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Coincidence best proximity point of F_g-weak contractive mappings in partially ordered metric spaces

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Abstract

The aim of this paper is to present coincidence best proximity point results of F_g-weak contractive mappings in partially ordered metric space. Some examples are presented to prove the validity of our results.

Keywords: Coincidence best proximity point, weak P-property, F_g-weak contraction mappings.
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1. Introduction and Preliminaries

Let A and B be nonempty subsets of a metric space X and $T : A \to B$. A fixed point problem $Fix(A, B, T)$ defined by a pair (A, B) of sets and a mapping T, is to find a point a^* in A such that $d(a^*, Ta^*) = 0$. A point a^* in A where $\inf\{d(a, Ta^*) : a \in A\}$ is attained, that is, a^* is best approximation to $Ta^* \in B$ in A. Such a point is called an approximate fixed point of T. If an operator equation $Ta = a$ does not admit a solution, it is a reasonable demand to settle down with $d(a^*, Ta^*) \leq d(a, Ta^*)$ for all a in A. The study of conditions that assure existence and uniqueness of approximate fixed point of a mapping T is an important area of research.

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Suppose that $\triangle_{AB} = d(A,B) = \inf(\{d(a,b) : a \in A, b \in B\})$ is the measure of a distance between two sets A and B. A point a^* is called a best proximity point of T, if $d(a^*, Ta^*) = \triangle_{AB}$. Thus a best proximity point problem defined by a mapping T and a pair of sets (A, B) is to find a point a^* in A such that $d(a^*, Ta^*) = \triangle_{AB}$. If $A \cap B = \phi$, fixed point problem defined by a pair (A, B) and a mapping T has no solution. If we take $A = B$, then a best proximity point problem reduces to fixed point problem. From this perspective, best proximity point problem can be viewed as a natural generalization of fixed point problem. Furthermore, results dealing with existence and uniqueness of best proximity point of certain mappings are more general than the ones dealing with approximate fixed point problem of those mappings. Recently, Kumam et al. [23] introduced the concept of coincidence best proximity point of a mapping in metric spaces. A coincidence best proximity point problem is defined as follows: Find a point a^* in A such that $d(ga^*, Ta^*) = \triangle_{AB}$ where g is a self mapping on A. This is an extension of a best proximity point problem. If g is an identity mapping on A, then a^* becomes a best proximity point of T. Existence of fixed points in partially ordered metric spaces has been initiated in 2004 by Ran et al. [33], and further studied by Nieto et al. [31]. Subsequently, several interesting and valuable results have appeared in this direction (see [3, 31, 32]). There are several results dealing with best proximity point problem in the setup of metric spaces and partial order metric space (see, [2, 4, 5, 6, 8, 9, 10, 11, 12, 14, 16, 18, 19, 20, 21, 22, 25, 27, 28, 29, 30] and references mentioned therein).

One of the basic and the most widely applied fixed point theorem in all of analysis is "Banach (or Banach-Caccioppoli) Contraction Principle" [7]. Due to its applications in mathematics and other related disciplines, it has been generalized in many directions (see, for example [15, 17, 24, 26] and references therein). Recently, Wardowski [35] first introduced the concept of F-contraction, then the concept of F-weak contraction [36] and proved a fixed point result as a generalization of Banach Contraction Principle. Abbas et al. [1] initiated the study of common fixed point theory introducing F-contraction mappings with respect to a self mapping on a complete metric space. They introduced a notion of generalized F-contraction mappings to prove a fixed point result for generalized nonexpansive mappings on star shaped subsets of normed linear spaces and initiated the study of invariant approximations in normed linear spaces for such mappings. Shukla et al. [34] obtained some common fixed point results for F-contraction type mappings in the framework of 0-complete partial metric spaces. Batra et al. [13] proved fixed point theorems for F-contraction on a metric space endowed with a graph.

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integer numbers, respectively.

In this paper, we prove coincidence best proximity point results for F_g-weak contraction in the context of a partially ordered metric space. We also present some examples to support the results proved herein. These results extend and strengthen various known comparable results in the literature.

