A Computational Study of Three Numerical Methods for Some Advection-Diffusion Problems

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Abstract

Three numerical methods have been used to solve two problems described by advection-diffusion equations with specified initial and boundary conditions. The methods used are the third order upwind scheme [4], fourth order upwind scheme [4] and Non-Standard Finite Difference scheme (NSFD) [9]. We considered two test problems. The first test problem has steep boundary layers near $x = 1$ and this is challenging problem as many schemes are plagued by non-physical oscillation near steep boundaries [15]. Many methods suffer from computational noise when modelling the second test problem especially when the coefficient of diffusivity is very small for instance 0.01. We compute some errors, namely L_2 and L_∞ errors, dissipation and dispersion errors, total variation and the total mean square error for both problems and compare the computational time when the codes are run on a matlab platform. We then use the optimization technique devised by Appadu [1] to find the optimal value of the time step at a given value of the spatial step which minimizes the dispersion error and this is validated by some numerical experiments.

Keywords: dispersion, dissipation, total variation, oscillations, advection-diffusion, optimization

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1. Introduction

The advection-diffusion equation is one of the most challenging equations in science as it represents a superposition of two different transport processes: advection and diffusion [11]. Thus, the numerical solution of convection diffusion problems is important. In practical applications, the advection-diffusion equation has been used to describe heat transfer in a draining film [7], water transport in soils [20], mass transfer [5], flow in porous media [8].

The 3D advection-diffusion equation is given by:

$$\frac{\partial u}{\partial t} + \beta_x \frac{\partial u}{\partial x} + \beta_y \frac{\partial u}{\partial y} + \beta_z \frac{\partial u}{\partial z} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} + \alpha_z \frac{\partial^2 u}{\partial z^2}$$

where $\beta_x, \beta_y, \beta_z$ are the velocity components of advection in the direction of x, y and z, respectively, and α_x, α_y, and α_z are diffusivities in the x, y and z directions, respectively.
In this study we solve the 1D advection diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, 0 < t \le T, \tag{1}$$

with initial condition

$$u(x, 0) = f(x),\ 0 \le x \le 1,$$

and boundary conditions

$$u(0, t) = g_0(t),\ 0 < t \le T$$
$$u(1, t) = g_1(t),\ 0 < t \le T.$$

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2. Organization of Paper

In section 3, we study the dissipative and dispersive characteristics of some numerical methods for the 1D advection diffusion equation. Section 4 describes the two test cases considered in this paper. In section 5, we show how to quantify errors from numerical results into dispersion and dissipation errors using the technique by Takacs [12]. In sections 6-8, we obtain the stability of the three schemes using the approach of Hindmarsh et al. [6]. Numerical results are presented in section 9. In section 10, we obtain the optimal temporal step size using a technique devised by Appadu [1] to minimize the dispersion error at a given Reynolds number and spatial step size and this is validated by some numerical experiments for problem 1. In section 11, we highlight the salient features of the paper.

3. Numerical Dissipation and Dispersion

Finite difference schemes used to solve partial differential equations will often lose energy as time t progresses, this property is called numerical dissipation [17]. In the case of dispersive schemes, oscillations are generated in regions of discontinuity. We let the elementary solution of Eq. (1) be

$$u(x,t) = \exp(\gamma t)\exp(I\theta\, x), \tag{2}$$

where θ is a wave number and γ is dispersion relation. On plugging Eq. (2) into Eq. (1), we get

$$\gamma = -\theta I - \alpha\theta^2 \tag{3}$$

Hence

$$u(x,t) = \exp\left[(-\theta I - \alpha\theta^2)t\right]\, \exp(I\theta x). \tag{4}$$

From Eq (4), we deduce that the partial differential equation given by (1) represents a wave with exponentially decaying amplitude travelling at a constant speed. The exact amplification factor is obtained as

$$\xi_{exact} = \frac{u(x,t+\Delta t)}{u(x,t)} = \exp((-I\theta - \alpha\theta^2)\Delta t). \tag{5}$$

The modulus of the exact amplification factor is then given by

$$|\xi_{exact}| = \exp(-\alpha\theta^2\Delta t). \tag{6}$$

The numerical amplification factor, ξ_{num} is obtained using von Neumann stability analysis. The relative phase error (RPE) is obtained as

$$RPE = \frac{\arg(\xi_{num})}{\arg(\xi_{exact})} = -\frac{arg(\xi_{num})}{\theta\,\Delta t} = -\frac{arg(\xi_{num})}{c\,\omega}, \tag{7}$$

where c is courant number and $\omega = \theta h$ is phase angle, with h being the spatial step size.

4. Test Cases

In this paper, we consider two test problems to compare the performance of the three numerical methods, namely third order, fourth order and NSFD.

4.1. Problem 1

We solve a problem from [16] which is described by the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \tag{8}$$

with initial condition

$$u(x,0) = 0,\ 0 < x < 1 \tag{9}$$

and boundary conditions

$$u(0,t) = 0,\ u(1,t) = 1,\ t > 0. \tag{10}$$

The analytical solution to this equation can be obtained by using the method of separation of variables

$$u(x,t) = \left[\frac{\exp(\text{Re } \times x) - 1}{\exp(\text{Re}) - 1}\right] + \sum_{m=1}^{\infty} \left\{ \frac{(-1)^m m\pi}{(m\pi)^2 + \frac{\text{Re}^2}{4}} \exp\left(\frac{\text{Re} \times (x-1)}{2}\right) \sin(m\pi x) \exp\left[-t\left(\frac{(m\pi)^2}{\text{Re}} + \frac{\text{Re}}{4}\right)\right]\right\}, \quad (11)$$

where Re is Reynolds number.

In [15], they have used two values of Re, namely 100 and 10,000 and two values of step-size (for coarse and fine grids), namely 0.1 and 0.025. The temporal step size was chosen as 0.01. This test case is quite challenging as for instance at Re $=$ 100, numerical solutions of Crank-Nicolson method show non-physical oscillations while the scheme constructed by Ding and Zhang [18] is not accurate on coarse mesh. Moreover, for Re $=$ 10000, the scheme by Ding and Zhang gives inaccurate solutions due to dispersion.

In this work, we consider three values of Re, say 10, 100, and 10000, and two values of h, say 0.1 and 0.025. The temporal step size is chosen as 0.01. We compute the L_2, L_∞,total variation, dissipation and dispersion errors when the three schemes are used to solve problem 1 at time, $T = 1$. Table (1) gives the regions of stability of the three schemes at some values of Re and h. The errors are shown in Tables (2) to (4) and the numerical and exact plots are shown in Figs. (1a) to (1e).

4.2. Problem 2

We consider the advection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0 \quad (12)$$

with the boundary conditions:

$$u(0,t) = u(1,t) = 0,$$

and the initial condition

$$u(x,0) = 3\sin(4\,\pi\, x).$$

The exact solution of the problem is given by [19]

$$u(x,t) = \exp\left[\frac{1}{2\alpha}\left(x - \frac{t}{2}\right)\right] \sum_{j=1}^{\infty} \zeta_j \exp\left(-\alpha\, j^2\, \pi^2\, t\right) \sin(j\pi x),$$

where

$$\zeta_j = \frac{3}{2\alpha}\left[1 + (-1)^{j+1}\exp(-\frac{1}{2\alpha})\right]\left[\frac{1}{(\frac{1}{2\alpha})^2 + (j-4)^2\,\pi^2} - \frac{1}{(\frac{1}{2\alpha})^2 + (j+4)^2\,\pi^2}\right].$$

In [19], they have used a one-parameter family of unconditionally stable third order time-integration scheme with temporal and spatial step sizes being $k = 0.25$ and $h = 0.05$, respectively and compared their results with Crank-Nicolson which is highly oscillatory. In this work, we consider four combinations of values of α and h; namely $\alpha = 0.01, h = 0.1$; $\alpha = 0.1, h = 0.05$; $\alpha = 1, h = 0.05$ and $\alpha = 1, h = 0.1$ and display the numerical results at time, $T = 1$. For each choice α and h, we consider four different values of the temporal step size for which the methods are stable and the regions of stability are depicted in Table 6.

5. Quantification of errors from numerical results

In this section, we describe how Takacs [12] quantifies errors from numerical results into dispersion and dissipation errors.

