32}$$

is dense in V, then by (14) it would follow that

$$u(x) > \lambda_{n(x_0)}(x) + \epsilon, \quad x \in V$$

which is contrary to (31). Consequently, we have that

$$\begin{aligned} &\exists U \subseteq V \text{ nonempty and open:} \\ &\forall x \in U: \\ &\quad \lambda_{n(x_0)}(x) \leqslant u(x) \leqslant \lambda_{n(x_0)}(x) + \epsilon. \end{aligned} \tag{33}$$

Since D_ϵ is dense in D, it follows from the relation (33) that

$$\begin{aligned} &\forall V \subseteq D \text{ nonempty, open:} \\ &\exists n_{V,\epsilon} \in \mathbb{N},\ U_{V,\epsilon} \subseteq V \text{ nonempty and open:} \\ &\forall x \in U_{V,\epsilon}: \\ &\quad \lambda_{n_{V,\epsilon}}(x) \leqslant u(x) \leqslant \lambda_{n_{V,\epsilon}}(x) + \epsilon. \end{aligned} \tag{34}$$

Since the sequence (λ_n) is increasing, relation (34) implies that

$$\begin{aligned} &\forall V \subseteq D \text{ nonempty, open:} \\ &\exists n_{V,\epsilon} \in \mathbb{N},\ U_{V,\epsilon} \subseteq V \text{ nonempty and open:} \\ &\forall n \in \mathbb{N},\ n \geqslant n_{V,\epsilon},\ x \in U_{V,\epsilon}: \\ &\quad \lambda_n(x) \leqslant u(x) \leqslant \lambda_n(x) + \epsilon. \end{aligned} \tag{35}$$

Define the set $C_\epsilon \subseteq X$ as

$$C_\epsilon = \bigcup \{U_{V,\epsilon}\colon\ V \subseteq D \text{ nonempty, open}\}. \tag{36}$$

Recalling that $D \subseteq X$ is open and dense, we see that (35) implies that C_ϵ is open and dense in X. Furthermore, the relation (35) also implies that

$$\begin{aligned}
&\forall x \in C_\epsilon:\\
&\exists N_\epsilon^0(x) \in \mathbb{N},\ U_\epsilon^0(x) \subseteq C_\epsilon \text{ a neighborhood of } x:\\
&\forall n \in \mathbb{N},\ n \geqslant N_\epsilon^0(x),\ y \in U_\epsilon^0(x):\\
&\quad \lambda_n(y) \leqslant u(y) \leqslant \lambda_n(y) + \epsilon.
\end{aligned} \tag{37}$$

In exactly the same way it follows that, for any $\epsilon > 0$, there exists an open and dense set $B_\epsilon \subseteq X$ with the property that

$$\begin{aligned}
&\forall x \in B_\epsilon:\\
&\exists N_\epsilon^1(x) \in \mathbb{N},\ U_\epsilon^1(x) \subseteq C_\epsilon \text{ a neighborhood of } x:\\
&\forall n \in \mathbb{N},\ n \geqslant N_\epsilon^1(x),\ y \in U_\epsilon^1(x):\\
&\quad \mu_n(y) - \epsilon \leqslant u(y) \leqslant \mu_n(y).
\end{aligned} \tag{38}$$

Setting $A_\epsilon = B_\epsilon \cap C_\epsilon$, the relation (23) follows from (24) and (37) to (38).

Conversely, assume that (23) holds for some sequence $(u_n) \subseteq \mathcal{NL}(X)$ and a function $u \in \mathcal{NL}(X)$. Consider the function

$$\varphi : X \ni x \mapsto \sup\{u_n(x) \colon n \in \mathbb{N}\} \in \overline{\mathbb{R}}. \tag{39}$$

Setting $\epsilon = 1$ in (23) we find that

$$\begin{aligned}
&\exists A_1 \subseteq X \text{ open and dense}:\\
&\forall x \in A_1:\\
&\exists V_1(x) \in \mathcal{V}(x) \text{ open},\ w \in \mathcal{NL}\big(V_1(x)\big):\\
&\forall y \in V_1(x):\\
&\quad \varphi(y) \leqslant w(y).
\end{aligned} \tag{40}$$