Consistent with [12], [23], [35] and [36] the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} \triangle_{AB} &= d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \textit{there exists some } b \in B \textit{ such that } d(a,b) = \triangle_{AB}\}, \\ B_0 &= \{b \in B : \textit{there exists some } a \in A \textit{ such that } d(a,b) = \triangle_{AB}\}. \end{aligned}$$

Definition 1.2. Let X be a nonempty set. Then $(X, \preceq, d)$ is called a partially ordered metric space if the following assertions hold:

(i) d is a metric on X;

(ii) $\preceq$ is a partial order on X.

Definition 1.3. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Kumam et al. [23] used the following property:

Definition 1.4. Let $(X, \preceq, d)$ be an partially ordered metric space, A and B two subsets of X such that A_0 is nonempty, $T : A \to B$ and $g : A \to A$. The triplet (A, B, g) has weak P-property of first kind if $d(ga_1, Ta_3) = \triangle_{AB}$ and $d(ga_2, Ta_4) = \triangle_{AB}$ implies that $d(ga_1, ga_2) \leq d(Ta_3, Ta_4)$ for any comparable elements $a_1, a_2, a_3, a_4 \in A_0$.

Wardowski in [35] introduced following class of functions and define a new type of contraction mapping:

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

(F1) F is strictly increasing;

(F2) for any sequence $\{\alpha_n\}$ in $\mathbb{R}^+$, $\lim_{n\to+\infty} \alpha_n = 0$ and $\lim_{n\to+\infty} F(\alpha_n) = -\infty$ are equivalent;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to+0^+} \alpha^k F(\alpha) = 0$.

Collection of all such functions will be denoted by $\mathfrak{F}$.

Definition 1.5 ([35])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is called an F-contraction if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1.1}$$

for any $x, y \in X$ with $d(Tx, Ty) > 0$.

Wardowski et al. [36] gave the following definition of an F-weak contraction:

Definition 1.6 ([36])**.** Let X be a metric space. A mapping $T : X \to X$ is said to be an F-weak contraction if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y))$$

for any $x, y \in X$ with $d(Tx, Ty) > 0$ where

$$M(x, y) = \max\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right).$$

The readers interested in fixed point results for F-contraction and F-weak contractions are referred to [35, 36].

To solve coincidence best proximity point problem defined by a pair of sets (A, B) in a partially ordered metric space and a mapping T, we give the following definitions.

Definition 1.7 ([10])**.** Let $(X, d, \preceq)$ be a partially ordered metric space, A and B nonempty subsets of X. A mapping $T : A \to B$ is called proximal increasing if for any $x_1, x_2, u_1, u_2 \in A$, the following condition holds:

$$x_1 \preceq x_2, d(u_1, Tx_1) = d(A, B), d(u_2, Tx_2) = d(A, B) \text{ implies that } u_1 \preceq u_2.$$

One can see that, for a self-mapping, the notion of proximally increasing mapping reduces to that of increasing mapping.

Example 1.8. Consider the Euclidean space $\mathbb{R}$ with the usual order $\leq$. Let $A = [0, 1]$ and $B = [2, 3]$, then $d(A, B) = 1$. Define mappings $T : A \to B$ by $T(x) = 3 - x$. Take $x_1 = 1 = x_2 = u_1 = u_2$, then it is clear that T is proximally increasing but not increasing.

Definition 1.9. Let A and B be two nonempty subsets of partially ordered metric space $(X, \preceq, d)$, and $g : A \to A$. A mapping $T : A \to B$ is said to be F_g-weak contraction if there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in A_0$, $x \preceq y$ with $d(gy, T(x)) = \triangle_{AB}$ and $d(T(x), T(y)) > 0$, we have

$$\tau + F(d(T(x), T(y))) \leq F(M^g(x, y)),$$

where

$$\begin{aligned} M^g(x, y) = \max(&d(gx, gy), d(gx, Tx) - \triangle_{AB}, d(gy, Ty) - \triangle_{AB}, \\ &\frac{d(gx, Ty) + d(gy, Tx)}{2} - \triangle_{AB}). \end{aligned} \tag{1.2}$$

2. Coincidence best proximity point of F_g-weak contraction mapping

In this section, we obtain coincidence best proximity point results of F_g-weak contraction mappings. We start with the following result.

