Finite element analysis of the stationary power-law Stokes equations driven by friction boundary conditions

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Abstract: In this work, we are concerned with the finite element approximation for the stationary power law Stokes equations driven by nonlinear slip boundary conditions of 'friction type'. After the formulation of the problem as mixed variational inequality of second kind, it is shown by application of a variant of Babuska–Brezzi's theory for mixed problems that convergence of the finite element approximation is achieved with classical assumptions on the regularity of the weak solution. Next, solution algorithm for the mixed varia-tional problem is presented and analyzed in details. Finally, numerical simulations that validate the theoret-ical findings are exhibited.

Keywords: Power-law Stokes equations, nonlinear slip boundary conditions, variational inequality, finite el-ement method, error estimate, regularization, penalization, long time behavior.

1 Introduction

This work is concerned with the finite element approximation of the power law Stokes flow governed by the partial differential equations

$$\begin{aligned} -\nu\ \mathrm{div}(|\mathbf{D}(\mathbf{u})|^{r-2}\mathbf{D}(\mathbf{u})) + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned} \tag{1.1}$$

where Ω is the flow region, a bounded domain in $\mathbb{R}^2$. The motion of our incompressible fluid is described by the velocity $\mathbf{u}(\mathbf{x}) = (u_1, u_2)$ and pressure $p(\mathbf{x})$. The external force per unit volume is $\mathbf{f}$ while the positive parameter ν is the viscosity of the fluid. Of course, $\mathbf{D}$ is the deformation tensor given as

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T).$$

The motion of the fluid at the boundary, say, $\partial\Omega$ is characterized by the presence of the Tresca type conditions which is described next. First, we assume that $\partial\Omega$ is made of two components S, and Γ such that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, and $S \cap \Gamma = \varnothing$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma. \tag{1.2}$$

We have chosen to work with homogeneous condition on the velocity in order to avoid the technical arguments linked to the Hopf lemma (see [24], Chapter 4, Lemma 2.3). In order to describe the motion of the fluid on S, we first assume the impermeability condition, that is

$$u_N = \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } S \tag{1.3}$$

where $\mathbf{n} = (n_1, n_2)$ is the outward unit normal on the boundary $\partial\Omega$, and $u_N\mathbf{n}$ is the normal component of the velocity while $\mathbf{u}_\tau = \mathbf{u} - u_N\mathbf{n}$ is its tangential component. In addition to (1.3) we also impose on S a threshold slip condition [15, 30] which is the principal ingredient of this work. The threshold slip condition can be formulated with the knowledge of a positive function $g : S \to (0, \infty)$ which is called barrier or threshold function and the tangential part of the traction force acting on S glue together in the following way

$$\left.\begin{array}{l} |(\sigma\mathbf{n})_\tau| \leq g \\ |(\sigma\mathbf{n})_\tau| < g \Rightarrow \mathbf{u}_\tau = \mathbf{0} \\ |(\sigma\mathbf{n})_\tau| = g \Rightarrow \mathbf{u}_\tau \neq \mathbf{0}, \quad -(\sigma\mathbf{n})_\tau = g\dfrac{\mathbf{u}_\tau}{|\mathbf{u}_\tau|} \end{array}\right\} \text{ on } S. \tag{1.4}$$

Of course, $(\sigma\mathbf{n})_\tau$ is the tangential component of the traction force $\sigma\mathbf{n}$ acting on the boundary S, and σ is the Cauchy stress tensor given by $\sigma = -p\mathbf{I} + \nu|\mathbf{D}(\mathbf{u})|^{r-2}\mathbf{D}(\mathbf{u})$, where $\mathbf{I}$ is the identity matrix. It should quickly be mentioned that (1.4) is equivalent to (see [13]):

$$(\sigma\mathbf{n})_\tau \cdot \mathbf{u}_\tau + g|\mathbf{u}_\tau| = 0 \quad \text{on } S$$

which is re-written with the use of sub-differential as

$$-(\sigma\mathbf{n})_\tau \in g\partial|\mathbf{u}_\tau| \quad \text{on } S \tag{1.5}$$

where $\partial|\cdot|$ is the sub-differential of the real valued function $|\cdot|$ with $|\mathbf{w}|^2 = \mathbf{w}\cdot\mathbf{w}$. We recall that if $\mathbf{X}$ is a Banach space, with $\mathbf{X}^*$ its dual, $\langle\cdot\rangle$ the duality pairing between $\mathbf{X}$ and $\mathbf{X}^*$, and $\Psi : \mathbf{X} \to [-\infty, \infty)$, and $x_0 \in \mathbf{X}$, then

$$y \in \partial\Psi(x_0) \text{ if and only if } \Psi(x) - \Psi(x_0) \geq \langle y, x - x_0\rangle \quad \forall x \in \mathbf{X}. \tag{1.6}$$

It should be mentioned that different boundary conditions describe different physical phenomena. The slip boundary condition of friction type (1.5) can be justified by the fact that frictional effects of the fluid at the pores of the solid can be very important. The class of boundary condition (1.5) was introduced by Fujita in [15], where he studied some hydrodynamics problems, such as the blood flow in a vein of an arterial sclerosis patient and the avalanche of water and rocks. Subsequently, many studies have focused on the properties of the solution of the resulting boundary value problem, for example, existence, uniqueness, regularity, and continuous dependence on data, for Stokes, Navier–Stokes and Brinkman–Forchheimer equations under such boundaries condition. Details can be found in [5, 6, 15–21, 30, 31, 37, 38] among others. But the combination of (1.5) with the p-Laplacian has not yet been considered in the literature, and in this work we give a detailed mathematical analysis on the existence and uniqueness of weak solution as well as finite element analysis. The main goal of this study is to contribute to the numerical analysis of flow problems driven by non-conventional boundary conditions. So, our focus is to analyze numerically (1.1)–(1.5) via finite element approximations, that is to establish stability and convergence of the finite element solution. It is manifest that (1.1)–(1.5) has many numerical challenges among others; the nonlinear operator associated to the p-Laplacian, the incompressible condition and the related pressure, and the nontrivial boundary condition (1.5) which brings a non-differentiable expression into the variational formulation of the problem. Hence, our second contribution in this work is to formulate and analyze an algorithm well adapted and easy to implement for the numerical challenges mentioned.

Even though many research work have been conducted for the approximations of variational inequalities [25, 27–29, 36] (just to mention a few), not much research in theoretical numerical analysis have been done for the kind of problem described by (1.1)–(1.5). Li and Li [32] proposed a penalty finite element approximation method for the Stokes equation with nonlinear slip boundary conditions (1.5). They proved the optimal order error estimate provided that the velocity is H^2 up the boundary, however, no numerical simulations are provided. An and Li [33] proposed a penalty finite element method for the steady Navier–Stokes equations. The Mathematical analysis of our work borrow heavily on the contribution of Reddy [36] and Han and Reddy [28], where sufficient conditions for existence and uniqueness are derived for the kind of weak formulations we analyze here, while the solution procedure we propose is divided in three steps. The first step is based on some works of R. Glowinski [25] in that, we associated to a steady problem an evolution problem in which only the long time effect is taken into consideration. Next, because of the incompressibility condition, and the non differentiable term appearing in the variational problem to solve, we approximated then the problem by a sequence of penalized/regularized 'better behaved' variational equations and justify the approximations by some convergence results. Thirdly, to improve the performance of our scheme, we add to the problem obtained in step 2 a viscosity term and show that the new 'perturbed' problem converges to the original variational formulation. All these theoretical results are supported by numerical simulations indicating the robustness of our algorithm. The rest of the paper is organized as follows. We introduce or recall some preliminaries in Section 2 and indicate how existence of weak solution is obtained. In Section 3, we formulate the finite element procedure, explain how existence and uniqueness of solution is obtained and derive error estimates. Section 4 is concerned with the algorithm, while Section 5 deals with numerical simulations. Some conclusions are drawn in the last section.

2 Preliminaries and variational formulations

In this section, we introduce notation and some results that will be used throughout our work. We also formulate various weak formulations and discuss (recall) some existence results.

2.1 Notations and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^2$ with a regular boundary $\partial\Omega$. The Lebesgue space is denoted as usual $L^r(\Omega)$, $1 \leq r \leq \infty$, with norms $\|\cdot\|_{L^r}$ (except the $L^2(\Omega)$-norm which is denoted by $\|\cdot\|$). For any non-negative integer m and real number $r \geq 1$, the classical Sobolev spaces [1, 2, 9, 14]:

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega);\ \partial^\alpha v \in L^r(\Omega)\ \forall |\alpha| \leq m\}$$

are equipped with the semi-norm

$$|v|_{m,r}(\Omega) = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^r \, \mathrm{d}x \right\}^{1/r} \tag{2.1}$$

and norm

$$\|v\|_{m,r}(\Omega) = \left\{ \sum_{0\leq|\alpha|=m} |v|^r_{W^{k,r}(\Omega)} \right\}^{1/r} \tag{2.2}$$

with the usual extension when $r = \infty$. Let $W^{m,2}(\Omega)$ be the Hilbert space $H^m(\Omega)$ with the scalar product

$$((v, w))_m = \sum_{|\alpha|\leq m} (\partial^\alpha v, \partial^\alpha w).$$

It should be mentioned that ∂^α stands for the derivative in the sense of distribution, while $\alpha = (\alpha_1, \alpha_2)$ denote a multi-index of length $|\alpha| = \alpha_1 + \alpha_2$. Here, $W_0^{m,r}(\Omega)$ consists of functions of $W^{m,r}(\Omega)$ that vanish on $\partial\Omega$. For $1 < r < \infty$, let r' denote the conjugate of r given by the relation $r' = r(r-1)^{-1}$; $W^{-m,r'}(\Omega)$ denotes the topological dual space of $W_0^{m,r}(\Omega)$, and $\|\cdot\|_{-m,r'}$ its norm. Since we will be working with the pressure, it is convenient to introduce the following space

$$L_0^r(\Omega) = \left\{ q \in L^r(\Omega)\,,\ \ (q, 1) = 0 \right\}.$$

The following result in Reddy [36], and Han and Reddy [28] will be used to claim solvability of the problem we have in hand.

