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A finite source perishable inventory system with second optional service and server interruptions

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Abstract

In this article, a service facility inventory system with server interruptions and a finite number of sources are considered. The inventory is replenished according to (s, S) ordering policy. Using the matrix methods, the stationary distribution of the stock level, server status and waiting area level is obtained in the steady state case. The Laplace-Stieltjes transform of the waiting time of the tagged customer is derived. Many impartment system performance measures are derived and the total expected cost rate is computed under a suitable cost structure. The results are illustrated numerically.

Key words: Essential and optional service, inventory with service time, service interruption, repair, finite source.

1 Introduction

Schwarz *et al.* [18] introduced the idea of inventory with positive service time. They have assumed that when the stock level is empty, each arriving customer enters into the queue. Berman *et al.* [2] derived deterministic approximations for a queueing-inventory system with a service facility. Berman & Kim [3] analyzed a single server inventory model with the assumption of instantaneous replenishment and a service facility. Berman & Sapna [5] considered a queueing-inventory model with single server and finite capacity. They assumed that customer arrival follows Poisson process, arbitrarily distributed service times and zero lead times. Berman & Kim [4] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a reorder policy that maximized the system proceeds. Krishnamoorthy

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& Anbazhagan [13] studied a queueing-inventory system with N policy and the finite waiting hall. They have assumed that if the customer level reaches a prefixed level N, then the server starts service immediately. Otherwise, server does not provide service to the waiting customers. Two other papers where a service facility inventory model is considered, are by Krishnamoorthy *et al.* [14, 15]. The paper [14] is analyzed for a queueing-inventory model with instantaneous replenishment and the service process is subject to interruptions. The discussion in [15] is a retrial queueing-inventory model with positive lead-time and the service process is subject to interruptions. Jeganathan & Periyasamy [11] studied a perishable inventory system with repeated attempts and the service process is subject to interruptions. They analyzed the system with the restriction that both the waiting area and orbit size are finite. Jeganathan *et al.* [9] discussed a service facility inventory system with multiple server vacations and server is subject to interruptions. For a detailed study in the service facility inventory models with an infinite number of sources the reader is referred to [1, 6, 8, 10, 12, 22].
An (s, S) inventory system with finite source was first initiated by Sivakumar [20]. He assumed that lifetime of each commodity is exponential and lead-time distribution is exponential. The author considered the constant retrial policy *i.e.*, the probability of retrial is independent of the number of demands in the orbit. Shophia Lawrence *et al.* [19] considered a service facility inventory management system with the service time, the lead time are assumed to have Phase type distribution and finite source. Yadavalli *et al.* [23] analysed a service facility retrial inventory model with finite source and multi homogeneous servers. Yadavalli *et al.* [21] studied a two heterogeneous queueing-inventory system including one server is perfectly reliable and another server is subject to interruptions. In a recent paper, Jeganathan [7] analysed a mixed priority retrial inventory system with additional optional service and finite source.

A finite source queueing-inventory model with server interruptions and extra optional service is motivated by the service facility system with limited customers such as military canteen providing service to soldiers or a company canteen serving the members of the specific working area in the business. In this paper, a continuous review (s, S) inventory model with server interruptions, second optional service and finite source simultaneously are considered.

In the next section, the mathematical model is explained and the notations used in this paper are defined. Model analysis and the steady state analysis are proposed in section 3. In section 4, the waiting time analysis of a customers in the waiting area is discussed. Various important performance measures are derived in section 5. In section 6, the total expected cost rate is derived. In section 7, optimality of the cost function and its sensitivity with respect to various parameters using numerical examples are presented. The last section is meant for conclusion.

2 Model description

In this work, finite-source queueing-inventory models with the following assumption are studied. Consider a service facility wherein perishable items are stored and the items are distributed to the demanding customers. The maximum stock level is S. The customers are generated by a finite number of identical sources N, $(1 < N < \infty)$ and the demand time points form a quasi-random distribution with rate $\lambda(> 0)$. An arriving customer finds the system either when the empty stock level or the server is busy or the server is on interruptions, then with probability r he/she enters into the waiting area. Otherwise, he/she balks (do not join) with probability $1 - r$. There is a single waiting area for the customers and the demand is for single item per customer. Before distributing items to the demanding customers, some primary service on the item is given first. In this article this type of service is referred to as first essential service (FES). This system has a single server who gives preliminary FES indicated by the rate μ_1 to all arriving customers one by one according to FIFO (first in first out) discipline. When the FES of a customer is completed, the server may offer a second optional service (SOS) indicated by the rate μ_2 with probability p to only those customers who opt for it otherwise leaves the system with the complementary probability q, where p+q=1. While the server is in working state it may be interrupted at any time with interruption rate α_1 during first essential service (FES) and α_2 during second optional service (SOS). When the server interruption occurs, it is immediately sent for repairing where repair time indicated by η_1 for FES and η_2 for SOS. After repairing, the server provides residual service of the customers of both of the phases (FES or SOS). It is assumed that if the server is in interruption, no more interruption can be caused on the server. The service times of the FES and SOS, and the interruption times are assumed to follow an exponential distribution.

The operating policy is (s, S) policy with exponential lead times for the ordered items. According to the ordering policy, when the stock level downfall to s, an order for $Q(= S-s > s+1)$ items are placed. Lifetime of each item has negative exponential distribution with rate $\gamma > 0$. The positive lead-time of the replenishment is assumed to be exponential with the rate $\beta(> 0)$. All stochastic processes involved in the system are independent of each other. The notation used in this paper follows below.

$\boldsymbol{e}$: a column vector of appropriate dimension containing all ones

$\mathbf{0}$: zero matrix

$[A]_{ij}$: entry at $(i, j)^{th}$ position of a matrix A

$$\delta_{ij} : \begin{cases} 1 & \text{if} j = i \\ 0 & \text{otherwise} \end{cases}$$

$\bar{\delta_{ij}}$: $1 - \delta_{ij}$

$$H(x) : \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$k \in V_i^j$: $k = i, i+1, \ldots j$

$$\overset{k}{\underset{i=j}{\Omega}} c_i : \begin{cases} c_j c_{j-1} \cdots c_k & \text{if } j \geq k \\ 1 & \text{if } j < k \end{cases}$$

3 Analysis

Let $L(t)$ and $X(t)$ respectively, denote the inventory level and the number of customers in the waiting area at time t, $L(t) \in \{0, 1, \ldots S\}$ and $X(t) \in \{0, 1, \ldots N\}$ and let $Y(t)$ denote the status of the server, given by

$$Y(t) = \begin{cases} 0, & \text{if the server is idle at time t,} \\ 1, & \text{if the server is providing first essential service to a customer at time t,} \\ 2, & \text{if the server is providing second optional service to a customer at time t,} \\ 3, & \text{if the server is on interruption during FES at time t,} \\ 4, & \text{if the server is on interruption during SOS at time t.} \end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $I(t) = \{(L(t), Y(t), X(t)), t \geq 0\}$ is a continuous time Markov chain with discrete state space given by $E = E_a \cup E_b \cup E_c \cup E_d$ where

$$\begin{aligned} E_a &: \{(0,0,i_3) \mid i_3 = 0,1,2,\ldots,N,\} \\ E_b &: \{(i_1,0,0) \mid i_1 = 1,2,\ldots,S,\} \\ E_c &: \{(i_1,i_2,i_3) \mid i_1 = 1,2,\ldots,S,\ i_2 = 1,3,\ i_3 = 1,2,\ldots,N,\ \} \\ E_d &: \{(i_1,i_2,i_3) \mid i_1 = 0,1,2,\ldots,S,\ i_2 = 2,4,\ i_3 = 1,2,\ldots,N.\ \} \end{aligned}$$

To determine the infinitesimal generator

$$\Theta = ((\ d((i_1,i_2,i_3),(j_1,j_2,j_3))\ )), \quad (i_1,i_2,i_3),(j_1,j_2,j_3) \in E$$

of this process the following arguments are used:

- Let $Y(t) = 0$;
  - a customer arrival in state $(0,0,i_3)$ will increases the number of customers in the waiting hall by one unit and the rate of the transition $d((0,0,i_3),(0,0,i_3+1))$ is given by $r(N-i_3)\lambda$, $i_3 = 0,1,2,\ldots,N-1$.
  - a transition from state $(i_1,0,0)$ to state $(i_1,1,1)$, $i_1 = 1,2,\ldots,S$, with intensity of transition $N\lambda$, when a customer arrives.
  - a transition from state $(i_1,0,0)$ to state $(i_1-1,0,0)$, will take place when any one of i_1 items fails at a rate of γ; thus intensity for this transition is $i_1\gamma$, $i_1 = 1,2,\ldots,S$.
- Let $Y(t)$= 1 or 3;
  - the arrival of the customer makes a transition from state (i_1,i_2,i_3) to (i_1,i_2,i_3+1) with intensity of transition $r(N-i_3)\lambda$, $i_1 = 1,2,\ldots,S$, $i_2 = 1,3$, $i_3 = 1,2,\ldots,N-1$.
  - when FES is completed, then with probability p the customer may ask for SOS, in this case his/her SOS will immediately commence. The system change from state $(i_1,1,i_3)$ to state $(i_1-1,2,i_3)$, $i_1 = 1,2,\ldots,S$, $i_3 = 1,2,\ldots,N$, with intensity of this transition $p\mu_1$ or with probability q he/she may opt to leave the system, in this case both the customer level in the waiting hall and the inventory level decrease by one unit and also if the server finds both $i_1 > 0$ and $i_3 > 0$, then server becomes busy, otherwise *i.e.*, if either $i_3 = 0$ or $i_1 = 0$, then server becomes idle. The state of the system change from state $(1,1,i_3)$ to state $(0,0,i_3-1)$, $i_3 = 1,2,\ldots,N$, or $(i_1,1,1)$ to state $(i_1-1,0,0)$, $i_1 = 2,\ldots,S$, or $(i_1,1,i_3)$ to state $(i_1-1,1,i_3-1)$, $i_1 = 2,\ldots,S$, $i_3 = 2,\ldots,N$, with intensity of this transition $q\mu_1$.

- a transition from state (i_1, i_2, i_3) to state $(i_1 - 1, i_2, i_3)$, takes place when any one of the $(i_1 - 1)$ items of the commodity perishes at a rate γ and the intensity for this transition is $(i_1 - 1)\gamma$, where $i_1 = 2, 3, \ldots, S$, $i_2 = 1, 3$, $i_3 = 1, 2, \ldots, N$.
- a transition from state $(i_1, 1, i_3)$ to state $(i_1, 3, i_3)$, $i_1 = 1, 2, 3, \ldots, S$, $i_3 = 1, 2, \ldots, N$ will take place with intensity of transition α_1, when the interruption occurs during FES.
- a transition from state $(i_1, 3, i_3)$ to state $(i_1, 1, i_3)$, $i_1 = 1, 2, 3, \ldots, S$, $i_3 = 1, 2, \ldots, N$ will take place with intensity of transition η_1, when the repair completion during FES.