Theorem 2.1. *Let $(X, \preceq, d)$ be a complete partially ordered metric space, A and B two closed subsets of X, g is continuous self mapping on A such that $\phi \neq A_0 \subseteq gA_0$ and its inverse is increasing, $T : A \to B$ a continuous F_g-weak contraction with $T(A_0) \subseteq B_0$. Suppose that the following conditions hold:*

(a) triplet (A, B, g) satisfies weak P-property;

(b) T is proximal increasing;

(c) if a sequence $\{z_n\}$ in A_0 is such that $\{gz_n\} \subseteq A_0$ is Cauchy, then $\{z_n\}$ is Cauchy;

(d) there exists $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$, $d(gx_1, Tx_0) = \triangle_{AB}$.

Then T has a coincidence best proximity point. In fact, there exists a convergent sequence $\{x_n\} \subseteq A_0$ which satisfies

$$d(gx_{n+1}, Tx_n) = \triangle_{AB} \text{ for all } n \geq 0, \tag{2.1}$$

and the limit of $\{x_n\}$ is a coincidence best proximity point of T.

Proof. As $x_1 \in A_0$, so $Tx_1 \in T(A_0) \subseteq B_0$. Hence there is $z_2 \in A$ such that $d(z_2, Tx_1) = \triangle_{AB}$ which implies that $z_2 \in A_0$. As $A_0 \subseteq gA_0$, there is $x_2 \in A_0$ such that $g(x_2) = z_2$, we conclude that $d(z_2, Tx_1) = d(gx_2, Tx_1) = \triangle_{AB}$ then by assumption (d) and Definition 1.7, $gx_1 \preceq gx_2$. Since inverse of g is increasing, it follows that $x_1 \preceq x_2$. In a similar way, there is $x_3 \in A_0$ such that $d(gx_3, Tx_2) = \triangle_{AB}$ with $x_2 \preceq x_3$. Inductively, we construct a sequence $\{x_n\} \subseteq A_0$ such that

$$d(gx_{n+1}, Tx_n) = \triangle_{AB} \quad \forall\, n \geq 0 \tag{2.2}$$

with $x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ....$ If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gx_{n_0+1}$, then $d(gx_{n_0}, Tx_{n_0}) = d(gx_{n_0+1}, Tx_{n_0}) = \triangle_{AB}$ implies that x_{n_0} is a coincidence best proximity point of T. If we define $x_m = x_{n_0}$ for all $m \geq n_0$, then $\{x_n\}$ converges to a coincidence best proximity point of T. The proof is complete. Assume that

$$d(gx_n, gx_{n+1}) > 0 \quad \forall\, n \in \mathbb{N}. \tag{2.3}$$

As for $x_n, x_{n+1}, x_{n+2} \in A_0$, we have

$$d(gx_{n+1}, Tx_n) = \triangle_{AB}, \quad d(gx_{n+2}, Tx_{n+1}) = \triangle_{AB}$$

for all $n \in \mathbb{N}$. So by weak P-property of first kind, we obtain that

$$d(gx_{n+1}, gx_{n+2}) \leq d(Tx_n, Tx_{n+1}). \tag{2.4}$$

Now by F_g-weak contractive property of T we have

$$F(d(gx_{n+1}, gx_{n+2})) \leq F(d(Tx_n, Tx_{n+1}) \leq F(M^g(x_n, x_{n+1})) - \tau \tag{2.5}$$

for all $n > 0$, where

$$\begin{aligned} M^g(x_n, x_{n+1}) =& \max(d(gx_n, gx_{n+1}), d(gx_n, Tx_n) - \triangle_{AB}, d(gx_{n+1}, Tx_{n+1}) - \triangle_{AB}, \\ & \frac{d(gx_n, Tx_{n+1}) + d(gx_{n+1}, Tx_n)}{2} - \triangle_{AB}) \\ \leq& \max(d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}) + d(gx_{n+1}, Tx_n) - \triangle_{AB}, \\ & d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, Tx_{n+1}) - \triangle_{AB}, \end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}[d(gx_n, gx_{n+2}) + d(gx_{n+2}, Tx_{n+1}) + d(gx_{n+1}, Tx_n)] - \triangle_{AB})\\
=&\max(d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}),\\
&\frac{1}{2}d(gx_{n+1}, gx_{n+2}) + \triangle_{AB} - \triangle_{AB})\\
=&\max(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})).
\end{aligned}
$$

