# A Generalized Formulation and Review of Piston Theory for Airfoils

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**The present work presents a brief review of some of the notable contributions to piston theory and of its theoretical basis. A generalized formulation of piston theory is given, applicable to both local and classical piston theory. A consistent generalized formulation of the downwash equation is given, accounting for arbitrary motion in the plane of the airfoil. The formulation reduces to established downwash equations through appropriate definition of the cylinder orientation. The theoretical range of validity of Lighthill's classical piston theory is examined, and the relative accuracy of a number of approximate theories encapsulated by the formulation as applied to a planar wedge is considered. The relative importance of higher-order terms in piston theory is examined, with the significance of recent literature extending the fidelity of the first-order term highlighted. It is subsequently suggested that current implementations of local piston theory may be improved through the use of a first-order term of suitable accuracy.**

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## Nomenclature

| | | |
|---|---|---|
| $a$ | = | speed of sound, m/s |
| $C_p$ | = | pressure coefficient |
| $C_{p_{cyl}}$ | = | cylinder reference pressure coefficient |
| $c$ | = | airfoil chord, m |
| $c_1$, $c_2$, $c_3$ | = | piston theory coefficients |
| $K$ | = | hypersonic similarity parameter, $K = M\tau$ |
| $k$ | = | reduced frequency, $k = \frac{\omega c}{2U_\infty}$ |
| $M$ | = | Mach number |
| $m$ | = | $\sqrt{M^2-1}$ |
| $\boldsymbol{n'}$ | = | normal vector of the deformed / perturbed surface |
| $\boldsymbol{n_s}$ | = | normal vector of the mean surface |
| $p$ | = | pressure on piston surface, N/m$^2$ |
| $\hat{p}$ | = | non-dimensionalized pressure |
| $t$ | = | time, s |
| $\tilde{t}$ | = | time variable in the earth-fixed frame, s |
| $U$ | = | flow velocity, m/s |
| $U_b$ | = | local velocity of the airfoil surface, m/s |
| $U_i$ | = | flow velocity in the $i$-coordinate, m/s |
| $v_i$ | = | perturbation velocity in the $i$-coordinate, m/s |
| $\bar{v}_i$ | = | scaled perturbation velocity in the $i$-coordinate, m/s |
| $w$ | = | downwash velocity, m/s |
| $\hat{w}$ | = | non-dimensionalized downwash velocity |
| $w_c$ | = | convective downwash component, m/s |
| $w_d$ | = | dynamic downwash component, m/s |
| $x$ | = | body-fixed coordinate parallel to the undisturbed flow, m |
| $x'$ | = | earth-fixed coordinate parallel to the undisturbed flow, m |
| $\bar{x}$ | = | scaled body-fixed coordinate parallel to the undisturbed flow, m |
| $\tilde{x}$ | = | scaled earth-fixed coordinate parallel to the undisturbed flow, m |
| $z$ | = | coordinate perpendicular to the undisturbed flow, m |
| $\alpha$ | = | angle of attack, rad |
| $\gamma$ | = | ratio of specific heats |
| $\delta$ | = | local surface inclination relative to freestream, rad |

$\epsilon$ = maximum amplitude of oscillation, m

$\zeta$ = coordinate parallel to the cylinder, m

$\theta$ = cylinder inclination relative to the freestream-normal, rad

$\xi$ = coordinate perpendicular to the cylinder, m

$\rho$ = air density, kg/m$^3$

$\tau$ = airfoil maximum thickness ratio

$\chi_i$ = $i^{\text{th}}$-order contribution to pressure coefficient

$\psi_i$ = $i^{\text{th}}$-order contribution to pressure coefficient derivative

$\omega$ = angular frequency, rad/s

**Subscripts**

$(\ )_\infty$ = freestream quantity

$(\ )_{cyl}$ = cylinder reference quantity

$(\ )_p$ = perturbation quantity

$(\ )_s$ = quantity at local mean steady conditions

$(\ )_x$ = component in the direction of the undisturbed flow

$(\ )_z$ = component perpendicular to the undisturbed flow

## I. Introduction

Piston theory has been used to refer broadly to a number of aerodynamic models which describe the pressure on a point of a body through analogy to the motion of a piston in a 1-dimensional cylinder. As a result, a number of flavours of piston theory exist, with variations in the basis of the pressure equation and in the reference frame used; however, all the variations assume supersonic flow at the point under consideration, with various limits of validity depending on the basis of the theory. In all cases, piston theory provides a quasi-steady, point-function relationship between the surface downwash and aerodynamic pressure at a point on a body. This renders piston theory a computationally inexpensive aerodynamic model.

Interest in piston theory as a method for modeling unsteady aerodynamic loads, in particular for aeroelastic analysis, rose in the 1950s [1–5]. The application of piston theory continued throughout the 1960s, with increasing studies of the application to shells and cylinders [6, 7]. Literature on the application and further development of piston theory in later years is comparatively sparse - this coincides with the increasing use of more computationally intensive aerodynamic calculations, such as the finite volume, finite difference, and finite element methods. However, piston theory enjoyed renewed attention in the 1990s, with the early implementation of piston theory with Euler solutions in CFD [8] and with continued development of aeroelastic panel codes [9]. The 2000s saw a marked resurgence in the interest in piston theory as a computationally inexpensive method for modelling supersonic and hypersonic aeroelastic problems, with applications being found in aero-servo-elasticity and aero-thermo-elasticity, with the further integration of piston theory with CFD; recent literature highlights the continued application of piston theory in reducing the computational cost of CFD for hypersonic aeroelasticity [10–15].

The recent renewal in interest in piston theory has led to the basic equations of piston theory being broadly cited, whilst the theoretical basis of the formulation of the theory has received little renewed attention. Furthermore, the variety of formulations which fall under the moniker of "piston theory" may be a source of confusion for a reader unfamiliar with the method. In this paper, a limited review of the developments in piston theory as applied to airfoils is given and the theoretical basis of piston theory is reviewed; the application to shell structures is not considered. A generalized formulation is put forward which is applicable to both classical and local piston theory, and which is only limited in the assumption that a binomial expansion of the piston pressure equation in terms of the dimensionless piston downwash exists.

## II. Developments in Piston Theory

### A. Lightill's Classical Piston Theory

Piston theory was originally developed by Lighthill [1], on the basis of the extension of Tsien's hypersonic similitude by Hayes [16]. Lighthill formulated piston theory in terms of a "slab" of fluid (oriented normal to the freestream velocity) being washed over an oscillating airfoil; the slab remains perpendicular to the freestream as it moves over the body at the freestream velocity. This formulation consequently modeled the effective velocity of the boundary of the fluid slab (i.e., the local downwash) as including the convective rate of displacement due to the surface gradient in the freestream direction, as well as the time-dependent local surface velocity arising from airfoil motion. In the notation of this paper, this was cast [1] as

$$w = \left[ U_\infty \left( \frac{\partial}{\partial x} \right) + \left( \frac{\partial}{\partial t} \right) \right] z \tag{1}$$

The airfoil surface was modeled [1] as a piston moving down the length of the "slab" into undisturbed fluid. Provided the piston motion produced no shock waves, Lighthill suggested that the piston pressure could be modeled by the insentropic simple wave formula:

$$\frac{p}{p_\infty} = \left( 1 + \frac{\gamma - 1}{2} \left( \frac{w}{a_\infty} \right) \right)^{\frac{2\gamma}{\gamma - 1}} \tag{2}$$

Lighthill recommended the use of the third-order binomial expansion of the equation, given in Eq. (3). The influence of entropy is of third-order in flow deflection; the series expansions of the pressure equations for oblique shocks and Prandlt-Meyer expansions differ in their third-order terms. Lighthill noted that the pressure given by third-order truncation, Eq. (3), was bounded by results from Eq. (2) and from oblique shock theory; hence the third-order expansion of the simple wave equation was deemed sufficiently accurate to use for both expansion and compression flows.

$$\frac{p}{p_\infty} = 1 + \gamma \left( \frac{w}{a_\infty} \right) + \frac{\gamma(\gamma + 1)}{4} \left( \frac{w}{a_\infty} \right)^2 + \frac{\gamma(\gamma + 1)}{12} \left( \frac{w}{a_\infty} \right)^3 \tag{3}$$

The assumptions inherent in the model were considered [1] to be a good approximation to the flow physics, provided that the piston velocity did not exceed the speed of sound in the freestream. The range of validity set by Lighthill was for piston motions conforming to Eq. (4), for Mach numbers in the range of $M \geq 4$. Lightill noted that enforcing the limits of validity allowed the piston pressure to be modeled as dependent on only the instantaneous piston velocity, neglecting the history of piston motion.

$$M_\infty \left[ \delta + 2k \left( \frac{\epsilon}{c} \right) \right] < 1 \tag{4}$$

Lighthill's original development of piston theory has been dubbed "classical piston theory" (CPT). In CPT, both the steady pressure distribution (due to airfoil shape and mean incidence) and the unsteady pressure distribution (due to airfoil motion and surface deformations) are computed.