- Let $Y(t)$= 2 or 4;

  - the arrival of the customer makes a transition from state (i_1, i_2, i_3) to $(i_1, i_2, i_3 + 1)$ with intensity of transition $r(N - i_3)\lambda$, $i_1 = 0, 1, 2, \ldots, S$, $i_2 = 2, 4$, $i_3 = 1, 2, \ldots, N - 1$.
  - a transition from state (i_1, i_2, i_3) to state $(i_1 - 1, i_2, i_3)$, takes place when any one of the i_1 items of the inventory perishes at a rate γ and the intensity for this transition is $i_1\gamma$, where $i_1 = 1, 2, 3, \ldots, S$, $i_2 = 2, 4$, $i_3 = 1, 2, \ldots, N$.
  - the completion of optional service for a customer makes a transition from state $(0, 2, i_3)$ to state $(0, 0, i_3 - 1)$, $i_3 = 1, 2, \ldots, N$, or from state $(i_1, 2, 1)$ to state $(i_1, 0, 0)$, $i_1 = 1, 2, 3, \ldots, S$, or from state $(i_1, 2, i_3)$ to state $(i_1, 1, i_3 - 1)$, $i_1 = 1, 2, 3, \ldots, S$, $i_3 = 2, \ldots, N$, with intensity of transition μ_2.
  - a transition from state $(i_1, 2, i_3)$ to state $(i_1, 4, i_3)$, $i_1 = 0, 1, 2, 3, \ldots, S$, $i_3 = 1, 2, \ldots, N$ will take place with intensity of transition α_2, when the interruption occurs during SOS.
  - a transition from state $(i_1, 4, i_3)$ to state $(i_1, 2, i_3)$, $i_1 = 0, 1, 2, 3, \ldots, S$, $i_3 = 1, 2, \ldots, N$ will take place with intensity of transition η_2, when the repair completion during SOS.

- A passage from $(0, 0, 0)$ to $(Q, 0, 0)$, where $Q(= S - s)$, or from $(0, 0, i_3)$ to $(Q, 1, i_3)$ for $i_3 = 1, 2, \ldots, N$, or from $(0, i_2, i_3)$ to (Q, i_2, i_3) for $i_2 = 2, 4$, $i_3 = 1, 2, \ldots, N$, or from (i_1, i_2, i_3) to $(i_1 + Q, i_2, i_3)$ for $i_1 = 1, 2, \ldots, s$, $i_2 = 1, 2, 3, 4$, $i_3 = 1, 2, \ldots, N$, or from $(i_1, 0, 0)$ to $(i_1 + Q, 0, 0)$ for $i_1 = 1, 2, \ldots, s$, will take place with intensity of transition β when a replenishment for Q items occurs.

- For other transition from (i_1, i_2, i_3) to (j_1, j_2, j_3), except $(i_1, i_2, i_3) \neq (j_1, j_2, j_3)$, the rate is zero.

- Finally, the intensity of passage for the state (i_1, i_2, i_3) is given by
$-\sum\limits_{(j_1,j_2,j_3)\neq(i_1,i_2,i_3)} d((i_1, i_2, i_3), (j_1, j_2, j_3)).$

Hence, we have $d((i_1, i_2, i_3), (j_1, j_2, j_3)) =$

$$
= \left\{ \begin{array}{llll}
\beta, & j_1 = Q, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 = 0, & i_2 = 0, & i_3 = 0, \\
 & & \text{or} & \\
 & j_1 = Q, & j_2 = 1, & j_3 = i_3, \\
 & i_1 = 0, & i_2 = 0, & i_3 \in V_1^N, \\
 & & \text{or} & \\
 & j_1 = Q, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 = 0, & i_2 = 2,4, & i_3 \in V_1^N, \\
 & & \text{or} & \\
 & j_1 = i_1 + Q, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 \in V_1^s, & i_2 = 0, & i_3 = 0, \\
 & & \text{or} & \\
 & j_1 = i_1 + Q, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 \in V_1^s, & i_2 \in V_1^4, & i_3 \in V_1^N, \\
 & & & \\
i_1\gamma, & j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 \in V_1^S, & i_2 = 0, & i_3 = 0, \\
 & & \text{or} & \\
 & j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 \in V_1^S, & i_2 = 2,4, & i_3 \in V_1^N, \\
 & & & \\
(i_1 - 1)\gamma, & j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, \\
 & i_1 \in V_2^S, & i_2 = 1,3, & i_3 \in V_1^N, \\
p\mu_1, & j_1 = i_1 - 1, & j_2 = 2, & j_3 = i_3, \\
 & i_1 \in V_1^S, & i_2 = 1, & i_3 \in V_1^N, \\
 & & & \\
q\mu_1, & j_1 = i_1 - 1, & j_2 = 0, & j_3 = i_3 - 1, \\
 & i_1 = 1, & i_2 = 1, & i_3 \in V_1^N, \\
 & & \text{or} & \\
 & j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3 - 1, \\
 & i_1 \in V_2^S, & i_2 = 1, & i_3 \in V_2^N, \\
 & & & \\
N\lambda, & j_1 = i_1, & j_2 = 1, & j_3 = i_3 + 1, \\
 & i_1 \in V_1^S, & i_2 = 0, & i_3 = 0, \\
 & & & \\
r(N - i_3)\lambda, & j_1 = i_1, & j_2 = 0, & j_3 = i_3 + 1, \\
 & i_1 = 0, & i_2 = 0, & i_3 \in V_0^{N-1}, \\
 & & \text{or} &
\end{array} \right.
$$

$$
\begin{cases}
 & j_1 = 0, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \\
 & i_1 = 0, \quad i_2 = 2, 4, \quad i_3 \in V_1^{N-1}, \\
 & \text{or} \\
 & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \\
 & i_1 \in V_1^S, \quad i_2 \in V_1^4, \quad i_3 \in V_1^{N-1}, \\
\mu_2, & j_1 = i_1, \quad j_2 = 0, \quad j_3 = i_3 - 1, \\
 & i_1 = 0, \quad i_2 = 2, \quad i_3 \in V_1^N, \\
 & \text{or} \\
 & j_1 = i_1, \quad j_2 = 0, \quad j_3 = 0, \\
 & i_1 \in V_1^S, \quad i_2 = 2, \quad i_3 = 1, \\
 & \text{or} \\
 & j_1 = i_1 - 1, \quad j_2 = 1, \quad j_3 = i_3 - 1, \\
 & i_1 \in V_1^S, \quad i_2 = 2, \quad i_3 \in V_2^N, \\
\alpha_1, & j_1 = i_1, \quad j_2 = 3, \quad j_3 = i_3, \\
 & i_1 \in V_1^S, \quad i_2 = 1, \quad i_3 \in V_1^N, \\
\alpha_2, & j_1 = i_1, \quad j_2 = 4, \quad j_3 = i_3, \\
 & i_1 \in V_0^S, \quad i_2 = 2, \quad i_3 \in V_1^N, \\
\eta_1, & j_1 = i_1, \quad j_2 = 1, \quad j_3 = i_3, \\
 & i_1 \in V_1^S, \quad i_2 = 3, \quad i_3 \in V_1^N, \\
\eta_2, & j_1 = i_1, \quad j_2 = 2, \quad j_3 = i_3, \\
 & i_1 \in V_0^S, \quad i_2 = 4, \quad i_3 \in V_1^N, \\
-(r(N - i_3)\lambda \bar{\bar{\delta}}_{i_3 N} + \beta), & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 = 0, \quad i_2 = 0, \quad i_3 \in V_0^N, \\
-(r(N - i_3)\lambda \bar{\bar{\delta}}_{i_3 N} + \beta + \delta_{i_2 2}\mu_2 + & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\
\delta_{i_2 2}\alpha_2 + \delta_{i_2 4}\eta_2), & i_1 = 0, \quad i_2 = 2, 4, \quad i_3 \in V_1^N, \\
-(N\lambda + H(s - i_1)\beta + i_1\gamma), & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\
 & i_1 \in V_1^S, \quad i_2 = 0, \quad i_3 = 0, \\
-(r(N - i_3)\lambda \bar{\bar{\delta}}_{i_3 N} + H(s - i_1)\beta + & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\
\delta_{i_2 1}\mu_1 + \delta_{i_2 2}\mu_2 + \delta_{i_2 1}\alpha_1 + \delta_{i_2 2}\alpha_2 + & i_1 \in V_1^S, \quad i_2 = 1, \quad i_3 \in V_1^N, \\
\delta_{i_2 1}(i_1 - 1)\gamma + \delta_{i_2 2} i_1 \gamma), & \\
-(r(N - i_3)\lambda \bar{\bar{\delta}}_{i_3 N} + H(s - i_1)\beta + & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\
\delta_{i_2 3}\eta_1 + \delta_{i_2 4}\eta_2 + \delta_{i_2 4} i_1 \gamma + \delta_{i_2 3}(i_1 - 1)\gamma), & i_1 \in V_1^S, \quad i_2 = 3, 4, \quad i_3 \in V_1^N, \\
0, & \text{Otherwise.}
\end{cases}
$$

Define the following ordered sets:

$$\ll i_1, i_2 \gg \;=\; \begin{cases} < i_1, 0, i_3 >, & i_1 = 0;\ i_3 = 0, 1, 2, \ldots, N; \\ < i_1, 0, i_3 >, & i_1 = 1, 2, \ldots, S;\ i_3 = 0; \\ < i_1, i_2, i_3 >, & i_1 = 1, 2, \ldots, S;\ i_2 = 1, 2, 3, 4;\ i_3 = 1, 2, \ldots, N; \\ < i_1, i_2, i_3 >, & i_1 = 0;\ i_2 = 2, 4;\ i_3 = 1, 2, \ldots, N; \end{cases}$$

$$\lll i_1 \ggg \;=\; \begin{cases} \ll i_1, 0, \gg, & i_1 = 0, 1, 2, \ldots S; \\ \ll i_1, i_2 \gg, & i_1 = 1, 2, \ldots, S;\ i_2 = 1, 2, 3, 4; \\ \ll i_1, i_2 \gg, & i_1 = 0;\ i_2 = 2, 4; \end{cases}$$

By ordering the sets of state space as $(\lll 0 \ggg, \lll 1 \ggg, \lll 2 \ggg, \ldots, \lll S \ggg)$, the infinitesimal generator Θ can be conveniently expressed in a block partitioned matrix with entries

$$\Theta_{i_1 j_1} = \begin{cases} Z_{i_1} & j_1 = i_1,\ \ i_1 = 0, 1, 2, \ldots, S, \\ W_{i_1} & j_1 = i_1 - 1,\ i_1 = 1, 2, \ldots, S-1, S, \\ C & j_1 = i_1 + Q,\ i_1 = 1, 2, \ldots, s, \\ C_1 & j_1 = i_1 + Q,\ i_1 = 0, \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