That is,

$$M^g(x_n, x_{n+1}) \leq \max(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})). \tag{2.6}$$

From (2.5) and (2.6), we have

$$F(d(gx_{n+1}, gx_{n+2}) \leq F(\max(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}))) - \tau \tag{2.7}$$

for all $n > 0$.

If there exists some $n_0 \in \mathbb{N}$ such that

$$\max(d(gx_{n_0}, gx_{n_0+1}), d(gx_{n_0+1}, gx_{n_0+2})) = d(gx_{n_0+1}, gx_{n_0+2})$$

that is, $d(gx_{n_0}, gx_{n_0+1}) \leq d(gx_{n_0+1}, gx_{n_0+2})$. Then by (2.7), we have

$$F(d(gx_{n_0+1}, gx_{n_0+2})) \leq F(d(gx_{n_0+1}, gx_{n_0+2})) - \tau$$

a contradiction since $\tau > 0$. Hence

$$\max(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})) = d(gx_n, gx_{n+1})$$

for all $n > 0$. So (2.7) implies that

$$F(d(gx_{n+1}, gx_{n+2})) \leq F(d(gx_n, gx_{n+1})) - \tau$$

for all $n > 0$.
In particular, for all $n \geq 1$, we have

$$
\begin{aligned}
F(d(gx_n, gx_{n+1})) &\leq F(d(gx_{n-1}, gx_n)) - 2\tau \leq F(d(gx_{n-2}, gx_{n-1})) - 3\tau\\
&\leq ... \leq F(d(gx_0, gx_1)) - n\tau
\end{aligned} \tag{2.8}
$$

for all $n \in \mathbb{N}$.

Set $\alpha_n = d(gx_n, gx_{n+1})$, for $n \in \mathbb{N}$. Then, $\alpha_n > 0$ for all n and taking the limit as $n \to \infty$ in (2.8), we get $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$. Thus, from (F2), we have $\lim\limits_{n\to\infty} \alpha_n = 0$. From (F3) there exists $k \in (0, 1)$ such that $\lim\limits_{n\to\infty} \alpha_n^k F(\alpha_n) = 0$. By (2.8), the following holds for all $n \in \mathbb{N}$

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -\alpha_n^k n\tau \leq 0. \tag{2.9}$$

Taking limit as $n \to \infty$ in (2.9), we get

$$\lim_{n\to\infty} n\alpha_n^k = 0. \tag{2.10}$$

From (2.10), there exists $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$. So we have

$$\alpha_n \leq \frac{1}{n^{\frac{1}{k}}} \tag{2.11}$$

for all $n \geq n_1$.

Let $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. By (2.8), we get

$$\begin{aligned}
d(gx_n, gx_m) \leq & d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) \\
& + d(gx_{n+2}, gx_{n+3}) + ... + d(gx_{m-2}, gx_{m-1}) + d(gx_{m-1}, gx_m) \\
= & \alpha_n + \alpha_{n+1} + ... + \alpha_{m-1} \\
= & \sum_{i=n}^{m-1} \alpha_i \leq \sum_{i=n}^{\infty} \alpha_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

By the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$, we get $d(gx_n, gx_m) \to 0$ as $m, n \to \infty$. This proves that $\{gx_n\}$ is a Cauchy sequence in X. Using the condition (c), $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists $z \in X$ such that $\{x_n\} \to z$ as $n \to \infty$. As $x_n \in A_0 \subseteq A$ for all n, so $z \in A$. Since T and g are continuous mappings, $\{Tx_n\} \to Tz$ and $\{gx_n\} \to gz$. Taking limit in (2.2) as $n \to \infty$, we conclude that z is a coincidence best proximity point of g and T. □