The Total Mean Square Error (TMSE) is calculated as

$$TMSE = \frac{1}{N}\sum_{i=1}^{N}(u_i - v_i)^2, \quad (13)$$

where u_i represents the analytical solution and v_i, the numerical (discrete) solution at a given grid point i and N is the number of discrete points.
The Total Mean Square Error can be expressed as

$$\frac{1}{N}\sum_{i=1}^{N}(u_i - v_i)^2 = \frac{1}{N}\sum_{i=1}^{N}(u_i - \bar{u})^2 + \frac{1}{N}\sum_{i=1}^{N}(v_i - \bar{v})^2 + \frac{2}{N}\sum_{i=1}^{N}u_i\bar{u} + \frac{2}{N}\sum_{i=1}^{N}v_i\bar{v} - \frac{1}{N}\sum_{i=1}^{N}(\bar{u})^2 - \frac{1}{N}\sum_{i=1}^{N}(\bar{v})^2 - \frac{2}{N}\sum_{i=1}^{N}u_i v_i. \tag{14}$$

The right hand side of Eq. (14) can be rewritten as

$$\sigma^2(u) + \sigma^2(v) + 2(\bar{u})^2 + 2(\bar{v})^2 - (\bar{u})^2 - (\bar{v})^2 - \frac{2}{N}\sum_{i=1}^{N}u_i v_i,$$

where $\sigma^2(u)$ and $\sigma^2(v)$ denote the variance of u and v, respectively, $\bar{u}$ and $\bar{v}$ denote the mean values of u and v, respectively. Then we have

$$TMSE = \sigma^2(u) + \sigma^2(v) + (\bar{u} - \bar{v})^2 - 2\,Cov(u, v),$$

where $Cov(u, v) = \frac{1}{N}\sum_{i=1}^{N}u_i v_i - \bar{u}\bar{v}$. The Total Mean Square Error can also expressed as

$$(\sigma(u) - \sigma(v))^2 + (\bar{u} - \bar{v})^2 + 2(1 - \rho)\sigma(u)\sigma(v), \tag{15}$$

where $\rho = Cov(u, v)/(\sigma(u)\sigma(v))$ is the coefficient of correlation. The expression $2(1 - \rho)\sigma(u)\sigma(v)$ measures the dispersion error and $(\sigma(u) - \sigma(v))^2 + (\bar{u} - \bar{v})^2$ measures the dissipation error.
We also obtain values of the L_2 and L_∞ errors which are obtained by the following formulae:

$$L_2 = \sqrt{h\sum_{i=1}^{N}(u_i - v_i)^2}, \tag{16}$$

$$L_\infty = \max|u_i - v_i|. \tag{17}$$

Numerical methods must have monotone and Total Variation Diminishing properties. A numerical scheme is said to be monotone if it produces a monotonic distribution after advection, given a distribution that is monotonic before advection. Monotonic methods neither create new extrema in the solution nor amplify existing extrema. Monotonic schemes are classified broadly into two classes: Flux-corrected transport (**FCT**) and Flax Limiter Method (**FLM**).

In **FCT**, the advective fluxes are essentially a weighted average of a lower-order monotonic scheme and a higher-order non-monotonic scheme. In **FLM**, the advective fluxes of a high-order scheme are modified so that the total variation of the solution does not increase with time. This property is called Total Variation Diminishing (**TVD**). The Total Variation (TV) of a function u is defined as

$$TV = \sum_{i=1}^{N-1}|u_{i+1} - u_i|.$$

A **TVD** scheme ensures that $TV(u^{n+1}) \leq TV(u^n)$ which signifies that the overall amount of oscillations is bounded [21]. All monotone schemes are TVD. All TVD schemes are monotonically preserving methods.

6. Third Order Upwind Explicit Scheme

In [4], a third order upwind and fourth order upwind schemes have been used to solve a 1-D problem described by a constant coefficient advection-diffusion equation with smooth initial conditions and quite good results with high accuracy have been obtained. In section 6-8 we describe the three methods and obtain the order of accuracy and the region of stability. We now describe how the third order upwind scheme is constructed.

$$\left.\frac{\partial u}{\partial t}\right|_i^n \simeq \frac{u_i^{n+1}-u_i^n}{\Delta t}, \tag{18}$$

$$\left.\frac{\partial u}{\partial x}\right|_i^n \simeq \left(\frac{2c^2+3c+12s-2}{12}\right)\left(\frac{u_i^n-u_{i-2}^n}{2\Delta x}\right)+\left(\frac{2c^2-3c+12s-2}{12}\right)\left(\frac{u_{i+2}^n-u_i^n}{2\Delta x}\right)$$
$$+\left(\frac{4-c^2-6s}{3}\right)\left(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right), \tag{19}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n \simeq \left(\frac{6s-12sc+2c-2c^3+3c^2}{6s}\right)\left(\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\right)$$
$$+\left(\frac{12sc-2c+2c^3-3c^2}{6s}\right)\left(\frac{u_{i+2}^n-2u_i^n+u_{i-2}^n}{4(\Delta x)^2}\right). \tag{20}$$

where $c=\frac{k}{h}$ and $s=\frac{\alpha k}{h^2}$.
On substitution of Eqs. (18)-(20) into Eq. (1), we have the following:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\left(\frac{2c^2+3c+12s-2}{12}\right)\left(\frac{u_i^n-u_{i-2}^n}{2\Delta x}\right)+\left(\frac{2c^2-3c+12s-2}{12}\right)\left(\frac{u_{i+2}^n-u_i^n}{2\Delta x}\right)$$
$$+\left(\frac{4-c^2-6s}{3}\right)\left(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right)=\alpha\left[\left(\frac{6s-12sc+2c-2c^3+3c^2}{6s}\right)\left(\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\right)\right.$$
$$\left.+\left(\frac{12sc-2c+2c^3-3c^2}{6s}\right)\left(\frac{u_{i+2}^n-2u_i^n+u_{i-2}^n}{4(\Delta x)^2}\right)\right].$$

On simplification we get

$$u_i^{n+1}=\frac{1}{24}[4c^3+24sc-4c]u_{i-2}^n+\frac{1}{6}(6c-3c^3-18sc+6s+3c^2)u_{i-1}^n+$$
$$\left[1-\frac{c^2}{4}-\frac{1}{12}(24s-36sc+6c-6c^3+9c^2)\right]u_i^n+\frac{1}{6}(-2c-c^3-6sc+6s+3c^2)u_{i+1}^n.$$

Therefore, the third order scheme is given by

$$u_i^{n+1}=A_1\,u_{i-2}^n+A_2\,u_{i-1}^n+A_3\,u_i^n+A_4\,u_{i+1}^n. \tag{21}$$

where

$$A_1=\frac{1}{6}c(c^2+6s-1), \qquad A_2=\frac{1}{2}(2c-c^3-6sc+2s+c^2),$$
$$A_3=\frac{1}{2}(2-2c^2-4s+6sc-c+c^3), \qquad A_4=\frac{1}{6}(1-c)(6s+c^2-2c).$$

The modified equation of the scheme is given by

$$u_t+u_x=\alpha\,u_{xx}+\frac{1}{24}\frac{h^3}{c}\left(12sc^2-2s-2c^3+2c+12s^2+6c^4-c^2-2cs\right)u_{xxxx}+\cdots, \tag{22}$$

and the leading error terms are dispersive in nature. The scheme is consistent and is third order accurate in space. The amplification factor of the scheme is given by

$$\xi=A_1\,e^{-2I\omega}+A_2\,e^{-I\omega}+A_3+A_4\,e^{I\omega}. \tag{23}$$

We use the Fourier analysis and the approach of Hindmarsh et al. [6] to obtain the stability region. When $\omega=\pi$ on simplification of Eq. (23), we obtain

$$\xi=1-4s-\frac{4}{3}c+8sc-2c^2+\frac{4}{3}c^3. \tag{24}$$

Then we have

$$2c+3c^2-2c^3-3\le s(12c-6)\le 2c+3c^2-2c^3. \tag{25}$$

When $\omega\to 0$, using Taylor's expansion and on neglecting higher order terms, we have

$$|\xi|^2\simeq 1-2s(\omega^2), \tag{26}$$

and therefore, we must have

$$s\ge 0. \tag{27}$$

Thus the scheme, is stable when both inequalities (25) and (27) are satisfied.

7. Fourth Order Upwind Explicit Scheme

For this scheme the following approximations are used [4];

$$\left.\frac{\partial u}{\partial t}\right|_i^n \simeq \frac{u_i^{n+1}-u_i^n}{\Delta t}, \tag{28}$$

$$\left.\frac{\partial u}{\partial x}\right|_i^n \simeq \left(\frac{12s+2c^2-3c-2}{12}\right)\left(\frac{u_{i+2}^n-u_i^n}{2\Delta x}\right)+\left(\frac{12s+2c^2+3c-2}{12}\right)\left(\frac{u_i^n-u_{i-2}^n}{2\Delta x}\right)$$
$$-\left(\frac{c^2+6s-4}{3}\right)\left(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right), \tag{29}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n \simeq \left(\frac{-c^4+4c^2-12s^2-12sc^2+8s}{6s}\right)\left(\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\right)$$
$$+\left(\frac{c^4-4c^2+12s^2+12sc^2-2s}{6s}\right)\left(\frac{u_{i+2}^n-2u_i^n+u_{i-2}^n}{4(\Delta x)^2}\right). \tag{30}$$

On substitution of Eqs. (28-30) into Eq. (1), we have the following:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\left(\frac{12s+2c^2-3c-2}{12}\right)\left(\frac{u_{i+2}^n-u_i^n}{2\Delta x}\right)+\left(\frac{12s+2c^2+3c-2}{12}\right)\left(\frac{u_i^n-u_{i-2}^n}{2\Delta x}\right)$$
$$-\left(\frac{c^2+6s-4}{3}\right)\left(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\right)=\alpha\left[\left(\frac{-c^4+4c^2-12s^2-12sc^2+8s}{6s}\right)\left(\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\right)\right.$$
$$\left.+\left(\frac{c^4-4c^2+12s^2+12sc^2-2s}{6s}\right)\left(\frac{u_{i+2}^n-2u_i^n+u_{i-2}^n}{4(\Delta x)^2}\right)\right],$$

which after some algebraic manipulation gives

$$\begin{aligned} u_i^{n+1} = {} & \frac{1}{24}[12s(s+c^2)+2s(6c-1)+c(c^3+2c^2-c-2)]u_{i-2}^n \\ & -\frac{1}{6}[12s(s+c^2)+2s(3c-4)+c(c^3+c^2-4c-4)]u_{i-1}^n \\ & +\frac{1}{4}[12s(s+c^2)-10s+((c)^2)^2-5c^2-4]u_i^n \\ & -\frac{1}{6}[12s(s+c^2)-2s(3c+4)+c(c^3-c^2-4c+4)]u_{i+1}^n \\ & +\frac{1}{24}[12s(s+c^2)-2s(6c+1)+c(c^3-2c^2-c+2)]u_{i+2}^n. \end{aligned}$$