More explicitly,

$$\Theta = \begin{array}{c} S \\ S-1 \\ \vdots \\ s+1 \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array} \left(\begin{array}{cccccccccc} Z_S & W_S & & & & & & & & \\ & Z_{S-1} & W_{S-1} & & & & & & & \\ & & \cdots & & & & & & & \\ & & & \cdots & Z_{s+1} & W_{s+1} & & & & \\ C & & & & & Z_s & W_s & & & \\ & C & & & & & Z_{s-1} & & & \\ & & \cdots & & & & \cdots & & & \\ & & C & & & & & \cdots & Z_1 & W_1 \\ & & & C_1 & & & & & & Z_0 \end{array} \right)$$

where

$$[C_1]_{i_2 j_2} \;=\; \begin{cases} C_0 & j_2 = i_2, & i_2 = 0, \\ C_2 & j_2 = 1, & i_2 = 0, \\ C_3 & j_2 = i_2, & i_2 = 2, 4, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_0]_{i_3 j_3} \;=\; \begin{cases} \beta & j_3 = 0, \quad i_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$[C_2]_{i_3 j_3} \;=\; \begin{cases} \beta & j_3 = i_3, \quad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[C]_{i_2 j_2} \;=\; \begin{cases} C_4 & j_2 = i_2, & i_2 = 0, \\ C_3 & j_2 = i_2, & i_2 = 1, 2, 3, 4, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_4]_{i_3 j_3} \;=\; \begin{cases} \beta & j_3 = 0, \quad i_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$[W_1]_{i_2 j_2} = \begin{cases} E_0 & j_2 = 2, \quad i_2 = 1, \\ D_0 & j_2 = 0, \quad i_2 = 1, \\ F_1 & j_2 = 2, \quad i_2 = 2, 4, \\ B_{00}^{(1)} & j_2 = i_2, \quad i_2 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_{00}{}^{(1)}]_{i_3 j_3} = \begin{cases} \gamma & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[E_0]_{i_3 j_3} = \begin{cases} p\mu_1, & j_3 = i_3, \quad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[D_0]_{i_3 j_3} = \begin{cases} q\mu_1, & j_3 = i_3 - 1, \quad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[F_1]_{i_3 j_3} = \begin{cases} \gamma, & j_3 = i_3, \quad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

For $i_1 = 2, 3, \ldots, S$

$$[W_{i_1}]_{i_2 j_2} = \begin{cases} E_0 & j_2 = 2, \quad i_2 = 1, \\ F_0 & j_2 = 0, \quad i_2 = 1, \\ E_{i_1} & j_2 = 0, \quad i_2 = 0 \\ D_{(i_1-1)} & j_2 = 1, \quad i_2 = 1 \\ F_{i_1} & j_2 = i_2, \quad i_2 = 2, 4 \\ G_{(i_1-1)} & j_2 = i_2, \quad i_2 = 3 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[F_0]_{i_3 j_3} = \begin{cases} q\mu_1, & j_3 = 0, \quad i_3 = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$[E_{i_1}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, \quad i_3 = 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$[D_{(i_1-1)}]_{i_3 j_3} = \begin{cases} (i_1 - 1)\gamma, & j_3 = i_3, \quad i_3 \in V_1^N \\ q\mu_1, & j_3 = i_3 - 1, \quad i_3 \in V_2^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[F_{i_1}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, \quad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[Z_0]_{i_2 j_2} = \begin{cases} V_0 & j_2 = i_2, \quad i_2 = 0, \\ V_1 & j_2 = i_2, \quad i_2 = 2, \\ V_2 & j_2 = 4, \quad i_2 = 4, \\ R & j_2 = 2, \quad i_2 = 4 \\ T_2 & j_2 = 4, \quad i_2 = 2 \\ U & j_2 = 0, \quad i_2 = 2 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[U]_{i_3j_3} = \begin{cases} \mu_3, & j_3 = i_3 - 1, \quad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[T_2]_{i_3j_3} = \begin{cases} \alpha_2, & j_3 = i_3, \qquad i_3 \in V_1^N \\ 0, & \text{otherwise.} \end{cases}$$

$$[V_0]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-1} \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta) & j_3 = i_3, \qquad i_3 \in V_0^N, \\ 0, & \text{otherwise.} \end{cases}$$

$$[V_1]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1} \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta + \mu_2 + \alpha_2) & j_3 = i_3, \qquad i_3 \in V_1^N, \\ 0, & \text{otherwise.} \end{cases}$$

$$[V_2]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1} \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta + \eta_2) & j_3 = i_3, \qquad i_3 \in V_1^N, \\ 0, & \text{otherwise.} \end{cases}$$

$$[H]_{i_3j_3} = \begin{cases} N\lambda & j_3 = 1, \qquad i_3 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[H_{i_1}]_{i_3j_3} = \begin{cases} -(N\lambda + i_1\gamma + H(s-i_1)\beta) & j_3 = i_3, \qquad i_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$[J]_{i_3j_3} = \begin{cases} \mu_2 & j_3 = 0, \qquad i_3 = 1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[J_0]_{i_3j_3} = \begin{cases} \mu_2 & j_3 = i_3 - 1, \quad i_3 \in V_2^N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $i_1 = 1, 2, 3, \ldots, S$

$$[Z_{i_1}]_{i_2j_2} = \begin{cases} H & j_2 = 1, \qquad i_2 = 0, \\ H_{i_1} & j_2 = i_2, \qquad i_2 = 0, \\ J & j_2 = 0, \qquad i_2 = 2, \\ J_0 & j_2 = 1, \qquad i_2 = 2, \\ J_{i_1} & j_2 = i_2, \qquad i_2 = 1, \\ K_{i_1} & j_2 = 2, \qquad i_2 = 2, \\ L_0 & j_2 = 1, \qquad i_2 = 3, \\ L_{i_1} & j_2 = i_2, \qquad i_2 = 3, \\ R & j_2 = 2, \qquad i_2 = 4, \\ R_{i_1} & j_2 = 2, \qquad i_2 = 4, \\ T_1 & j_2 = 3, \qquad i_2 = 1, \\ T_2 & j_2 = 4, \qquad i_2 = 2, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[J_{i_1}]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \mu_1 + H(s-i_1)\beta + (i_1-1)\gamma + \alpha_1) & j_3 = i_3, \qquad i_3 \in V_1^N, \\ 0, & \text{otherwise.} \end{cases}$$

$$[K_{i_1}]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \mu_2 + H(s-i_1)\beta + i_1\gamma + \alpha_2) & j_3 = i_3, \qquad i_3 \in V_1^N, \\ 0, & \text{otherwise.} \end{cases}$$

$$[L_0]_{i_3j_3} = \begin{cases} \eta_1 & j_3 = i_3, \qquad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[R]_{i_3j_3} = \begin{cases} \eta_2 & j_3 = i_3, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[T_1]_{i_3j_3} = \begin{cases} \alpha_1 & j_3 = i_3, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[L_{i_1}]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda & j_3 = i_3+1, \quad i_3 \in V_1^{N-1}, \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \eta_1 + H(s-i_1)\beta + (i_1-1)\gamma) & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & \text{otherwise.} \end{cases}$$

$$[R_{i_1}]_{i_3j_3} = \begin{cases} r(N-i_3)\lambda & j_3 = i_3+1, \quad i_3 \in V_1^{N-1}, \\ -(r(N-i_3)\lambda\bar{\delta}_{i_3N} + \eta_2 + H(s-i_1)\beta + i_1\gamma) & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & \text{otherwise.} \end{cases}$$

It can be noted that Z_{i_1}, W_{i_1}, $i_1 = 2, \ldots, S$, Z_1, and C are square matrices of order $(4N+1)$. C_1 is of size $(3N+1) \times (4N+1)$, Z_0 is a square matrices of order $(3N+1)$, W_1 is of size $(4N+1) \times (3N+1)$. The sub matrices C_3, E_0, F_1, T_1, T_2, R, J_0, L_0, R_1, K_1, J_1, L_1, $D_{(i_1-1)}$, F_{i_1}, $G_{(i_1-1)}$, R_{i_1}, K_{i_1}, J_{i_1}, L_{i_1}, $i_1 = 2, 3, \ldots, S$, are square matrices of order N. C_4, H_1, E_{i_1}, H_{i_1} $i_1 = 2, 3, \ldots, S$, are square matrices of order 1. C_0 and J are matrices of order $(N+1) \times 1$. D_0, C_2, $B_{00}^{(1)}$, F_0 and H are matrices of order $N \times (N+1)$, $(N+1) \times N$, $1 \times (N+1)$, $N \times 1$ and $1 \times N$ respectively.

3.1 Steady state analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), Y(t), X(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[L(t) = i_1, Y(t) = i_2, X(t) = i_3 | L(0), Y(0), X(0)] \text{ exists.}$$

Let $\mathbf{\Phi} = (\mathbf{\Phi}^{(0)}, \mathbf{\Phi}^{(1)}, \ldots, \mathbf{\Phi}^{(S)})$ be the steady state probability vector of the Markov chain I. Each vector $\mathbf{\Phi}^{(i_1)}$ being partitioned as follows

$$\begin{aligned} \mathbf{\Phi}^{(0)} &= (\mathbf{\Phi}^{(0,0)}, \mathbf{\Phi}^{(0,2)}, \mathbf{\Phi}^{(0,4)}), \\ \mathbf{\Phi}^{(i_1)} &= (\mathbf{\Phi}^{(i_1,0)}, \mathbf{\Phi}^{(i_1,1)}, \mathbf{\Phi}^{(i_1,2)}, \mathbf{\Phi}^{(i_1,3)}, \mathbf{\Phi}^{(i_1,4)}), \quad i_1 = 1, 2, 3, \ldots, S; \end{aligned}$$

where

$$\begin{aligned} \mathbf{\Phi}^{(0,0)} &= (\phi^{(0,0,0)}, \phi^{(0,0,1)}, \ldots, \phi^{(0,0,N)}), \\ \mathbf{\Phi}^{(0,i_2)} &= (\phi^{(0,i_2,0)}, \phi^{(0,i_2,1)}, \ldots, \phi^{(0,i_2,N)}), i_2 = 2, 4; \\ \mathbf{\Phi}^{(i_1,0)} &= (\phi^{(i_1,0,0)}), \quad i_1 = 1, 2, 3, \ldots, S; \\ \mathbf{\Phi}^{(i_1,i_2)} &= (\phi^{(i_1,i_2,1)}, \phi^{(i_1,i_2,2)}, \ldots, \phi^{(i_1,i_2,N)}), \ i_1 = 1, 2, 3, \ldots, S; \ i_2 = 1, 2, 3, 4; \end{aligned}$$

The computation of steady state probability vector $\mathbf{\Phi} = (\mathbf{\Phi}^{(0)}, \mathbf{\Phi}^{(1)}, \ldots, \mathbf{\Phi}^{(S)})$, by solving the following set of equations,

$$\begin{aligned} \mathbf{\Phi}^{i_1} W_{i_1} + \mathbf{\Phi}^{i_1-1} Z_{i_1-1} &= \mathbf{0}, \quad i_1 = 1, 2, \ldots, Q, \\ \mathbf{\Phi}^{i_1} W_{i_1} + \mathbf{\Phi}^{i_1-1} Z_{i_1-1} + \mathbf{\Phi}^{(i_1-1-Q)} C_1 &= \mathbf{0}, \quad i_1 = Q+1, \\ \mathbf{\Phi}^{i_1} W_{i_1} + \mathbf{\Phi}^{i_1-1} Z_{i_1-1} + \mathbf{\Phi}^{(i_1-1-Q)} C &= \mathbf{0}, \quad i_1 = Q+2, Q+3, \ldots, S, \\ \mathbf{\Phi}^{S} Z_S + \mathbf{\Phi}^{s} C &= \mathbf{0}. \end{aligned}$$

subject to conditions $\mathbf{\Phi}\Theta = \mathbf{0}$ *and* $\sum\sum\sum_{(i_1,i_2,i_3)} \phi^{(i_1,i_2,i_3)} = 1.$

This is done by the following algorithm.