Theorem 2.1. *Let Ψ and N be two Hilbert spaces, and A a nonlinear operator from Ψ to its dual $\Psi^\star$, $b : \Psi \times N \to \mathbb{R}$ a bilinear form, and $J : \Psi \to \mathbb{R}$ a functional. The bilinear form $b(\cdot,\cdot)$ is assumed to have the following properties:*

(i) $b(\cdot,\cdot)$ *is continuous, so that there exists a bounded linear operator* B *defined by* $B : \Psi \to N^*$, $\langle B\psi, m\rangle = b(\psi, m)$ *for all* $\psi \in \Psi$ *and* $m \in N$*;*
(ii) $b(\cdot,\cdot)$ *is inf-sup stable, that is there exists a constant* $\beta \geq 0$ *such that*

$$\beta \|m\|_{N/\mathrm{ker}B^T} \leq \sup_{\psi\in\Psi} \frac{b(\psi, m)}{\|\psi\|_\Psi}. \tag{2.3}$$

Next, we also assume that the operator A *is coercive, strictly monotone, hemi-continuous and bounded.*

Finally, we assume that the functional J *is convex, nonnegative and lower semi-continuous. Then the problem: Find* $(\varphi, m) \in \Psi \times N$ *such that for all* $(\psi, q) \in \Psi \times N$,

$$\begin{aligned} \langle A\varphi, \psi - \varphi\rangle - b(\psi - \varphi, m) + J(\psi) - J(\varphi) &\geq \langle \boldsymbol{f}, \psi - \varphi\rangle \\ b(\varphi, q) &= 0 \end{aligned} \tag{2.4}$$

has a unique solution.

2.2 Mixed variational formulations

In this subsection, we formulate variational models associated to problem (1.1)–(1.5). We also indicate how existence and uniqueness of solution is obtained.

We first introduce the following spaces

$$\begin{gathered} \mathbb{W}^{m,r}(\Omega) = [W^{m,r}(\Omega)]^2, \quad \mathbb{V}_0 = \mathbb{W}_0^{1,r}(\Omega) \\ \mathbb{V} = \{\mathbf{v} \in \mathbb{W}^{1,r}(\Omega),\ \mathbf{v}|_\Gamma = 0,\ \mathbf{v}\cdot\mathbf{n}|_S = 0\} \\ \mathbb{V}_{\mathrm{div}} = \{\mathbf{v} \in \mathbb{V},\ \mathrm{div}\,\mathbf{v} = 0\} \\ M = L_0^{r'}(\Omega). \end{gathered}$$

Throughout the paper, bold notation will represent the vector value function. The following inequality, known as Poincaré-Fredrichs's inequality will be used frequently and state that; there exists a positive constant C, such that

$$\int_\Omega |\mathbf{v}|^r \,\mathrm{d}x \leq C \int_\Omega |\mathbf{D}(\mathbf{v})|^r \,\mathrm{d}x \quad \forall \mathbf{v} \in \mathbb{W}_0^{m,r}(\Omega) \tag{2.5}$$

which implies that the semi-norm (2.1) defines a norm which is equivalent to the norm (2.2). Thus one can equip $\mathbb{V}$ with $\|\cdot\|_\mathbb{V} = \|\mathbf{D}(\cdot)\|_{L^r}$ because $\|\mathbf{D}(\cdot)\|_{L^r}$ is equivalent to $\|\cdot\|_{1,r}$. Next, we define the operators A, $b(\cdot,\cdot)$ and J as follows

$$A : \mathbb{W}_0^{1,r}(\Omega) \to \mathbb{W}^{-1,r'}(\Omega), \quad \langle A\mathbf{u}, \mathbf{v}\rangle = \nu \int_\Omega |\mathbf{D}(\mathbf{u})|^{r-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})\,\mathrm{d}x$$

$$b : \mathbb{V} \times M \to \mathbb{R}, \quad b(\mathbf{u}, p) = \int_\Omega p\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x$$

and

$$J : \mathbb{V} \to \mathbb{R}^+, \quad J(\mathbf{v}) = (g, |\mathbf{v}_\tau|)_S.$$

Given $\mathbf{f} \in \mathbb{W}^{-1,r'}(\Omega)$ and $g \in L^{r'}(S)$ with $g \geq 0$ on S. We multiply the first equation in (1.1) by $\mathbf{v} - \mathbf{u}$ for all $\mathbf{v} \in \mathbb{V}$, integrate the resulting equation over Ω, after application of Green's formula, we obtain

$$\langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle - b(\mathbf{v} - \mathbf{u}, p) - \int_S \sigma\mathbf{n}\cdot(\mathbf{v} - \mathbf{u})\,\mathrm{d}s = \int_\Omega \mathbf{f}\cdot(\mathbf{v} - \mathbf{u})\,\mathrm{d}x. \tag{2.6}$$

Next, we briefly recall that

$$\sigma\mathbf{n} = \sigma_N\mathbf{n} + \sigma_\tau, \quad \mathbf{v} - \mathbf{u} = (v_N - u_N)\mathbf{n} + (\mathbf{v}_\tau - \mathbf{u}_\tau)$$

then we have

$$\int_S \sigma\mathbf{n}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}s = \int_S \sigma_\tau\cdot(\mathbf{v}_\tau-\mathbf{u}_\tau)\,\mathrm{d}s, \quad \text{since} \quad v_N-u_N|_\Gamma = 0.$$

On the other hand, according to (1.6), one then obtains

$$\int_S g(|\mathbf{v}_\tau|-|\mathbf{u}_\tau|)\,\mathrm{d}s \geq -\int_S \sigma_\tau\cdot(\mathbf{v}_\tau-\mathbf{u}_\tau)\,\mathrm{d}s$$

which together with (2.6) leads to the following weak formulation formulation of (1.1)–(1.5): Find $(\mathbf{u},p)\in\mathbb{V}\times M$ such that for all $(\mathbf{v},q)\in\mathbb{V}\times M$,

$$\begin{aligned}\langle A\mathbf{u},\mathbf{v}-\mathbf{u}\rangle - b(\mathbf{v}-\mathbf{u},p) + J(\mathbf{v}) - J(\mathbf{u}) &\geq \langle \mathbf{f},\mathbf{v}-\mathbf{u}\rangle \\ b(\mathbf{u},q) &= 0.\end{aligned} \tag{2.7}$$

Note that because the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (2.3), the variational inequality problem (2.7) is equivalent to

$$\begin{cases}\text{Find } \mathbf{u}\in\mathbb{V}_{\mathrm{div}} \text{ such that} \\ \langle A\mathbf{u},\mathbf{v}-\mathbf{u}\rangle + J(\mathbf{v}) - J(\mathbf{u}) \geq \langle\mathbf{f},\mathbf{v}-\mathbf{u}\rangle \text{ for all } \mathbf{v}\in\mathbb{V}_{\mathrm{div}}\end{cases} \tag{2.8}$$

which is also equivalent to the following optimization problem

$$\begin{cases}\text{Find } \mathbf{u}\in\mathbb{V}_{\mathrm{div}} \text{ such that} \\ Q(\mathbf{u}) \leq Q(\mathbf{v}) \text{ for all } \mathbf{v}\in\mathbb{V}_{\mathrm{div}}\end{cases} \tag{2.9}$$

where

$$Q(\mathbf{v}) = \frac{\nu}{r}\|\nabla\mathbf{v}\|^r_{L^r} + J(\mathbf{v}) - \langle\mathbf{f},\mathbf{v}\rangle.$$

To show the existence and uniqueness of solution of (2.7) it suffices to show that the assumptions of Theorem 2.1 (see [28, 36]) are satisfied.

The coercivity, monotonicity and boundedness of A are proved by Barrett and Liu [8], see also Chow [11]. In fact, for all $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{W}^{1,r}_0(\Omega)$ one has

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2_{1,r} &\leq C\langle A(\mathbf{u})-A(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle\,(\|\mathbf{u}\|_{1,r}+\|\mathbf{v}\|_{1,r})^{2-r} \\ \|A(\mathbf{u})-A(\mathbf{v})\|_{-1,r'} &\leq C\|\mathbf{u}-\mathbf{v}\|^{r-1}_{1,r}, \quad 1<r\leq 2\end{aligned} \tag{2.10}$$

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^r_{1,r} &\leq C\langle A(\mathbf{u})-A(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle \\ \|A(\mathbf{u})-A(\mathbf{v})\|_{-1,r'} &\leq C\|\mathbf{u}-\mathbf{v}\|_{1,r}\,(\|\mathbf{u}\|_{1,r}+\|\mathbf{v}\|_{1,r})^{r-2}, \quad 2\leq r<\infty\end{aligned} \tag{2.11}$$

where $C>0$ denotes a generic constant independent of $\mathbf{u}$ and $\mathbf{v}$. The inf-sup condition (2.3) has been proved by Baranger and Najib [7], Amrouche and Girault [4]. The functional $J(\cdot)$ is easily shown to be convex, nonnegative and continuous. However J is not differentiable. We thus conclude this subsection with the following result.

Lemma 2.1. *The mixed variational problem (2.7) admits a unique solution $(\boldsymbol{u},p)\in\mathbb{V}\times M$, which satisfies the following bound*

$$\|\boldsymbol{u}\|_{1,r} \leq C(\|\boldsymbol{f}\|_{-1,r'}+\|g\|_{L^{r'}(S)})^{1/(r-1)} \tag{2.12}$$

$$\|p\|_{r'} \leq C(\|\boldsymbol{f}\|_{-1,r'}+\|g\|_{L^{r'}(S)}). \tag{2.13}$$

Proof. Let $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = 2\mathbf{u}$ in (2.8), one has

$$\langle A\mathbf{u}, \mathbf{u}\rangle + J(\mathbf{u}) = \langle \mathbf{f}, \mathbf{u}\rangle$$

then, we have by (2.11),

$$\|\mathbf{u}\|_{1,r}^{r} \leq C\langle A\mathbf{u}, \mathbf{u}\rangle = C(\langle \mathbf{f}, \mathbf{u}\rangle - J(\mathbf{u})) \leq C(\|\mathbf{f}\|_{-1,r'}\|\mathbf{u}\|_{1,r} + \|g\|_{L^{r'}(S)}\|\mathbf{u}\|_{1,r})$$

for $2 \leq r < \infty$. Similarly, by (2.10),

$$\begin{aligned}\|\mathbf{u}\|_{1,r}^{2} &\leq C\|\mathbf{u}\|_{1,r}^{2-r}\langle A\mathbf{u}, \mathbf{u}\rangle = C\|\mathbf{u}\|_{1,r}^{2-r}(\langle \mathbf{f}, \mathbf{u}\rangle - J(\mathbf{u})) \\ &\leq C\|\mathbf{u}\|_{1,r}^{2-r}(\|\mathbf{f}\|_{-1,r'}\|\mathbf{u}\|_{1,r} + \|g\|_{L^{r'}(S)}\|\mathbf{u}\|_{1,r})\end{aligned}$$

for $1 < r \leq 2$. Thus the two relations above give the same result (2.12).