### B. Further Developments of Classical Piston Theory

The widely cited review of piston theory by Ashley and Zartarian [2] summarises the work of Hayes [16] and Lighthill [1], and considers the application to a number aeroelastic problems, including airfoil flutter, wing flutter, and panel flutter. In the review by Ashley and Zartarian, linearized piston theory as applied to "small motions of thin airfoils" was determined to be valid [2] for any of the conditions listed in Eqs. (5 − 7).

$$M_\infty^2 \gg 1 \tag{5}$$

$$kM_\infty^2 \gg 1 \tag{6}$$

$$k^2 M_\infty^2 \gg 1 \tag{7}$$

Whilst no extensions to the formulation of piston theory were made in the review by Ashley and Zartarian [2], the applications of contemporary research were covered. This was shortly followed by the work of Chawla [4], which was significant in conducting a parametric study of airfoil flutter at high Mach numbers using piston theory.

The theoretical basis for piston theory was revisited by Bird [3], who noted that the limiting assumptions on high Mach number and small airfoil thickness may be avoided in certain special flow cases. Bird noted that the equations of motion of two-dimensional steady flow reduce identically to those of one-dimensional unsteady flow when the velocity component of the flow in any direction remains constant throughout the flow field. Examples of simple flows for which this condition arises include an oblique shock over a planar wedge (velocity component parallel to the shock) and a Prandtl-Meyer expansion around a planar corner. Bird noted that for these special cases, the piston acts perpendicular to the direction of constant velocity, rather than perpendicular to the freestream velocity, and provided an amendeded equation for the convective component of the piston downwash. Of significance is Bird's recommendation to define the cylinder orientation as perpendicular to the surface - this recommendation was followed in many subsequent applications of piston theory.

Rodden et al [17] used third-order CPT to extract aerodynamic influence coefficients for an aerodynamic modeling routine for swept wings. The formulation represented an extension of the application of piston theory, with a sweep correction being introduced. Of particular interest is the generalized formulation of the equation for the pressure coefficient in CPT, first put forward by Rodden et al [17] as:

$$C_p = \frac{2}{M_\infty^2}\left[c_1\left(\frac{w}{a_\infty}\right) + c_2\left(\frac{w}{a_\infty}\right)^2 + c_3\left(\frac{w}{a_\infty}\right)^3\right] \tag{8}$$

It was noted that both Lighthill's CPT and Van Dyke's second-order theory could be described by Eq. (8), with the coefficients $c_1$, $c_2$, and $c_3$ being defined by which theory was implemented.

Wood [18] reviewed methods for unsteady hypersonic flows, including CPT and hypersonic small disturbance theory (SDT). Wood identified limitations in the validity of the theoretical basis for CPT, where the assumed conditions of CPT did not match the physical flow conditions. Examples included the large entropy gradients along the length of the cylinder, and the acceleration and deceleration of the piston, which are not accounted for in CPT. The mathematical treatment of the reduction of a $n$-dimensional spatial flow to unsteady flow in $n-1$ dimensions is given [18] using hypersonic SDT, with the differences and similarites between hypersonic SDT, Sychev's SDT [19], and CPT being reviewed.

Ericsson et al [20] considered the differences between CPT using simple-wave pressures, the tangent-wedge method (which is equivalent to CPT using simple-wave expansion and oblique shock compression pressures), and Newtonian impact theory. It was shown that the third-order term in CPT has a pronounced effect on the aerodynamic derivatives relative to second-order CPT as $K \to 1$. Ericsson et al further showed that for the range $0.5 \leq K \leq 2.5$, similar values of the derivatives were obtained using all of the theories; it was suggested that the flutter boundaries predicted by the theories should be similar.

The review by Liu et al [9] examined various formulations of the pressure equation in CPT. Notably, the review by Liu et al was formulated upon the generalized equation for CPT originally put forward by Rodden et al [17] as in Eq. (8), with the review by Liu et al collecting various pressure equations upon which piston theory may be based. The coefficients $c_1$, $c_2$, and $c_3$ varied according to the pressure equation expanded. The review included the work of several authors, and noted some of the differences in the third-order terms derived by various authors. The work of Liu et al was further significant in the combination of CPT with supersonic lifting surface theory. The unsteady three-dimensional influence on the wing surface pressures was given by the lifting surface theory, whilst local quasi-steady nonlinear thickness effects were modeled by third-order CPT.

Dowell and Bliss [21] recently extended first-order CPT through consideration of two-dimensional unsteady potential flow in terms of Laplace transform analysis for simple harmonic airfoil motions. The formulation is shown to reduce to first-order CPT in the limit of high Mach numbers. Several formulations of the extended piston theory are offered by Dowell and Bliss, in terms of arbitrary Mach number and reduced frequency. In the extended theory, the coefficient of the downwash in the pressure equation is cast as a series in inverse powers of Mach number (squared) and oscillation frequency, with integrations over the airfoil chord of the airfoil deflection. The extended theory of Dowell and Bliss represents an extension [21] of the validity of piston theory to lower Mach numbers ($M \leq 1.6$).

Classical piston theory is a mature aerodynamic method. However, a number of formulations exist, with differences in the coefficients used and in the definition of the downwash and the direction of the piston action. The works cited here are representative of the main developments to CPT as applied to airfoils.

### C. Local Piston Theory

The essence of local piston theory (LPT) is the dynamic linearization of the flow about a mean steady state. The mean steady flow field is computed by a suitable aerodynamic model, and the perturbations from the mean steady state are computed using piston theory. As the pertubations are small relative to the mean steady state, the local flow conditions are used for the cylinder conditions in LPT, with inclinations to the local steady flow being modeled. The method was first suggested by Morgan et al [5] towards the end of the 1950s as "local-flow piston theory". Local-flow piston theory was relatively sparsely used throughout the 1970s; the method received renewed attention in the 1990s, with further implementation in CFD studies in the 2000s.

Morgan et al [5] suggested use of local flow conditions for the cylinder reference conditions in piston theory. This was motivated by the inherent locality of piston theory — the piston pressure is a point-function independent of the conditions at surrounding locations on the airfoil. In the approach of Morgan et al, a computation using the exact oblique shock relations was made to determine the local flow conditions; these conditions were used in a CPT formulation to compute both the mean steady and perturbation aerodynamic loads.

Yates [22] similarly used the steady flow conditions from shock-expansion theory in LPT. Yates offered a formulation for the downwash on a rectangular diamond wing in pitch-plunge motion. A clear definition of the application of LPT was given, with only perturbation motions modeled, and with the cylinder oriented normal to the local mean surface. The variation of the flutter boundary with angle-of-attack was studied by Yates for variations in Mach number and airfoil thickness. Yates noted that the predicted trends of first-order LPT based on the simple-wave pressures (Eq. (2)) differed notably near shock-detachment from those of LPT based on Van Dyke's second-order theory due to the differences in formulation at the low local Mach numbers ($M_s \approx 1$).

Ericsson et al [20] used first-order LPT to model the perturbation from steady inviscid flow caused by viscous effects. The steady inviscid flow was calculated using CPT with tangent-wedge coefficients; the viscous effects were modelled as a perturbation slope arising from the boundary layer. This analysis was extended [20, 23] to modeling the effects of the dynamic boundary layer thickness on the pressure derivatives from piston theory.

Hunter [8] applied LPT relative to an Euler computation of the mean steady flow to find small steady and unsteady perturbations. A number of test cases were considered in comparison of the LPT-Euler method with a full unsteady Euler solution and with CPT. Hunter found that the LPT-Euler method gave good correlation with the full unsteady Euler solution for $M_\infty \geq 1.5$ for the majority of the cases considered. This was achieved at a significant reduction in computational cost.

Zhang et al [10] similarly utilized LPT with a steady Euler solution to compute unsteady aerodynamic loads on airfoils and on a three-dimensional wing. A comprehensive comparison between CPT, LPT-Euler, and unsteady Euler was made for a range of circular arc airfoils, as well as for a NACA0012 profile (to which CPT is not applicable). Zhang et al demonstrated that using LPT relative to an Euler solution removed the limitations on surface geometry associated with CPT, provided the perturbations about the mean steady Euler solution were small.