Step 1. Solve the following system of equations to find the value of $\mathbf{\Phi}^Q$

$$\mathbf{\Phi}^Q \left[\left\{ (-1)^Q \sum_{j=0}^{s-1} \left[\left(\Omega_{k=Q}^{s+1-j} W_k Z_{k-1}^{-1} \right) C Z_{S-j}^{-1} \left(\Omega_{l=S-j}^{Q+2} W_l Z_{l-1}^{-1} \right) \right] \right\} W_{Q+1} + Z_Q + \left\{ (-1)^Q \Omega_{j=Q}^{1} W_j Z_{j-1}^{-1} \right\} C \right] = \mathbf{0},$$

and

$$\mathbf{\Phi}^Q \left[\sum_{i_1=0}^{Q-1} \left((-1)^{Q-i_1} \Omega_{j=Q}^{i_1+1} W_j Z_{j-1}^{-1} \right) + I + \sum_{i_1=Q+1}^{S} \left((-1)^{2Q-i_1+1} \sum_{j=0}^{S-i_1} \left[\left(\Omega_{k=Q}^{s+1-j} W_k Z_{k-1}^{-1} \right) C Z_{S-j}^{-1} \left(\Omega_{l=S-j}^{i_1+1} W_l Z_{l-1}^{-1} \right) \right] \right) \right] \boldsymbol{\pi} = 1.$$

Step 2. Compute the values of

$$\begin{aligned} \mathbf{\Omega}_{i_1} &= (-1)^{Q-i_1} \mathbf{\Phi}^Q \Omega_{j=Q}^{i_1+1} W_j Z_{j-1}^{-1}, && i_1 = Q-1, Q-2, \ldots, 0 \\ &= (-1)^{2Q-i_1+1} \mathbf{\Phi}^Q \sum_{j=0}^{S-i_1} \left[\left(\Omega_{k=Q}^{s+1-j} W_k Z_{k-1}^{-1} \right) C Z_{S-j}^{-1} \left(\Omega_{l=S-j}^{i_1+1} W_l Z_{l-1}^{-1} \right) \right], \\ & && i_1 = S, S-1, \ldots, Q+1 \\ &= I, && i_1 = Q \end{aligned}$$

Step 3. Using, Step 1 and Step 2, calculate the value of $\mathbf{\Phi}^{(i_1)}, i_1 = 0, 1, \ldots, S$. That is,

$$\mathbf{\Phi}^{(i_1)} = \mathbf{\Phi}^{(Q)} \mathbf{\Omega}_{i_1}, \quad i_1 = 0, 1, \ldots, S.$$

3.2 Waiting time analysis

In this section, the aim is to derive the waiting time for the customer. The specific as the time between the arrival times of the customer and immediate upon which he gets service. We will symbolize this continuous time random variable as W. The aim is to derive the probability distribution of W and to derive n^{th} order moments of W. Note that W is zero when the is in the state $(i_1, 0, 0), i_1 \in V_1^S$. Consequently, the probability that the customer does not have to wait is given by

$$P\{W = 0\} = \sum_{i_1=1}^{S} \phi^{(i_1,0,0)}.$$

To obtain the distribution of W, some auxiliary variables are defined. Let us consider the Markov process at an arbitrary time t and assume that the system in the state $(i_1,i_2,i_3), i_3 > 0$. We tag any of those waiting customer and $W_{(i_1,i_2,i_3)}$ denotes the time until the selected customer gets the desired item. Let $W^*(y) = E[e^{-yW}]$ and $W^*_{(i_1,i_2,i_3)}(y) = E[e^{-yW_{(i_1,i_2,i_3)}}]$ respectively, denote the unconditional and conditional waiting time. Then, we have

$$\begin{aligned} W^*(y) &= \sum_{i_1=1}^{S}\phi^{(i_1,0,0)} + \sum_{i_3=0}^{N-1}\phi^{(0,0,i_3)}W^*_{(0,0,i_3+1)}(y) \\ &+ \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1}(\phi^{(i_1,1,i_3)}W^*_{(i_1,1,i_3+1)}(y) + \phi^{(i_1,3,i_3)}W^*_{(i_1,3,i_3+1)}(y)) \\ &+ \sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1}(\phi^{(i_1,2,i_3)}W^*_{(i_1,2,i_3+1)}(y) + \phi^{(i_1,4,i_3)}W^*_{(i_1,4,i_3+1)}(y)). \end{aligned} \tag{1}$$

To derive $W^*_{(i_1,i_2,i_3)}$, we introduce an auxiliary Markov chain on the state space $E^* = E_a \cup E_c \cup E_d \cup \{*\}$, where $\{*\}$ represents an absorbing state. The chain is on a state (i_1,i_2,i_3), we apply a first-step argument in the auxiliary chain to resolve $W^*_{(i_1,i_2,i_3)}(y)$. Then (see [16], Theorem 6.21) the functions $W^*_{(i_1,i_2,i_3)}(y)$, $(i_1,i_2,i_3) \in E$ are the smallest non-negative solution to the system

For $i_1 = 0, \quad i_2 = 0, \quad 1 \le i_3 \le N$,

$$w_1 W^*_{(0,0,i_3)}(y) - r(N-i_3)\lambda\bar{\delta}_{i_3N}W^*_{(0,0,i_3+1)}(y) - \beta\delta_{i_30}W^*_{(Q,0,0)}(y) - \beta\bar{\delta}_{i_30}W^*_{(Q,1,i_3)}(y) = 0 \tag{2}$$

where

$$w_1 = y + r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta\delta_{i_30} + \beta\bar{\delta}_{i_30}$$

For $1 \le i_1 \le S \quad i_2 = 1,3, \quad 1 \le i_3 \le N$

$$\begin{aligned} &w_2 W^*_{(i_1,i_2,i_3)}(y) - r(N-i_3)\lambda\bar{\delta}_{i_3N}W^*_{(i_1,i_2,i_3+1)}(y) - (i_1-1)\gamma\beta\delta_{i_11}W^*_{(i_1-1,i_2,i_3)}(y) \\ &-\beta H(s-i_1)W^*_{(i_1+Q,i_2,i_3)}(y) - p\mu_1\delta_{i_11}\delta_{i_21}W^*_{(i_1-1,2,i_3-1)}(y) - q\mu_1\delta_{i_11}\delta_{i_21}W^*_{(i_1-1,0,i_3-1)}(y) \\ &-p\mu_1\bar{\delta}_{i_11}\bar{\delta}_{i_21}\delta_{i_31}W^*_{(i_1-1,2,1)}(y) - q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\delta_{i_31}W^*_{(i_1-1,0,0)}(y) - p\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31}W^*_{(i_1-1,i_2,i_3)}(y) \\ &-q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31}W^*_{(i_1-1,i_2,i_3-1)}(y) - \alpha_1\delta_{i_21}W^*_{(i_1,3,i_3)}(y) - \eta_1\delta_{i_23}W^*_{(i_1,1,i_3)}(y) = 0 \end{aligned} \tag{3}$$

where

$$\begin{aligned} w_2 = \; & y + r(N-i_3)\lambda\bar{\delta}_{i_3N} + (i_1-1)\gamma\beta\delta_{i_11} + \beta H(s-i_1) + p\mu_1\delta_{i_11}\delta_{i_21} + q\mu_1\delta_{i_11}\delta_{i_21} \\ &+p\mu_1\bar{\delta}_{i_11}\bar{\delta}_{i_21}\delta_{i_31} + q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\delta_{i_31} + p\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31} + q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31} + \alpha_1\delta_{i_21} + \eta_1\delta_{i_23} \end{aligned}$$

For $0 \le i_1 \le S, \quad i_2 = 2,4, \quad 1 \le i_3 \le N$,

$$\begin{aligned} &w_3 W^*_{(i_1,i_2,i_3)}(y) - r(N-i_3)\lambda\bar{\delta}_{i_3N}W^*_{(i_1,i_2,i_3+1)}(y) - \beta H(s-i_1)W^*_{(i_1+Q,i_2,i_3)}(y) \\ &-i_1\gamma W^*_{(i_1-1,i_2,i_3)}(y) - \mu_2\delta_{i_10}\delta_{i_22}W^*_{(i_1,0,i_3-1)}(y) - \mu_2\bar{\delta}_{i_10}\delta_{i_22}\delta_{i_31}W^*_{(i_1,0,0)}(y) \\ &-\mu_2\bar{\delta}_{i_10}\delta_{i_22}\bar{\delta}_{i_31}W^*_{(i_1,1,i_3-1)}(y) - \alpha_2\delta_{i_22}W^*_{(i_1,4,i_3)}(y) - \eta_2\delta_{i_24}W^*_{(i_1,2,i_3)}(y) = 0 \end{aligned} \tag{4}$$

where

$$\begin{aligned} w_3 = \; & y + r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta H(s-i_1) + i_1\gamma + \mu_2\delta_{i_10}\delta_{i_22} \\ &+\mu_2\bar{\delta}_{i_10}\delta_{i_22}\delta_{i_31} + \mu_2\bar{\delta}_{i_10}\delta_{i_22}\bar{\delta}_{i_31} + \alpha_2\delta_{i_22} + \eta_2\delta_{i_24}. \end{aligned}$$

Using the linear equations (2)–(4), we can compute the values of $W^*(y)$ for a given y and also we can utilize the system of linear equations to obtain a recursive algorithm for calculating moments for the waiting times. By differentiating (n+1) times (2)–(4) the system of linear equations, and evaluating at $y = 0$, we arrive at
For $i_1 = 0, \quad i_2 = 0, \quad 1 \leq i_3 \leq N$

$$\begin{aligned} w_4 E\left[W^{(n+1)}_{(0,0,i_3)}\right] - r(N-i_3)\lambda\bar{\delta}_{i_3N} E\left[W^{(n+1)}_{(0,0,i_3+1)}\right] - \beta\delta_{i_30} E\left[W^{(n+1)}_{(Q,0,0)}\right] \\ -\beta\bar{\delta}_{i_30} E\left[W^{(n+1)}_{(Q,1,i_3)}\right] = (n+1)E\left[W^{(n)}_{(0,0,i_3)}\right] \end{aligned} \tag{5}$$

where

$$w_4 = r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta\delta_{i_30} + \beta\bar{\delta}_{i_30}$$