Remark 2.2. If we assume that CB_T, the set of coincidence best proximity point of T is well ordered. Then coincidence best proximity point of T is unique. Let $x_1, x_2 \in CB_T$ be two distinct coincidence best proximity points of T. Using the weak P-property of the first kind and the given assumption, we obtain that

$$F(d(gx_1, gx_2)) \leq F(d(Tx_1, Tx_2)) \leq F(M^g(x_1, x_2)) - \tau,$$

where

$$\begin{aligned}
M^g(x_1, x_2) = & \max(d(gx_1, gx_2), d(gx_1, Tx_1) - \triangle_{AB}, d(gx_2, Tx_2) - \triangle_{AB}, \\
& \frac{d(gx_1, Tx_2) + d(gx_2, Tx_1)}{2} - \triangle_{AB}) \\
= & \max(d(gx_1, gx_2), \frac{1}{2}[d(gx_1, Tx_2) + d(gx_2, Tx_1)] - \triangle_{AB}) \\
\leq & \max(d(gx_1, gx_2), \frac{1}{2}[d(gx_1, gx_2) + d(gx_2, Tx_2) + d(gx_2, gx_1) + d(gx_1, Tx_1)] - \triangle_{AB}) \\
= & d(gx_1, gx_2).
\end{aligned}$$

This further implies that

$$F(d(gx_1, gx_2)) \leq F(d(gx_1, gx_2)) - \tau,$$

a contradiction. Hence $x_1 = x_2$.

Remark 2.3. In Theorem 2.1, if $g = I_A$ (an identity map on A), then we obtain best proximity point of F-weak contraction mapping T. Furthermore, if $M^g(x, y) = d(x, y)$, then we have best proximity point of F-contraction mapping.

Example 2.4. Let $X = \mathbb{R}^2$. Define $\preceq$ on X as follows: $(x, y) \preceq (z, t) \Leftrightarrow x < z, y < t$. Then $(X, \preceq, d)$ is a complete metric space with metric d defined as:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Suppose that $A = \{(3, 2), (5, 6), (8, 9)\}$ and $B = \{(4, 5), (6, 7), (9, 8)\}$. Then $d(A, B) = \triangle_{AB} = 2$. So, $A_0 = \{(5, 6), (8, 9)\}$ and $B_0 = \{(4, 5), (6, 7), (9, 8)\}$. Define $g : A \to A$ by

$$g(3, 2) = (3, 2), \ g(5, 6) = (8, 9), \ g(8, 9) = (5, 6).$$

Note that g is continuous, $\phi \neq A_0 = g(A_0)$, the triplet (A, B, g) satisfies the weak P-property. Define $T : A \to B$ as follows:

$$T(3, 2) = (9, 8), \ T(5, 6) = (4, 5), \ T(8, 9) = (6, 7).$$

Obviously T is continuous, $T(A_0) \subseteq B_0$ and is proximal increasing. Also, T is F_g-weak contraction with $F(\alpha) = \ln(\alpha)$. Only comparable points in A_0 are $u = (5,6) \preceq v = (8,9)$. Indeed

$$\begin{aligned}
d(Tu,Tv) &= d(T(5,6),T(8,9)) = d((4,5),(6,7)) = 4,\\
d(gu,gv) &= d(g(5,6),g(8,9)) = d((8,9),(5,6)) = 6,\\
d(gu,Tu) - \triangle_{AB} &= d(g(5,6),T(5,6)) - 2 = d((8,9),(4,5)) - 2 = 6,\\
d(gv,Tv) - \triangle_{AB} &= d(g(8,9),T(8,9)) - 2 = d((5,6),(6,7)) - 2 = 0 \text{ and}\\
\frac{d(gu,Tv)+d(gv,Tu)}{2} - \triangle_{AB} &= \frac{d(g(5,6),T(8,9)) + d(g(8,9),T(5,6))}{2} - 2 = 4
\end{aligned}$$