By rearranging, we get the following finite difference scheme

$$u_i^{n+1}=A\,u_{i-2}^n+B\,u_{i-1}^n+C\,u_i^n+D\,u_{i+1}^n+E\,u_{i+2}^n, \tag{31}$$

where

$$\begin{aligned} A &= \frac{1}{24}\left(12s(s+c^2)+2s(6c-1)+c(c-1)(c+1)(c+2)\right), \\ B &= -\frac{1}{6}\left(12s(s+c^2)+2s(3c-4)+c(c-2)(c+1)(c+2)\right), \\ C &= \frac{1}{4}\left(12s(s+c^2)-10s+(c-1)(c-2)(c+1)(c+2)\right), \\ D &= -\frac{1}{6}\left(12s(s+c^2)-2s(3c+4)+c(c-2)(c-1)(c+2)\right), \\ E &= \frac{1}{24}\left(12s(s+c^2)-2s(6c+1)+c(c-1)(c+1)(c-2)\right). \end{aligned}$$

The modified equation of the scheme is given by

$$u_t+u_x=\alpha\,u_{xx}+\frac{h^4}{120}\left(60s^2+20sc^2+c^4-5c^2+4-30s\right)\,u_{xxxxxx}+\ldots. \tag{32}$$

The scheme is essentially dispersive as the leading error terms are dispersive in nature due to the presence of the odd-order derivative term, u_{xxxxx}. The scheme is consistent and is fourth order accurate in space. The amplification factor of the scheme is given by

$$\xi = A\, e^{-2I\omega} + B\, e^{-I\omega} + C + D\, e^{I\omega} + E e^{2I\omega}. \tag{33}$$

Using the approach of [6], we obtain the region of stability as

$$0 \leq s \leq \frac{1}{3} - \frac{1}{6}c^2 + \frac{1}{6}\sqrt{4+6c^4}. \tag{34}$$

8. Non-Standard Finite Difference Scheme (NSFD)

In this section, we describe how NSFD has been constructed by Mickens [10] for the 1-D convection-diffusion equation.

The equation $u_t + u_x = \alpha\, u_{xx}$ has three sub-equations [9] which are given by

$$u_t + u_x = 0, \tag{35}$$

$$u_x = \alpha\ u_{xx}, \tag{36}$$

$$u_t = \alpha\ u_{xx}. \tag{37}$$

Eqs. (35) and (36) have known exact finite difference scheme which are

$$\frac{u_i^{n+1} - u_i^n}{k} + \frac{u_i^n - u_{i-1}^n}{h} = 0, \tag{38}$$

and

$$\frac{u_i - u_{i-1}}{h} = \alpha\Big(\frac{u_{i+1} - 2\ u_i + u_{i-1}}{\alpha h(\exp(h/\alpha) - 1)}\Big), \tag{39}$$

respectively.

The NSFD is given by [9, 10]

$$\frac{u_i^{n+1} - u_i^n}{k} + \frac{u_i^n - u_{i-1}^n}{h} = \alpha\Big(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\alpha h(\exp(h/\alpha) - 1)}\Big), \tag{40}$$

which on re-arranging gives

$$u_i^{n+1} = \beta_1\ u_{i+1}^n + (1 - c - 2\beta_1)\ u_i^n + (c + \beta_1)\ u_{i-1}^n, \tag{41}$$

where $c = \frac{k}{h}$ and $\beta_1 = \frac{c}{\exp(h/\alpha)-1}$.

The modified equation of NSFD scheme is given by

$$u_t + u_x = \frac{1}{2}h\left(-c + \frac{\beta_1}{c} + 1\right) u_{xx} + \frac{h^2}{6}(6s + c^2 - 1)u_{xxx} + \ldots, \tag{42}$$

and the leading error terms are dissipative in nature.
The square of the modulus of the amplification factor is given by

$$|\xi|^2 = \Big((1 - c - 2\beta_1) + (c + 2\ \beta_1)\ \cos(w)\Big)^2 + (c\ \sin(w))^2. \tag{43}$$

For stability, $0 < |\xi| \leq 1$ and this implies that $0 < |\xi|^2 \leq 1$. We now obtain the region of stability using the approach used in [6].

We consider the case when $w = \pi$. The square of the modulus of the amplification factor is given by

$$|\xi|^2 = (1 - 2\ c - 4\ \beta_1)^2. \tag{44}$$

and therefore,

$$0 \leq c + 2\ \beta_1 \leq 1. \tag{45}$$

Since c and β_1 are positive, we have $c + 2\beta_1 \geq 0$. Hence, we have the inequality

$$c + 2\beta_1 \leq 1. \tag{46}$$

We next consider the case when $w \to 0$. When $w \to 0$, $\cos(w) \approx 1 - \frac{1}{2}w^2$ and $\sin(w) \approx w$ and we get

$$c + 2\ \beta_1 - c^2 \geq 0. \tag{47}$$

Thus the scheme is stable if it satisfies inequalities (46) and (47).

Table 1 displays the regions of stability of the three schemes at three values of Re:10, 100, 10000 and also for two values of h namely 0.1 and 0.025. The three schemes are stable at Re=100 and Re=10000, when $h = 0.1$ and 0.025, with $k = 0.01$. However, the fourth order and NSFD schemes are not stable at Re=10, $h = 0.025$ and $k = 0.01$.

9. Numerical Results

In this section, the numerical results obtained from the test problems, Problem 1 and Problem 2, at $T = 1$ are presented. For each test problem, the numerical profiles obtained using the three methods; third order, fourth order and NSFD are plotted and some types of errors are tabulated.

9.1. Problem 1

We tabulated the L_2, L_∞, dissipation, dispersion, Total Mean Square errors and total variation at temporal step size 0.01 and Reynolds number: 10, 100 and 10000 for two values of h namely 0.1 and 0.025 for the three different methods in Table (2) to (5). For each method, it is seen that the errors are larger at higher Reynolds number for same values of h and k. The NSFD is by far the best scheme in terms of L_2, L_∞, dissipation and dispersion errors followed by the third order and fourth order. The profiles from the three methods and exact profile are shown in Figs. (1a) to (1e). At $Re = 100$ and $Re = 10000$, with $h = 0.1$ (coarse grid), the results obtained from the third order and the fourth order are very oscillatory. However, the profile obtained using NSFD is very close to the exact profile. With $h = 0.025$, the profiles are less oscillatory as compared to $h = 0.1$ as expected. Again the NSFD is the most efficient shock-capturing scheme.

From Table 5, one can observe that at $Re = 100$ and $Re = 10000$, the total variation obtained from the third order and the fourth order are larger when $h = 0.1$ as compared to the case $h = 0.025$. In all cases considered the total variation for the NSFD scheme is much less than for the other methods.

9.2. Problem 2

Test case 2 has been described in section 4.2 where we consider the following values of α and h:

(i) $\alpha = 0.01$, $h = 0.1$
(ii) $\alpha = 0.1$, $h = 0.05$
(iii) $\alpha = 1$, $h = 0.05$
(iv) $\alpha = 1$, $h = 0.1$

Fig. 2 compares the profile when $\alpha = 0.01$, $h = 0.1$ at four different values of k. The fourth order scheme is very oscillatory and the third order is quite oscillatory. Table 7 compares the errors. It is seen that the NSFD performs the best in regard to L_2, L_∞ and Total Mean Square errors. The dissipation error is greater than dispersion error for the third and fourth order schemes.

For the case $\alpha = 0.1$ and $h = 0.05$, the exact profile is quite smooth as the coefficient of diffusivity is larger. The results from NSFD are very close to the exact profile. There is some dispersion and an overshoot in the peak from the third order and fourth order methods as shown Fig. (3). From Table 8, we can deduce the NSFD is the most effective scheme.
When $\alpha = 1$, the errors obtained using the three methods are very small as shown in Tables 9 and 10 and again the NSFD performs the best.

10. Optimization

From the numerical results obtained for Problem 1, it is observed that all errors obtained from NSFD are very small. The results obtained from third order shows that the dispersive errors are much greater than the dissipation errors for the case $h = 0.1$. In the case of fourth order upwind scheme both dispersion and dissipation errors are very large and are almost the same in magnitude.

Our aim in this section is to find an optimal value of k that minimizes the dispersion error when $h = 0.1$ and Re=100, using test problem 1. Tam and Webb [13], Bogy and Bailly [3] and Berland et al.[22] have implemented techniques which enable coefficients to be determined in high order numerical methods for Computational Aeroacoustics applications. In Appadu [2], these techniques were modified into equivalent forms so that the optimal CFL is computed for some known numerical schemes where the parameters were CFL and the phase angle.