Whilst local piston theory is not a new method, the use of the method with CFD is a comparitively recent approach which continues to be developed [24].

## III. Theoretical Basis

### A. Piston Analogy

The basis for piston theory lies in the hypersonic equivalence between steady flow in $n$ spatial dimensions and unsteady flow in $n-1$ spatial dimensions. This basis was estabilished by Hayes [16] and was implemented as piston theory by Lighthill [1]. In particular, Lighthill noted that for thin airfoils in high Mach number flows at small incidence, the gradients and velocity components *perpendicular to the undisturbed flow* are large in comparison to the gradients and disturbances to the velocity in the direction of the undisturbed flow. These are key assumptions in hypersonic small-disturbance theory (SDT) (such as that of Van Dyke [25]). It may be shown [18, 25] that hypersonic SDT effectively supports the assumptions made by Lighthill that the contributions to the pressure from perturbations in the direction of the undisturbed flow may be neglected.

The exposition of hypersonic small disturbance theory by Van Dyke [25] gives an order of magnitude analysis of the Euler equations. Wood [18] gives a simplified treatment, and also considers Sychev's [19] SDT. The approach of Wood in reducing the Euler equations is followed here, with an emphasis on the transformations for a two-dimensional airfoil.

Consider the two-dimensional flow around a slender airfoil in a body-fixed reference frame $z$-$x$, as depicted in Fig. 1. The velocity components are given as

$$U_x = U_\infty + v_x \tag{9}$$

$$U_z = v_z \tag{10}$$

The Euler equations in two dimensions written in the $z$-$x$ frame are then given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_x)}{\partial x} + \frac{\partial (\rho U_z)}{\partial z} = 0 \tag{11}$$

$$\rho \frac{\partial U_x}{\partial t} + \rho U_x \frac{\partial U_x}{\partial x} + \rho U_z \frac{\partial U_x}{\partial z} = -\frac{\partial p}{\partial x} \tag{12}$$

$$\rho \frac{\partial U_z}{\partial t} + \rho U_x \frac{\partial U_z}{\partial x} + \rho U_z \frac{\partial U_z}{\partial z} = -\frac{\partial p}{\partial z} \tag{13}$$

$$\frac{\partial s}{\partial t} + U_x \frac{\partial s}{\partial x} + U_z \frac{\partial s}{\partial z} = 0 \tag{14}$$

In order for the airfoil dimensions to be of the same order of magnitude, the airfoil is shrunk in the $x$-direction, as depicted in Fig. 2, through the transform

$$\bar{x} = \tau x \tag{15}$$

To bring the the perturbation velocities to the same order of magnitude [25], the following transform [18] may be used

$$\bar{v}_x = \frac{1}{\tau} v_x \tag{16}$$

As a result of the transformation, the velocity in the $x$-direction in the transformed $z$-$\bar{x}$ plane is also scaled as in Eq. (15). Noting the relationship between derivatives of the two planes, the Euler equations as written in the $z$-$x$ plane Eqs. (11 – 14) transform to the $z$-$\bar{x}$ plane as

$$\frac{\partial \rho}{\partial t} + \tau U_\infty \frac{\partial \rho}{\partial \bar{x}} + \frac{\partial\left(\rho v_z\right)}{\partial z} = -\tau^2 \frac{\partial\left(\rho \bar{v}_x\right)}{\partial \bar{x}} \tag{17}$$

$$\frac{\partial \bar{v}_x}{\partial t} + \tau U_\infty \frac{\partial \bar{v}_x}{\partial \bar{x}} + v_z \frac{\partial \bar{v}_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial \bar{x}} = -\tau^2 \bar{v}_x \frac{\partial \bar{v}_x}{\partial \bar{x}} \tag{18}$$

$$\frac{\partial v_z}{\partial t} + \tau U_\infty \frac{\partial v_z}{\partial \bar{x}} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -\tau^2 \bar{v}_x \frac{\partial v_z}{\partial \bar{x}} \tag{19}$$

$$\frac{\partial s}{\partial t} + \tau U_\infty \frac{\partial s}{\partial \bar{x}} + v_z \frac{\partial s}{\partial z} = -\tau^2 \bar{v}_x \frac{\partial s}{\partial \bar{x}} \tag{20}$$

If terms of relative smallness $\tau^2$ are neglected, then the order of the flow problem has reduced to one dimension, with Eqs. (17, 19 – 20) giving three equations in three unknowns ($\rho$, $P$, and $v_z$). If the transformation is made to a earth-fixed reference frame $z$-$\tilde{x}$ in still air, as depicted in Fig. 3, the following relationships hold

$$\tilde{t} = t \tag{21}$$

$$\tilde{x} = \bar{x} - \tau U_\infty t \tag{22}$$

It is evident that the local derivative $\frac{\partial}{\partial \tilde{t}}$ in the $z$-$\tilde{x}$ reference frame consists of both a local derivative as well as a convective derivative in the $z$-$\bar{x}$ frame; from Eqs. (21, 22) it follows that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} - \tau U_\infty \frac{\partial}{\partial \tilde{x}} \tag{23}$$

Hence, the Euler equations in an earth-fixed reference frame for a two-dimensional airfoil under the assumptions of hypersonic small disturbance theory [25] are given by

$$\frac{\partial \rho}{\partial \tilde{t}} + \frac{\partial (\rho v_z)}{\partial z} = -\tau^2 \frac{\partial (\rho \bar{v}_x)}{\partial \tilde{x}} \tag{24}$$

$$\frac{\partial \bar{v}_x}{\partial \tilde{t}} + v_z \frac{\partial \bar{v}_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial \tilde{x}} = -\tau^2 \bar{v}_x \frac{\partial \bar{v}_x}{\partial \tilde{x}} \tag{25}$$

$$\frac{\partial v_z}{\partial \tilde{t}} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -\tau^2 \bar{v}_x \frac{\partial v_z}{\partial \tilde{x}} \tag{26}$$

$$\frac{\partial s}{\partial \tilde{t}} + v_z \frac{\partial s}{\partial z} = -\tau^2 \bar{v}_x \frac{\partial s}{\partial \tilde{x}} \tag{27}$$

Neglecting terms of relative smallness $\tau^2$, it is seen that Eqs. (24, 26 – 27) represent the unsteady Euler equations for a one-dimensional flow perpendicular to the freestream. This represents the mathematical basis for Lighthill's physical interpretation of the flow as "slabs" of fluid (oriented perpendicular to the freestream) being washed over the surface of an oscillating airfoil. The flow in each "slab" is considered indepedent of the flow in adjacent slabs — the dependency on the axial coordinate $x$ is removed. Lighthill [1] described the slab of fluid as a one-dimensional cylinder of fluid, with the airfoil surface acting as a piston to displace the fluid down the length of the cylinder.

It is important to note that this development of hypersonic SDT assumes small flow incidences; this is in contrast to Sychev's [19] SDT, which is applicable to slender bodies at large incidence — here the small parameter is the transverse dimension of the body. Both the hypersonic SDT of Van Dyke and the large incidence SDT of Sychev, yield a form of the hypersonic equivalence principle ("law of plane sections" in Sychev's work), or piston analogy; the steady flow in $n$ spatial dimensions is modeled by the unsteady flow in $n-1$ spatial dimensions. However, important differences in the formulation arise from the definition of the spatial dimension which is reduced, and the terms which are neglected. In Sychev's formulation, the reduction is made to unsteady flow in planes perpendicular to the body axis (which is at large incidence); however, in piston theory, the reduction is made to unsteady flow in planes perpendicular to the undisturbed velocity vector. In terms of the piston analogy, it may be seen that the orientation of the cylinder plays an important role.

The development of small-disturbance theories is summarized in the authoritative text of Hayes and Probstein [26], with particular consideration given to the differences between hypersonic SDT and the linearized slender body theory for compressible flows. Hayes and Probstein also considered the application of the hypersonic equivalence principle to unsteady flows, citing Lighthill's [1] piston theory, and noted that its applicability in hypersonic flows is dependent on the reduced frequency, $k$, being small. Dowell et al [27], working from a unsteady potential flow formulation, noted that first-order piston theory is obtained not only for small $k$ at hypersonic Mach numbers, but is also obtained in the limit of very large reduced frequencies for any Mach number. In both of these limits, the piston theory formulation is obtained as a result of the contribution of spatial derivatives in the direction of the flow becoming negligible relative to derivatives perpendicular to the flow. This reduction in the spatial dimension of the problem is characteristic of the piston analogy, and has been exploited in reducing the cost of computational fluid dynamics, as was noted in the introduction. The resulting spatial locality of the flow renders the formulation similar to other aerodynamic models at low order; this is elaborated on in [28].