Indeed, the relation (23) implies that

$$\begin{aligned}
&\forall n \in \mathbb{N},\ n \geqslant N_1(x),\ y \in V_1(x):\\
&\quad u_n(y) \leqslant u(y) + 1.
\end{aligned} \tag{41}$$

The function $v = u + 1$ is clearly nearly finite and normal lower semi-continuous. Since $V_1(x)$ is open, it follows from (12) that

$$v_{|V_1(x)}, u_{n|V_1(x)} \in \mathcal{NL}\big(V_1(x)\big), \quad n \in \mathbb{N}, \tag{42}$$

while (41) implies that

$$\begin{aligned}
&\forall n \in \mathbb{N},\ n \geqslant N_1(x):\\
&\quad u_{n|V_1(x)} \leqslant v_{V_1(x)}.
\end{aligned}$$

The relation (40) now follows upon setting

$$w = \sup\{u_{1|V_1(x)}, \dots, u_{N_1(x)|V_1(x)}, v_{|V_1(x)}\}.$$

The monotonicity (10) of the operators I and S now implies that

$$u_0(y) = I\big(S(\varphi)\big)(y) \leqslant I\big(S(w)\big)(y) = w(y), \quad y \in V_1(x). \tag{43}$$

Since (43) holds for every $x \in A_1$, where $A_1 \subseteq X$ is open and dense, and since w is finite on an open and dense subset of $V_1(x)$, it follows that $u_0 \in \mathcal{NL}(X)$. In particular, applying (13) we find that

$$u_0 = \sup_{n \in \mathbb{N}} u_n \in \mathcal{NL}(X). \tag{44}$$

In the same way, we may show that

$$v_0 = \inf_{n \in \mathbb{N}} u_n \in \mathcal{NL}(X). \tag{45}$$

Therefore the sequence (u_n) is bounded from above and below in $\mathcal{NL}(X)$. Therefore, since $\mathcal{NL}(X)$ is Dedekind complete, it now follows from (44) and (45) that the sequences (μ_n) and (λ_n), defined as

$$\lambda_n = \inf\{u_k\colon k \geqslant n\}, \qquad \mu_n = \sup\{u_k\colon k \geqslant n\} \tag{46}$$

are well defined in $\mathcal{NL}(X)$. Furthermore, it follows immediately from the respective definitions in (46) that

$$u_0 \leqslant \lambda_n \leqslant \lambda_{n+1} \leqslant u_{n+1} \leqslant \mu_{n+1} \leqslant \mu_n \leqslant v_0, \quad n \in \mathbb{N} \tag{47}$$

so that

$$w_0 = \sup\{\lambda_n\colon n \in \mathbb{N}\} \leqslant \inf\{\mu_n\colon n \in \mathbb{N}\} = w_1. \tag{48}$$

We claim that $w_0 = w_1$. Assume that this is not the case. Then (14) and (48) imply that

$$\begin{aligned}
&\exists V \subseteq X \text{ nonempty and open:}\\
&\forall x \in V:\\
&\quad w_0(x) < w_1(x).
\end{aligned} \tag{49}$$

Without loss of generality, we may assume that $V \subseteq u_0^{-1}(\mathbb{R}) \cap v_0^{-1}(\mathbb{R})$. Then (47) and (48) imply that w_0, w_1 and each of the functions λ_n and μ_n are real valued on V, so that the usual pointwise operations are well defined. Without loss of generality, the relation (49) may be strengthened to

$$\begin{aligned}
&\exists M \in \mathbb{R},\ \epsilon > 0,\ V \subseteq X \text{ nonempty and open:}\\
&\forall x \in V:\\
&\quad w_0(x) + \epsilon < M < w_1(x) - \epsilon.
\end{aligned} \tag{50}$$

Since $\lambda_n \leqslant w_0 \leqslant w_1 \leqslant \mu_n$ for every $n \in \mathbb{N}$, it follows from (50) that

$$\begin{aligned}
&\forall x \in V,\ n \in \mathbb{N}:\\
&\quad \lambda_n(x) + \epsilon < M < \mu_n(x) - \epsilon.
\end{aligned} \tag{51}$$