For $1 \leq i_1 \leq S \quad i_2 = 1, 3, \quad 1 \leq i_3 \leq N$

$$\begin{aligned} w_5 E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] - r(N-i_3)\lambda\bar{\delta}_{i_3N} E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] \\ -(i_1-1)\gamma\bar{\delta}_{i_11} E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] - \beta H(s-i_1) E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] \\ -p\mu_1\delta_{i_11}\delta_{i_21} E\left[W^{(n+1)}_{(i_1-1,2,i_3-1)}\right] - q\mu_1\delta_{i_11}\delta_{i_21} E\left[W^{(n+1)}_{(i_1-1,0,i_3-1)}\right] \\ -p\mu_1\bar{\delta}_{i_11}\delta_{i_21}\delta_{i_31} E\left[W^{(n+1)}_{(i_1-1,2,1)}\right] - q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\delta_{i_31} E\left[W^{(n+1)}_{(i_1-1,0,0)}\right] \\ -p\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31} E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] - q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31} E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] \\ -\alpha_1\delta_{i_21} E\left[W^{(n+1)}_{(i_1,3,i_3)}\right] - \eta_1\delta_{i_23} E\left[W^{(n+1)}_{(i_1,1,i_3)}\right] \\ = (n+1)E\left[{}^{(1)}W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{6}$$

where

$$\begin{aligned} w_5 = {} & r(N-i_3)\lambda\bar{\delta}_{i_3N} + (i_1-1)\gamma\bar{\delta}_{i_11} + \beta H(s-i_1) + p\mu_1\delta_{i_11}\delta_{i_21} + q\mu_1\delta_{i_11}\delta_{i_21} \\ & +p\mu_1\bar{\delta}_{i_11}\bar{\delta}_{i_21}\delta_{i_31} + q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\delta_{i_31} + p\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31} + q\mu_1\bar{\delta}_{i_11}\delta_{i_21}\bar{\delta}_{i_31} + \alpha_1\delta_{i_21} + \eta_1\delta_{i_23} \end{aligned}$$

For $0 \leq i_1 \leq S, \quad i_2 = 2, 4, \quad 1 \leq i_3 \leq N,$

$$\begin{aligned} w_6 E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] - r(N-i_3)\lambda\bar{\delta}_{i_3N} E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - \beta H(s-i_1) E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] \\ -i_1\gamma E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] - \mu_2\delta_{i_10}\delta_{i_22} E\left[W^{(n+1)}_{(i_1,0,i_3-1)}\right] - \mu_2\bar{\delta}_{i_10}\delta_{i_22}\delta_{i_31} E\left[W^{(n+1)}_{(i_1,0,0)}\right] \\ -\mu_2\bar{\delta}_{i_10}\delta_{i_22}\bar{\delta}_{i_31} E\left[W^{(n+1)}_{(i_1,1,i_3-1)}\right] - \alpha_2\delta_{i_22} E\left[W^{(n+1)}_{(i_1,4,i_3)}\right] - \eta_2\delta_{i_24} E\left[W^{(n+1)}_{(i_1,2,i_3)}\right] \\ = (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{7}$$

where

$$\begin{aligned} w_6 = {} & r(N-i_3)\lambda\bar{\delta}_{i_3N} + \beta H(s-i_1) + i_1\gamma + \mu_2\delta_{i_10}\delta_{i_22} \\ & +\mu_2\bar{\delta}_{i_10}\delta_{i_22}\delta_{i_31} + \mu_2\bar{\delta}_{i_10}\delta_{i_22}\bar{\delta}_{i_31} + \alpha_2\delta_{i_22} + \eta_2\delta_{i_24} \end{aligned}$$

Equations (5)–(7) are used to determine the unknowns $E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right]$, $(i_1, i_2, i_3) \in E$ in terms of the moments of one order less. Noticing that $E\left[W^{(n)}_{(i_1,i_2,i_3,)}\right] = 1$, for $n = 0$, we can obtain the moments up to a desired order in a recursive way.

For determine the moments of W we differentiate $W^*(y)$ and evaluate at $y = 0$, we have

$$E[W^{(n)}] = \delta_{0n} + \left[\sum_{1_3=0}^{N-1} \phi^{(0,0,i_3)} E\left[W^{(n)}_{(0,0,i_3+1)}\right] + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1} (\phi^{(i_1,1,i_3)} E\left[W^{(n)}_{(i_1,1,i_3+1)}\right]\right.$$
$$+\phi^{(i_1,3,i_3)} E\left[W^{(n)}_{(i_1,3,i_3+1)}\right]) + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1} (\phi^{(i_1,2,i_3)} E\left[W^{(n)}_{(i_1,2,i_3+1)}\right]$$
$$\left. +\phi^{(i_1,4,i_3)} E\left[W^{(n)}_{(i_1,4,i_3+1)}\right])\right] (1 - \delta_{0n})$$

which provides the n^{th} moments of the unconditional waiting time in terms of conditional moments of the same order.

4 System performance measures

In this section, some measures of system performance in the steady state are derived. Using this, the total expected cost rate is derived.

4.1 Expected inventory level

Let η_I denote the excepted inventory level in the steady state, then

$$\eta_I = \sum_{i_1=1}^{S} i_1\phi^{(i_1,0,0)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{4}\sum_{i_2=1}^{N} i_1\phi^{(i_1,i_2,i_3)}.$$

4.2 Expected reorder rate

Let η_R denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s + 1$ to s. This may occur in the following three cases:

- The server completes a first essential service for the customer.
- Any one of the s items fails when the server is busy/interruption during FES.
- Any one of the $(s + 1)$ items fails when the server is idle/busy/interruption during SOS.

Hence, we get

$$\eta_R = \sum_{i_3=1}^{N} \mu_1\phi^{(s+1,1,i_3)} + (s + 1)\gamma\phi^{(s+1,0,0)}$$
$$+\sum_{i_3=1}^{N} (s + 1)\gamma(\phi^{(s+1,2,i_3)} + \phi^{(s+1,4,i_3)}) + \sum_{i_3=1}^{N} s\gamma(\phi^{(s+1,1,i_3)} + \phi^{(s+1,3,i_3)}).$$

4.3 Expected perishable rate

Let η_P denote the expected perishable rate in the steady state, then

$$\begin{aligned}\eta_P &= \sum_{i_1=1}^{S} i_1\gamma\phi^{(s+1,0,0)} + \sum_{i_1=2}^{S}\sum_{i_3=1}^{N}(i_1-1)\gamma(\phi^{(i_1,1,i_3)}+\phi^{(i_1,3,i_3)}) \\ &\quad + \sum_{i_3=1}^{N}\sum_{i_1=1}^{S} i_1\gamma(\phi^{(i_1,2,i_3)}+\phi^{(i_1,4,i_3)})\end{aligned}$$

4.4 Expected number of customers in the waiting area

Let Γ_1 denote the expected number of customers in the steady state, then

$$\begin{aligned}\Gamma_1 &= \sum_{i_3=1}^{N} i_3\phi^{(0,0,i_3)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} i_3(\phi^{(i_1,1,i_3)}+\phi^{(i_1,3,i_3)}) \\ &\quad + \sum_{i_3=1}^{N}\sum_{i_1=0}^{S} i_3(\phi^{(i_1,2,i_3)}+\phi^{(i_1,4,i_3)}).\end{aligned}$$

4.5 Expected waiting time

Let η_W denote the expected waiting time of the customers in the waiting area. Then by Little's formula

$$\eta_W = \frac{\Gamma_1}{\Gamma_2},$$

where Γ_1 is the expected number of customers in the waiting area and the effective arrival rate of the customer [17], Γ_2 is given by

$$\begin{aligned}\Gamma_2 &= \sum_{i_3=0}^{N-1} r(N-i_3)\lambda\phi^{(0,0,i_3)} + \sum_{i_1=1}^{S}(N-i_3)\lambda\phi^{(i_1,0,0)} \\ &\quad + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1} r(N-i_3)\lambda(\phi^{(i_1,1,i_3)}+\phi^{(i_1,3,i_3)}) \\ &\quad + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1} r(N-i_3)\lambda(\phi^{(i_1,2,i_3)}+\phi^{(i_1,4,i_3)}).\end{aligned}$$

4.6 Expected loss rate for customers

Let η_L denote the expected loss rate for the customers in the steady state, then

$$\begin{aligned}\eta_L &= \sum_{i_3=0}^{N-1}(1-r)(N-i_3)\lambda\phi^{(0,0,i_3)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1}(1-r)(N-i_3)\lambda(\phi^{(i_1,1,i_3)}+\phi^{(i_1,3,i_3)}) \\ &\quad + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1}(1-r)(N-i_3)\lambda(\phi^{(i_1,2,i_3)}+\phi^{(i_1,4,i_3)}).\end{aligned}$$

4.7 Effective interruption rate

Let η_{INTR} denote the effective interruption rate which is given by

$$\eta_{INTR} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} \alpha_1 \phi^{(i_1,1,i_3)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} \alpha_2 \phi^{(i_1,2,i_3)}.$$

4.8 Effective repair rate

Let η_{RR} denote the effective repair rate which is given by

$$\eta_{RR} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} \eta_1 \phi^{(i_1,3,i_3)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} \eta_2 \phi^{(i_1,4,i_3)}.$$

4.9 Probability that server is idle

Let η_{PI} denote the probability that server is idle is given by

$$\eta_{PI} = \sum_{i_3=0}^{N} \phi^{(0,0,i_3)} + \sum_{i_1=1}^{S} \phi^{(i_1,0,0)}.$$

4.10 Probability that server is working

Let η_{PW} denote the probability that server is working is given by

$$\eta_{PW} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} (\phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)}) + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} (\phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)}).$$

4.11 Probability that server is on FES

Let η_{PFES} denote the probability that server is providing FES is given by

$$\eta_{PFES} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} \phi^{(i_1,1,i_3)}.$$

4.12 Probability that server is on SOS

Let η_{SOS} denote the probability that server is providing SOS is given by

$$\eta_{PSOS} = \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} \phi^{(i_1,2,i_3)}.$$

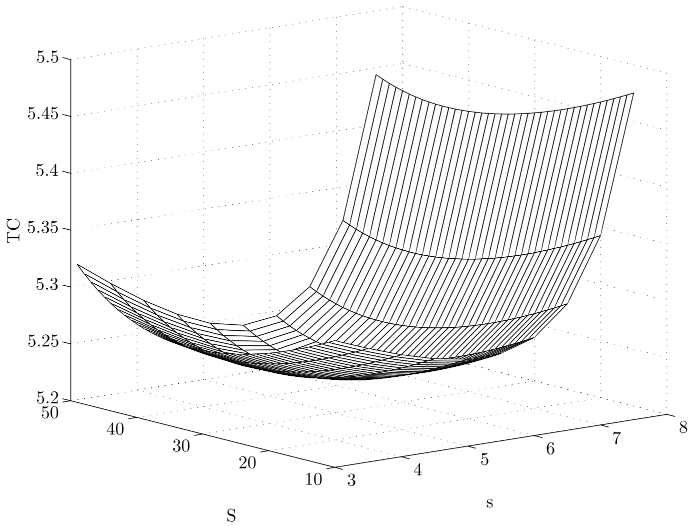


Figure 1: *A three dimensional plot of the cost function $TC(s, S)$*

5 Cost analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(S, s, N) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_w\eta_W + c_l\eta_L + c_i\eta_{INTR} + c_r\eta_{RR}$$

where

c_h : The inventory carrying cost per unit item per unit time
c_s : Setup cost per order
c_p : Perishable cost per unit item per unit time
c_w : Waiting cost of a customer per unit time
c_l : Cost per customer lost
c_i : Cost per interruption per unit time
c_r : Cost per repair per unit time