The Dispersion-Relation-Preserving (DRP) scheme was constructed [13] such that the dispersion relation of the finite difference scheme is formally the same as that of the original partial differential equation. The integrated error is calculated as

$$E = \int_{-\eta}^{\eta} |\theta^* h - \theta h|^2 d(\theta\, h),$$

where the quantities $\theta^* h$ and $\theta\, h$ represent the numerical and exact wave numbers respectively. The dispersion and dissipation error are obtained as $|\Re(\theta^* h) - \theta\, h|$ and $|\Im(\theta^* h)|$, respectively. Tam and Shen [14] set η as 1.1.

In Appadu [2], the following integrals were defined namely,

$$\text{IETAM} = \int_{0}^{\omega_1} |1 - RPE|^2 \, d\omega,$$
$$\text{IEBOGEY} = \int_{0}^{\omega_1} |1 - RPE| \, d\omega,$$

with $\omega_1 = 1.1$.

In Appadu [1], optimization techniques based on minimization of the dispersion error have been used to obtain the optimal k at a fixed value of h for the Lax-Wendroff and NSFD discretising a 1-D advection-diffusion equation and these techniques have been validated.

In this work, we use the same approach as in Appadu [1]. The amplification factor of the third order scheme approximating Eq. (1) with $h = 0.1$ is

$$\xi_{\text{Third}} = \Re(\xi) + I\Im(\xi),$$

and the relative phase error is given by

$$RPE_{\text{Third}} = \frac{-1}{10k\omega} \arctan(\frac{\Im(\xi)}{\Re(\xi)}),$$

where $\Re(\xi)$ and $\Im(\xi)$ are given by

$$\begin{aligned}
\Re(\xi) &= 1 - 5.33\, k - 80k^2 + 60k^2 \cos(\omega) + 20k^2 \cos^2(\omega) + 333.33\, k^3 \cos^2(\omega) + 8.67\, k \cos(\omega) \\
&\quad - 666.67\, k^3 \cos(\omega) + 333.33k^3, \\
\Im(\xi) &= 3.33\, k \sin(\omega) \cos(\omega) - 13.33\, k \sin(\omega) - 333.33\, k^3 \sin(\omega) \cos(\omega) + 20k^2 \sin(\omega) \\
&\quad + 333.33\, k^3 \sin(\omega) - 20k^2 \sin(\omega) \cos(\omega).
\end{aligned}$$

3D plots of the exact RPE versus $k \in [0, 0.1675]$ vs ω are shown in Figs. (6a) and (6b) and we have phase wrapping phenomenon. We thus obtain an approximation for the RPE till the terms $\mathcal{O}(\omega^4)$ using Taylor's series for $\omega \in [0, 1.1]$. The 3D plot of the approximated RPE vs k vs phase angle is shown in Fig. (6c).

The approximated RPE is obtained as

$$RPE_{\text{Approx}} = 1 - (0.0333 + 5k^2 - k + 50k^3 - 333.3333)\omega^4.$$

The integrated error is obtained as

$$\int_0^{1.1} (RPE_{\text{Approx}} - 1)^2 d\,\omega, \tag{48}$$

and is a function of k. A plot of the integrated error vs k is shown in Fig. 8a. The integrated error decreases as k increases from a value close to 0 to 0.05 and then it oscillates with local minimum near $k = 0.1$ and $k = 0.1414$ and then increases again as as shown in Fig. 8b. The scheme is stable if $k \in [0, 0.1675765067]$ and using NLPSolve, the optimal value of k is 0.1.

Table 11 compares the TV, L_2, L_∞, dissipation and dispersion errors, Total Mean Square Error and CPU time using the Third order upwind scheme at some different values of k with $h = 0.1$, Re= 100 and in Fig. (9), we obtain the plots of these error vs k.

It is seen that the errors initially decreases and reach a minimum when $k = 0.1$ and then increases again. We conclude that the variation of the integrated error in Fig. (8a) mimic the actual variation of the errors in Fig. (9). We conclude that the time step, $k = 0.1$ is indeed the optimal time step size which allows the method to perform at its best. Moreover, it can be observed from Table (11), there is no big change in CPU time when different values of k are used.
We also plot the exact RPE vs $\omega \in [0, \pi]$ at five different values of k namely $0.001, 0.08, 0.1, 0.125$ and $\frac{1}{6}$ as shown in Fig. (7) and it is seen that the scheme has best dispersion properties when $k = 0.1$.

11. Conclusion

In this paper, three numerical methods have been used to solve two test problems. The NSFD is much better than third order and fourth order for all the cases considered. An optimization technique has been implemented for the third order scheme when Re=100 and h is fixed as 0.1 to find an optimal temporal step that minimizes the dispersion error. This optimal value is validated using numerical experiments. The computational time using NSFD is in general less as compared to the times using the Fourth Order Upwind and Third Order Upwind schemes.

Nomenclature

NSFD= Non-Standard Finite Difference scheme
TMSE= total mean square error
Diss. Error= dissipation error
Disp. Error= dispersion error
α = coefficient of diffusivity
Re=Reynolds number
h =spatial step size
k =temporal step size
$\theta\, h$ = phase angle
$c = \frac{k}{h}$ = CFL number
$s = \frac{\alpha k}{h^2}$
ω = phase angle.

Acknowledgments

Dr A.R. Appadu is grateful to the Research Development Programme of the University of Pretoria and the DST/NRF SARChI Chair in Mathematical Models and Methods in Bioengineering and Biosciences for funding. Prof J.K. Djoko is funded through the incentive fund N00 401 Project 85796. Mr Hagos is grateful to the University of Pretoria, African Institute for Mathematical Sciences (AIMS)-South Africa and Aksum University (Ethiopia) for their financial support for his PhD studies.

References

[1] A. R. Appadu, "Numerical Solution of the 1D Advection-Diffusion Equation Using Standard and Nonstandard Finite Difference Schemes", *Journal of Applied Mathematics*, Article ID 734374, 14 pages, 2013.

[2] A.R. Appadu, "Comparison of Some Optimisation techniques for Numerical Schemes Discretising Equations with Advection Terms", *International Journal of Innovative Computing and Applications*, 4, 1,12-27, 2012.

[3] C. Bogey and C. Bailly, "A Family of Low Dispersive and Low Dissipative Explicit Schemes for Computing the Aerodynamic Noise", *Journal of Computational Physics*, 194, 194-214, 2004.

[4] M. Dehghan, "On the Numerical Solution of the One-Dimensional Convection-Diffusion Equation", *Mathematical Problems in Engineering*, 1, 61-74, 2005.

[5] V. Guvanasen and RE Volker, "Numerical Solutions for Solute Transport in Unconfined Aquifers", *International Journal Numerical methods in Fluids*, 3, 103-123, 1983.

[6] A. C. Hindmarsh, P. M. Gresho and D. F. Griffiths, "The Stability of Explicit Euler time-integration for Certain Finite Difference Approximations of the Multi-dimensional Advection-Diffusion equation", *International Journal Numerical Methods in Fluids*, 853-897, 1984.

[7] J. Isenberg and Gutfinger, "Heat Transfer to a draining film," *International Journal of Heat Transfer*, 16, 505-513, 1972.

[8] N. Kumar, "Unsteady flow against dispersion in finite porous media", *Journal of Hydrology*, 63, 343-358, 1983.

[9] R.E. Mickens, "Analysis of a new finite-difference scheme for the linear advection-diffusion equation", *Journal of Sound and Vibration*, 146, 342-344, 1991.

[10] R.E. Mickens, "Applications of Nonstandard Finite Difference Schemes", *World Scientific Publishing Co. Pte. Ltd*, ISBN: 981-02-4133-X, 2000.

[11] Romuald Szymkiewicz,"Numerical Modelling in Open Channel Hydraulics," *Water Science and Technology Library*, 2010.

[12] L. Takacs, "A Two-Step Scheme for the Advection Equation with Minimized Dissipation and Dispersion errors", *Monthly Weather Review*, 113, 1050-1065, 1985.

[13] C.K.W. Tam and J.C. Webb, "Dispersion-Relation-Preserving Finite Difference Schemes for Computational Acoustics", *International Journal of Computational Physics*, 107, 262-281, 1993.

[14] C.K.W. Tam and H. Shen, "Direct Computation of Nonlinear Acoustic pulses using High-order Finite Differences schemes", *AIAA Paper 93-4325*, 1993.

[15] Zhen F. Tian and P. X. Yu, "A high-order exponential scheme for solving 1D convection-diffusion equations", *Journal of Applied Mathematics*, 235, 2477-2491,2011.

[16] Feng, Z. F. Tian, "Alternating group explicit method with exponential-type for the diffusion-convection equation", *International Journal of Computer Mathematics*, 83, 765-775, 2006.

[17] Lloyd N. Trefethen, "Finite Difference and Spectral Methods for Ordinary and Partial Differential Equations", New York 14853, USA, 1996.

[18] H. Ding and Y. Zhang, "A new difference scheme with high accuracy and absolute stability for solving convection- diffusion equations", *Journal of Computational and Applied Mathematics*, 230, 600-606, 2009.

[19] M. M. Chawla, M.A. Al-Zanaidi and M. G. Al-Aslab,"Extended One-Step Time-Integration Schemes for Convection-Diffusion Equations", *Computers and Mathematics with Applications*, 39,71-84,2000.

[20] J. Y. Parlarge,"Water transport in soils", *Annual Review of Fluid Mechanics*, 2,77-102,1980.