Without loss of generality, we may assume that $V \subseteq A_\epsilon$, where A_ϵ is the open and dense subset of X associated with (u_n) and ϵ through (23). For each $n \in \mathbb{N}$, consider the functions φ_n and ψ_n defined as

$$\varphi_n : X \ni x \mapsto \sup\bigl\{u_k(x)\colon k \geqslant n\bigr\} \in \overline{\mathbb{R}}$$

and

$$\psi_n : X \ni x \mapsto \inf\bigl\{u_k(x)\colon k \geqslant n\bigr\} \in \overline{\mathbb{R}}.$$

Using the techniques employed in the proof of Lemma 4, we find that

$$\begin{aligned}
&\forall n \in \mathbb{N}:\\
&\exists V_n \subseteq V \text{ open and dense:}\\
&\forall x \in V_n:\\
&\quad \psi_n(x) + \epsilon < M < \varphi_n(x) - \epsilon.
\end{aligned} \tag{52}$$

Since $V \subseteq A_\epsilon$, it follows from (23) that

$$\begin{aligned}
&\forall x \in V:\\
&\exists N_\epsilon(x) \in \mathbb{N},\ V_\epsilon(x) \subseteq V \text{ an open neighborhood of } x:\\
&\forall y \in V_\epsilon(x),\ \in \mathbb{N},\ n \geqslant N_\epsilon(x):\\
&\quad 0 < \varphi_n(y) - \psi_n(y) < \epsilon.
\end{aligned} \tag{53}$$

Since V_n is open and dense in V for every $n \in \mathbb{N}$, it follows that

$$V_\epsilon(x) \cap V_n \neq \emptyset, \quad n \geqslant N_\epsilon(x). \tag{54}$$

But according to (52) we have

$$\varphi_n(y) - \psi_n(y) > 2\epsilon, \quad y \in V_\epsilon(x) \cap V_n. \tag{55}$$

In view of (54), the relation (55) contradicts (53). Therefore the assumption that $w_0 \neq w_1$ is false, which verifies our claim that $w_0 = w_1$. The relation (47) now implies that (u_n) order converges to w_0.

We now show that $w_0 = u$. Suppose that this were not the case. Then (14) implies that

$$\begin{aligned}&\exists V \subseteq X \text{ nonempty and open:}\\&\forall x \in V:\\&\quad w_0(x) \neq u(x).\end{aligned}$$

Without loss of generality, we may assume that

$$\begin{aligned}&\exists \epsilon > 0:\\&\forall x \in V:\\&\quad w_0(x) + \epsilon < u(x).\end{aligned} \tag{56}$$

Since (u_n) order converges to w_0, it follows that (23) is satisfied. In particular,

$$\begin{aligned}&\exists V_\epsilon \subseteq V \text{ open in } V,\ N_\epsilon \in \mathbb{N}:\\&\forall x \in V_\epsilon,\ n \in \mathbb{N},\ n \geqslant N_\epsilon:\\&\quad \left|w_0(x) - u_n(x)\right| < \frac{\epsilon}{3}.\end{aligned} \tag{57}$$

Since (u_n) and u satisfy (23) by assumption, we also have

$$\begin{aligned}&\exists V'_\epsilon \subseteq V_\epsilon \text{ open in } V_\epsilon,\ N'_\epsilon \in \mathbb{N}:\\&\forall x \in V'_\epsilon,\ n \in \mathbb{N},\ n \geqslant N'_\epsilon:\\&\quad \left|u(x) - u_n(x)\right| < \frac{\epsilon}{3}.\end{aligned} \tag{58}$$

It follows from (57) and (58) and the triangle inequality that

$$\left|w_0(x) - u(x)\right| < \frac{2\epsilon}{3}, \quad x \in V'_\epsilon. \tag{59}$$