### B. Downwash Equation

A number of formulations of the downwash equation exist, which are as a result of differences in the definition of the cylinder orientation and the direction of the piston action. Generally, the formulations are approximately equivalent for small perturbations and small surface inclinations. A variety expressions for the downwash with the corresponding cylinder orientations are noted here.

In the original formulation of Lighthill, the cylinder (and piston action) are perpendicular to the undisturbed (freestream) velocity vector. This is reflected by the theoretical basis, in which the motion is reduced to unsteady one-dimensional in the $z$ direction (in the earth-fixed reference frame) for a body passing through the cylinder at the reference speed $U_{cyl}$. In the cylinder (earth-fixed) reference frame, the piston downwash is then defined as

$$w \equiv \frac{\partial z}{\partial \tilde{t}} \tag{28}$$

Noting the relationships between derivatives in the various reference frames (through Eqs. (15, 22 – 23)), the following expression for the piston downwash is obtained in the original body-fixed frame of the airfoil

$$w = U_{cyl}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} \tag{29}$$

This is equal to the expression given by Lighthill in Eq. (1). The convective and dynamic downwash components are cast, respectively, as

$$w_c = U_{cyl}\frac{\partial z}{\partial x} = U_{cyl}\tan\delta \tag{30}$$

$$w_d = \frac{\partial z}{\partial t} \tag{31}$$

Bird [3] noted that if the velocity component in any direction in a flowfield remains constant, then the flow equations reduce identically to unsteady flow in a plane normal to the direction of unchanged velocity — this may be viewed as the special, exact case of piston theory. Bird notes that two special cases where this occurs are the steady planar oblique shock, and steady planar Prandtl-Meyer expansion. For these cases, it may be shown that the Euler equations reduced identically to one-dimensional unsteady flow in an earth-fixed frame with cylinder oriented normal to the oblique shock or the line of constant velocity components. The axes for the oblique shock case are shown in Fig. 4. As the special case requires a steady two-dimensional problem, the dynamic downwash component is zero, and Bird [3] notes that the convective downwash component is given by

$$w_c = U_\infty\left(\frac{\sin\delta}{\cos(\theta-\delta)}\right) \tag{32}$$

In Bird's formulation, no assumption is made of the smallness of parameters in the flow problem, as is made in the SDT formulation and in Lighthill's CPT formulation. It is noted that for a cylinder defined perpendicular to the freestream ($\theta = 0$), the expression of Bird reduces to that of Lighthill in Eq. (30). Bird [3] consequently recommended that the modeling of CPT for large deflections could be improved by replacing the $\tan\delta$ term in Eq. (30) with $\sin\delta$ on the basis of Bird's downwash expression. From Eq. (32), it is evident that this is equivalent to defining the cylinder as acting perpendicular to the surface ($\theta = \delta$) of the airfoil.

Bird's recommendations are based on the approximation that the term $\theta - \delta$ may be replaced by the free-stream Mach angle. This results in the dimensionless downwash term being equivlaent to the first-order term in Van Dyke's second-order pressure equation, with $\tan\delta$ replaced by $\sin\delta$. It must be noted, however, that the definition of the cylinder as acting normal to the surface of the airfoil is essentially a semi-empirical approximation.

Nevertheless, this approximation was widely implemented based on Bird's recommendation, with several formulations of local piston theory (LPT) effectively being based on the cylinder acting normal to the perturbed surface of the airfoil. This may be shown through consideration of the downwash formulations used by Hunter [8] and Zhang et al [10]. The downwash equation of Hunter [8], with the sign of the flow-induced term corrected, is given as

$$w = -\boldsymbol{U_s} \cdot \boldsymbol{n'} + \boldsymbol{U_b} \cdot \boldsymbol{n'} \tag{33}$$

The downwash equation of Zhang et al [10] is given through

$$w = \boldsymbol{U_s} \cdot \delta\boldsymbol{n'} + \boldsymbol{U_b} \cdot \boldsymbol{n'} \tag{34}$$

$$\delta\boldsymbol{n'} = \boldsymbol{n_s} - \boldsymbol{n'} \tag{35}$$

Noting that for local flow $\vec{U}_s \cdot \vec{n}_s = 0$, the flow-induced downwash components for no airfoil motion ($\boldsymbol{U_b} = 0$) of Eqs. (33, 34) are seen to reduce to

$$w_c = -\boldsymbol{U_s} \cdot \boldsymbol{n'} = U_s \sin\delta \tag{36}$$

with $\delta$ being the deflection of the perturbation surface relative to the mean steady surface.

Whilst the variety of downwash formulations resulting from different cylinder orientations is evident, it should be remembered that the theoretical basis for the piston-cylinder analogy is the reduction of steady flow in $n$ spatial dimensions to unsteady flow in $n-1$ dimensions. This may be achieved through neglecting gradients of certain flow quantities in a particular directions in the fluid equations of motion [1, 16, 21]. In the case of hypersonic small disturbance theory and CPT, the approximations made define the cylinder to act perpendicular to the undisturbed (freestream) flow. Bird's approximation of the cylinder acting perpendicular to the surface of the airfoil is only theoretically valid in the quasi-steady treatment of a tangent-wedge approximation.

### C. Local Piston Theory

Consider an airfoil undergoing small perturbations about a mean steady state. The mean steady position and inclination of the local airfoil surface are considered time-invariant; its trajectory in time sweeps out a flat surface parallel to the local velocity at mean steady conditions. This is depicted in Fig. 5. Perturbations in the relative position of the local airfoil surface in the cylinder and the local surface inclination are considered to vary in time. Their trajectories in time define curved walls. The crux of local piston theory (LPT) is to model the local perturbation pressures over the curved wall using piston theory. The cylinder reference conditions for the flat surface prior to the curved wall are then given by the local mean steady flow conditions.

The basis for LPT follows from an adaptation of classical piston theory (CPT). In LPT, the local mean steady flow at a point on an airfoil is considered to be the undisturbed flow; perturbations in a direction perpendicular to the local mean flow sweep out a two dimensional perturbation surface in time. The unsteady local perturbations relative to the mean steady state may be modeled using piston theory, provided that the perturbation flow swept out in time meets the equivalent validity conditions of Eq. (4), where $\delta$ and $\epsilon$ are measured relative to the mean steady surface (i.e. are perturbation quantities).

### D. Pressure Equation

The original pressure relation used by Lighthill [1] is that for the pressure on the face of a piston moving at uniform speed, generating isentropic simple waves, which for arbitrary cylinder reference conditions is given as

$$\frac{p}{p_{cyl}} = \left(1 + \frac{\gamma - 1}{2}\frac{w}{a_{cyl}}\right)^{\frac{2\gamma}{\gamma-1}} \tag{37}$$

Lighthill deemed this formulation as suitable for the validity condition given in Eq. (4), i.e. for piston motions that do not generate shocks.

As noted by Wood [18], the reduction of the dimension of the flow in hypersonic small disturbance theory does not directly yield the piston theory formulation — the Euler equations must be subjected to a further set of simplifying assumptions together with isentropic gas relations in order for Lighthill's piston theory to be obtained. These include assumptions of constant piston velocity and zero entropy gradients along the cylinder. Alternative formulations of the pressure equation used in piston theory were introduced to circumvent the limitations of the isentropic simple wave equation. For example, for compression flows, the equation for the pressure behind an oblique shock over an equivalent tangent wedge in hypersonic flow has been used

$$\frac{p}{p_{cyl}} = 1 + \gamma K^2 \left[ \frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}} \right] \tag{38}$$

Ericsson et al [20] note that for flows of $K \geq 1$, Eq. (38) is equivalent to what was termed "strong shock" piston theory. The use of the equation leads to a tangent-wedge approximation. Ericsson et al further note that the tangent-wedge approximation yields more accurate results for the surface pressure than the isentropic pressure relation for flow with $K > 1$.

The use of further pressure relations in a piston-theory formulation was investigated by Liu et al [9]. The pressure relations typically vary from third-order terms and higher — this was noted by Lighthill [1] and Liu et al [9] to be associated with the differences in flow physics between isentropic expansion and shock waves (e.g. rotationality introduced by the shock wave). Further examples of pressure relations include the use of Van Dyke's second-order theory [29] and the pressure relation derived by Dowell and Bliss [21] for potential flow with arbitrary reduced frequency and Mach number.