gives $M^g(x,y) = 6$. Thus for $0 < \tau \le ln(\frac{3}{2})$ we have

$$\tau + \ln(4) \le \ln(6).$$

Hence,

$$\tau + F(4) \le F(6)$$

$$\tau + F(d(Tx,Ty)) \le F(M^g(x,y)).$$

Also, there exists $(x_0,x_1) \in A_0 \times A_0$ such that $d(gx_1,Tx_0) = \triangle_{AB} = 2$. Thus all the conditions of Theorem 2.1 are satisfied. Furthermore, we have

$$\begin{aligned}
d(g(8,9),T(8,9)) &= d((5,6),(6,7))\\
&= d(A,B) = 2.
\end{aligned}$$

Hence $(8,9)$ is the coincidence best proximity point of g and T.

Example 2.5. Let $X = [0,1]\times[0,1]$. Define $\preceq$ on X as follows: $(x,y) \preceq (z,t) \Leftrightarrow x \le z, y < t$. Thus $(X,\preceq)$ is a partially ordered set and $(X,\preceq,d)$ is a complete metric space with metric d defined as:

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Let

$$A = \{(0,x) : 0 \le x \le 1\} \quad \text{and} \quad B = \{(1,y) : 0 \le y \le 1\}.$$

Then $d(A,B) = \triangle_{AB} = 1$. Let $A_0 = A$ and $B_0 = B$. Define $g : A \to A$ by $g(0,x) = (0, \frac{2x}{1+x})$. Obviously g is continuous, $\phi \ne A_0 = g(A_0)$. Define $T : A \to B$ as follows:

$$T(0,x) = (1, \frac{x}{2}) \text{ for } (0,x) \in A.$$

The triplet (A,B,g) satisfies the weak P-property. Note that T is continuous, $T(A_0) \subseteq B_0$ and is proximal increasing. Also, T is F_g-weak contraction with $F(\alpha) = \ln(\alpha)$. Note that, for $0 < \tau < 1$ and for any comparable $u = (0,x_1), v = (0,x_2) \in A_0$

$$d(Tu,Tv) \le e^{-\tau} M^g(u,v).$$

Hence, we have

$$\tau + F(d(Tu,Tv)) \le F(M^g(u,v)).$$

Also, there exists $(x_0,x_1) \in A_0 \times A_0$ such that $d(gx_1,Tx_0) = \triangle_{AB} = 1$. Thus all the conditions of Theorem 2.1 are satisfied. Furthermore, we have

$$\begin{aligned}
d(g(0,0),T(0,0)) &= d((0,0),(1,0))\\
&= d(A,B) = 1.
\end{aligned}$$

Hence $(0,0)$ is the unique coincidence best proximity point of T.

Any nondecreasing sequence $\{x_n\}$ in a partially ordered metric space X satisfy the following condition: **(H)** If $x_n \to x$ then $x_n \preceq x$.

Theorem 2.6. *Conclusion of* Theorem 2.1 *also holds if we replace the continuity of T by condition (H).*

Proof. Following similar arguments to those given in proof of Theorem 2.1, we deduce that $\{gx_n\}$ and $\{x_n\}$ are Cauchy sequences in a closed subset A of X. There exists $x \in A$ such that $\{x_n\} \to x$ and $\{gx_n\} \to gx$. Then by the given assumption, we have $x_n \preceq x$ and $gx_n \preceq gx$. Note that $x \in A_0$. Now $Tx \in B_0$ gives that $d(gx, Tx) = d(A, B)$. Using F_g-weak contractive property of T we have