[21] H. Trac and U. Pen, "A primer on Eulerian Computational Fluid Dynamics for Astrophysics", *Astronomical Society of the Pacific*, 115, 303-321, 2003.

[22] J. Berland, C. Bogey, O. Marsden, C. Bailly, " High-order, low dispersive and low dissipative explicit schemes for multiple-scale and boundary problems", *Journal of Computational Physics*, 224, 637-662, 2007.

Scheme	Re	h	Range of k for stability
Fourth Order	10	0.1	[0,0.0541]
	100	0.1	[0,0.1578]
		0.025	[0,0.0237]
	10000	0.1	[0,0.1995]
		0.025	[0,0.0495]
Third order	10	0.1	[0,0.0760]
	100	0.1	[0,0.1675]
		0.025	[0,0.0275]
	10000	0.1	[0,0.1996]
		0.025	[0,0.0496]
NSFD	10	0.1	[0,0.0333]
	100	0.1	[0,0.0833]
		0.025	[0,0.0138]
	10000	0.1	[0,0.0998]
		0.025	[0,0.0248]

Table 1: Stability regions when Re= 10, Re= 100 and Re= 10000 for Problem 1.

h	Re	L_2	L_∞	Diss. Error	Disp. Error	TMSE
0.1	10	0.0012	0.0300	3.3114×10^{-7}	9.0556×10^{-7}	1.2370×10^{-6}
0.1	100	0.0635	0.1969	3.3448×10^{-4}	0.0033	0.0037
0.025	100	0.0011	0.0067	3.9951×10^{-8}	1.0893×10^{-6}	1.1293×10^{-6}
0.1	10000	0.1013	0.3052	9.0851×10^{-4}	0.0084	0.0093
0.025	10000	0.0252	0.1572	1.7086×10^{-5}	6.0090×10^{-4}	6.1798×10^{-4}

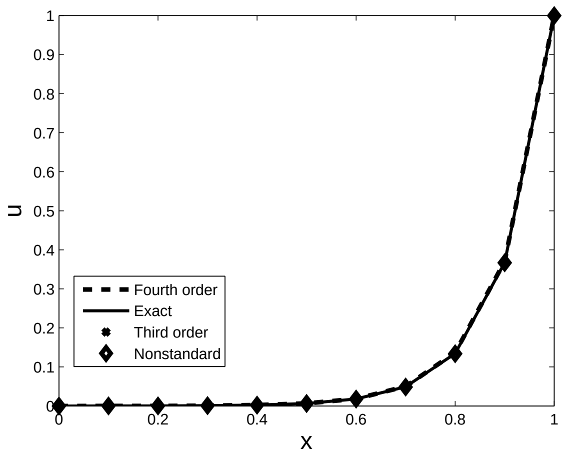
Table 2: Errors obtained from third order when $k = 0.01$, Re= 10, Re= 100 and Re= $10,000$ for Problem 1 at $T = 1$.

h	Re	L_2	L_∞	Diss. Error	Disp. Error	TMSE
0.1	10	0.0012	0.0030	3.5526×10^{-7}	8.8273×10^{-7}	1.2380×10^{-6}
0.1	100	0.1659	0.3973	0.0027	0.0224	0.0250
0.025	100	0.0029	0.0182	1.4444×10^{-7}	8.2459×10^{-6}	8.3904×10^{-6}
0.1	10000	0.3860	0.6031	0.0448	0.0906	0.1354
0.025	10000	0.0591	0.3102	1.4821×10^{-4}	0.0033	0.0034

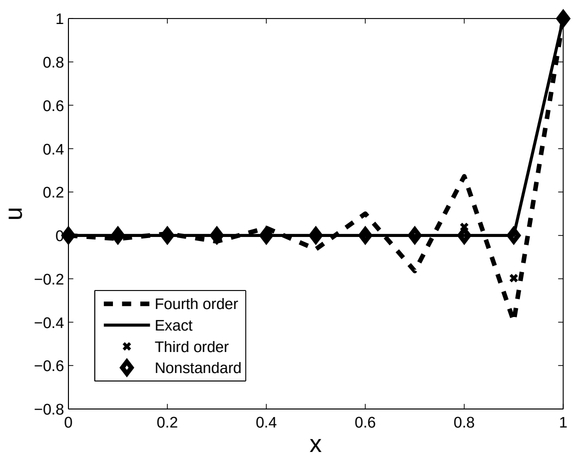
Table 3: Errors obtained from fourth order when $k = 0.01$, Re= 10, Re= 100 and Re= $10,000$ for Problem 1 at $T = 1$.

h	Re	L_2	L_∞	Diss. Error	Disp. Error	TMSE
0.1	10	3.6413×10^{-4}	6.7422×10^{-4}	5.8429×10^{-8}	6.2106×10^{-8}	1.2054×10^{-7}
0.1	100	3.8220×10^{-10}	1.2086×10^{-9}	1.3293×10^{-20}	3.6701×10^{-17}	1.3280×10^{-19}
0.025	100	1.4133×10^{-12}	8.8194×10^{-12}	7.0285×10^{-26}	1.0591×10^{-17}	1.9488×10^{-24}
0.1	10000	0	0	0	0	0
0.025	10000	4.2204×10^{-110}	2.6692×10^{-109}	0	5.2836×10^{-18}	5.2836×10^{-18}

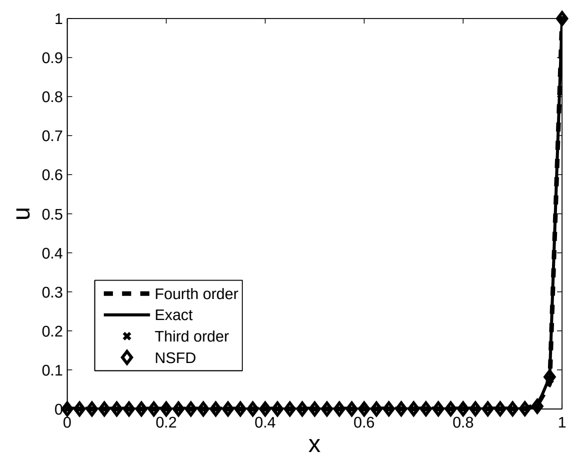
Table 4: Errors obtained from NSFD when $k = 0.01$, Re= 10, Re= 100 and Re= $10,000$ for Problem 1 at $T = 1$.