Since $V'_\epsilon \subseteq V$ it is clear that (59) contradicts (56). Therefore the assumption that $u \neq w_0$ is false. Therefore (u_n) order converges to $u = w_0$. This completes the proof. □

One may make the following comparison between Theorem 5 and Egoroff's Theorem 2: Whenever a sequence in $\mathcal{NL}(X)$ order converges to some $u \in \mathcal{NL}(X)$, one can obtain a uniform estimate of convergence locally on a topologically large set, namely, on an open and dense set. Thus Theorem 5 relates order convergence of normal lower semi-continuous functions to uniform convergence estimates, while Theorem 2 relates order convergence in $\mathcal{M}(\Omega)$ to uniform convergence on sets of arbitrarily large measure.

As a first application of Theorem 5 we obtain a characterization of order convergence in $\mathcal{NL}(X)$, for X a Baire space, in terms of pointwise convergence. In this regard, we show that a sequence (u_n) in $\mathcal{NL}(X)$ order converges to u if and only if (u_n) converges to u pointwise on a residual set. That is, order convergence in $\mathcal{NL}(X)$ is equivalent to pointwise convergence on a set which is topologically large, in the sense of Baire category. This result should be compared with order convergence in the Archimedean Riesz space $\mathcal{M}(\Omega)$ of nearly finite measurable functions on a σ-finite measure space Ω. As mentioned in Section 1, order convergence in $\mathcal{M}(\Omega)$ is equivalent to pointwise convergence almost everywhere.

Theorem 6. *Let X be a Baire space. Then a sequence (u_n) in $\mathcal{NL}(X)$ order converges to $u \in \mathcal{NL}(X)$ if and only if (u_n) converges to u pointwise on a residual set.*

Proof. Suppose that (u_n) order converges to u in $\mathcal{NL}(X)$. Then (u_n) converges to u pointwise on a residual set by Theorem 5.

Conversely, suppose that (u_n) converges to $u \in \mathcal{NL}(X)$ pointwise on a residual set $R \subseteq X$. Then (u_n) is pointwise bounded on R. Thus Lemma 4 implies that (u_n) is order bounded in $\mathcal{NL}(X)$. Since $\mathcal{NL}(X)$ is Dedekind complete, it follows that the sequences

$$\lambda_n = \inf\{u_k:\ k \geqslant n\}, \qquad \mu_n = \sup\{u_k:\ k \geqslant n\} \tag{60}$$

are well defined in $\mathcal{NL}(X)$. Furthermore, we clearly have

$$\lambda_n \leqslant \lambda_{n+1} \leqslant u_{n+1} \leqslant \mu_{n+1} \leqslant \mu_n, \quad n \in \mathbb{N}. \tag{61}$$

Therefore we have

$$w_0 = \sup\{\lambda_n\colon n \in \mathbb{N}\} \leqslant \inf\{\mu_n\colon n \in \mathbb{N}\} = w_1 \tag{62}$$

in $\mathcal{NL}(X)$. We claim that $w_0 = w_1$. Assume that this is not the case. Then (14) implies that

$$\begin{aligned}
&\exists V \subseteq X \text{ open and nonempty:}\\
&\forall x \in V\colon\\
&\quad w_0(x) < w_1(x).
\end{aligned} \tag{63}$$

In fact, without loss of generality, we may strengthen (63) to

$$\begin{aligned}
&\exists M \in \mathbb{R},\ \epsilon > 0\colon\\
&\forall x \in V\colon\\
&\quad w_0(x) + \epsilon < M < w_1(x) - \epsilon.
\end{aligned} \tag{64}$$

For each $n \in \mathbb{N}$, consider the functions

$$\varphi_n : X \ni x \mapsto \inf\{u_k(x)\colon k \geqslant n\} \in \overline{\mathbb{R}}$$

and

$$\psi_n : X \ni x \mapsto \sup\{u_k(x)\colon k \geqslant n\} \in \overline{\mathbb{R}}.$$

According to (13), for each $n \in \mathbb{N}$, the functions φ_n and ψ_n determine λ_n and μ_n, respectively, through the identities

$$\lambda_n = I\big(S(\varphi_n)\big), \qquad \mu_n = I\big(S(\psi_n)\big). \tag{65}$$