$$F(d(Tx_n, Tx) \leq F(M^g(x_n, x)) - \tau$$

for all $n > 0$, where

$$\begin{aligned} M^g(x_n, x) =& \max(d(gx_n, gx), d(gx_n, Tx_n) - \triangle_{AB}, d(gx, Tx) - \triangle_{AB}, \\ & \frac{d(gx_n, Tx) + d(gx, Tx_n)}{2} - \triangle_{AB}) \\ \leq& \max(d(gx_n, gx), d(gx_n, gx_{n+1}) + d(gx_{n+1}, Tx_n) - \triangle_{AB}, \\ & d(gx, Tx) - \triangle_{AB}, \frac{1}{2}[d(gx_n, gx) + d(gx, Tx) + d(gx, gx_{n+1}) \\ & + d(gx_{n+1}, Tx_n)] - \triangle_{AB}) \\ =& \max(d(gx_n, gx), d(gx_n, gx_{n+1}), d(gx, Tx) - \triangle_{AB}, \\ & \frac{1}{2}[d(gx_n, gx) + d(gx, Tx) + d(gx, gx_{n+1}) + \triangle_{AB}] - \triangle_{AB}). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M^g(x_n, x) = d(gx, Tx) - \triangle_{AB}. \tag{2.12}$$

Since

$$\begin{aligned} d(gx, Tx) \leq& d(gx, gx_{n+1}) + d(gx_{n+1}, Tx_n) + d(Tx_n, Tx) \\ =& d(gx, gx_{n+1}) + d(A, B) + d(Tx_n, Tx). \end{aligned}$$

Hence,

$$\begin{aligned} F(d(gx, Tx) - d(gx, gx_{n+1}) - d(A, B)) &\leq F(d(Tx_n, Tx)) \\ &\leq F(M^g(x_n, x)) - \tau. \end{aligned}$$

Therefore

$$\begin{aligned} F(d(gx, Tx) - d(A, B)) =& \lim_{n\to\infty} F(d(gx, Tx) - d(gx, gx_{n+1}) - d(A, B)) \\ \leq& \lim_{n\to\infty} F(d(Tx_n, Tx) \\ \leq& \lim_{n\to\infty} F(M^g(x_n, x)) - \tau \\ =& F(d(gx, Tx) - d(A, B)) - \tau, \end{aligned}$$

a contradiction. Hence $d(gx, Tx) = d(A, B)$. □

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References

[1] M. Abbas, B. Ali, S. Romaguera, *Fixed and periodic points of generalized contractions in metric spaces*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 1

[2] M. Abbas, A. Hussain, P. Kumam, *A coincidence best proximity point problem in G-metric spaces*, Abstr. Appl. Anal., **(2015)** 2015, 12 pages. 1

[3] M. Abbas, T. Nazir, S. Radenovic, *Common fixed points of four maps in partially ordered metric spaces*, Appl. Math. Lett., **24** (2011), 1520–1526. 1

[4] M. Abbas, V. Rakočevič, A. Hussain, *Proximal cyclic contraction of perov type on regular cone metric space*, J. Adv. Math. Stud., **9** (2016), 65–71. 1

[5] A. Almeida, E. Karapinar, K. Sadarangani, *A Note on Best Proximity Point Theorems under Weak P-Property*, Abstr. Appl. Anal., **2014** (2014), 4 pages. 1

[6] A. Amini-Harandi, N. Hussain, F. Akbar, *Best proximity point results for generalized contractions in metric spaces*, Fixed Point Theory Appl., **2013** (2013), 13 pages. 1

[7] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae, **3** (1922), 133–181. 1

[8] S. S. Basha, *Best proximity points: global optimal aproximate solutions*, J. Globbal Optim., **49** (2011), 15–21. 1

[9] S. S. Basha, *Best proximity point theorems on partially ordered sets*, Optim. Lett., **7** (2013), 1035–1043. 1

[10] S. S. Basha, *Discrete optimization in partially ordered sets*, J. Global Optim., **54** (2012), 511–517. 1, 1.7

[11] S. S. Basha, *Global optimization in metric spaces with partial orders*, Optimization, **63** (2014), 817–825. 1

[12] S. S. Basha, P. Veeramani, *Best proximity pair theorems for multifunctions with open bres*, J. Approx. Theory, **103** (2000), 119–129. 1

[13] R. Batra, S. Vashisthab, *Fixed points of an F-contraction on metric spaces with a graph*, Int. Jour. Comp. Math., **91** (2014), 2483–2490. 1