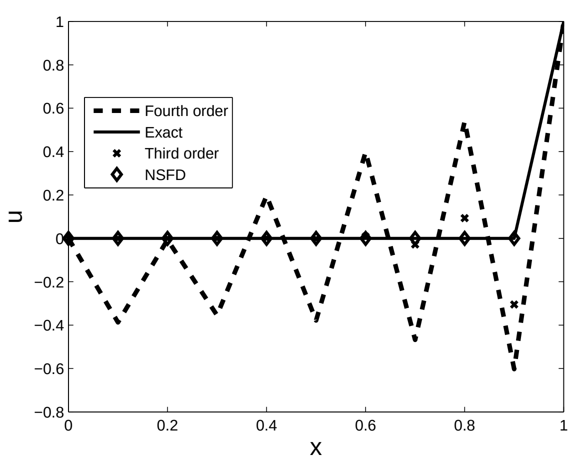
(a) $h = 0.1$ and Re=10



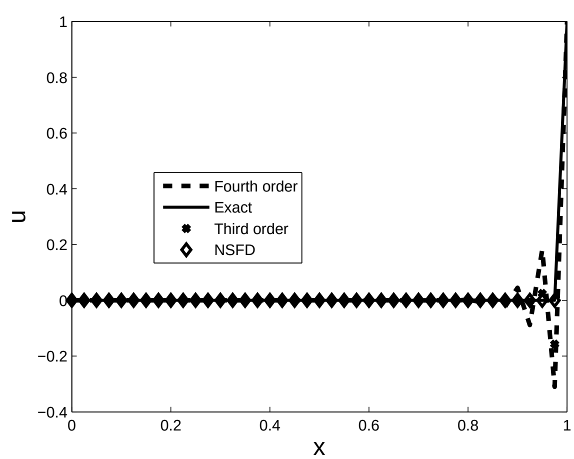
(b) $h = 0.1$ and Re=100



(c) $h = 0.025$ and Re=100



(d) $h = 0.1$ and Re=10000



(e) $h = 0.025$ and Re=10000

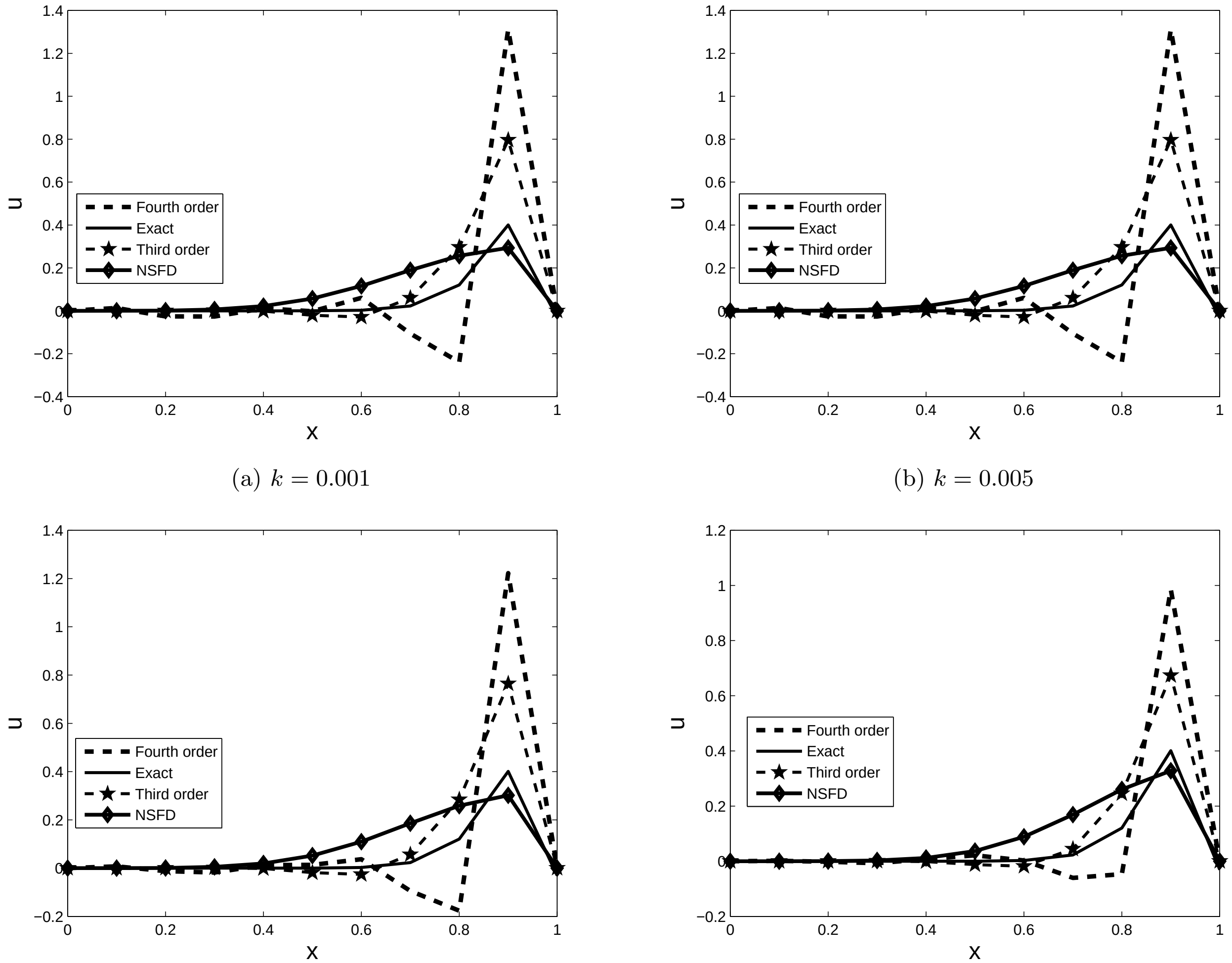
Figure 1: Comparison of the numerical schemes when Re=10, Re=100 and Re=10000 for Problem 1 at $T = 1$.

Numerical methods	Re	h	Total Variation	CPU time to solve and compute all errors	CPU time to solve u
Fourth Order	10	0.1	1	4.246	0.016
	100	0.1	3.1664	4.157	0.085
		0.025	1	8.718	0.099
	10000	0.1	7.6374	3.703	0.071
		0.025	2.3420	4.186	0.081
Third Order	10	0.1	1	4.363	0.072
	100	0.1	1.4902	4.106	0.084
		0.025	1	8.521	0.098
	10000	0.1	1.8783	3.545	0.074
		0.025	1.3730	3.961	0.095
NSFD	10	0.1	1	4.127	0.084
	100	0.1	1	4.262	0.073
		0.025	1	8.754	0.078
	10000	0.1	1	3.662	0.030
		0.025	1	4.295	0.075

Table 5: Total Variation and CPU time for different values of h when $k = 0.01$ for Problem 1 at $T = 1$.

Scheme	α	h	Range of k for stability
Fourth Order	0.01	0.1	[0,0.01578]
	0.1	0.05	[0,0.0155]
	1	0.05	[0,0.0016]
	1	0.1	[0,0.0066]
Third order	0.01	0.1	[0,0.1675]
	0.1	0.05	[0,0.0314]
	1	0.05	[0,0.0012]
	1	0.1	[0,0.0053]
NSFD	0.01	0.1	[0,0.0999]
	0.1	0.05	[0,0.0122]
	1	0.05	[0,0.0012]
	1	0.1	[0,0.0049]

Table 6: Stability regions for three different values of α namely $0.01, 0.1$ and 1 for Problem 2.



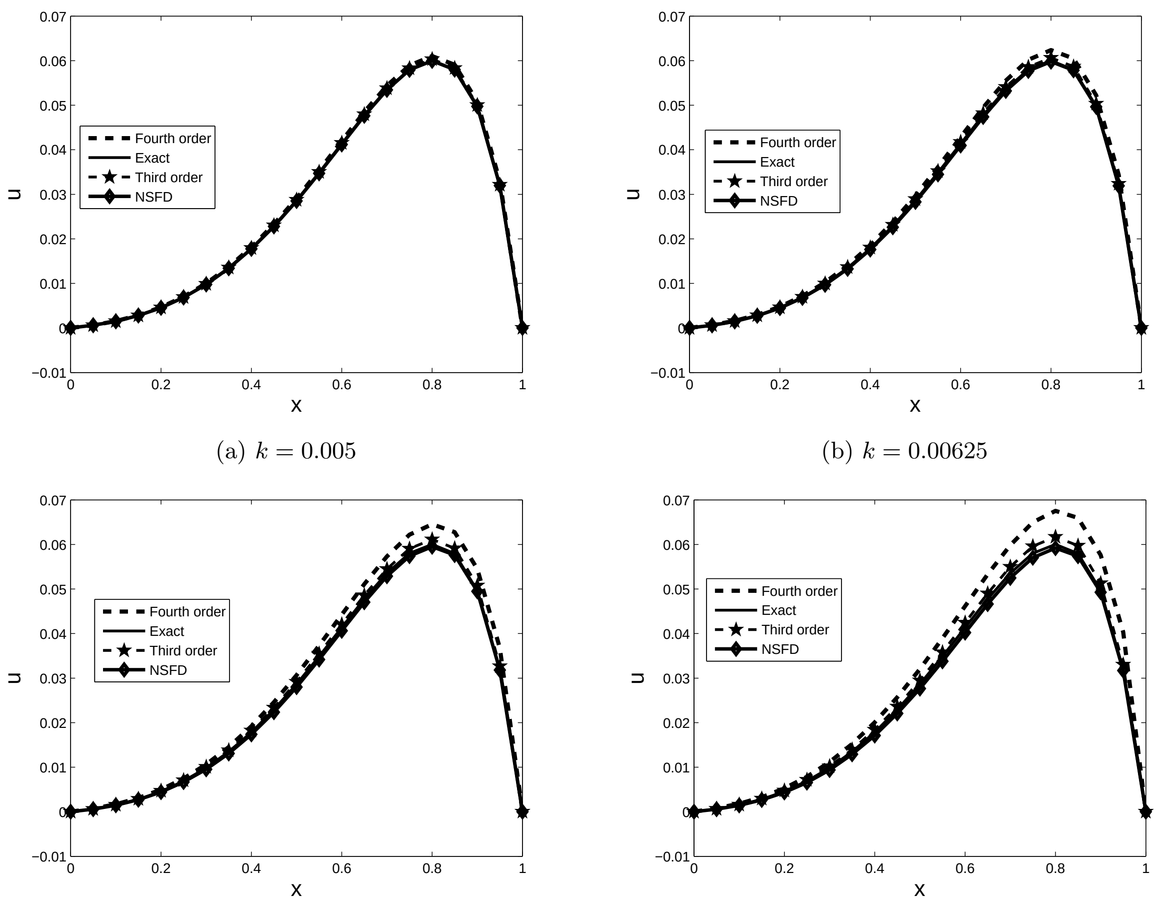
(a) $k = 0.001$ (b) $k = 0.005$

(c) $k = 0.01$ (d) $k = 0.025$

Figure 2: Comparison of the numerical schemes when $\alpha = 0.01$ and $h = 0.1$ for Problem 2 at $T = 1$.

Scheme	k	L_2	L_∞	Diss. Error	Disp. Error	TMSE
Third Order	0.001	0.1475	0.4226	0.0194	3.9638×10^{-4}	0.0198
	0.005	0.1384	0.3958	0.0170	3.7389×10^{-4}	0.0174
	0.01	0.1274	0.3636	0.0144	3.4468×10^{-4}	0.0147
	0.025	0.0960	0.2733	0.0081	2.5683×10^{-4}	0.0084
Fourth Order	0.001	0.3436	0.9866	0.0939	0.0134	0.1073
	0.005	0.3133	0.9108	0.0780	0.0112	0.0892
	0.01	0.2792	0.8215	0.0619	0.0090	0.0709
	0.025	0.1944	0.5845	0.0297	0.0046	0.0344
NSFD	0.001	0.0888	0.1696	0.0017	0.0055	0.0072
	0.005	0.0868	0.1670	0.0016	0.0052	0.0069
	0.01	0.0842	0.1630	0.0015	0.0049	0.0064
	0.025	0.0748	0.1462	0.0012	0.0039	0.0051

Table 7: Comparison of the numerical schemes when $\alpha = 0.01$ and $h = 0.1$ for Problem 2 at $T = 1$.