Next, we derive the *a priori* bound for the pressure. For that purpose, let $\mathbf{w} \in \mathbb{V}_0$, and replace $\mathbf{v}$ in (2.7) successively by $\mathbf{u} + \mathbf{w}$ and $\mathbf{u} - \mathbf{w}$, and observe that $J(\mathbf{v}) = J(\mathbf{u} \pm \mathbf{w}) = J(\mathbf{u})$. Then one obtains

$$\langle A\mathbf{u}, \mathbf{w}\rangle - b(\mathbf{w}, p) = \langle \mathbf{f}, \mathbf{w}\rangle \quad \forall\, \mathbf{w} \in \mathbb{V}_0. \tag{2.14}$$

Next, from the compatibility condition (2.3) and (2.14), one has

$$\begin{aligned}\beta\|p\|_{r'} &\leq \sup_{\mathbf{w}\in\mathbb{V}_0} \frac{b(\mathbf{w}, p)}{\|\mathbf{w}\|_{1,r}} = \sup_{\mathbf{w}\in\mathbb{V}_0} \frac{|\langle A\mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{f}, \mathbf{w}\rangle|}{\|\mathbf{w}\|_{1,r}} \\ &\leq \|A\mathbf{u}\|_{-1,r'} + \|\mathbf{f}\|_{-1,r'}.\end{aligned} \tag{2.15}$$

Thirdly from (2.10) and (2.11), there holds that for all $r > 1$, $\|A\mathbf{u}\|_{-1,r'} \leq C\|\mathbf{u}\|_{1,r}^{r-1}$, which when combined with (2.15) and (2.12) leads to result announced in (2.13). $\square$

3 Finite element approximation of the variational inequality (2.7)

3.1 Preliminaries and existence of solution

In this section, we analyze the finite element discretization of the variational inequality (2.7). We assume that $\mathcal{T}_h$ is a regular partition of Ω in the sense introduced in Cialert [12]. The diameter of an element $K \in \mathcal{T}_h$ is denoted by h_K, and the mesh size h is defined by $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathbb{V}_h \subset \mathbb{V}$ and $M_h \subset M$ be two conforming finite element spaces that will be made precise later.

Let us introduce the following subspaces:

$$\mathbb{V}_{\sigma h} = \{\mathbf{v}_h \in \mathbb{V}_h, \quad b(\mathbf{v}_h, q_h) = 0 \quad \forall\, q_h \in M_h\}$$

$$\mathbb{V}_{0h} = \mathbb{V}_h \cap \mathbb{V}_0.$$

The mixed weak formulation for the finite element discretization of the variational inequality (2.7) reads:

$$\begin{cases}\text{Find } (\mathbf{u}_h, p_h) \in \mathbb{V}_h \times M_h \text{ such that} \\ \langle A\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h\rangle - b(\mathbf{v}_h - \mathbf{u}_h, p_h) + J(\mathbf{v}_h) - J(\mathbf{u}_h) \geq \langle \mathbf{f}, \mathbf{v}_h - \mathbf{u}_h\rangle \\ b(\mathbf{u}_h, q_h) = 0 \quad \text{for all } (\mathbf{v}_h, q_h) \in \mathbb{V}_h \times M_h.\end{cases} \tag{3.1}$$

For the existence and uniqueness of solution of (3.1), we apply Theorem 2.1, with the special requirement among others that constant in the discrete counterpart of the inf-sup condition (2.3) be independent of h. Indeed, the reader can consult [10, 23, 24], where examples of elements pair that satisfies the discrete version of (2.3) are given. To summarize, we have the following result.

Lemma 3.1. *The finite element formulation (3.1) has a unique solution $(\boldsymbol{u}_h, p_h)$ which moreover satisfies:*

$$\|\boldsymbol{u}_h\|_{1,r} \le C\left(\|\boldsymbol{f}\|_{-1,r'} + \|g\|_{L^{r'}(S)}\right)^{1/(r-1)} \tag{3.2}$$

$$\|p_h\|_{r'} \le C\left(\|\boldsymbol{f}\|_{-1,r'} + \|g\|_{L^{r'}(S)}\right). \tag{3.3}$$

where $C > 0$ is a generic constant independent on h.

Proof. The proof follows the same arguments presented in the proof of Lemma 2.1.

Let $\mathbf{v}_h = \mathbf{0}$ and $\mathbf{v}_h = 2\mathbf{u}_h$ in the first relation in (3.1) and exploiting the second relation in (3.1), one has

$$\langle A\mathbf{u}_h, \mathbf{u}_h\rangle + J(\mathbf{u}_h) = \langle \mathbf{f}, \mathbf{u}_h\rangle.$$

Hence one obtains (3.2) by (2.11) and (2.10).

For the a priori bound (3.3), let $\mathbf{w}_h \in \mathbb{V}_{0h}$, and $\mathbf{v}_h = \mathbf{u}_h + \mathbf{w}_h$, $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$ respectively in (3.1) and observing that $J(\mathbf{v}_h) = J(\mathbf{u}_h \pm \mathbf{w}_h) = J(\mathbf{u}_h)$, one has

$$\langle A\mathbf{u}_h, \mathbf{w}_h\rangle - b(\mathbf{w}_h, p_h) = \langle \mathbf{f}, \mathbf{w}_h\rangle \quad \forall\, \mathbf{w}_h \in \mathbb{V}_{0h}. \tag{3.4}$$

Then, from the inf-sup condition (2.3), we conclude (3.3) by using (3.2). □

3.2 *A priori* error estimate

In this subsection, our goal is to measure the difference between the solution $(\mathbf{u}, p)$ of the continuous problem and the solution $(\mathbf{u}_h, p_h)$ of the finite element solution. But before doing so, we first recall the following inequality which will be use later

$$\left|\sum_{i=1}^{m} a_i\right|^{\beta} \le C(m,\beta)\sum_{i=1}^{m}|a_i|^{\beta} \quad \forall\, a_i, \beta \ge 0 \tag{3.5}$$

where $C(m,\beta)$ is a positive constant depending on m and β. The main result of this paragraph can be stated as follows.

Theorem 3.1. *Suppose that Ω is a bounded convex domain in $\mathbb{R}^2$ and assume that $\boldsymbol{f} \in \mathbb{W}^{-1,r'}(\Omega)$. Let $(\boldsymbol{u}, p)$ be the unique solution of* (2.7) *and $(\boldsymbol{u}_h, p_h)$ the unique solution of* (3.1)*. Then there exists a generic positive constant C independent on h such that for all $\boldsymbol{v}_h \in \mathbb{V}_h$ and $q_h \in M_h$, there hold.*

For $1 < r \le 2$ we have

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r}^2 &\le C\left\{\begin{array}{l}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^r(S)} + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^2\\ +\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^r + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^{2(r-1)}\\ +\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^{2/(3-r)} + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^r(S)}^{2/r}\\ +\|p-q_h\|_{r'}^{2/(r-1)} + \|p-q_h\|_{r'}^2\end{array}\right\}\\
\|p-p_h\|_{r'} &\le C\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r}^{(r-1)} + \|p-q_h\|_{r'}\right\}.
\end{aligned} \tag{3.6}$$

For $2 \le r < \infty$ we have

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r}^r &\le C\left\{\begin{array}{l}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{L^r(S)} + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^2\\ +\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^r + \|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,r}^{r'}\\ +\|p-q_h\|_{r'}^2 + \|p-q_h\|_{r'}^{r'}\end{array}\right\}\\
\|p-p_h\|_{r'} &\le C\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,r} + \|p-q_h\|_{r'}\right\}.
\end{aligned} \tag{3.7}$$

Proof. Subtracting (2.14) from (3.4) with $\mathbf{w} = \mathbf{w}_h$ we obtain

$$\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{w}\rangle - b(p - p_h, \mathbf{w}_h) = 0 \quad \forall\, \mathbf{w}_h \in \mathbb{V}_{0h}$$

from which we easily deduce that

$$b(p_h - q_h, \mathbf{w}_h) = \langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{w}_h\rangle + b(p - q_h, \mathbf{w}_h)$$

which together with the discrete version of the inf-sup condition (2.3), gives

$$\begin{aligned} \beta\|p_h - q_h\|_{r'} &\le \sup_{\mathbf{w}_h \in \mathbb{V}_{0h}} \frac{|\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{w}_h\rangle + b(\mathbf{w}_h, p - q_h)|}{\|\mathbf{w}_h\|_{1,r}} \\ &\le C(\|A\mathbf{u} - A\mathbf{u}_h\|_{-1,r'} + \|p - q_h\|_{r'}) \end{aligned}$$

so that,

$$\|p - p_h\|_{r'} \le \|p - q_h\|_{r'} + \|q_h - p_h\|_{r'} \le C\|A\mathbf{u} - A\mathbf{u}_h\|_{-1,r'} + C\|p - q_h\|_{r'}$$

which after application of properties of the operator A (see (2.10) and (2.11)) yields

$$C\|p - p_h\|_{r'} \le \begin{cases} \|\mathbf{u} - \mathbf{u}_h\|_{1,r}^{r-1} + \|p - q_h\|_{r'}, & 1 < r \le 2 \\ \|\mathbf{u} - \mathbf{u}_h\|_{1,r} + \|p - q_h\|_{r'}, & 2 \le r < \infty. \end{cases} \tag{3.8}$$

