Comment

Kinetic models – mathematical models of everything? Comment on "Collective learning Modelling Based on the Kinetic Theory of Active Particles" by D. Burini et al.

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Since the emergence of systematic science it has been recognized that a natural phenomenon can be described by different models that vary in their complexity and their ability to capture the details of the features relevant at the required level of the resolution. It has been tacitly assumed that whenever two such models are applicable at the same level, they must provide equivalent descriptions of the phenomenon. One of the earliest and most celebrated examples of this type is offered by gas flow which can be described either by the Boltzmann equation at a suitably understood molecular level or by the Euler or Navier-Stokes equations at the level of continuum. More precisely, the flow of a gas as a continuous medium, or, in other words, at the macro level, can be explained in more detail by analysing elementary collisions between pairs of molecules. Thus, the Boltzmann equation is often recognized as a more detailed equation of gas at the so-called mesoscopic, or kinetic, level from which macroscopic properties of gas, such as density, momentum or temperature, can be derived. It should be noted that one can model gas at an even more fundamental, or micro, level by tracing the motion of individual molecules by solving the system of the Newton equations that describe their interactions, [11].

The idea behind the derivation of the Boltzmann equation is to use an appropriate conservation law to derive a type of Master equation describing evolution in time of the density of objects that have specific attributes (in the Boltzmann equation context, the density of particles with specific velocity and position in space). The idea behind the derivation of the abstract Master equation is quite basic. First, we divide the set of investigated objects into classes with respect to a certain attribute we are interested in. Then, the rate of change of the number of objects in the class with a given attribute, say A, equals the rate at which the objects from all other classes are recruited to this class by changing their attributes to A, minus the rate at which the objects with attribute A change and move to some other class. Additionally, nonconservative processes such as birth/creation or death/annihilation of objects can be easily included into this framework.

This type of argument, leading to what is commonly referred to as a kinetic type model, is very general and can be used to model a variety of (all?) dynamical processes involving interactions of objects and the interchange of properties between them. Kinetic models describe the reality at the so-called mesoscale that, by focusing on statistical properties of a sample of the population (the density function), is less detailed than the microscale that involves the individually based models. On the other hand, at the mesoscale we get much more detailed information of the system than at the macroscale that only involves macroscopic observables, [4, Chapter 8]. As such, kinetic models have been extensively used in physics and chemistry and recently they have been gaining ground in biological and social sciences by providing a convenient framework for their mathematization as evidenced in several monographs and articles [5, 6, 12]. It is interesting that even the network structure can be included in such a description, [2, 9, 14].

The process of learning, understood as acquiring a more desirable attribute (increasing one's knowledge) by interactions with other agents such as peers, instructors, manuals, is formally similar to processes like swarming [1], opinion forming or neural networks training [8]. Thus it can be modelled within the framework of kinetic equations like, in the latter case, by the Ising model in the Glauber kinetic formulation, [13]. The paper under consideration, [10], systematizes this approach along the lines of [6, 7] by building a model consisting of a complex network in which the learners, treated as active agents, are divided into nodes and functional subsystems composed of individuals with the same learning strategy. The learning process is then modelled by kinetic operators describing the transfer of knowledge between the learners, or from teachers or media, in terms of the evolution of the density of individuals with a particular level of knowledge. This can be regarded as a conservation law. The crux of the matter is to provide the constitutive relations that, in this case, are given by the transition kernels of the kinetic operators. In the paper, the authors provide one example of the realization of the model and show that it gives consistent results with the one based on the Ising model developed in [8] and that, in turn, agree with qualitative results reported in the literature quoted there. This, in my opinion, should determine the future direction of research in this field. Kinetic models are universal but, for them to be practical, there should be a systematic way of constructing transition kernels that, on one hand, give experimentally verifiable results and, on the other, ensure that they are consistent with other established models in a similar way as the Boltzmann and the Navier-Stokes equations, or the kinetic and partial differential description of swarming, see e.g. [3, 4].

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