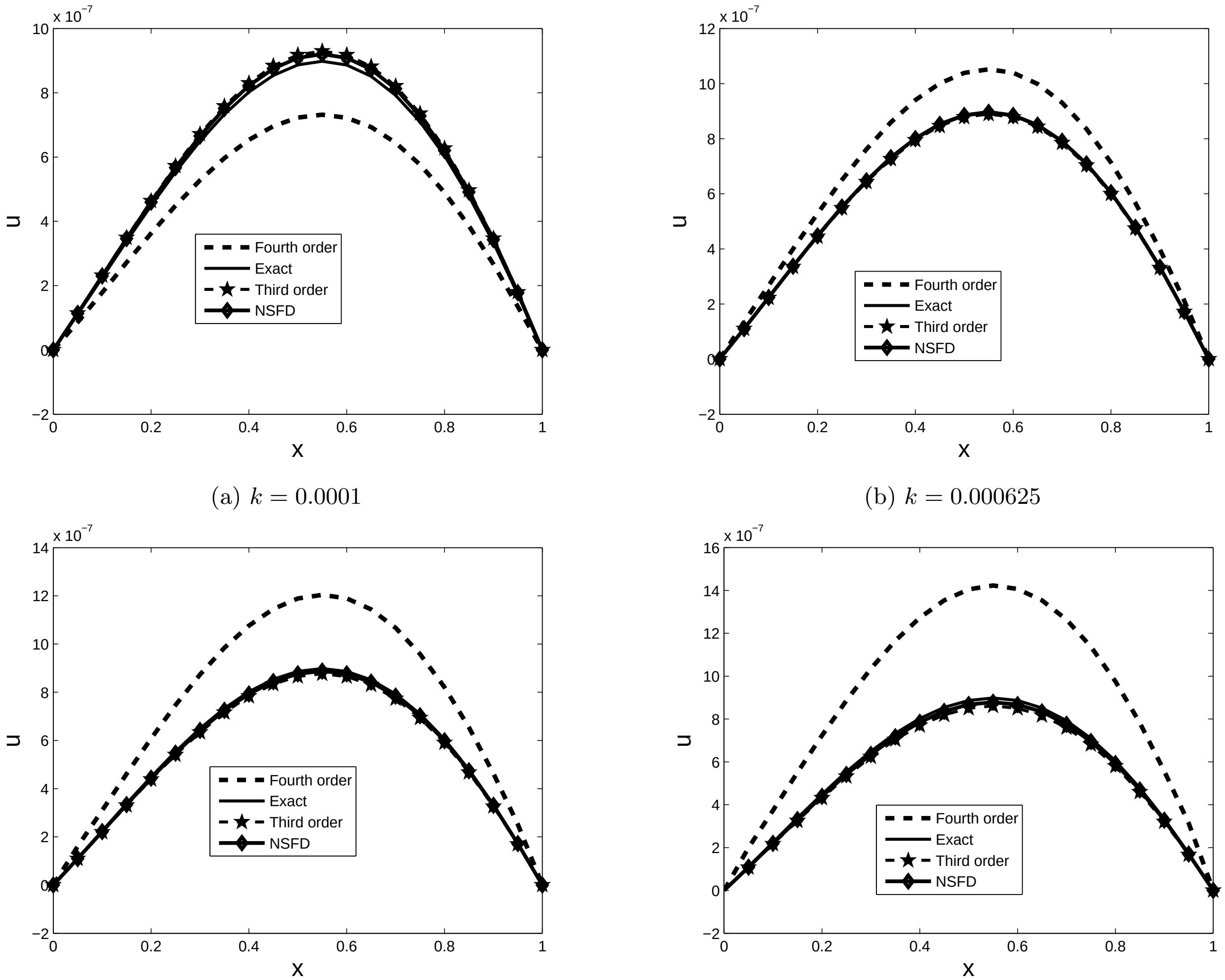
(a) $k = 0.005$ (b) $k = 0.00625$

(c) $k = 0.008$ (d) $k = 0.01$

Figure 3: Comparison of the numerical schemes when $\alpha = 0.1$ and $h = 0.05$ for Problem 2 at $T = 1$.

Scheme	k	L_2	L_∞	Diss. Error	Disp. Error	TMSE
Third Order	0.005	2.1355×10^{-4}	3.7906×10^{-4}	3.6010×10^{-8}	7.4241×10^{-9}	4.3434×10^{-8}
	0.00625	3.9782×10^{-4}	6.8387×10^{-4}	1.4248×10^{-7}	8.2386×10^{-9}	1.5072×10^{-7}
	0.008	6.7694×10^{-4}	0.0012	4.2648×10^{-7}	9.9494×10^{-9}	4.3643×10^{-7}
	0.01	0.0010	0.0018	9.7346×10^{-7}	1.2862×10^{-8}	9.8632×10^{-7}
Fourth Order	0.005	6.4476×10^{-4}	0.0012	3.6733×10^{-7}	2.8590×10^{-8}	3.9592×10^{-7}
	0.00625	0.0015	0.0026	1.9726×10^{-6}	1.0343×10^{-7}	2.0761×10^{-6}
	0.008	0.0028	0.0050	7.2691×10^{-6}	3.49477×10^{-7}	7.6186×10^{-6}
	0.01	0.0047	0.0086	2.0288×10^{-5}	1.0097×10^{-6}	2.1297×10^{-5}
NSFD	0.005	1.1822×10^{-4}	3.3094×10^{-4}	1.4911×10^{-9}	1.1819×10^{-8}	1.3310×10^{-8}
	0.00625	2.4401×10^{-4}	3.8193×10^{-4}	3.2902×10^{-8}	2.3803×10^{-8}	5.6706×10^{-8}
	0.008	4.5680×10^{-4}	7.2895×10^{-4}	1.4871×10^{-7}	5.0027×10^{-8}	1.9873×10^{-7}
	0.01	7.1049×10^{-4}	0.0011	3.8799×10^{-7}	9.2768×10^{-8}	4.8075×10^{-7}

Table 8: Comparison of the numerical schemes when $\alpha = 0.1$ and $h = 0.05$ for Problem 2 at $T = 1$.



(a) $k = 0.0001$

(b) $k = 0.000625$

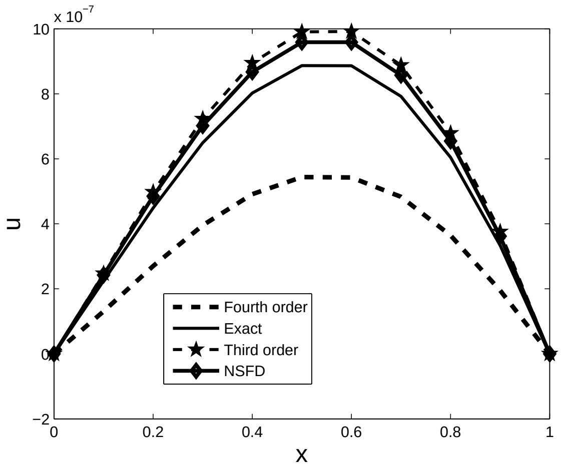
(c) $k = 0.0008$

(d) $k = 0.001$

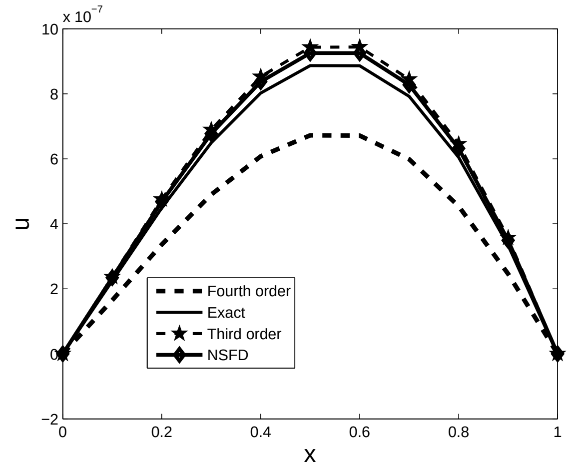
Figure 4: Comparison of the numerical schemes when $\alpha = 1$ and $h = 0.05$ for Problem 2 at $T = 1$.