Substituting the values of η's, we get TC(S, s, N)=

$$c_s \sum_{i_3=1}^{N} \mu_1 \phi^{(s+1,1,i_3)} + c_s(s+1)\gamma\phi^{(s+1,0,0)} + c_s \sum_{i_3=1}^{N} (s+1)\gamma(\phi^{(s+1,2,i_3)} + \phi^{(s+1,4,i_3)})$$
$$+c_s \sum_{i_3=1}^{N} s\gamma(\phi^{(s+1,1,i_3)} + \phi^{(s+1,3,i_3)}) + c_h \sum_{i_1=1}^{S} i_1\phi^{(i_1)}\boldsymbol{e} + c_p \sum_{i_1=1}^{S} i_1\gamma\phi^{(s+1,0,0)}$$

$$+c_p\sum_{i_1=2}^{S}\sum_{i_3=1}^{N}(i_1-1)\gamma(\phi^{(i_1,1,i_3)}+\phi^{(i_1,3,i_3)})+c_p\sum_{i_3=1}^{N}\sum_{i_1=1}^{S}i_1\gamma(\phi^{(i_1,2,i_3)}+\phi^{(i_1,4,i_3)})$$
$$+c_w\frac{\Gamma_1}{\Gamma_2}+c_l\sum_{i_3=0}^{N-1}(1-r)(N-i_3)\lambda\phi^{(0,0,i_3)}+c_l\sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1}(1-r)(N-i-3)\lambda(\phi^{(i_1,1,i_3)}$$
$$+\phi^{(i_1,3,i_3)})+c_l\sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1}(1-r)(N-i-3)\lambda(\phi^{(i_1,2,i_3)}+\phi^{(i_1,4,i_3)})$$

$$+c_i\sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\alpha_1\phi^{(i_1,1,i_3)}+c_i\sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\alpha_2\phi^{(i_1,2,i_3)}+c_r\sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\eta_1\phi^{(i_1,3,i_3)}$$
$$+c_r\sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\eta_2\phi^{(i_1,4,i_3)}$$

6 Numerical illustrations

In this section, some numerical examples that reveal the possible convexity of the total expected cost rate are discussed. A typical 3-dimensional plot of $TC(S,s)$ is presented in Figure 1. The numerical search procedure is employed to obtain the optimal values of S, s and TC (say S^*, s^* and TC^*). the effect of varying the cost and other system parameters on the optimal values and the results agreed with what one would expect, have been studied. Some of the results are presented in Tables 4 through 11 where the lower entry in each cell gives the optimal expected cost rate and the upper entries the corresponding S^* and s^*.

Example 1 *First, the behaviour of the cost function is explored by considering as the function of two variables by fixing the others at a constant level. Tables* $1-3$, *give the total expected cost rate as a function of* $TC(S,s,10)$, $TC(50,s,N)$ *and* $TC(S,7,N)$. *All the costs and other parameters are assigned fixed values which are indicated in each Table. The value that is shown* **bold** *is the least among the values in that row and the value that is shown* <u>underlined</u> *is the least in that column. It may be observed that, these values in each Table exhibit a (possibly) local minimum of the function of the two variables. Also it may be observed that, the total expected cost rate function* $TC(S,s,N)$ *is more sensitive to changes in* N *than to changes in* S *and* s.

Example 2 *In this example, the impact of the setup cost* c_s, *holding cost* c_h, *waiting cost* c_w, *shortage cost* c_l, *perishable cost* c_p, *interruption cost* c_i *and repair cost* c_r *on the optimal values (possibly local)* S^*, s^* *and* TC^* *is studied. Towards this end, the parameter values as* $\lambda=0.7$, $\beta=0.1$, $\gamma=1$, $\alpha_1=0.3$, $\alpha_2=0.14$, $\mu_1=0.8$, $\mu_2=0.9$, $\eta_1=9$, $\eta_2=7$, $p=0.7$, $q=0.5$, $r=0.4$, $N=10$ *are first fixed. The following from Table* 4 *to* 11 *are observed:*

s S	4	5	6	7	8	9
46	5.311090	5.255928	5.232628	**5.226150**	5.229322	5.238366
47	5.310921	5.255662	5.232194	**5.225500**	5.228419	5.237180
48	5.310959	5.255614	5.231989	**5.22509**3	5.227775	5.236270
49	5.311192	5.255771	5.232003	**5.224918**	5.227378	5.235623
50	5.311612	5.256124	5.232223	**5.224962**	5.227214	5.235224
51	5.312209	5.256663	5.232640	**5.225214**	5.227270	5.235059
52	5.312976	5.257380	5.233244	**5.225664**	5.227536	5.235117

Table 1: *Total expected cost rate as a function of S and s*
$\lambda = 0.7$, $\beta = 0.1$, $\gamma = 1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.14$, $\mu_1 = 0.8$, $\mu_2 = 9$, $\eta_1 = 9$, $\eta_2 = 0.7$ $p = 0.7$, $q = 0.3$, $r = 0.5$, $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $C_r = 0.4$.

N s	9	10	11	12	13	14
9	1.955756	0.936119	**0.684408**	2.182527	3.843321	4.744828
10	1.820187	0.767302	**0.526258**	2.079604	3.792323	4.726229
11	1.710271	0.647613	**0.441452**	2.033204	3.768393	4.719874
12	1.622339	0.570300	**0.418326**	2.033742	3.767116	4.723682
13	1.553450	0.529465	**0.446032**	2.072297	3.784541	4.735940
14	1.531084	0.579806	**0.514680**	2.140807	3.817140	4.755185
15	1.542921	0.656414	**0.615366**	2.232128	3.861753	4.780126

Table 2: *Total expected cost rate as a function of s and N*
$\lambda = 0.07$, $\beta = 0.1$, $\gamma = 1$, $\alpha_1 = 3$, $\alpha_2 = 0.14$, $\mu_1 = 0.8$, $\mu_2 = 9$, $\eta_1 = 9$, $\eta_2 = 7$ $p = 0.7$, $q = 0.3$, $r = 0.5$, $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $C_r = 0.4$.

1. *The total expected cost rate monotonically increases when c_h, c_s, c_w, c_p, c_l, c_i and c_r increase. The optimal cost is more sensitive to c_h than to c_s, c_w, c_p, c_l, c_i and c_r.*
2. *As is to be expected, as c_h increases, the optimal values S^* and s^* decrease monotonically. This is because, if the holding cost increases, we resort to maintain low stock in the inventory.*
3. *If the setup cost increases, it is a common decision that we have to maintain more stock to avoid frequent ordering. This fact is also observed in the model.*
4. *If the waiting cost c_w of customers increases then the optimal values S^* and s^* monotonically increase. This is because if waiting cost of customers increases then we have to maintain high inventory to reduce the number of waiting customers. Also, we note that c_p, c_l, c_i and c_r monotonically increase when S^* decreases.*
5. *As is to be expected as s^* decreases, c_p increases. We cannot predict the behaviour of s^* when each of c_l, c_i and c_r increases.*

Example 3 *In this example, we look to the impact of the demand rate λ, essential service rate mu_1, second optional service rate μ_2, the reorder rate β and the perishable rate γ on the total expected cost rate. Towards this end, we first fix the cost values as $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$. From Figures 2 to 6, we observe the following.*

N / S	5	6	7	8	9	10
147	17.705136	17.530296	**17.485661**	17.492800	17.516362	17.544374
148	17.704412	17.529726	**17.485141**	17.492294	17.515861	17.543874
149	17.703868	17.529346	**17.484814**	17.491982	17.515553	17.543569
150	17.703501	17.529151	**17.484676**	17.491861	17.515437	17.543454
151	17.703307	17.529140	**17.484725**	17.491926	17.515507	17.543526
152	17.703284	17.529307	**17.484955**	17.492175	17.515761	17.543782
153	17.703427	17.529650	**17.485365**	17.492604	17.516195	17.544218

Table 3: *Total expected cost rate as a function of S and N*
$\lambda = 0.07$, $\beta = 0.1$, $\gamma = 1$, $\alpha_1 = 3$, $\alpha_2 = 0.14$, $\mu_1 = 0.8$, $\mu_2 = 9$, $\eta_1 = 9$, $\eta_2 = 7$ $p = 0.7$, $q = 0.3$, $r = 0.5$, $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $C_r = 0.4$.

c_s / c_h	50	55	60	65	70
0.01	49 | 7	54 | 7	58 | 6	62 | 6	66 | 6
	5.224918	5.689087	6.149771	6.607357	7.062127
0.02	46 | 7	50 | 6	54 | 6	58 | 6	61 | 6
	5.259759	5.727077	6.190898	6.651575	7.109379
0.03	43 | 6	47 | 6	50 | 6	54 | 5	57 | 5
	5.292203	5.762469	6.229193	6.692709	7.153369
0.04	41 | 6	44 | 5	47 | 5	51 | 4	54 | 4
	5.322580	5.795585	6.265046	6.731239	7.194521
0.05	39 | 5	42 | 5	45 | 4	48 | 4	51 | 4
	5.351155	5.826746	6.298730	6.767464	7.233248

Table 4: *Variation in optimal values for different values of c_h and c_s*
$c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$

- *The optimal expected cost rate increases when λ and γ increase.*
- *As is to be expected, the optimal cost rate decreases, when β, μ_1 and μ_2 decrease*

Example 4 *In this example, we look to the impact of the demand rate λ, essential service rate mu_1, second optional service rate μ_2, waiting hall size N, the interruption rates α_1 and α_2 during FES and SOS respectively, repair rates η_1 and η_2 during FES and SOS respectively, on the expected waiting time. Towards this end, we first fix the cost values as $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$. From Figures 7 to 11, we observe the following.*

- *The expected waiting time η_W is an increasing function of arrival rate (see Figure 6) and this behaviour is maintained for various values of $N = 10, 20, 30$. However, the expected waiting time is higher if N is larger.*
- *The expected waiting time increases when α_1 and α_2 decrease.*
- *As is to be expected, waiting time cost η_W, decreases, when η_1, η_2, μ_1 and μ_2 increase.*

c_p / c_h	0.1	0.13	0.16	0.19	0.22
0.01	49 | 7	41 | 7	35 | 7	31 | 6	28 | 6
	5.224918	5.313074	5.38522	5.446292	5.499214
0.02	46 | 7	39 | 7	34 | 6	30 | 6	27 | 5
	5.259759	5.341743	5.409672	5.467673	5.518337
0.03	43 | 6	37 | 6	32 | 5	29 | 5	27 | 5
	5.292203	5.368787	5.433019	5.488201	5.536658
0.04	41 | 6	35 | 6	31 | 5	28 | 4	26 | 4
	5.322580	5.394421	5.455245	5.507947	5.554338
0.05	39 | 5	34 | 5	30 | 4	27 | 4	25 | 4
	5.351155	5.418815	3.476555	5.526988	5.571504