From (62) and (64) we obtain the relation

$$\begin{aligned}
&\forall x \in V,\ n \in \mathbb{N}\colon\\
&\quad \lambda_n(x) + \epsilon < M < \mu_n(x) - \epsilon.
\end{aligned} \tag{66}$$

Note that $\mu_n = I(S(\psi_n)) \leqslant S(\psi_n)$ for all $n \in \mathbb{N}$. Therefore (66) implies

$$\begin{aligned}
&\forall x \in V,\ n \in \mathbb{N}\colon\\
&\quad M + \epsilon < S(\psi_n)(x).
\end{aligned} \tag{67}$$

Then it follows from (7) that

$$\begin{aligned}
&\forall x \in V,\ n \in \mathbb{N}\colon\\
&\exists U \in \mathcal{V}(x),\ x_M \in U \cap V\colon\\
&\quad M + \epsilon < \psi_n(x_M).
\end{aligned} \tag{68}$$

Since ψ_n is the pointwise supremum of a set of lower semi-continuous functions, ψ_n is lower semi-continuous itself. Thus (68) implies that

$$\begin{aligned}
&\forall n \in \mathbb{N}\colon\\
&\exists D_n \subseteq V \text{ open and dense in } V\colon\\
&\forall x \in D_n\colon\\
&\quad M + \epsilon < \psi_n(x).
\end{aligned} \tag{69}$$

In the same way, we obtain

$$\begin{aligned}
&\forall n \in \mathbb{N}\colon\\
&\exists D'_n \subseteq V \text{ open and dense in } V\colon\\
&\forall x \in D'_n\colon\\
&\quad \varphi_n(x) < M - \epsilon.
\end{aligned} \tag{70}$$

Since X is a Baire space, the open subset V of X is a Baire space in the subspace topology [6, Exercise 46.9]. Therefore the set $R' = \bigcap_{n\in\mathbb{N}}(D_n \cap D'_n) \subseteq V$ is a residual set in V, being the intersection of countable many open and dense sets in V. According to (69) and (70) we have

$$\forall n \in \mathbb{N},\ x \in R':$$
$$\varphi_n(x) + \epsilon < M < \psi_n(x) - \epsilon \tag{71}$$

which implies that

$$\liminf_{n\in\mathbb{N}} u_n(x) \leqslant M - \epsilon < M + \epsilon \leqslant \limsup_{n\in\mathbb{N}} u_n(x), \quad x \in R'.$$

Therefore $R' \cap (V \cap R) = \emptyset$ so that $R \cap V$ is contained in a set of first Baire category in V. This contradicts the assumption that R is a residual set in X. Therefore $w_0 = w_1$.

Using essentially the same method as above, we may show that $u = w_0 = w_1$. Then (61) implies that (u_n) order converges to u in $\mathcal{NL}(X)$. This completes the proof. □

Recall [5, Definition 1.1.5] that a filter on $\mathcal{C}(X)$ converges *continuously* to $u \in \mathcal{C}(X)$ whenever the filter $\omega_{X,\mathbb{R}}(\mathcal{F}, \mathcal{V}(x))$ converges to $u(x)$ for every $x \in X$. Here

$$\omega_{X,\mathbb{R}} : \mathcal{C}(X) \times X \ni (u, x) \mapsto u(x) \in \mathbb{R}$$

is the evaluation mapping. Continuous convergence is perhaps the most important non-topological convergence structure, and has important applications in duality theory for locally convex spaces. If $\mathcal{F}$ is the Frechét filter associated with a sequence (u_n), that is,

$$\mathcal{F} = \bigl[\bigl\{\{u_n\colon n \geqslant k\}\colon k \in \mathbb{N}\bigr\}\bigr]$$

then the definition of continuous convergence is easily seen to be equivalent to

$$\forall \epsilon > 0,\ x \in X:$$
$$\exists N_\epsilon(x) \in \mathbb{N},\ V_\epsilon \in \mathcal{V}(x):$$
$$\forall y \in V_\epsilon,\ n \in \mathbb{N},\ n \geqslant N_\epsilon(x):$$
$$\bigl|u_n(y) - u(x)\bigr| < \epsilon. \tag{72}$$

In view of these remarks, we introduce the following definition.