The selection of the pressure equation used in the piston theory formulation effectively only determines the coefficients $c_1$, $c_2$, $c_3$, etc. used. That is to say, although the theoretical basis for the pressure equations may differ, the equations yield an equivalent relationship between the piston pressure and piston downwash, albeit with different coefficients. This has led to some ambiguity, as various formulations with different validities have fallen under the broad term of "piston theory". As an example, the isentropic simple wave formulation of Lighthill is only valid in a small disturbance formulation in the approximate range of $0.34 < K < 1$, whereas the potential-flow basis for Van Dyke's formulation extends to lower values of $K$.

The pressure equations further vary in the parameter upon which the pressure is dependent: for the simple wave formulation, the pressure is dependent on the velocity of the piston (perpendicular to the undisturbed stream); for the tangent-wedge, Van Dyke, and other formulations, the pressure is dependent on the surface inclination $\delta$. The pressure formulations dependent on $\delta$ may only be used in the form of Eq. (8) under limiting assumptions of small $\delta$, such that $\delta \approx \sin\delta \approx \tan\delta$, so that the convective downwash approximation of Eqs. (30, 36) may be made.

## IV. Generalized Formulation

### A. Generalized Pressure Equation

The unification of several formulations of classical piston theory (CPT) through the introduction of a generalized pressure equation by Rodden et al (Eq. (8)) is extended here to include local piston theory (LPT). As noted previously, the differences between CPT and LPT chiefly arise from differences in the reference frame used; however, in both theories the piston-cylinder analogy is used. The equation of Rodden et al may be extended through formulation in terms of the cylinder reference conditions. It is assumed that the pressure equation upon which the piston theory is based may be expanded in powers of the piston downwash as

$$\frac{p}{p_{cyl}} = 1 + \gamma \left[ c_1 \left( \frac{w}{a_{cyl}} \right) + c_2 \left( \frac{w}{a_{cyl}} \right)^2 + c_3 \left( \frac{w}{a_{cyl}} \right)^3 \right] \tag{39}$$

The pressure coefficient is referenced to freestream quantities, and is therefore given by

$$C_p = C_{p_{cyl}} + \left( \frac{p_{cyl} M_{cyl}^2}{p_\infty M_\infty^2} \right) \frac{2}{M_{cyl}^2} \left[ c_1 \left( \frac{w}{a_{cyl}} \right) + c_2 \left( \frac{w}{a_{cyl}} \right)^2 + c_3 \left( \frac{w}{a_{cyl}} \right)^3 \right] \tag{40}$$

where

$$C_{p_{cyl}} = \frac{p_{cyl} - p_\infty}{\frac{1}{2} \gamma p_\infty M_\infty^2} \tag{41}$$

The extension to LPT offered by Eq. (40) requires generalization of the downwash expression, $w$, to account for the differences in modeling betweeen various piston theories and between CPT and LPT.

### B. Generalized Downwash Equation

A generalized formulation of the downwash equation for arbitrary cylinder orientation and for in-plane motion of the local airfoil surface is presented here, applicable to both CPT and LPT. The downwash is considered as arising from a dynamic component and from a convective component. The dynamic component represents the flow or body motion in the direction of the cylinder. The convective component gives the downwash arising from local surface inclination relative to the cylinder-normal. The nomenclature is defined in Fig. 6, with downwash defined as positive for compression.

The downwash equation is formulated in the $\zeta$-$\xi$ axes of Fig. 6b, in which the cylinder is oriented at an angle $\theta$ relative to the freestream-normal ($\theta$ positive for inclination into the freestream). Note that when applied to LPT, the *perturbation* surface is modeled: that is, the surface is defined by $z = z_p$, $\zeta = \zeta_p$. A wind-tunnel reference frame is used, with the observer and cylinder fixed in space, and with the airfoil on a flexible mount, allowing general motion about a position of static equilibrium. The downwash on the upper surface of the airfoil is given by

$$w = w_c + w_d \tag{42}$$

where

$$w_c = \left(U_\xi - U_{b\xi}\right)\frac{\partial \zeta}{\partial \xi} \tag{43}$$

$$w_d = -U_\zeta + U_{b\zeta} \tag{44}$$

The flow and body motion components follow as

$$U_\xi = U_{cyl}\cos\theta \tag{45}$$

$$U_\zeta = -U_{cyl}\sin\theta \tag{46}$$

$$U_{b\xi} = \frac{\partial \xi}{\partial t} \tag{47}$$

$$U_{b\zeta} = \frac{\partial \zeta}{\partial t} \tag{48}$$

The equations may then be rearranged to give

$$w_c = \left(-U_{cyl}\cos\theta + \frac{\partial \xi}{\partial t}\right)\tan\left(\theta - \delta\right) \tag{49}$$

$$w_d = U_{cyl}\sin\theta + \frac{\partial \zeta}{\partial t} \tag{50}$$

Leading to the generalized downwash expression

$$w = U_{cyl}\frac{\sin\delta}{\cos\left(\theta - \delta\right)} + \frac{\partial \xi}{\partial t}\tan\left(\theta - \delta\right) + \frac{\partial \zeta}{\partial t} \tag{51}$$

The generalized equation may be shown to reduce to the downwash expression used in various formulations of both CPT and LPT, using the appropriate assumptions in formulation given in Table 1, and noting that $\frac{\partial \zeta}{\partial t} \equiv \boldsymbol{U_b} \cdot \boldsymbol{n}'$. In applying the generalized downwash expression to LPT, it should be remembered that the *perturbation* surface is modeled.

Thus, the generalized downwash formulation gives a consistent description of the downwash on the piston surface for arbitrary cylinder conditions and orientation, and for arbitrary motion of the body. The equation reduces to the established special cases in literature for both CPT and LPT.

## V. Discussion

### A. Limits of Validity

The generalized formulation presented here which gathers various aerodynamic methods which have a formulation resembling that of Lighthill's piston theory, extending the work of Rodden et al [17] to include local piston theory. Whilst the formulations resulting from the various methods may be expressed in a generalized form, the theoretical basis of the pressure equations differ and must be evaluated before applying the method under consideration.

The theoretical basis of Donov's [30] pressure equation lies in small perturbations from a characteristics development of inviscid supersonic flows; thus, no limiting assumption is made regarding Mach number, other than that the flowfield is everywhere supersonic. Donov's formulation of the pressure equation is in terms of a power series in $\delta$. The error introduced through using the coefficients in a power series in terms of $\tan\delta$ instead is negligible for small $\delta$ of approximately $\delta \leq 10°$.

The physical basis of Lighthill's [1] piston theory, in which he assumed that flow in the fluid cylinders was independent between cylinders, has a mathematical foundation similar to Van Dyke's hypersonic small disturbance theory [25] (SDT). Lighthill's pressure equation further limits the validity to flows where the piston downwash is subsonic ($K < 1$). The lower limit on the value of the hypersonic similarity $K$ may not be immediately obvious in Lighthill's formulation, as it follows from Van Dyke's order of magnitude analysis in hypersonic SDT. (It should be noted that the coefficients from Van Dyke's unified supersonic-hypersonic theory do not share the same lower limit.) Van Dyke introduces [25] the non-dimensionalized quantities below, assuming that both have bounded growth of $O(1)$ as $\tau \to 0$ for fixed $K$:

$$w = \tau U_\infty \hat{w} \qquad (\hat{w} = O(1)) \tag{52}$$

$$p = \gamma M_\infty^2 \tau^2 p_\infty \hat{p} \qquad (\hat{p} = O(1)) \tag{53}$$

Lighthill's pressure equation may then be recast in terms of the non-dimensionalized quantities as

$$\hat{p} = \frac{1}{\gamma K^2}\left(1 + \frac{\gamma - 1}{2} K\hat{w}\right)^{\frac{2\gamma}{\gamma - 1}} \tag{54}$$

The behaviour of the function with $\gamma = 1.4$ and $\hat{w} = 1$ is shown in Fig. 7, and it is noted that the assumption of $\hat{p} = O(1)$ is violated in the limit as $K \to 0$ and as $\tau \to \infty$. If an approximate limit is set to confine $\hat{p}$ to be within order of magnitude of $\hat{w} = 1$, the function is limited to the range $0.336 \le K \le 8.38$ for air ($\gamma = 1.4$). This is in agreement with Van Dyke's more general formulation, in which he notes that the reduction in flow dimension offered by hypersonic SDT breaks down in the range of linearized supersonic flow [25].