Next, replacing successively $\mathbf{v}$ in the first equation of (2.7) by $\mathbf{v} = \mathbf{u}_h$ and $\mathbf{v} = 2\mathbf{u} - \mathbf{v}_h$, one gets

$$\langle A\mathbf{u}, \mathbf{u}_h - \mathbf{u}\rangle - b(\mathbf{u}_h - \mathbf{u}, p) + J(\mathbf{u}_h) - J(\mathbf{u}) \ge \langle \mathbf{f}, \mathbf{u}_h - \mathbf{u}\rangle \tag{3.9}$$

and

$$\langle A\mathbf{u}, \mathbf{u} - \mathbf{v}_h\rangle - b(\mathbf{u} - \mathbf{v}_h, p) + J(2\mathbf{u} - \mathbf{v}_h) - J(\mathbf{u}) \ge \langle \mathbf{f}, \mathbf{u} - \mathbf{v}_h\rangle. \tag{3.10}$$

Relations (3.9) and (3.10) yield

$$\langle A\mathbf{u}, \mathbf{u}_h - \mathbf{v}_h\rangle - b(\mathbf{u}_h - \mathbf{v}_h, p) + J(2\mathbf{u} - \mathbf{v}_h) + J(\mathbf{u}_h) - 2J(\mathbf{u}) \ge \langle \mathbf{f}, \mathbf{u}_h - \mathbf{v}_h\rangle. \tag{3.11}$$

Note that the first relation in (3.1) can be re-written as

$$-\langle A\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h\rangle + b(\mathbf{u}_h - \mathbf{v}_h, p_h) + J(\mathbf{v}_h) - J(\mathbf{u}_h) \ge -\langle \mathbf{f}, \mathbf{u}_h - \mathbf{v}_h\rangle. \tag{3.12}$$

Equations (3.11) and (3.12) yields

$$\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h\rangle - b(\mathbf{u}_h - \mathbf{v}_h, p - p_h) + J(2\mathbf{u} - \mathbf{v}_h) + J(\mathbf{v}_h) - 2J(\mathbf{u}) \ge 0. \tag{3.13}$$

Note that

$$\langle A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}_h\rangle = \langle A\mathbf{u} - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle - \langle A\mathbf{u}_h - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle. \tag{3.14}$$

Now, from the both the second relations in (2.7) and (3.1), one has

$$\begin{aligned} b(\mathbf{u}_h - \mathbf{v}_h, p - p_h) &= b(\mathbf{u}_h - \mathbf{u}, p - q_h) + b(\mathbf{u}_h - \mathbf{u}, q_h - p_h) + b(\mathbf{u} - \mathbf{v}_h, p - p_h) \\ &= b(\mathbf{u}_h - \mathbf{u}, p - q_h) + b(\mathbf{u} - \mathbf{v}_h, p - p_h). \end{aligned} \tag{3.15}$$

Substitute the equalities (3.14) and (3.15) into (3.13) yields

$$\begin{aligned} \langle A\mathbf{u}_h - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle \le{}& \langle A\mathbf{u} - A\mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h\rangle - b(\mathbf{u}_h - \mathbf{u}, p - q_h) \\ & -b(\mathbf{u} - \mathbf{v}_h, p - p_h) + J(2\mathbf{u} - \mathbf{v}_h) + J(\mathbf{v}_h) - 2J(\mathbf{u}). \end{aligned} \tag{3.16}$$

Now, inserting the first and second relation in (2.10), (3.16) becomes:

$$\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r}^2 \le C\,(\|\mathbf{u}_h\|_{1,r} + \|\mathbf{v}_h\|_{1,r})^{2-r} \left\{ \begin{aligned} &\|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{r-1}\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r} \\ &+\|p - q_h\|_{r'}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ &+\|p - p_h\|_{r'}\|\mathbf{u} - \mathbf{v}_h\|_{1,r} \\ &+\|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} \end{aligned} \right\} \tag{3.17}$$

for $1 < r \le 2$. Similarly, from the first and second relation in (2.11), (3.16) becomes:

$$\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r}^r \le \left\{ \begin{array}{l} C(\|\mathbf{u}_h\|_{1,r} + \|\mathbf{v}_h\|_{1,r})^{r-2}\|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u}_h - \mathbf{v}_h\|_{1,r} \\ +\|p - q_h\|_{r'}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ +\|p - p_h\|_{r'}\|\mathbf{u} - \mathbf{v}_h\|_{1,r} + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} \end{array} \right\} \tag{3.18}$$

for $2 \le r < \infty$. Finally, from the triangle inequality, the inequalities (3.5), (2.12), (3.2), and from the first relation in (3.8) to (3.17), we obtain for $1 < r \le 2$:

$$\begin{aligned} C\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^2 \le\ & \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^r + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{r-1}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ & + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|p - q_h\|_{r'} + \|\mathbf{u} - \mathbf{u}_h\|_{1,r}\|p - q_h\|_{r'} \\ & + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^{r-1} + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} \\ & + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} + \|p - q_h\|_{r'}\|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{3-r} \\ & + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{2-r}\|p - q_h\|_{r'}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \\ & + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{2-r}\|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{3-r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^{r-1} \end{aligned} \tag{3.19}$$

which with Young's inequality, leads to the following inequalities

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^2 \le C \left\{ \begin{array}{l} \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^r + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{2(r-1)} \\ +\|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{2/(3-r)} + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)}^{2/r} + \|p - q_h\|_{r'}^{2/(r-1)} + \|p - q_h\|_{r'}^2 \end{array} \right\}.$$

Likewise, when $2 \le r < \infty$, using the triangle inequality, the inequalities (3.5), (2.12), (3.2) and from the second relation in (3.8) to (3.18), we obtain

$$\begin{aligned} C\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^r \le\ & \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^r + \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} \\ & + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{r-1}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|p - q_h\|_{r'} \\ & + \|\mathbf{u} - \mathbf{u}_h\|_{1,r}\|p - q_h\|_{r'} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}\|\mathbf{u} - \mathbf{u}_h\|_{1,r} \end{aligned} \tag{3.20}$$

which again with the help of Young's inequality yields for $2 \le r < \infty$:

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,r}^r \le C \left\{ \begin{array}{l} \|\mathbf{u} - \mathbf{v}_h\|_{L^r(S)} + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^2 + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^r + \|\mathbf{u} - \mathbf{v}_h\|_{1,r}^{r'} \\ +\|p - q_h\|_{r'}^2 + \|p - q_h\|_{r'}^{r'} \end{array} \right\}.$$

□

3.3 Rate of convergence

In this paragraph, we derive rate of convergence by considering classical assumptions on regularity of the solution $(\mathbf{u}, p)$, and adopting well known finite elements spaces $\mathbb{V}_h$ and M_h.

We first consider finite element approximations defined in J. W. Barrett and W. B. Liu [8]. We state the following assumptions for an integer $m \ge 1$ and any $\alpha \in [1, \infty]$.

(H1): Approximation property of $\mathbb{V}_h$.

There is a continuous linear operator $\pi_h : W_0^{1,\alpha}(\Omega)^2 \to \mathbb{V}_h$ such that for $k = 0, \dots, m$ we have

$$\|\mathbf{w} - \pi_h\mathbf{w}\|_{1,\alpha} \le Ch^k\|\mathbf{w}\|_{k+1,\alpha} \quad \forall\, \mathbf{w} \in \mathbb{W}^{k+1,\alpha}(\Omega) \cap \mathbb{W}_0^{1,\alpha}(\Omega)). \tag{3.21}$$

(H2): Approximation property of M_h.

There is a continuous linear operator $\rho_h : L^\alpha(\Omega) \to M_h$ such that for all $k = 0, \dots, m$ we have

$$\|q - \rho_h q\|_{L^\alpha} \le Ch^k\|q\|_{k,\alpha} \quad \forall\, q \in W^{k,\alpha}(\Omega). \tag{3.22}$$

(H3): The spaces M_h and $\mathbf{V}_h$ satisfy the inf-sup condition (2.3).

Then the following result holds.

Theorem 3.2. *Let* $(\boldsymbol{u}, p)$ *be the unique solution of* (2.7) *and* $(\boldsymbol{u}_h, p_h)$ *the unique solution of* (3.1). *Then if* (H1), (H2)*, and* (H3) *hold for* $k = 0, \dots, m$ *we have: For* $1 < r \leq 2$

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,r} &\leq C(\|\boldsymbol{u}\|_{k+1,r}, \|p\|_{k,r'})h^{k\min\{1/r,(r-1)\}} \\ \|p - p_h\|_{r'} &\leq C(\|\boldsymbol{u}\|_{k+1,r}, \|p\|_{k,r'})h^{k\min\{(r-1)/r,(r-1)^2\}}. \end{aligned} \tag{3.23}$$

Next, for $2 \leq r < \infty$ *we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,r} + \|p - p_h\|_{r'} \leq C(\|\boldsymbol{u}\|_{k+1,r}, \|p\|_{k,r'})h^{k/r} \tag{3.24}$$

where $C(\|\boldsymbol{u}\|_{k+1,r}, \|p\|_{k,r'})$ *is generic constant depending on* $\|\boldsymbol{u}\|_{k+1,r}, \|p\|_{k,r'}$ *and independent on* h.

Proof. Let $\mathbf{w}_h = \pi_h\mathbf{u}$ and $q_h = \rho_h u$ in (3.6), (3.7), and applying the usual trace theorem, (3.21) and (3.22), we obtain: For $1 < r \leq 2$

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,r}^2 &\leq C(\|\mathbf{u}\|_{k+1,r}, \|p\|_{k,r'})\left\{h^k + h^{2k} + h^{kr}\right. \\ &\quad \left. + h^{2k(r-1)} + h^{2k/(3-r)} + h^{2k/(r-1)} + h^{2k/r}\right\} \\ &\leq C(\|\mathbf{u}\|_{k+1,r}, \|p\|_{k,r'})\left\{h^{2k(r-1)} + h^{2k/r}\right\} \\ &\leq C(\|\mathbf{u}\|_{k+1,r}, \|p\|_{k,r'})h^{2k\min\{1/r,(r-1)\}} \end{aligned}$$

where Young's inequality has been used.