Definition 7. A sequence (u_n) of extended real valued functions on a topological space X converges continuously to $u : X \to \overline{\mathbb{R}}$ at $x \in X$ if

$$\forall \epsilon > 0:$$
$$\exists N_\epsilon(x) \in \mathbb{N},\ V_\epsilon \in \mathcal{V}(x):$$
$$\forall y \in V_\epsilon,\ n \in \mathbb{N},\ n \geqslant N_\epsilon(x):$$
$$\bigl|u_n(y) - u(x)\bigr| < \epsilon.$$

We should note that Definition 7 implies that the function u is real valued at $x \in X$, and the sequence (u_n) is eventually real valued in a neighborhood of x, whenever (u_n) converges continuously to u at x. Also note that a sequence of continuous functions that converges continuously at every point in X converges continuously in the sense of (72). The following is an immediate consequence of Theorem 5.

Corollary 8. *Let X be a Baire space. Then a sequence (u_n) in $\mathcal{NL}(X)$ order converges to $u \in \mathcal{NL}(X)$ if and only if (u_n) converges to u continuously at every point of a residual set.*

An immediate consequence of Theorem 6 and Corollary 8 is the following rather surprising result relating continuous convergence to pointwise convergence.

Corollary 9. *Let X be a Baire space. Then a sequence (u_n) in $\mathcal{NL}(X)$ converges to $u \in \mathcal{NL}(X)$ pointwise on a residual set if and only if (u_n) converges to u continuously at every point of a residual set.*

3. Convergence in $\mathcal{C}(X)$

We now apply the results obtained in the previous section to convergence in $\mathcal{C}(X)$. In this regard, we note that $\mathcal{NL}(X)$ contains $\mathcal{C}(X)$ as a sublattice. Therefore any sequence that order converges in $\mathcal{C}(X)$ also order converges in $\mathcal{NL}(X)$. The following are therefore immediate consequences of Theorem 6 and Corollary 8.

Corollary 10. *Let X be a Baire space, and let (u_n) order converge to u in $\mathcal{C}(X)$. Then there is a residual set $R \subseteq X$ such that (u_n) converges continuously to u at every $x \in R$.*

Corollary 11. *Let X be a Baire space, and let (u_n) order converge to u in $\mathcal{C}(X)$. Then there is a residual set $R \subseteq X$ such that (u_n) converges pointwise to u on R.*

Tucker [15] investigated the relationship between order convergence and pointwise convergence in $\mathcal{C}(X)$. The main result states that if a sequence (u_n) of continuous functions on a completely regular Baire space X with the countable chain property is uniformly bounded, and converges pointwise to $u \in \mathcal{C}(X)$, then (u_n) order converges to u in $\mathcal{C}(X)$. Therefore Corollary 11 is a partial converse to Tucker's result.

The converses of Corollary 10 and 11, respectively, may fail, as is shown in the following.

Example 12. Consider the sequence (u_n) of continuous functions on $\mathbb{R}$ defined as

$$u_n(x) = \begin{cases} n - n^2|x| & \text{if } |x| \leqslant \frac{1}{n}, \\ 0 & \text{if } |x| > \frac{1}{n}. \end{cases}$$

The sequence converges continuously, and hence also pointwise, to 0 on $\mathbb{R} \setminus \{0\}$. However, (u_n) is unbounded (in the sense of order) in $\mathcal{C}(\mathbb{R})$, and hence it cannot order converge to 0.

We can, however, obtain partial converses to Corollaries 10 and 11. In this regard we recall the following [1,7]. By $\mathbb{H}_{ft}(X)$ we denote the space of finite Hausdorff continuous interval valued functions on X. This space is order isomorphic to the set $\mathcal{NL}_{ft}(X) \subseteq \mathcal{NL}(X)$ of finite normal lower semi-continuous functions. This order isomorphism of $\mathbb{H}_{ft}(X)$ onto $\mathcal{NL}_{ft}(X)$ also preserves arbitrary suprema. A topological space is called a weak *cb*-space if every locally bounded lower semi-continuous function on X is bounded above by a continuous function. The following result is due to Dăneţ [7].