Thus, Lighthill's classical piston theory (CPT) is theoretically only valid in the range $0.336 < K < 1$ for Van Dyke's assumption of $O(\hat{w}) = 1$. For the simple case of a wedge, the limit of an attached oblique shock adds a third bound on the validity of Lighthill's CPT in terms of $\delta$ and $M_\infty$. The bounds of validity are shown in Fig. 8. Refering to the figure, the region above the shock-detachment line may be divided into three regions (A, B, and C). Lighthill's CPT is theoretically valid in region B; region A represents flows in which supersonic SDT is more appropriate; region C defines flow for which the Mach number of the perturbation down the cylinder length is greater than unity.

The extended piston theory of Dowell and Bliss [21] is based on linearized supersonic potential flow — however, in the limit of high Mach number, the extended theory tends to linear classical piston theory. Thus, it may be expected that the extended theory shares the same upper limit of validity of $K < 1$, whilst enjoying an extension of the lower limit into low supersonic Mach numbers - in terms of Fig. 8, this represents an extension of the validity to include region A in addition to B.

In the discussion of the validity made thus far, no mention has been made of the reduced frequency $k$ of vibration. The reduced frequency gives a measure of the unsteadiness of the flow, as it represents the ratio between the rate at which disturbances are introduced into the flow (through structural vibration) and the rate at which they are washed downstream. In assuming that flows in fluid cylinders are independent of one another, the influences are neglected. This renders piston theory a quasi-steady theory — the aerodynamic loading is dependent on the instantaneous flow field. In a similar sense, local piston theory is limited to perturbations about the mean steady state that are sufficiently small that the change in the mean flow field may be neglected relative to the local perturbations. It is noted that the validity of piston theory in terms of reduced frequency generally [9] appears in terms coupled with Mach number, such as $kK < 1$ or $kM^2 \gg 1$. For flows with significant unsteady effects, the validity of piston theory is limited to modeling very short durations of flow, with successive updates of the reference "quasi-steady" state. A quantitative treatment of the validity range of linear piston theory for small values as $k$, based in potential flow analysis, is given in [28]; the treatment serves as a quantitative estimate of the validity range for the extended piston theory of Dowell and Bliss [21] for small reduced frequencies.

### B. Accuracy of Theories

The various aerodynamic methods covered by the generalized pressure equation of Eq. (40) are identified by the coefficients $c_1$, $c_2$, and $c_3$, and by the order of the equation. The relative merits of one set of coefficients over another or the justification of higher-order theories is dependent on the problem being analyzed — the accuracy and validity in part depend on the application. This should be considered in the discussion that follows, in which the relative accuracy of the theories as applied to flow over a planar wedge is considered; it can not be expected that Donov's development for planar flows can be extended with the same accuracy to three-dimensional flows, for example.

Consider the flow over a planar wedge, modeled in the sense of classical piston theory, with the generalized formulation of Eq. (40) used to find the total pressure. The equation may be recast in terms of the non-dimensionalized downwash of Eq. (52) into the hypersonic similarity parameter of

$$\frac{C_p}{\tau^2} = 2\left(c_1 \hat{w} K^{-1} + c_2 \hat{w}^2 + c_3 \hat{w}^3 K\right) \tag{55}$$

It is noted that in general (for Van Dyke's theory and for Donov's analysis), the coefficients $c_1$, $c_2$, and $c_3$ are functions of $M$. For a cylinder orientation perpendicular to the freestream, it may be shown that for the stationary wedge, $\hat{w} = 1$; the differences in $\hat{w}$ for cylinder orientation perpendicular to the surface are negligible for a wedge semi-angle of $\delta = 10°$. With this consideration, the wedge surface pressure coefficient predicted by the various theories from Eq. (55) is shown in Fig. 9 for $\delta = 10°$ and $\hat{w} = 1$. The angle $\delta = 10°$ has been chosen for direct comparison with the similar figure of Liu et al [9]. The error in the pressure predictions relative to exact shock results are shown in Fig. 10a, and the error in the slope prediction is shown in Fig. 10b.

It is well known that the wedge pressure coefficient becomes Mach independent for large $K$; this is reflected in the exact shock results. However, consideration of the behaviour of Eq. (55) in the limit of large $K$ shows that Mach independence is only achieved if third-order terms are discarded for large $K$, or if $c_3$ is developed such that $c_3 K = O(1)$ in the limit $K \to \infty$. The second-order theories are seen to have the correct behaviour in the limit of large $K$. It is of interest to note that, with the exception of Donov's compression series, the theories tend to the corresponding order of Lighthill's classical piston theory in the limit of large $K$; these theories are all based on the assumption of isentropic flow disturbances, where Donov's compression series tends to the hypersonic limit for oblique shocks. In considering the pressure prediction for flows with $K < 1$, it is noted that for all the theories, successively better modeling is achieved through the inclusion of higher-order terms. Physically, these higher-order contributions are from linear (second-order term) and nonlinear (third-order term) thickness effects. It is noted that near $K = 1$, the contribution from nonlinear thickness effects becomes important, whilst for small $K$ (in the supersonic range), linear thickness effects dominate.

The order of magnitude of contribution by the higher-order terms relative to the first-order terms of the theories are compared in Fig. 11, with the nomenclature defined by recasting Eq. (55) as

$$\frac{C_p}{\tau^2} = \chi_1 \hat{w} + \chi_2 \hat{w}^2 + \chi_3 \hat{w}^3 \tag{56}$$

and introducing,

$$\frac{\partial \left( \frac{C_p}{\tau^2} \right)}{\partial K} = \psi_1 \hat{w} + \psi_2 \hat{w}^2 + \psi_3 \hat{w}^3 \tag{57}$$

In contrast to Lighthill's CPT, the contribution of the second-order term from Van Dyke's [29] theory to the slope is non-zero. (However, it should be noted that for the symmetrical wedge, the second-order downwash contribution of any theory has zero effect on the damping-in-pitch.) The third-order terms only begin to make an order-of-magnitude contribution at high $K$, in the region of $K > 0.8$. Nonetheless, the contribution of the third-order term at high $K$ is important in correctly modeling the pressure derivative, as is noted from the difference in accuracy between

Van Dyke's second-order theory and Donov's third-order theories (the theories are equivalent up to second-order).

From this consideration of the contributions from the various order terms, the following generalization regarding higher-order terms is made: the inclusion of second-order terms is important in modeling the pressure for all $K$, but make no contribution to the modeling of damping; third-order terms are important in modeling the effect of nonlinear thickness effects on the pressure slope, and become increasingly important as $K \to 1$.

Whilst the discussion thus far has been in terms of the application of CPT, important conclusions may also be drawn for local piston theory (LPT). The LPT equivalent of $K$ is $M_s \tan \delta_p$, and for sufficiently small perturbations $\delta_p$ it is seen that the perturbation $K$ may in turn be small enough that the contribution from terms higher than first-order may reasonably be neglected.

A second implication for LPT is drawn from the relative performance in modeling of Van Dyke's [29] theory compared to Lighthill's theory at small $K$. That is to say, the calculation of the coefficient $c_1$ at low $K$ becomes important for accurate modeling. This suggests that LPT may be further improved by utilising the coefficient $c_1$ from Van Dyke's theory or from the extended piston theory of Dowell and Bliss; however, this would come at the cost of calculating the local Mach number $M_s$ on the surface of the airfoil. This, in turn, could serve as a useful check to investigate regions of the flow (subsonic) in which LPT may be theoretically invalid.

## VI. Conclusions

The generalized formulation of piston theory in terms of cylinder reference conditions put forward in the present work offers a consistent approach to piston theory applicable to both local and piston theory. The accompanying generalized formulation of the downwash equation in terms of arbitrary cylinder orientation and arbitrary motion in the plane of the airfoil serves a dual purpose: it highlights the subtle differences between implementations of piston theory in literature, whilst giving a general basis from which the implementations may be derived. The review of the theoretical basis for Lighthill's classical piston theory provides a theoretical basis for the suggested lower limit of validity as approximately $K > 0.34$.

A comparison of the modeling accuracy of Lighthill's formulation, Van Dyke's formulation, and Donov's third-order formulations for a planar highlights the relative importance of the coefficients used in piston theory, as well as the relative necessity of higher-order terms. It is shown that for the static wedge, third-order terms only become important in the lower limit of $K$ — that is to say, as the shock-detachment Mach number is approached. When considering first-order piston theory in the limit of small $K$, it is seen that the use of Van Dyke's coefficient yields significantly better modeling relative to the use of Lighthill's first-order coefficient. Hence, considering the small value of the perturbation $K$ associated with local piston theory, it is suggested that local piston theory may be further improved through using a first-order coefficient which is calculated for the local flow conditions.