For $2 \leq r < \infty$ we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,r}^r &\leq C(\|\mathbf{u}\|_{k+1,r}, \|p\|_{k,r'})\left\{h^k + h^{2k} + h^{kr} + h^{kr'}\right\} \\ &\leq C(\|\mathbf{u}\|_{k+1,r}, \|p\|_{k,r'})h^k \end{aligned}$$

where the last expression is obtained using the same arguments as above. □

4 Numerical algorithm

In this section, we formulate and analyze the algorithm for the implementation of (3.1). We first regularize the formulation (3.1) by replacing the non differentiable functional J by a 'better behaved' approximation J_ε, where ε is a small positive parameter (see B. D. Reddy [36]). It should be mentioned that the introduction of the new functional J_ε transforms the variational inequality problem into a variational equation. The next step in our strategy consists of eliminating the incompressibility condition by penalizing the regularized problem by adding a coercive-like term in the form $\eta(p, q)$, where η is a small positive parameter. We recall that the transformed problem is very close to the original one in the sense that when ε and η tend to zero, one recovers the original problem. Next, we consider an evolution problem in $\mathbb{V}_H \times M_h$ and show that the solution of the later problem converges as the time is big enough to the solution of the former problem. Hence this result allow us to approximate instead the time dependent problem and considered only the long time behavior. One of the advantage of using this approach is that a linear scheme can be formulated for a nonlinear problem. We next present the details of our approach.

The non-differentiable functional J is replaced in (3.1) by the regularized functional J_ε defined by

$$J_\varepsilon(\mathbf{v}) = \int_S g\sqrt{|\mathbf{v}|^2 + \varepsilon^2}\,\mathrm{d}s. \tag{4.1}$$

Note that J_ε satisfies the following properties:

(i) J_ε is convex and differentiable, with Gateaux derivative and

$$\langle J_\varepsilon'(\mathbf{v}), \mathbf{w}\rangle = \int_S g\frac{\mathbf{v}\cdot\mathbf{w}}{\sqrt{|\mathbf{v}|^2 + \varepsilon^2}}\,\mathrm{d}s. \tag{4.2}$$

(ii)

$$0 \le J_\varepsilon(\mathbf{w}) - J(\mathbf{w}) \le C_1\varepsilon \quad \forall\, \mathbf{w}. \tag{4.3}$$

(iii)

$$\|J_\varepsilon'(\mathbf{w})\| \le C_2\|g\|_{L^{r'}(S)} \quad \forall\, \mathbf{w}. \tag{4.4}$$

The constants C_1 and C_2 are both independent of ε, and furthermore C_2 is independent of ψ.

With the introduction of the more smooth functional J_ε, the regularized problem reads

$$\begin{cases} \text{Find } (\mathbf{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}_h \times M_h \text{ such that} \\ \langle A\mathbf{u}_h^\varepsilon, \mathbf{v}_h - \mathbf{u}_h^\varepsilon\rangle - b(\mathbf{v}_h - \mathbf{u}_h^\varepsilon, p_h^\varepsilon) + J_\varepsilon(\mathbf{v}_h) - J_\varepsilon(\mathbf{u}_h^\varepsilon) \ge \langle \mathbf{f}, \mathbf{v}_h - \mathbf{u}_h^\varepsilon\rangle \\ b(\mathbf{u}_h^\varepsilon, q_h) = 0 \quad \text{for all } (\mathbf{v}_h, q_h) \in \mathbb{V}_h \times M_h. \end{cases} \tag{4.5}$$

The solution of (4.5) is related to the one of (3.1) by the following result.

Lemma 4.1. *Let $(\boldsymbol{u}_h, p_p)$ be the solution of (3.1), and $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ the solution of (4.5), then there is positive constant C, independent of ε such that*

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon\|_{1,r} \le C \begin{cases} \varepsilon^{1/2}, & 1 < r \le 2 \\ \varepsilon^{1/r}, & r \le r < \infty. \end{cases}$$

Proof. For $\mathbf{v}_h = \mathbf{u}_\varepsilon^\varepsilon$ in (3.1), $\mathbf{v}_h = \mathbf{u}_h$ in (4.5) and $q_h = p_h^\varepsilon - p_h$, one obtains

$$J_\varepsilon(\mathbf{u}_h^\varepsilon) - J(\mathbf{u}^\varepsilon) + \langle A\mathbf{u}_h^\varepsilon - A\mathbf{u}_h, \mathbf{u}_h^\varepsilon - \mathbf{u}_h\rangle \le J_\varepsilon(\mathbf{u}_h) - J(\mathbf{u}_h).$$

Now using (4.3), and the properties of the operator A (see (2.10) and (2.11)), one obtains the desired results. □

Remark 4.1. The convergence result in Lemma 4.1 as $\varepsilon \to 0$ ensure that one can approximate the solution of (3.1) by the one of (4.5). Moreover since J_ε is differentiable, it can be shown (see [13]) that (4.5) is equivalent to

$$\begin{cases} \text{Find } (\mathbf{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}_h \times M_h \text{ such that} \\ \langle A\mathbf{u}_h^\varepsilon, \mathbf{v}_h\rangle - b(\mathbf{v}_h, p_h^\varepsilon) + \langle J_\varepsilon'(\mathbf{u}_h), \mathbf{v}_h\rangle = \langle \mathbf{f}, \mathbf{v}_h\rangle \\ b(\mathbf{u}_h^\varepsilon, q_h) = 0 \quad \text{for all } (\mathbf{v}_h, q_h) \in \mathbb{V}_h \times M_h. \end{cases} \tag{4.6}$$

Next, to eliminate the incompressible condition, we introduce a penalization term by considering instead

$$b(\mathbf{u}_h^{\varepsilon,\eta}, q_h) + \eta\, c(p_h^{\varepsilon,\eta}, q_h) = 0$$

where $c(p, q) = (p, q)$. Then we 'approximate' the problem (4.6) by

$$\begin{cases} \text{Find } (\mathbf{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}) \in \mathbb{V}_h \times M_h \text{ such that} \\ \langle A\mathbf{u}_h^{\varepsilon,\eta}, \mathbf{v}_h\rangle - b(p_h^{\varepsilon,\eta}, \mathbf{v}_h) + \langle J_\varepsilon'(\mathbf{u}_h^{\varepsilon,\eta}), \mathbf{v}_h\rangle = \langle \mathbf{f}, \mathbf{v}_h\rangle \\ b(\mathbf{u}_h^{\varepsilon,\eta}, q_h) + \eta\, c(p_h^{\varepsilon,\eta}, q_h) = 0 \quad \text{for all } (\mathbf{v}_h, q_h) \in M_h \times \mathbb{V}_h. \end{cases} \tag{4.7}$$

Following [28, 36], (4.7) admits a unique solution $(\mathbf{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}) \in \mathbb{V}_h \times M_h$ which converges to $(\mathbf{u}_h, p_h) \in \mathbb{V}_h \times M_h$ solution of (4.6) as $\eta \to 0$.

The numerical resolution of (4.7) remains quite challenging because of the presence of the nonlinear expressions $\langle A\mathbf{u}_h^{\varepsilon,\eta}, \mathbf{v}_h\rangle$, and $\langle J_\varepsilon'(\mathbf{u}_h^{\varepsilon,\eta}), \mathbf{v}_h\rangle$ among other. We next then introduce a solution strategy of (4.7) by adopting a suitable time evolution in which the long term behavior of the solution of the later problem will be 'close enough in some sense' to solution of the former problem (see the pioneering work [26]). Hence the following steps will be adopted to solve (4.7):

1) associate to the weak formulation (4.7) an initial value problem in $\mathbb{V}_h \times M_h$;
2) time discretize the initial value problem formulated in step 1.

Applying the above methodology, we obtain step 1 by associating to (4.7) the following initial value problem: Given $\mathbf{u}_0 \in \mathbb{L}^2(\Omega)$, and assuming that $\mathbf{f} \in \mathbb{V}^\star$ independent of time, we consider the following problem:

$$\begin{cases} \text{Find } (\mathbf{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t)) \in \mathbb{V}_h \times M_h \text{ such that} \\ \left(\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{v}_h\right) + \langle A\mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{v}_h\rangle - b(p_h^{\varepsilon,\eta}(t), \mathbf{v}_h) \\ +\langle J_\varepsilon'(\mathbf{u}_h^{\varepsilon,\eta}(t)), \mathbf{v}_h\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \\ b(\mathbf{u}_h^{\varepsilon,\eta}(t), q_h) + \eta\, c(p_h^{\varepsilon,\eta}(t), q_h) = 0 \\ \mathbf{u}_h^{\varepsilon,\eta}(0) = \mathbf{0} \\ \text{for all } (q_h, \mathbf{v}_h) \in M_h \times \mathbb{V}_h. \end{cases} \tag{4.8}$$

Following J. L. Lions [34] and Theorem 2.1, it can be shown that the initial value problem (4.8) admits a unique solution $(\mathbf{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t)) \in \mathbb{V}_h \times M_h$ which is bounded independent on time.

The next result tells us why it is important to consider only the long time behavior of the solution $(\mathbf{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t))$ of (4.8).

Theorem 4.1. *The evolution problem (4.8) admits a unique solution $(\boldsymbol{u}_h^{\varepsilon,\eta}(t), p_h^{\varepsilon,\eta}(t)) \in \mathbb{V}_h \times M_h$ which is bounded independent on time. Furthermore $\boldsymbol{u}_h^{\varepsilon,\eta}(t)$ converges to $\boldsymbol{u}_h^{\varepsilon,\eta}$ solution of (4.7) exponentially as t goes to infinity. More precisely, we have:*

$$\|\boldsymbol{u}_h^{\varepsilon,\eta}(t) - \boldsymbol{u}_h^{\varepsilon,\eta}\|_r^2 \le \|\boldsymbol{u}_0 - \boldsymbol{u}_h^{\varepsilon,\eta}\|_{1,r}^2 e^{-2Cvt}, \quad \forall\, t \ge 0 \tag{4.9}$$

where C is a generic positive constant independent of ε and η.