Proposition 13. *Let X be a completely regular weak cb-space. Then $\mathbb{H}_{ft}(X)$ is the Dedekind completion of $\mathcal{C}(X)$.*

Theorem 14. *Let X be a completely regular Baire space which is also a weak cb-space. Assume that $\mathcal{C}(X)$ is order separable. Assume that (u_n) is order bounded in $\mathcal{C}(X)$. If (u_n) converges pointwise to $u \in \mathcal{C}(X)$ on a residual subset of X, then (u_n) order converges to u in $\mathcal{C}(X)$.*

Proof. Since (u_n) converges pointwise to $u \in \mathcal{C}(X)$ on a residual subset of X, and X is a Baire space, it follows from Theorem 6 that (u_n) order converges to u in $\mathcal{NL}(X)$. Let (λ_n) and (μ_n) be the increasing, respectively decreasing, sequences associated with (u_n) through (1). Since (u_n) is order bounded in $\mathcal{C}(X)$ we have

$$\begin{aligned} &\exists w_0, w_1 \in \mathcal{C}(X): \\ &\forall n \in \mathbb{N}: \\ &\quad w_0 \leqslant u_n \leqslant w_1. \end{aligned} \tag{73}$$

For $n \in \mathbb{N}$ set

$$\lambda_n' = \lambda_n \vee w_0, \qquad \mu_n' = \mu_n \wedge w_1.$$

Then (1) and (73) imply that

$$\begin{aligned} &\forall n \in \mathbb{N}: \\ &\quad 1)\ \lambda_n \leqslant \lambda_n' \leqslant u_n \leqslant \mu_n' \leqslant \mu_n, \\ &\quad 2)\ \lambda_n' \leqslant \lambda_{n+1}' \leqslant \mu_{n+1}' \leqslant \mu_n'. \end{aligned} \tag{74}$$

The relation (74) implies that (λ_n') increases to u and (μ_n') decreases to u. Furthermore, since $w_0, w_1 \in \mathcal{C}(X) \subseteq \mathcal{NL}_{ft}(X)$, it follows from

$$w_0 \leqslant \lambda_n' \leqslant \mu_n' \leqslant w_1, \quad n \in \mathbb{N}$$

that $(\lambda_n'), (\mu_n') \subseteq \mathcal{NL}_{ft}(X)$. Thus (u_n) order converges to u in $\mathcal{NL}_{ft}(X)$. Since X is a completely regular weak *cb*-space, Proposition 13 implies that $\mathcal{NL}_{ft}(X)$ is the Dedekind completion of $\mathcal{C}(X)$. Since $\mathcal{C}(X)$ is order separable by assumption, it follows that

$$\begin{aligned}
&\forall n \in \mathbb{N}:\\
&\exists (\lambda_m^n),\ (\mu_m^n) \subseteq \mathcal{C}(X):\\
&\quad 1)\ (\lambda_m^n) \text{ increases to } \lambda_n',\\
&\quad 2)\ (\mu_m^n) \text{ decreases to } \lambda_n'.
\end{aligned}$$

Using [3, Lemma 36] we can construct sequences (λ_n'') and (μ_n'') in $\mathcal{C}(X)$ that increases and decreases, respectively, to u and satisfy $\lambda_n'' \leqslant \lambda_n' \leqslant u_n \leqslant \mu_n' \leqslant \mu_n''$, $n \in \mathbb{N}$. Therefore (u_n) order converges to u in $\mathcal{C}(X)$. □

Remark 15. The assumption of order separability of $\mathcal{C}(X)$ in Theorem 14 is a technical one. We should note that this assumption is true whenever X is a metric space or a separable space.