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**Table 1 Assumptions present in piston theory downwash formulations**

| Formulation | $U_{cyl}$ | $\theta$ | $\frac{\partial \xi}{\partial t}$ | $\frac{\partial \zeta}{\partial t}$ |
|---|---|---|---|---|
| Lighthill (1953) | $U_\infty$ | 0 | 0 | – |
| Bird (1958) | $U_\infty$ | $\beta$ | 0 | 0 |
| Hunter (1997) | $U_s$ | $\delta$ | – | – |
| Current work | – | – | – | – |

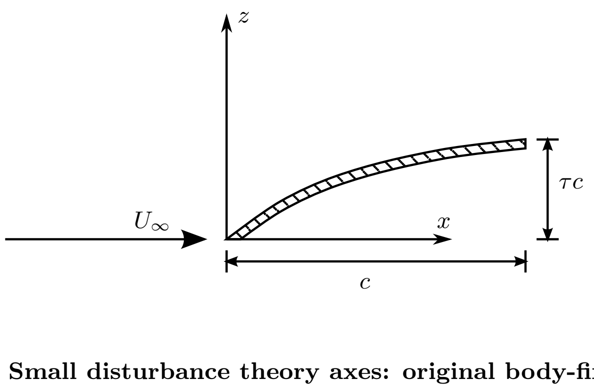



**Fig. 1 Small disturbance theory axes: original body-fixed axes.**

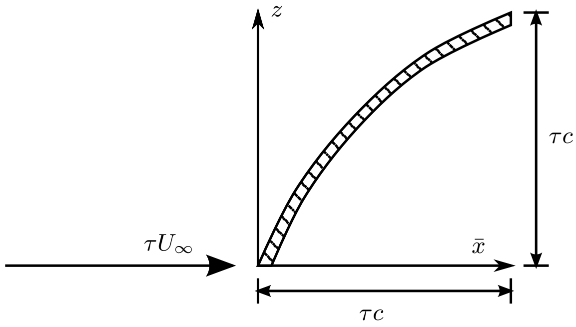



**Fig. 2 Small disturbance theory axes: transformed body-fixed axes.**

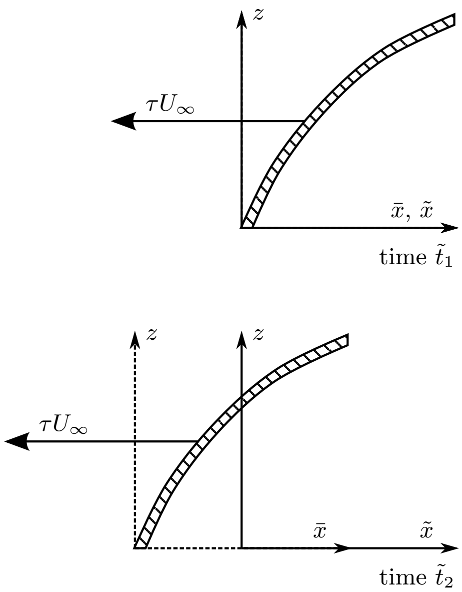



**Fig. 3 Small disturbance theory axes: transformed earth-fixed axes.**

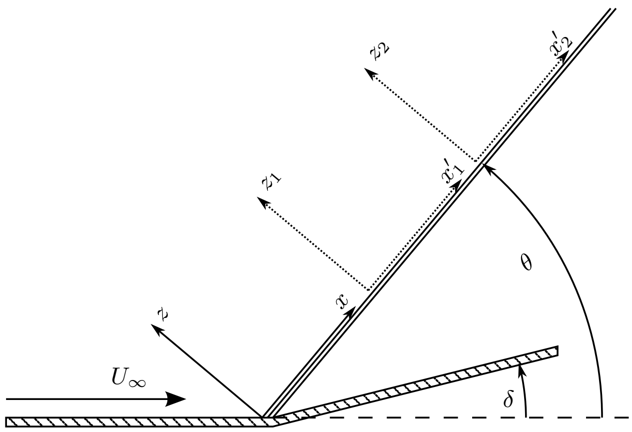



**Fig. 4 Axes in Bird's formulation for an oblique shock.**

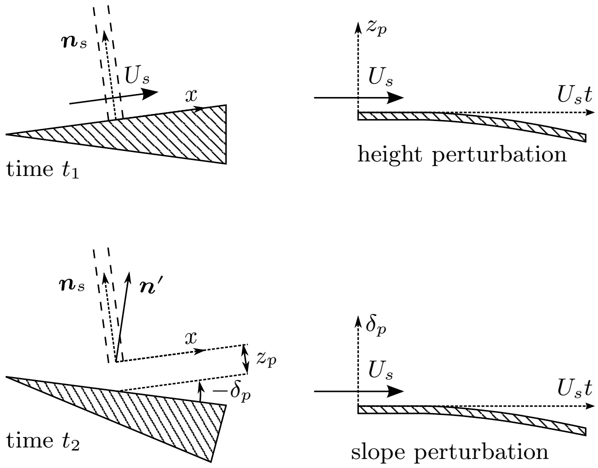



**Fig. 5 Time-swept surface in local piston theory.**

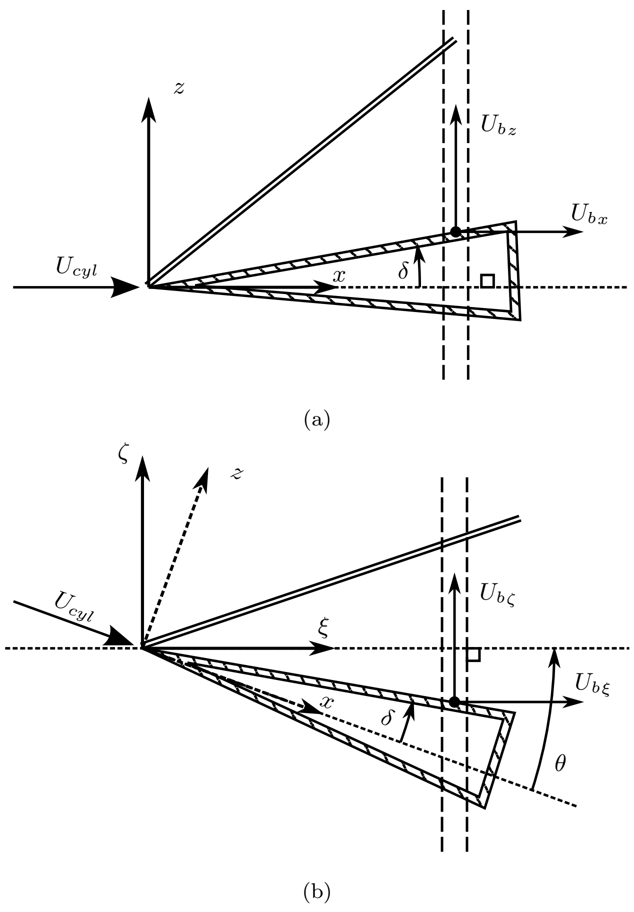



**Fig. 6 Basis for the downwash equation for: (a) Lighthill's cylinder orientation (b) an arbitrary cylinder orientation.**