Table 5: *Variation in optimal values for different values of c_h and c_p*
$c_s = 50$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$

c_w / c_h	0.03	0.05	0.07	0.09	0.11
0.01	49 | 7	53 | 7	56 | 8	57 | 8	58 | 8
	5.224918	5.245588	5.299176	5.352610	5.405885
0.02	46 | 7	50 | 7	52 | 8	53 | 8	54 | 8
	5.259759	5.276286	5.329274	5.382070	5.434706
0.03	43 | 6	47 | 6	50 | 7	52 | 7	53 | 7
	5.292203	5.304724	5.357099	5.409325	5.461395
0.04	41 | 6	45 | 6	48 | 6	50 | 7	51 | 7
	5.322580	5.331203	5.383033	5.434664	5.486144
0.05	39 | 5	41 | 6	44 | 6	46 | 7	50 | 7
	5.351155	5.355956	5.455956	5.458339	5.509280

Table 6: *Variation in optimal values for different values of c_w and c_h*
$c_s = 50$, $c_p = 0.1$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$

c_p / c_s	0.1	0.13	0.16	0.19	0.22
50	49 | 7	41 | 7	35 | 7	31 | 6	28 | 6
	5.224918	5.313074	5.38522	5.446292	5.499214
55	53 | 7	44 | 7	38 | 6	34 | 6	30 | 5
	5.689087	5.784839	5.864172	5.931427	5.989827
60	58 | 6	48 | 6	41 | 5	36 | 5	33 | 4
	6.149771	6.254176	6.340618	6.413965	6.477715
65	62 | 6	51 | 6	44 | 6	39 | 5	35 | 4
	6.607357	6.720251	6.813785	6.893204	6.962172
70	66 | 6	54 | 5	46 | 5	41 | 4	37 | 4
	7.062270	7.183441	7.283953	7.369280	7.443527

Table 7: *Variation in optimal values for different values of c_p and c_s*
$c_h = 0.01$, $c_w = 0.03$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$

c_w / c_s	0.03	0.10	0.17	0.24	0.31
50	49 | 7	52 | 7	54 | 7	55 | 8	56 | 8
	5.224918	5.379287	5.564689	5.747695	5.927866
55	53 | 7	54 | 7	55 | 7	56 | 7	59 | 8
	5.689087	5.835793	6.024168	6.210580	6.394741
60	58 | 6	59 | 7	60 | 7	62 | 7	63 | 8
	6.149771	6.288396	6.479336	6.668610	6.856006
65	62 | 6	63 | 7	64 | 7	64 | 7	65 | 8
	6.607357	6.737558	6.930748	7.124850	7.312606
70	66 | 6	64 | 6	65 | 7	66 | 7	66 | 8
	7.062270	7.183638	7.378815	7.572691	7.765185

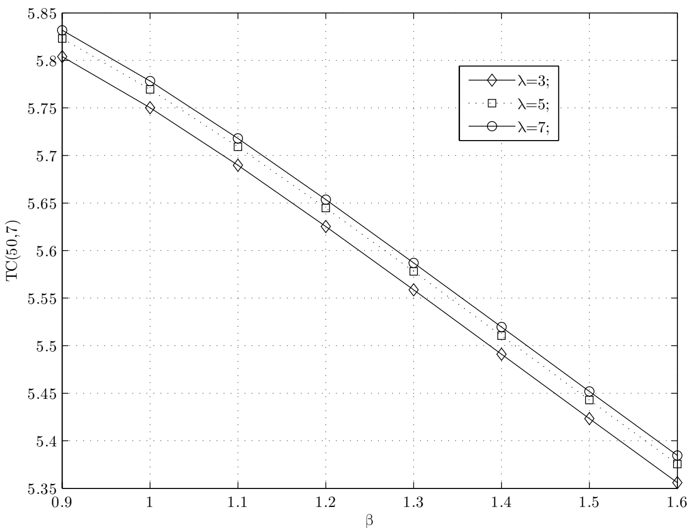
Table 8: *Variation in optimal values for different values of c_w and c_s*
$c_h = 0.01$, $c_p = 0.1$, $c_l = 0.2$, $c_i = 0.1$, $c_r = 0.4$

c_l / c_s	0.2	0.4	0.6	0.8	1.0
50	49 | 7	48 | 7	47 | 7	46 | 7	45 | 7
	5.224918	5.284571	5.345084	5.405360	5.465388
55	53 | 7	52 | 7	51 | 7	50 | 7	49 | 7
	5.689087	5.749647	5.811082	5.872305	5.933310
60	58 | 6	57 | 6	56 | 6	55 | 6	54 | 6
	6.149771	6.211140	6.273398	6.335465	6.397338
65	62 | 6	61 | 6	60 | 6	59 | 6	58 | 6
	6.607357	6.669455	6.732450	6.795274	6.857922
70	66 | 6	65 | 6	64 | 6	63 | 6	62 | 6
	7.062270	7.124889	7.188556	7.252066	7.315416

Table 9: *Variation in optimal values for different values of c_s and c_l*
$c_h = 0.01$, $c_p = 0.1$, $c_w = 0.03$, $c_i = 0.1$, $c_r = 0.4$

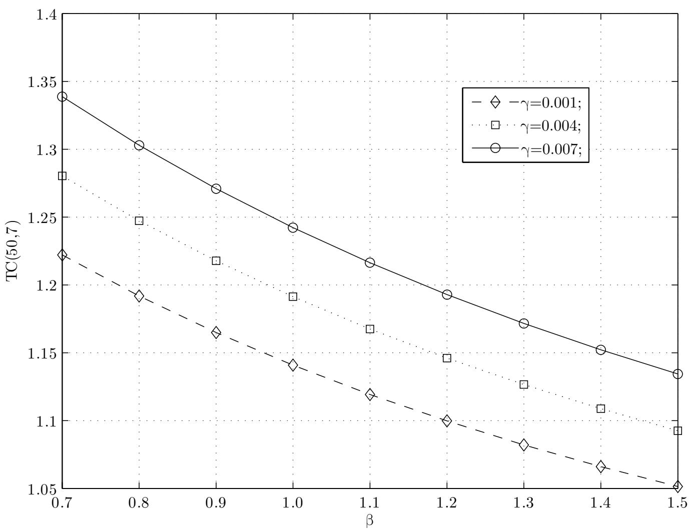
c_l / c_p	0.2	0.4	0.6	0.8	1.0
0.1	49 | 7	48 | 7	47 | 7	46 | 7	45 | 7
	5.224918	5.284571	5.345084	5.405360	5.465388
0.13	41 | 7	40 | 7	39 | 7	38 | 7	38 | 7
	5.313074	5.370578	5.428971	5.487130	5.545032
0.16	35 | 7	35 | 7	34 | 7	33 | 7	33 | 7
	5.385220	5.440983	5.497561	5.553937	5.610105
0.19	31 | 6	31 | 6	30 | 6	30 | 6	29 | 6
	5.446292	5.500497	5.555580	5.610514	5.665009
0.22	28 | 6	28 | 6	27 | 6	27 | 6	26 | 6
	5.499214	5.552043	5.605838	5.659340	5.712563

Table 10: *Variation in optimal values for different values of c_p and c_l*
$c_h = 0.01$, $c_s = 50$, $c_w = 0.03$, $c_i = 0.1$, $c_r = 0.4$



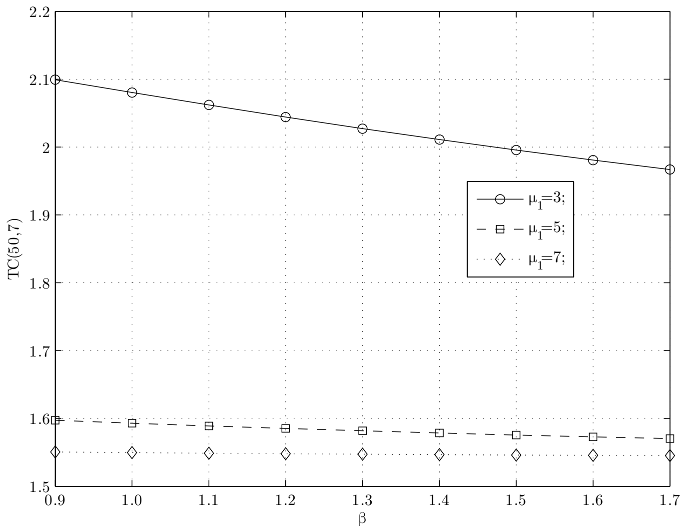
$\gamma = 0.1,\ \alpha_1 = 0.3,\ \alpha_2 = 0.1,\ \mu_1 = 3,\ \mu_2 = 9,\ \eta_1 = 9,\ \eta_2 = 7,$
$N = 10,\ p = 0.3,\ q = 0.7,\ r = 0.5$

Figure 2: *TC versus β for different values of λ*



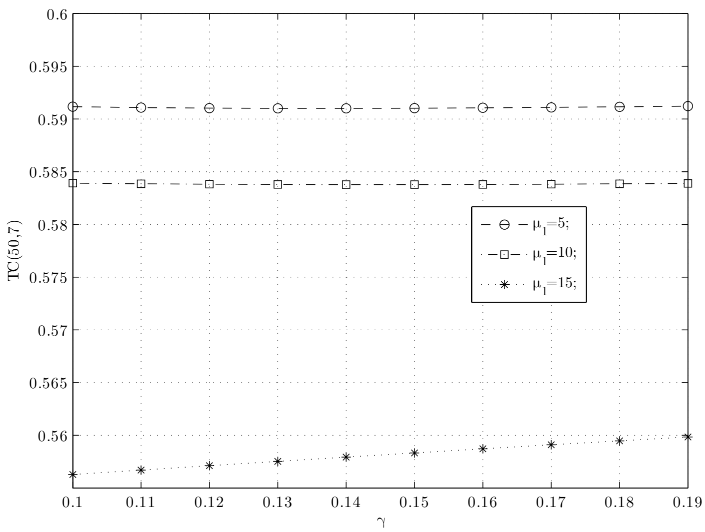
$N = 10,\ \lambda = 7,\ \alpha_1 = 0.003,\ \alpha_2 = 0.1,\ \mu_1 = 0.8,\ \mu_2 = 9,\ \eta_1 = 9,$
$\eta_2 = 7,\ p = 0.3,\ q = 0.7,\ r = 0.5$

Figure 3: *TC versus β for different values of γ*



$$\lambda = 0.7,\ \gamma = 0.1,\ \alpha_1 = 0.3,\ \alpha_2 = 0.1,\ \mu_2 = 9,\ \eta_1 = 9,\ \eta_2 = 0.7,$$
$$p = 0.3,\ q = 0.7,\ r = 0.5, N = 10$$