We should note that Theorem 14 is a partial strengthening of Tucker's result [15, Theorem 1] mentioned above. In particular, the conditions on the convergence and boundedness of the sequence (u_n) in Theorem 14 is weaker than those imposed on the sequence in [15, Theorem 1]. Thus, if the space X is assumed to satisfy the conditions of Theorem 14 and [15, Theorem 1], then Theorem 14 is a stronger result than Tucker's.

4. Order convergence in Riesz spaces

The results obtained in Sections 2 and 3 may, to some extent, be applied to *all Archimedean Riesz spaces*. Indeed, using well-known representation theorems [11, Chapter 7], we show that a large class of Riesz spaces are Riesz isomorphic to sublattice of $\mathcal{NL}(\Theta)$, for some topological space Θ.

In this regard, we recall [11, Chapter 7] that for a given topological space X, the set of all continuous functions $u : X \to \overline{\mathbb{R}}$ that are finite on a dense set, where $\overline{\mathbb{R}}$ carries its usual topology, is denoted by $\mathcal{C}^\infty(X)$. With respect to the pointwise order, $\mathcal{C}^\infty(X)$ is a lattice, but not necessarily a Riesz space. The Johnson–Kist Representation Theorem [11, Theorem 44.4] states that every Archimedean Riesz space is Riesz isomorphic to a Riesz space which is a sublattice of $\mathcal{C}^\infty(X)$, for some topological space X. The Yosida Representation Theorem [11, Theorem 45.3] states that an Archimedean Riesz space with strong order unit is Riesz isomorphic to a Riesz subspace of $\mathcal{C}(X)$ for some compact Hausdorff space X. We summarize these results in the following.

Theorem 16. *Let F be an Archimedean Riesz space. Then there exists a topological space Θ and a Riesz space $\hat{F}$ which is a sublattice of $\mathcal{C}^\infty(\Theta)$ such that F is Riesz isomorphic to $\hat{F}$.*

If F has a strong order unit, then Θ is a compact Hausdorff space and $\hat{F}$ is a Riesz subspace of $\mathcal{C}(\Theta)$.

Since each $u \in \mathcal{C}^\infty(X)$ is continuous, it is also normal lower semi-continuous. Furthermore, the set of points where u is finite is dense in X so that u is nearly finite, that is, $u \in \mathcal{NL}(X)$. The inclusion

$$\mathcal{C}^\infty(X) \subseteq \mathcal{NL}(X)$$

is easily seen to be an inclusion of lattices. Combining these remarks with Theorem 16, we arrive at the following.

Proposition 17. *Let F be an Archimedean Riesz space. Then there exists a topological space Θ and a Riesz space $\hat{F}$ which is a sublattice of $\mathcal{NL}(\Theta)$ such that F is Riesz isomorphic to $\hat{F}$. If F has a strong order unit, then Θ is a compact Hausdorff space and $\hat{F} \subseteq \mathcal{C}(\Theta)$.*

In this way we come to realize how the results of Sections 2 and 3 may be applied to a large class of Archimedean Riesz spaces. In particular, Corollaries 10 and 11 apply to all Archimedean Riesz spaces with strong order unit, while Theorem 5 applies to all Archimedean Riesz spaces. Note that only one direction of Theorem 5 may be used for an arbitrary Archimedean Riesz space. Indeed, a sequence $(u_n) \subseteq \hat{F}$ that order converges to $u \in \hat{F}$ in $\mathcal{NL}(\Theta)$ may fail to order converge to u in $\hat{F}$.

5. Conclusion

We have obtained a number of characterizations of order convergence of sequences in $\mathcal{NL}(X)$. In particular, it was shown that order convergence is equivalent to pointwise convergence on a residual set, provided that X is a Baire space. Furthermore, order convergence is also equivalent to continuous convergence at every point of a residual set, provided that X is a Baire space. Similar results are obtained for convergence of sequences of continuous functions, although additional assumptions on the topological space X as well as the sequence of functions are necessary. Lastly, it is indicated how these results may be applied to order convergence in Archimedean Riesz spaces. Here we may also mention that these results have a significant application in nonlinear theories of generalized functions, see [20], which are used in the solution of large classes of nonlinear partial differential equations [12,13,17–19].

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