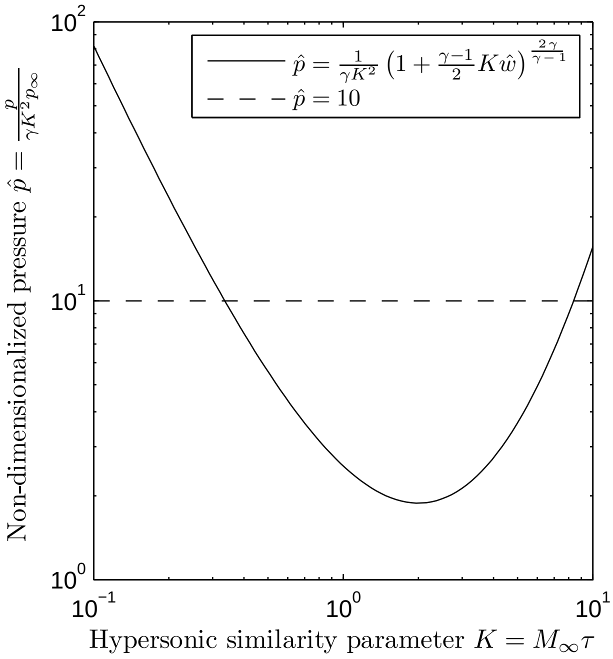



**Fig. 7 Variation of non-dimensionalized piston pressure in the simple wave equation.**

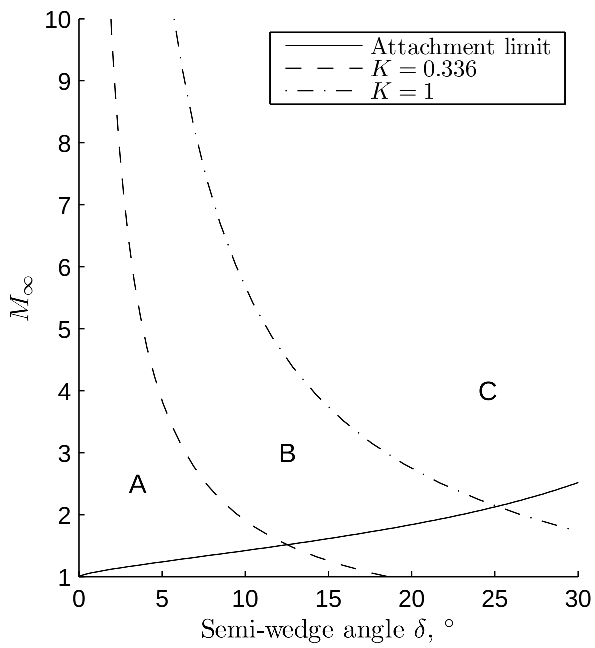



**Fig. 8 Validity bounds of Lighthill's CPT.**

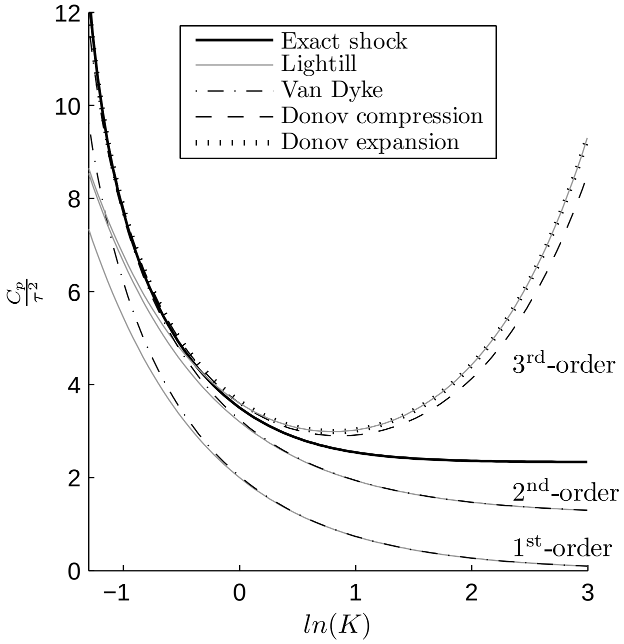
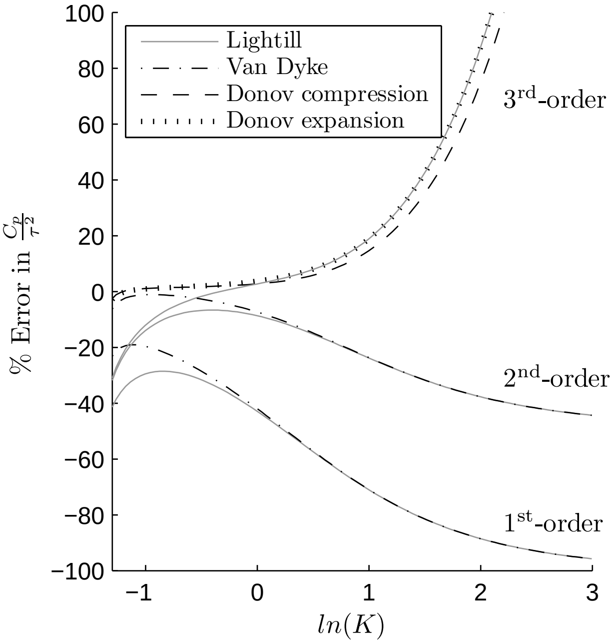
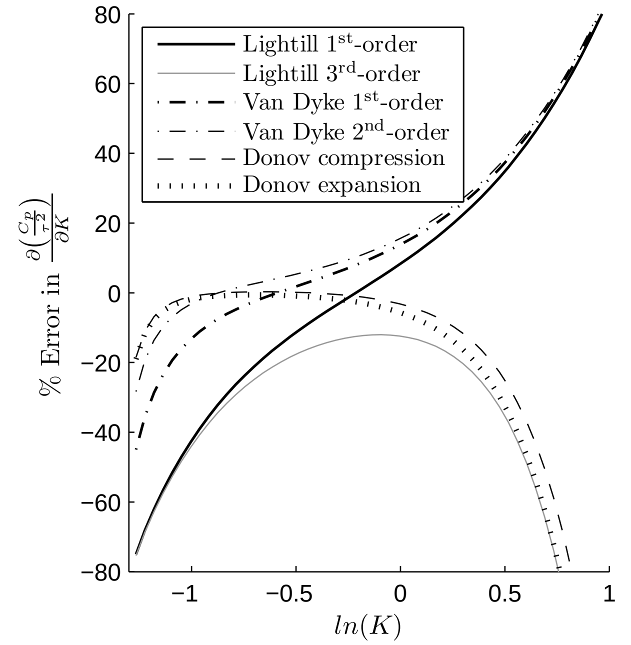



**Fig. 9 Pressure coefficient predictions for semi-wedge angle $10^{\circ}$.**




(a)




(b)

**Fig. 10 Prediction error for semi-wedge angle $10^{\circ}$ in: (a) pressure coefficient (b) pressure coefficient slope.**

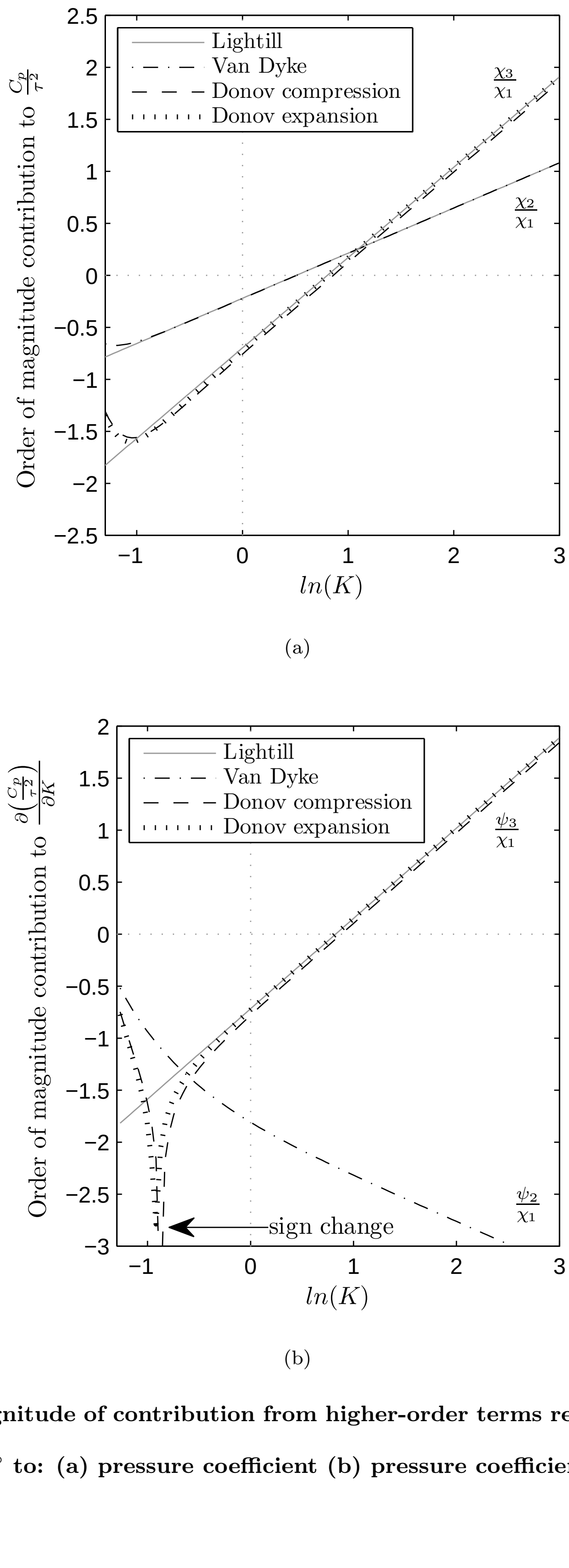



**Fig. 11 Order of magnitude of contribution from higher-order terms relative to first-order for semi-wedge angle $10^{\circ}$ to: (a) pressure coefficient (b) pressure coefficient slope.**