Proof. Note both the first equations in (4.7) and (4.8) are equivalent to; for all $\mathbf{v}_h \in \mathbb{V}_h$ we have

$$(A\mathbf{u}_h^{\varepsilon,\eta}, \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}) - b(p_h^{\varepsilon,\eta}, \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}) + J_\varepsilon(\mathbf{v}_h) - J_\varepsilon(\mathbf{u}_h^{\varepsilon,\eta}) \ge \langle \mathbf{f}, \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}\rangle \tag{4.10}$$

and

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)\right) + (A\mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)) - b(p_h^{\varepsilon,\eta}(t), \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)) \\ &\qquad +J_\varepsilon(\mathbf{v}_h) - J_\varepsilon(\mathbf{u}_h^{\varepsilon,\eta}(t)) \ge \langle \mathbf{f}, \mathbf{v}_h - \mathbf{u}_h^{\varepsilon,\eta}(t)\rangle. \end{aligned} \tag{4.11}$$

Let $\mathbf{v}_h = \mathbf{u}_h^{\varepsilon,\eta}(t)$ in the first equation of (4.10) and $\mathbf{v}_h = \mathbf{u}_h^{\varepsilon,\eta}$ in (4.11) and adding the resulting inequalities, it follows that

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_h^{\varepsilon,\eta}(t), \mathbf{u}_h^{\varepsilon,\eta} - \mathbf{u}_h^{\varepsilon,\eta}(t)\right) - (A\mathbf{u}_h^{\varepsilon,\eta}(t) - A\mathbf{u}_h^{\varepsilon,\eta}, \mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}) \\ &\qquad +b(p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}, \mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}) \ge 0. \end{aligned} \tag{4.12}$$

Subtracting the second relation in (4.7) from the second relation in (4.8), and taking $q_h = p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}$ one obtains

$$b(\mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}) = -\eta\, c(p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}) \le 0.$$

Let us set $\mathbf{w}_h(t) = \mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}$, using the fact that $-b(\mathbf{u}_h^{\varepsilon,\eta}(t) - \mathbf{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta}(t) - p_h^{\varepsilon,\eta}) \ge 0$, the monotonicity properties (2.10) and (2.11), one obtains from (4.12) that: for $1 < r \le 2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}_h(t)\|_r^2 + 2Cv(\|\mathbf{u}_h^{\varepsilon,\eta}(t)\|_{1,r} + \|\mathbf{u}_h^{\varepsilon,\eta}\|_{1,r})^{r-2}\|\mathbf{w}_h(t)\|_{1,r}^2 \le 0$$

for $2 \le r < \infty$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}_h(t)\|_r^2 + 2Cv\|\mathbf{w}_h(t)\|_{1,r}^r \le 0.$$

Note that $\|\mathbf{u}_h^{\varepsilon,\eta}\|_{1,r} \le C$ and $\|\mathbf{u}_h^{\varepsilon,\eta}(t)\|_{1,r} \le C$ when t goes to infinity, then for $1 < r \le 2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}_h(t)\|_r^2 + 2Cv\|\mathbf{w}_h(t)\|_{1,r}^2 \le 0.$$

We obtain the result via Poincare's inequality and Gronwall's lemma. □

Finally, concerning step 2, we first consider the following discrete linear scheme: Let $N \in \mathbb{N}^\star$ and set $k = T/N$. Given $(\mathbf{u}_h^{\varepsilon,\eta,0}, p_h^{\varepsilon,\eta,0})$ which is a suitable approximation of $(\mathbf{u}_0, p_0)$. Knowing $(\mathbf{u}_h^{\varepsilon,\eta,m-1}, p_h^{\varepsilon,\eta,m-1})$, compute $(\mathbf{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m})$ in $\mathbb{V}_h \times M_h$ solution of the problem

$$\begin{cases} \dfrac{1}{k}\,(\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}, \mathbf{v}_h) + \nu((|\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2})\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h) \\ -b(p_h^{\varepsilon,\eta,m}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - \langle J'_\varepsilon(\mathbf{u}_h^{\varepsilon,\eta,m-1}), \mathbf{v}_h\rangle \\ b(\mathbf{u}_h^{\varepsilon,\eta,m}, q_h) + \eta\, c(p_h^{\varepsilon,\eta,m}, q_h) = 0 \\ \forall\,(q_h, \mathbf{v}_h) \in M_h \times \mathbb{V}_h. \end{cases} \tag{4.13}$$

But in order to obtain better numerical results, we instead adopted the following scheme

$$\begin{cases} \dfrac{1}{k}\,(\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}, \mathbf{v}_h) + \nu((|\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2})\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h) \\ +\mu(k)(\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h) - b(p_h^{\varepsilon,\eta,m}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - \langle J'_\varepsilon(\mathbf{u}_h^{\varepsilon,\eta,m-1}), \mathbf{v}_h\rangle \\ b(\mathbf{u}_h^{\varepsilon,\eta,m}, q_h) + \eta\, c(p_h^{\varepsilon,\eta,m}, q_h) = 0 \\ \forall\,(q_h, \mathbf{v}_h) \in M_h \times \mathbb{V}_h \end{cases} \tag{4.14}$$

where $\mu(k)$ should be regarded as artificial viscosity, given as follows

$$0 < \mu(k) < 1 \qquad \text{such that} \qquad \lim_{k\to 0}\mu(k) = 0. \tag{4.15}$$

Remark 4.2. (i) Since (4.14) is a linear system of equations, by rearranging terms, one sees that it is a square linear system in finite dimension; hence uniqueness implies existence of solution. Uniqueness of $\mathbf{u}_h^{\varepsilon,\eta,m}$ follows from the energy estimate (4.16), while uniqueness of $p_h^{\varepsilon,\eta,m}$ is the consequence of the discrete version of the inf-sup condition (2.3).
(ii) The introduction of a coercive term $\mu(k)(\nabla \mathbf{u}_h^{\varepsilon,\eta,m}, \nabla \mathbf{v}_h)$ has the effect of bringing more stability/smoothness to the system. The results of our numerical computations support that intuition.

Theorem 4.2. *Suppose that $\boldsymbol{u}_h^{\varepsilon,\eta,0}$ is chosen to satisfy*

$$\|\boldsymbol{u}_h^{\varepsilon,\eta,0}\| \le C\|\boldsymbol{u}_0\|$$

where C denote a constant independent of k and h. Let $(\boldsymbol{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta})$ be the solution of (4.8), then the iterative solution $(\boldsymbol{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m})$ of (4.14) converges to $(\boldsymbol{u}_h^{\varepsilon,\eta}, p_h^{\varepsilon,\eta})$ as m tends to infinity.

Proof. For that, we follow Girault and Gonzalez [22]. The proof is obtained in two steps. First, one obtains some energy estimates, next we use compactness results and pass to the limit.

Step 1: energy estimates

Lemma 4.2. *There exists $C > 0$ and δ which verify $0 < \delta^2 < \mu(k) < 1$ such that:*

$$\sup_{0\le m\le N} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|^2 \le C, \quad k(\mu(k) - \delta^2)\sum_{m=1}^{N} \|\boldsymbol{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \le C \tag{4.16}$$

$$\sum_{m=1}^{N} \|\boldsymbol{u}_h^{\varepsilon,\eta,m} - \boldsymbol{u}_h^{\varepsilon,\eta,m-1}\|^2 \le C \tag{4.17}$$

$$k\sum_{m=1}^{N} \int_\Omega |\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m-1}|^{r-2}|\nabla \boldsymbol{u}_h^{\varepsilon,\eta,m}|^2\,\mathrm{d}x \le C. \tag{4.18}$$

Proof. Take $\mathbf{v}_h = \mathbf{u}_h^{\varepsilon,\eta,m}$, $q_h = p_h^{\varepsilon,\eta,m}$ in (4.14), use (4.4) and the fact

$$-b(\mathbf{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m}) = \eta c(p_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m}) \geq 0$$

and $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ we have:

$$\begin{aligned}
\|\mathbf{u}_h^{\varepsilon,\eta,m}\|^2 \; - \; & \|\mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 + \|\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 \\
& + 2\nu k \int_\Omega |\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \mathbf{u}_h^{\varepsilon,\eta,m}|^2 \, \mathrm{d}x + 2k\mu(k) \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \\
& \leq \; 2k(\mathbf{f}, \mathbf{u}_h^{\varepsilon,\eta,m}) - 2k\langle J_\varepsilon'(\mathbf{u}_h^{\varepsilon,\eta,m-1}), \mathbf{u}_h^{\varepsilon,\eta,m}\rangle \\
& \leq \; 2k\|\mathbf{f}\|_{-1,r'} \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r} + 2k\|g\|_{L^{r'}(S)} \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r} \\
& \leq \; k\frac{\|\mathbf{f}\|_{-1,r'}^2}{\delta^2} + k\frac{\|g\|_{L^{r'}(S)}^2}{\delta^2} + 2k\delta^2 \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 .
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|\mathbf{u}_h^{\varepsilon,\eta,m}\|^2 \; - \; & \|\mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 + \|\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 \\
& + 2\nu k \textstyle\int_\Omega |\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \mathbf{u}_h^{\varepsilon,\eta,m}|^2 \, \mathrm{d}x + 2k(\mu(k) - \delta^2) \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \\
\leq \; & \frac{k}{\delta^2} \left(\|\mathbf{f}\|_{-1,r'}^2 + \|g\|_{L^{r'}(S)}^2 \right) .
\end{aligned} \tag{4.19}$$

Summing (4.19) for $m = 1, \ldots, N$ and use the fact that $\sum_{m=1}^N k = T$ yields

$$\begin{aligned}
\|\mathbf{u}_h^{\varepsilon,\eta,N}\|^2 \; + \; & \sum_{m=1}^N \|\mathbf{u}_h^{\varepsilon,\eta,m} - \mathbf{u}_h^{\varepsilon,\eta,m-1}\|^2 + 2\nu k \sum_{m=1}^N \int_\Omega |\nabla \mathbf{u}_h^{\varepsilon,\eta,m-1}|^{r-2} |\nabla \mathbf{u}_h^{\varepsilon,\eta,m}|^2 \, \mathrm{d}x \\
& + \; 2k(\mu(k) - \delta^2) \sum_{m=1}^N \|\mathbf{u}_h^{\varepsilon,\eta,m}\|_{1,r}^2 \leq \frac{T}{\delta^2} \left(\|\mathbf{f}\|_{-1,r'}^2 + \|g\|_{L^{r'}(S)}^2 \right) + \|\mathbf{u}_h^{\varepsilon,\eta,0}\|^2
\end{aligned}$$

therefore we obtain the second relation in (4.16), (4.17) and (4.18). The first inequality in (4.16) is readily obtained by summing (4.19) for $m = 1, \ldots, n$ and use the fact that $\sum_{m=1}^n k \leq \sum_{m=1}^N k = T$. □