Figure 4: *TC versus β for different values of μ_1*

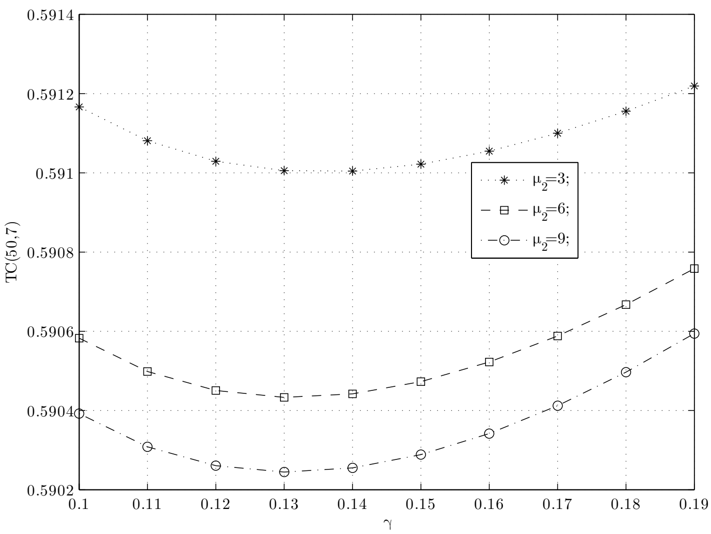


$$\lambda = 5,\ \beta = 0.01,\ \alpha_1 = 0.3,\ \alpha_2 = 0.1,\ \mu_2 = 3,\ \eta_1 = 9,\ \eta_2 = 7,$$
$$p = 0.3,\ q = 0.7,\ r = 0.5, N = 10$$

Figure 5: *TC versus γ for different values of μ_1*

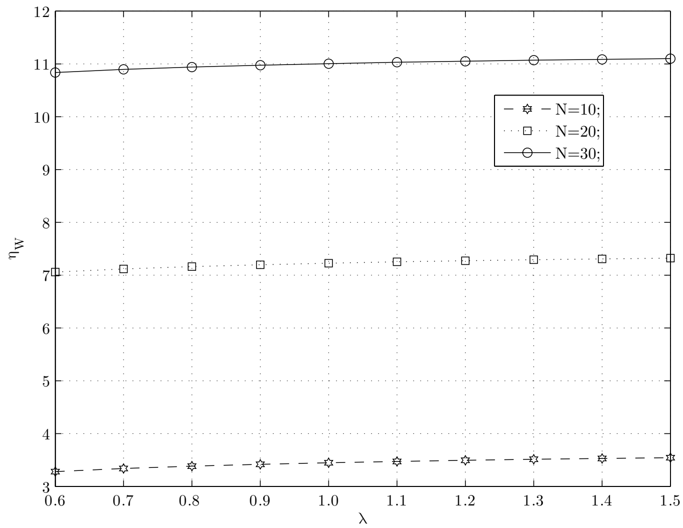
c_i / c_r	0.1	1.0	1.9	2.8	3.7
0.4	49 | 7	48 | 7	46 | 7	45 | 7	43 | 7
	5.223824	5.313229	5.398148	5.484535	5.570391
0.8	49 | 7	47 | 7	46 | 7	44 | 7	43 | 7
	5.262584	5.349762	5.436460	5.52260	5.608212
1.2	48 | 7	46 | 7	45 | 7	43 | 7	42 | 7
	5.301226	5.388220	5.474647	5.560585	5.645896
1.6	47 | 7	46 | 7	44 | 7	43 | 7	41 | 7
	5.339795	5.426532	5.512753	5.598406	5.683491
2.0	47 | 7	45 | 7	44 | 7	42 | 7	41 | 7
	5.378260	5.464758	5.550743	5.636134	5.720959

Table 11: *Variation in optimal values for different values of* c_r *and* c_i $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_l = 0.2$.



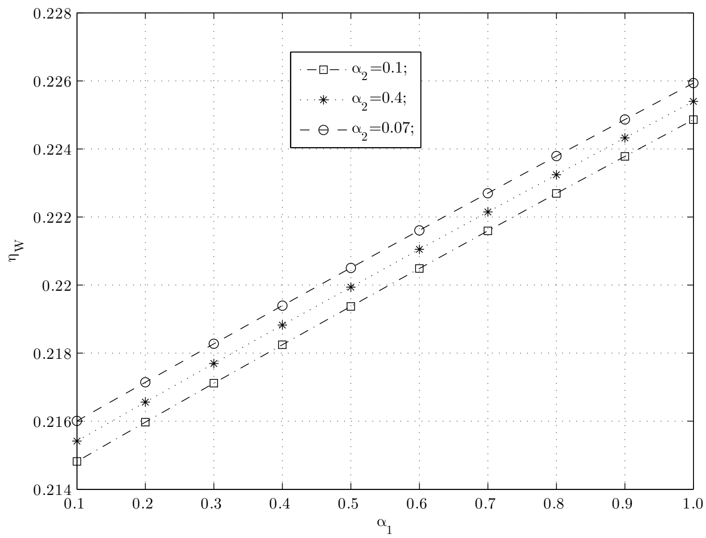
$$\lambda = 5,\ \beta = 0.01,\ \alpha_1 = 0.3,\ \alpha_2 = 0.1,\ \mu_1 = 5,\ \mu_2 = 9,\ \eta_1 = 9,$$
$$\eta_2 = 7,\ p = 0.3,\ q = 0.7,\ r = 0.5, N = 10$$

Figure 6: TC *versus* γ *for different values of* μ_2



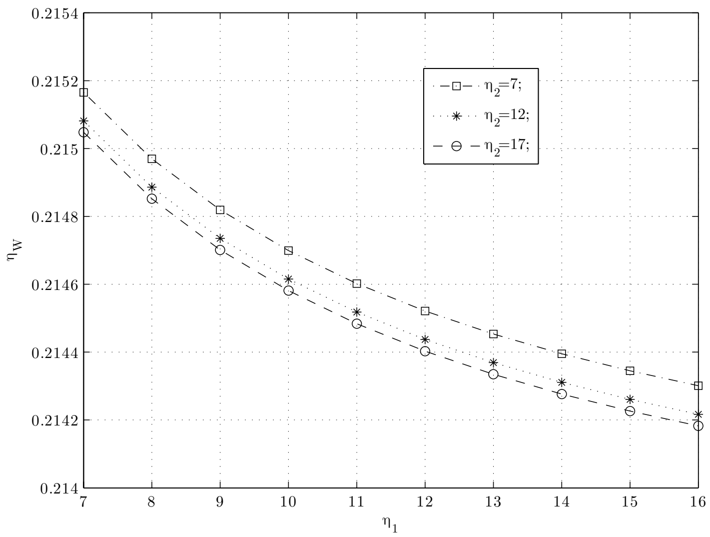
$\beta = 0.1,\ \gamma = 1,\ \alpha_1 = 0.3,\ \alpha_2 = 0.14,\ \mu_1 = 0.8,\ \mu_2 = 9,\ \eta_1 = 9,$
$\eta_2 = 0.7,\ p = 0.7,\ q = 0.3,\ r = 0.5, N = 10$

Figure 7: η_W *versus* λ *for different values of* N



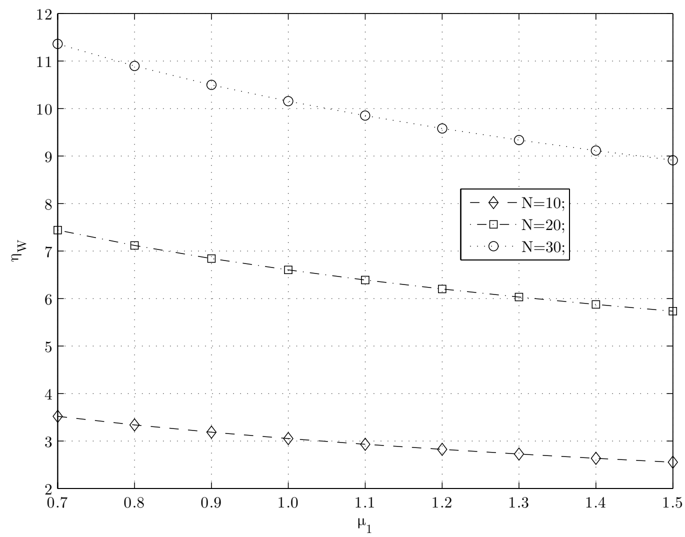
$\lambda = 0.7,\ \beta = 0.1,\ \gamma = 1,\ \mu_1 = 0.8,\ \mu_2 = 9,\ \eta_1 = 9,\ \eta_2 = 0.7,$
$p = 0.7,\ q = 0.3,\ r = 0.5, N = 10$

Figure 8: η_W *versus* α_1 *for different values of* α_2



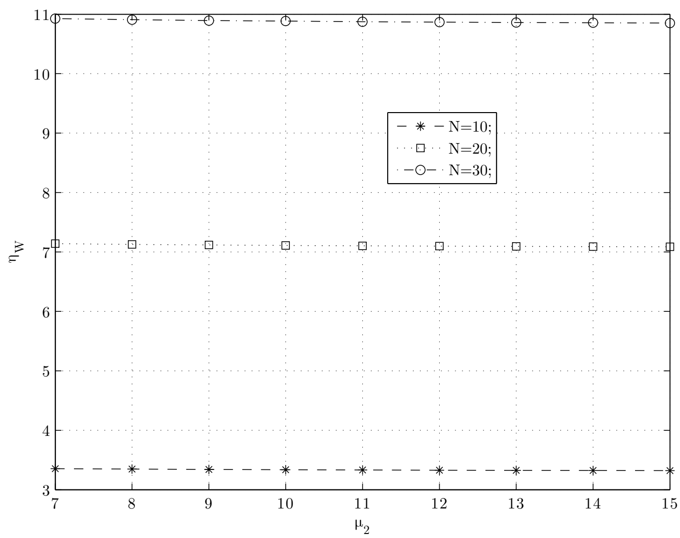
$$\lambda = 0.7,\ \beta = 0.1,\ \gamma = 1,\ \alpha_1 = 0.3,\ \alpha_2 = 0.14,\ \mu_1 = 0.8,\ \mu_2 = 9,$$
$$p = 0.7,\ q = 0.3,\ r = 0.5, N = 10$$

Figure 9: *η_W versus η_1 for different values of η_2*



$$\lambda = 0.7,\ \beta = 0.1,\ \gamma = 1,\ \alpha_1 = 0.3,\ \alpha_2 = 0.14,\ \mu_2 = 9,\ \eta_1 = 9,\ \eta_2 = 0.7,$$
$$p = 0.7,\ q = 0.3,\ r = 0.5$$

Figure 10: *η_W versus μ_1 for different values of N*



$$\lambda = 0.7,\ \beta = 0.1,\ \gamma = 1,\ \alpha_1 = 0.3,\ \alpha_2 = 0.14,\ \mu_1 = 0.8,\ \eta_1 = 9,\ \eta_2 = 0.7,$$
$$p = 0.7,\ q = 0.3,\ r = 0.5$$

Figure 11: *η_W versus μ_2 for different values of N*

7 Summary and conclusion

In this article, a continuous review stochastic queueing-inventory system with (s, S) control policy, server interruptions and finite source was analyzed. The model is analyzed within the framework of Markov processes. Stationary distribution of the number of customers in the waiting area, the server status and the inventory level is obtained in the steady state. Various system performance measures are derived and the long-run total expected cost rate is derived. The waiting time distribution is derived. A sensitivity analysis is numerically performed on the expected total cost function with respect to various parameters of the model. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times follow PH-distributions.

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