[14] N. Bilgili, E. Karapinar, K. Sadarangani, *A generalization for the best proximity point of Geraghty-contractions*, J. Inequal. Appl., **2013** (2013), 9 pages. 1

[15] N. Hussain, A. Latif, I. Iqbal, *Fixed point results for generalized F-contractions in modular metric and fuzzy metric spaces*, Fixed Point Theory Appl., **2015** (2015), 20 pages. 1

[16] N. Hussain, A Latif, P. Salimi, *Best proximity point results for modified Suzuki α-ψ-proximal contractions*, Fixed Point Theory Appl., **2014** (2014), 16 pages. 1

[17] N. Hussain, P. Salimi, *Suzuki-Wardowski type fixed point theorems for α-GF-contractions*, Taiwanese J. Math., **18** (2014), 1879–1895. 1

[18] M. Jleli, E. Karapinar, B. Samet, *Best proximity points for generalized α -ψ-proximal contractive type mappings*, J. Appl. Math., **2013** (2013), 10 pages. 1

[19] M. Jleli, B. Samet, *Best proximity points for α -ψ-proximal contractive type mappings and applications*, Bull. Sci. Math., **137** (2013), 977–995. 1

[20] E. Karapinar, *Best Proximity Points Of Cyclic Mappings*, Appl. Math. Lett., **25** (2012), 1761–1766. 1

[21] E. Karapinar, *Best proximity points of Kannan type cylic weak phi-contractions in ordered metric spaces*, An. St. Univ. Ovidius Constanta, **20** (2012), 51–64. 1

[22] E. Karapinar, *On best proximity point of ψ-Geraghty contractions*, Fixed Point Theory and Appl., **2013** (2013), 9 pages. 1

[23] P. Kumam, A. F. Roldn Lpez de Hierro, *On existence and uniqueness of g-best proximity points under $(\varphi, \theta, \alpha, g)$-contractivity conditions and consequences*, Abstr. Appl. Anal., **2014** (2014), 14 pages. 1, 1.3

[24] M. A. Kutbi, N. Hussain, S. Khaleghizadeh, *New PPF dependent fixed point theorems for Suzuki type GF-contractions*, J. Function Spaces, **2015** (2015), 13 pages. 1

[25] A. Latif, M. Hezarjaribi, P. Salimi, N. Hussain, *Best proximity point theorems for α-ψ-proximal contractions in intuitionistic fuzzy metric spaces*, J. Inequal. Appl., **2014** (2014), 19 pages. 1

[26] K. Latrach, M. A. Taoudi, A. Zeghal, *Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations*, J. Diff. Eq., **221** (2006), 256–271. 1

[27] C. Mongkolkeha, P. Kumam, *Best proximity point theorems for generalized cyclic contractions in ordered metric Spaces*, J. Optim. Theory Appl., **155** (2012), 215–226. 1

[28] C. Mongkolkeha, P. Kumam, *Best proximity points for asymptotic proximal pointwise weaker Meir-Keeler-type ψ-contraction mappings*, J. Egyptian Math. Soc., **21** (2013), 87–90. 1

[29] C. Mongkolkeha, P. Kumam, *Some common best proximity points for proximity commuting mappings*, Optim. Lett., **7** (2013), 1825–1836. 1

[30] H. K. Nashine, P. Kumam, C. Vetro, *Best proximity point theorems for rational proximal contractions*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 1

[31] J. J. Nieto, R. L. Pouso, R. Rodríguez-López, *Fixed point theorems in ordered abstract sets*, Proc. Amer. Math. Soc., **135** (2007), 2505–2517. 1

[32] J. J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin., **23** (2007), 2205–2212. 1

[33] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1

[34] S. Shukla, S. Radenović, *Some Common Fixed Point Theorems for F-contraction Type Mappings in* 0*-Complete Partial Metric Spaces*, J. Math., **2013** (2013), 7 pages. 1

[35] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), 6 pages. 1, 1.4, 1.5, 1

[36] D. Wardowski, N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math., **47** (2014), 146–155. 1, 1, 1.6, 1