Step 2: weak convergence/passage to the limit

Let $\mathbf{u}_{hk}^{\varepsilon,\eta} \in \mathcal{C}^0([0,T], \mathbb{V})$ be affine in each subinterval $[t_{m-1}, t_m]$ with $\mathbf{u}_{hk}^{\varepsilon,\eta}(t_m) = \mathbf{u}_h^{\varepsilon,\eta,m}$ for $1 \leq m \leq N$. Let $\mathbf{u}_{hk}^{\varepsilon,\eta\, r}$, $\mathbf{u}_{hk}^{\varepsilon,\eta l}$ be the piecewise constant function such that

$$\mathbf{u}_{hk}^{\varepsilon,\eta\, r}|_{[t_{m-1},t_m[} = \mathbf{u}_h^{\varepsilon,\eta,m}, \quad \mathbf{u}_{hk}^{\varepsilon,\eta\, l}|_{]t_{m-1},t_m]} = \mathbf{u}_h^{\varepsilon,\eta,m-1}.$$

Using *a priori* obtained in Lemma 4.2, one gets

$$\begin{aligned}
& \mathbf{u}_{hk}^{\varepsilon,\eta}, \;\; \mathbf{u}_{hk}^{\varepsilon,\eta\, r}, \;\; \mathbf{u}_{hk}^{\varepsilon,\eta\, l} \;\text{ remain in a bounded set of }\; L^\infty(\mathbb{L}^2(\Omega)) \text{ and} \\
& \mathbf{u}_{hk}^{\varepsilon,\eta\, r} \;\text{ remain in a bounded set of }\; L^2(\mathbb{W}_0^{1,r}(\Omega)).
\end{aligned} \tag{4.20}$$

Furthermore, from (4.17), we have

$$\|\mathbf{u}_{hk}^{\varepsilon,\eta\, r} - \mathbf{u}_{hk}^{\varepsilon,\eta\, l}\|_{L^2(\mathbb{L}^2(\Omega))} \leq Ck^{1/2}, \quad \|\mathbf{u}_{hk}^{\varepsilon,\eta} - \mathbf{u}_{hk}^{\varepsilon,\eta\, r}\|_{L^2(\mathbb{L}^2(\Omega))} \leq Ck^{1/2}. \tag{4.21}$$

Then we can extract a subsequence $k' \subset k$ still denoted k such that

$$\begin{aligned}
\mathbf{u}_{hk}^{\varepsilon,\eta\, r} &\to \mathbf{u}_h^{\varepsilon,\eta\, r} \;\text{ weakly* in }\; L^\infty(\mathbb{L}^2(\Omega)) \\
\mathbf{u}_{hk}^{\varepsilon,\eta\, l} &\to \mathbf{u}_h^{\varepsilon,\eta\, l} \;\text{ weakly* in }\; L^\infty(\mathbb{L}^2(\Omega)) \\
\mathbf{u}_{hk}^{\varepsilon,\eta} &\to \mathbf{u}_h^{\varepsilon,\eta} \;\text{ weakly* in }\; L^\infty(\mathbb{L}^2(\Omega)) \\
\mathbf{u}_{hk}^{\varepsilon,\eta\, r} &\to \mathbf{u}_h^{\varepsilon,\eta\, r} \;\text{ weakly in }\; L^2(\mathbb{W}_0^{1,r}(\Omega))
\end{aligned}$$

and from (4.21), one obtains $\mathbf{u}_h^{\varepsilon,\eta\, r} = \mathbf{u}_h^{\varepsilon,\eta\, l} = \mathbf{u}_h^{\varepsilon,\eta}$.

Note that $\mathbf{u}_{hk}^{\varepsilon,\eta\, r}$, $\mathbf{u}_{hk}^{\varepsilon,\eta\, l}$ and $\mathbf{u}_{hk}^{\varepsilon,\eta}$ verify:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_{hk}^{\varepsilon,\eta}, \mathbf{v}_h\right) &+ \nu((|\nabla \mathbf{u}_{hk}^{\varepsilon,\eta\, l}|^{r-2})\nabla \mathbf{u}_{hk}^{\varepsilon,\eta\, r}, \nabla \mathbf{v}_h) + \mu(k)(\nabla \mathbf{u}_{hk}^{\varepsilon,\eta\, r}, \nabla \mathbf{v}_h) \\ &+ \frac{1}{\eta}\left(\rho_h(\nabla\cdot \mathbf{u}_h^{\varepsilon,\eta\, r}), \rho_h(\nabla\cdot\mathbf{v}_h)\right) = (\mathbf{f}, \mathbf{v}_h) - \langle J_\varepsilon'(\mathbf{u}_{hk}^{\varepsilon,\eta\, l}), \mathbf{v}_h\rangle \quad \forall \mathbf{v}_h \in \mathbb{V}_h\end{aligned} \tag{4.22}$$

where ρ_h is the orthogonal projection of $L^2(\Omega)$ onto M_h. The weak convergence above allows us to pass to the limit in all bilinear terms in (4.22). But as far as the nonlinear term is concerned, we advice the reader to see the results in [34], where similar expression has been analyzed. □

Remark 4.3. Note that establishing convergence of the pressure is more delicate because it involves convergence of the time derivative of the velocity whose proof is fairly long and intricate; cf. Lions [34] and Temam [40].

The initialization of the flow defined by (4.8) and of its time discrete counterpart defined by (4.14) is important. Let us observe that since one has well-posedness of (4.7) for all values of r in $[1, \infty)$, in order to consolidate the convergence of (4.7), we suggest the solution of Stokes equations

$$\begin{cases}\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - b(p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbb{V}_h \\ b(\mathbf{u}_h, q_h) + \eta\, c(p_h, q_h) = 0 & \forall q_h \in M_h\end{cases} \tag{4.23}$$

as initial condition for our algorithm.

5 Numerical experiments

In this section our goal is to test numerically the convergence and rate of convergence obtained in Section 3, and to illustrate the performance of our algorithm via benchmark examples. The implementation is done by extending the Matlab code developed in [3]. In all the examples presented, the velocity and pressure will be approximated by the continuous $P2 - P1$ element.

We recall that the different steps of our algorithm are as follows:

(a) compute the initial flow by solving (4.23);

(b) knowing $(\mathbf{u}_h^{\varepsilon,\eta,m-1}, p_h^{\varepsilon,\eta,m-1})$, compute $(\mathbf{u}_h^{\varepsilon,\eta,m}, p_h^{\varepsilon,\eta,m})$ solution of (4.14).

Numerical accuracy check

We consider $\nu = 0.4$, $k = 1/100$, $\mu(k) = 1/500$, $\eta = \varepsilon = 1/1000$, the exterior force to be unity and $g = 0.1$.

The convergence result obtained in Section 3, is tested by computing the rate of convergence using (4.14). In this first example, we take $\Omega = (0, 1)^2$, with $\partial\Omega = \Gamma \cup S$, where

$$\begin{aligned}\Gamma &= \{(x, 0)\,,\ 0 < x < 1\} \cup \{(0, y)\,,\ 0 < y < 1\} \\ S &= \{(1, y)\,,\ 0 < y < 1\} \cup \{(x, 1)\,,\ 0 < x < 1\}.\end{aligned}$$

Since we do not know the exact solution, we employ the approximate solutions with $N = 60$ as the reference solutions $(\mathbf{u}_{\mathrm{ref}}, p_{\mathrm{ref}})$, and we compute the L^r-norm and $\mathbb{W}_{1,r}$-norm of the difference of the reference solution and the approximate solution $(\mathbf{u}_h, p_h)$. The results are presented in Tables 1, 2, and 3. The predicted convergence rate $O(h^{1/r})$ for $W^{1,r}$ is noted.

Next, to see the effect of the 'stabilizing term' added, we repeat the same exercise with $\mu(k) = 0$. The contribution of the added term is made visible (see Table 4). Indeed, the convergence is much faster when $\mu(k)$ is not zero.

Table 1. Velocity convergence results for $r = 3$.

h	$\|\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|\|_{L^r}$	$\|\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.6696E-5	4.4945E-4		
1/10	7.9862E-6	3.6173E-4	1.4437	0.4351
1/12	6.1507E-6	3.3163E-4	1.4324	0.4765
1/15	4.5180E-6	2.9887E-4	1.3825	0.4661
1/20	2.9500E-6	2.4827E-4	1.4368	0.5430

Table 2. Velocity convergence results for $r = 7/2$.

h	$\|\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|\|_{L^r}$	$\|\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.7653E-5	4.8807E-4		
1/10	8.6259E-6	4.0641E-4	1.4019	0.3584
1/12	6.6953E-6	3.7694E-4	1.3897	0.4129
1/15	4.9884E-6	3.4468E-4	1.3188	0.4010
1/20	3.3116E-6	2.9126E-4	1.4241	0.4854

Driven cavity

It is a benchmark test problem that has been considered by many researchers [35, 39, 41] among others. We assume $\Omega = (0, 1)^2$, the boundary of which consists of two portions Γ and S is given by

$$\Gamma = \{(0, y)/0 < y < 1\} \cup \{(x, 0)/0 < x < 1\}$$
$$S_1 = \{(x, 1)/0 < x < 1\}, \quad S_2 = \{(1, y)/0 < y < 1\}, \quad S = S_1 \cup S_2.$$

For the triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, we employ a uniform $N \times N$ mesh, where N denotes the division number of each side of the domain.

Let us consider

$$\begin{cases} u_1(x, y) = -x^2 y(x-1)(3y-2) \\ u_2(x, y) = xy^2(y-1)(3x-2) \\ p(x, y) = (2x-1)(2y-1) \end{cases} \tag{5.1}$$

which turns out to be the exact solution of the problem (1.1)–(1.5) under the appropriate choice of $\mathbf{f}$ and g. It is easy to verify that the exact solution $\mathbf{u}$ satisfies $\mathbf{u} = \mathbf{0}$ on Γ, $\mathbf{u} \cdot \mathbf{n} = u_1 = 0$, $u_2 \neq 0$ on S_2, and $u_1 \neq 0$, $\mathbf{u} \cdot \mathbf{n} = u_2 = 0$ on S_1. Thus

$$\begin{aligned} \sigma_\tau &= -2\mu x^2(x-1)[2(x^2-2x)^2 + 8x^4(x-1)^2]^{(r-2)/2} \quad \text{on } S_1 \\ \sigma_\tau &= -2\mu y^2(y-1)[2(y^2-2y)^2 + 8y^4(y-1)^2]^{(r-2)/2} \quad \text{on } S_2. \end{aligned} \tag{5.2}$$

On the other hand, from the slip boundary conditions (1.5), we have

$$|\sigma_\tau| \leq g \quad \text{on } S = S_1 \cup S_2$$

then we find that with g constant:
- $g \geq \max_S |\sigma_\tau| \Rightarrow$ (5.1) remains a solution;
- $g < \max_S |\sigma_\tau| \Rightarrow$ (5.1) is no longer a solution and a non-trivial slip occurs.

We indeed observe some of the above mentioned phenomena in our numerical computation, as indicated in the plots of the velocity field shown in Figs. 1–5.

In addition, we find that:
(a) the bigger the threshold g of tangential stress becomes, the more difficult it becomes for a non-trivial slip to occur;
(b) the smaller the threshold g of tangential stress becomes, the more easier it becomes for a non-trivial slip to occur,

Table 3. Velocity convergence results for $r = 4$.

h	$\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|_{L^r}$	$\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.8528E-5	5.2119E-4		
1/10	9.2218E-6	4.4538E-4	1.3659	0.3077
1/12	7.2046E-6	4.1679E-4	1.3542	0.3639
1/15	5.4342E-6	3.8538E-4	1.2636	0.3511
1/20	3.6620E-6	3.4006E-4	1.3720	0.4349

Table 4. Velocity convergence results for $r = 3$ and $\mu(k) = 0$.

h	$\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|_{L^r}$	$\|\mathbf{u}_{\text{ref}} - \mathbf{u}_h\|_{1,r}$	convergence rate L^r	convergence rate $W_{1,r}$
1/6	1.8428E-5	5.2009E-4		
1/10	9.3218E-6	4.4538E-4	1.3341	0.3036
1/12	7.4046E-6	4.2080E-4	1.2629	0.3114
1/15	5.534E-6	3.9438E-4	1.3049	0.2906
1/20	3.7620E-6	3.6120E-4	1.3417	0.3151

which is in agreement with the predicted outcome.

For all the numerical results which follow, $\nu = 0.4$, $k = 1/100$, $\mu(k) = 1/500$, $\eta = \varepsilon = 1/1000$, and g is indicated on the pictures.

To be more precise, one observes in Fig. 1, that in some region of the boundaries, the fluid slips (see the right and top part of the boundary for picture on the right), where as in the left hand side, the fluid adheres at the boundaries. Similar pattern are observed in Figs. 2–5 below.

Lastly, the role of the 'stabilizing term' is also justify in this test problem. Indeed, when $\mu(k)(\nabla \mathbf{u}_h^{\eta,\varepsilon,m}, \nabla \mathbf{v}_h)$ is neglected, that is $\mu(k) = 0$ the slip is more pronounce in Fig. 3 than Fig. 6.

6 Conclusions

In this work, we have formulated and analyzed the finite element approximation for the power law Stokes flow driven by slip boundary conditions of friction type. Also, we have discussed and implemented a particular algorithm combining vanishing viscosity method and stationary solution of an initial value problem (flow in the dynamical system terminology). The well posedness of the finite element approximation is obtained by using the generalized version of Babuska–Brezzi's theory of mixed formulation introduced by [28, 36]. As far as the implementation of the finite element formulated is concerned, we have adapted the well known methodology consisting to associate to a stationary problem an initial value problem in which the focus is on the behavior of the solution of the later problem when the time is big enough. But in order to improve the rate of convergence, we have added a stabilizing term to the initial value problem (numerical computations confirm the predictions). This approach leads naturally to a solution method based on time discretization; it has also an advantage of being easily implementable, but much progress has to be made for a systematic way of choosing the initial flow.

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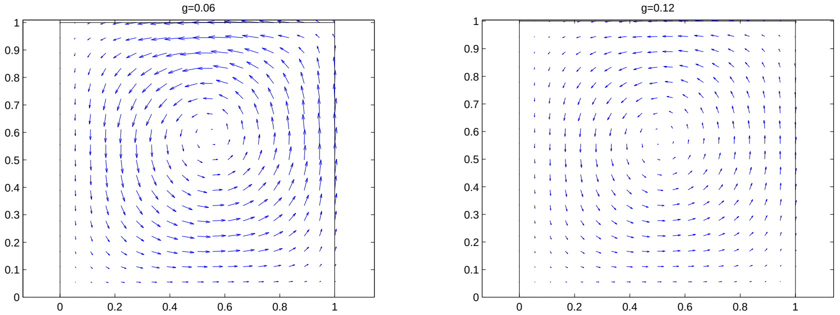

Figure 1. Velocity field for $r = 3/2$ and $\max_S |\sigma_T| = 0.1035$.

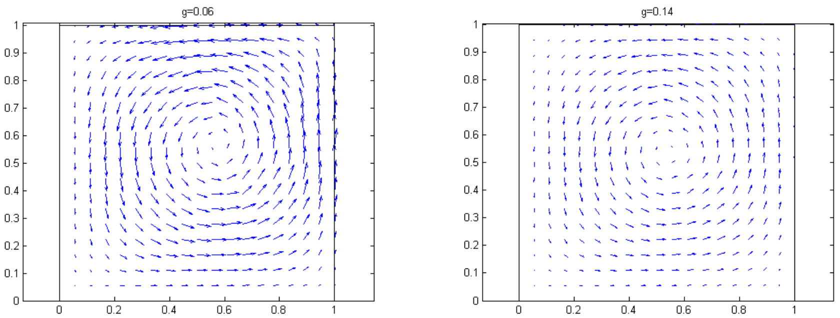

Figure 2. Velocity field for $r = 2$ and $\max_S |\sigma_T| = 0.1185$.

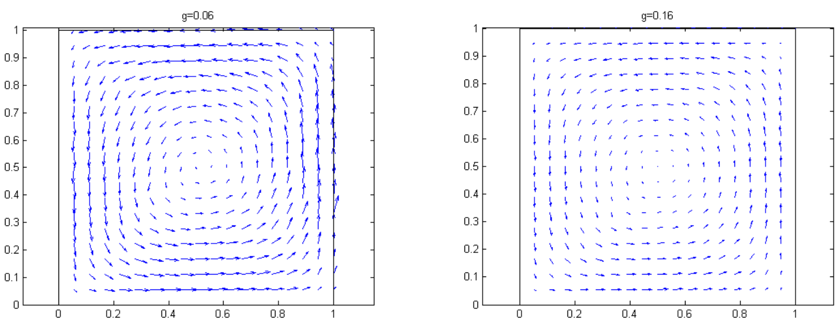

Figure 3. Velocity field for $r = 3$ and $\max_S |\sigma_T| = 0.1591$.

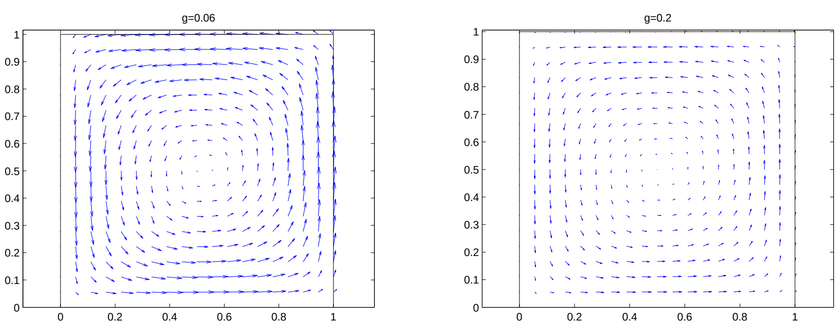

Figure 4. Velocity field for $r = 7/2$ and $\max_S |\sigma_T| = 0.1855$.

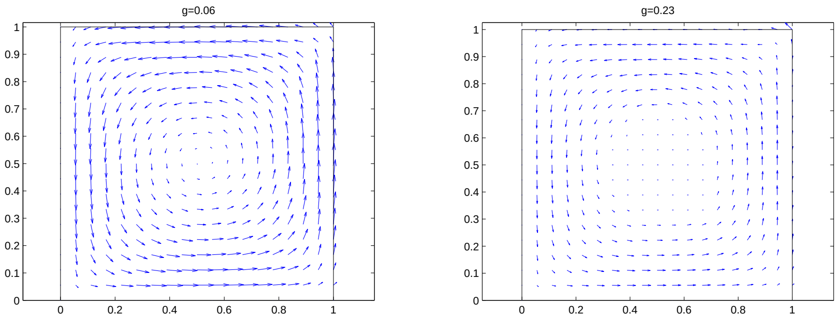

Figure 5. Velocity field for $r = 4$ and $\max_S |\sigma_T| = 0.2170$.

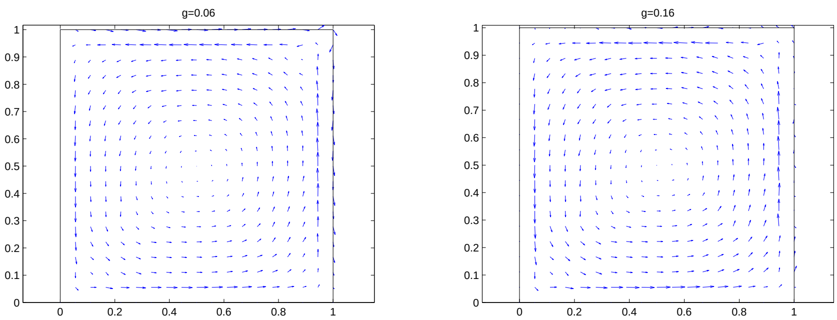

Figure 6. Velocity field for $r = 3$ and $\max_S |\sigma_T| = 0.1591$ with $\mu(k) = 0$.

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