Scheme	k	L_2	L_∞	Diss. Error	Disp. Error	TMSE
Third Order	0.0001	2.2371×10^{-8}	3.1736×10^{-8}	4.7587×10^{-16}	7.7443×10^{-19}	4.7665×10^{-16}
	0.000625	5.7507×10^{-9}	8.4119×10^{-9}	3.1399×10^{-17}	9.7311×10^{-20}	3.1496×10^{-17}
	0.0008	1.5026×10^{-8}	2.1688×10^{-8}	2.1497×10^{-16}	6.7603×10^{-20}	2.1504×10^{-16}
	0.001	2.5597×10^{-8}	3.6796×10^{-8}	6.2384×10^{-16}	1.4553×10^{-19}	6.2399×10^{-16}
Fourth Order	0.0001	1.1829×10^{-7}	1.6605×10^{-7}	1.3322×10^{-14}	3.8972×10^{-18}	1.3326×10^{-14}
	0.000625	1.1035×10^{-7}	1.5428×10^{-7}	1.1593×10^{-14}	4.7517×10^{-18}	1.1597×10^{-14}
	0.0008	2.1916×10^{-7}	3.0609×10^{-7}	4.5725×10^{-14}	1.8239×10^{-17}	4.5743×10^{-14}
	0.001	3.7680×10^{-7}	5.2541×10^{-7}	1.3516×10^{-13}	5.6404×10^{-17}	1.3522×10^{-13}
NSFD	0.0001	1.5801×10^{-8}	2.2254×10^{-8}	2.3752×10^{-16}	2.6164×10^{-19}	2.3778×10^{-16}
	0.000625	1.5277×10^{-9}	2.2411×10^{-9}	1.9681×10^{-18}	2.5469×10^{-19}	2.2228×10^{-18}
	0.0008	7.0858×10^{-9}	1.0205×10^{-8}	4.7565×10^{-17}	2.5241×10^{-19}	4.7818×10^{-17}
	0.001	1.3492×10^{-8}	1.9322×10^{-8}	1.7311×10^{-16}	2.4982×10^{-19}	1.7336×10^{-16}

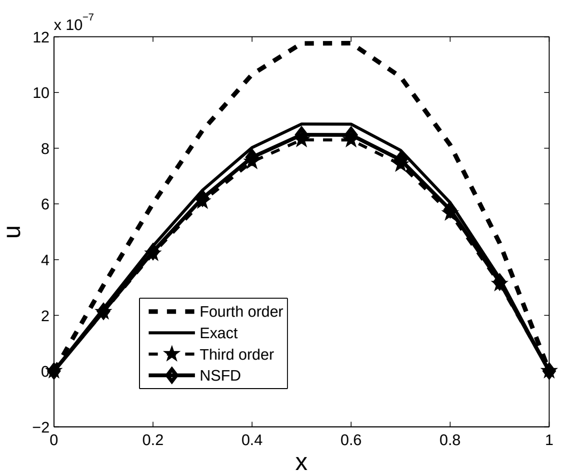
Table 9: Comparison of the numerical schemes when $\alpha = 1$ and $h = 0.05$ for Problem 2 at $T = 1$.



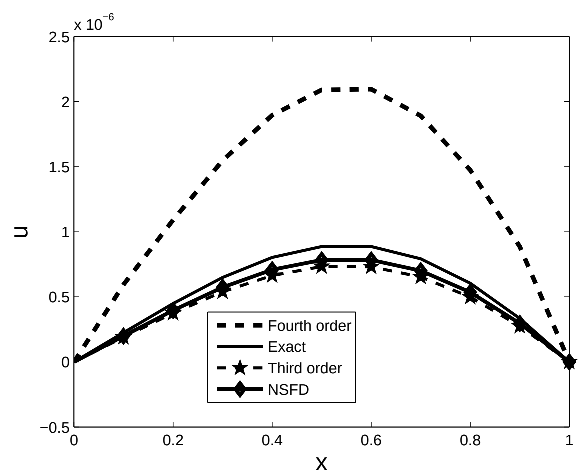
(a) $k = 0.0001$



(b) $k = 0.0008$



(c) $k = 0.0025$



(d) $k = 0.004$

Figure 5: Comparison of the numerical schemes when $\alpha = 1$ and $h = 0.1$ for Problem 2 at $T = 1$.

Scheme	k	L_2	L_∞	Diss. Error	Disp. Error	TMSE
Third Order	0.0001	1.0558×10^{-7}	5.0247×10^{-15}	5.3754×10^{-18}	5.0301×10^{-15}	5.0301×10^{-15}
	0.0008	4.0576×10^{-8}	5.7541×10^{-8}	1.4946×10^{-15}	2.1423×10^{-18}	1.4968×10^{-15}
	0.0025	4.0199×10^{-8}	5.7023×10^{-8}	1.4687×10^{-15}	3.8842×10^{-19}	1.4690×10^{-15}
	0.004	1.0981×10^{-7}	1.5554×10^{-7}	1.0958×10^{-14}	4.3849×10^{-18}	1.0962×10^{-14}
Fourth Order	0.0001	2.4713×10^{-7}	3.4362×10^{-7}	5.5504×10^{-14}	1.7528×10^{-17}	5.5522×10^{-14}
	0.0008	1.5477×10^{-7}	2.1495×10^{-7}	2.1770×10^{-14}	7.3444×10^{-18}	2.1777×10^{-14}
	0.0025	2.1010×10^{-7}	2.9041×10^{-7}	4.0109×10^{-14}	1.9375×10^{-17}	4.0128×10^{-14}
	0.004	8.7923×10^{-7}	1.2089×10^{-7}	7.0240×10^{-13}	3.7892×10^{-16}	7.0278×10^{-13}
NSFD	0.0001	5.1603×10^{-8}	7.2322×10^{-8}	2.4205×10^{-15}	2.8981×10^{-19}	2.4208×10^{-15}
	0.0008	2.7793×10^{-8}	3.8923×10^{-8}	7.0193×10^{-16}	2.7971×10^{-19}	7.0221×10^{-16}
	0.0025	2.7513×10^{-8}	3.8847×10^{-8}	6.8789×10^{-16}	2.5627×10^{-19}	6.8815×10^{-16}
	0.004	7.3389×10^{-8}	1.0321×10^{-7}	4.8961×10^{-15}	2.3682×10^{-19}	4.8963×10^{-15}

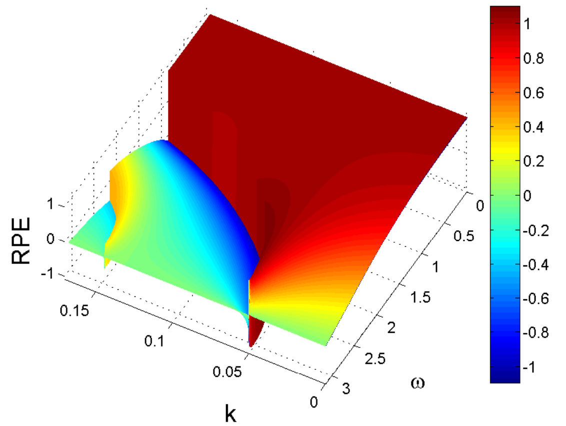
Table 10: Comparison of the numerical schemes when $\alpha = 1$ and $h = 0.1$ for Problem 2 at $T = 1$.

k	TV	L_2	L_∞	Diss. Error	Disp. Error	TMSE	CPU time
0.0001	1.6291	0.0779	0.2393	5.1174×10^{-4}	0.0050	0.0055	4.807
0.00025	1.6267	0.0777	0.2386	5.0844×10^{-4}	0.0050	0.0055	4.333
0.0005	1.6228	0.0773	0.2375	5.0299×10^{-4}	0.0049	0.0054	4.198
0.0008	1.5156	0.0662	0.2050	3.6451×10^{-4}	0.0036	0.0040	4.178
0.001	1.6150	0.0765	0.2352	4.9227×10^{-4}	0.0048	0.0053	4.199
0.0025	1.5922	0.0742	0.2285	4.6152×10^{-4}	0.0045	0.0050	4.128
0.005	1.5560	0.0705	0.2176	4.1457×10^{-4}	0.0041	0.0045	4.140
0.01	1.4903	0.0635	0.1969	3.3448×10^{-4}	0.0033	0.0037	4.124
0.025	1.3333	0.0457	0.1429	1.7264×10^{-4}	0.0017	0.0019	4.139
0.05	1.1612	0.0237	0.0746	4.7694×10^{-5}	4.6144×10^{-4}	5.0914×10^{-4}	4.106
0.0625	1.1026	0.0155	0.0488	2.0753×10^{-5}	1.9668×10^{-4}	2.1743×10^{-4}	4.128
0.08	1.0426	0.0066	0.0209	3.8832×10^{-6}	3.5798×10^{-5}	3.9681×10^{-5}	4.138
0.1	1	1.4357×10^{-5}	4.5400×10^{-5}	1.8739×10^{-11}	1.6864×10^{-10}	1.8738×10^{-10}	4.126
0.125	1	0.0025	0.0079	5.6869×10^{-7}	5.0676×10^{-6}	5.6363×10^{-6}	4.165
1/7	1.0060	9.6247×10^{-4}	0.0030	8.3942×10^{-8}	7.5819×10^{-7}	8.4213×10^{-7}	4.125
1/6	1.1319	0.0196	0.0619	3.3062×10^{-5}	3.1675×10^{-4}	3.4981×10^{-4}	4.148

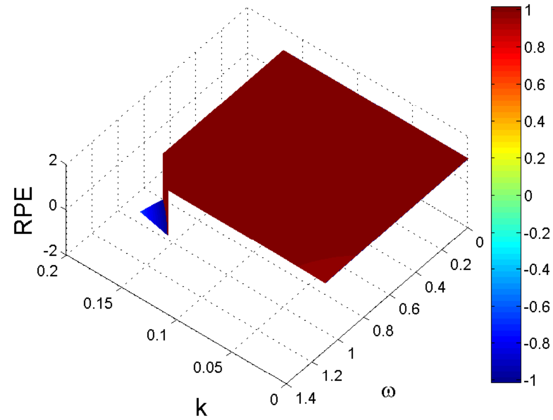
Table 11: Errors obtained from third order when $h = 0.1$, Re= 100 for different values of k for Problem 1 at $T = 1$.

Numerical methods	α	h	k	Total Variation	CPU time to solve and compute all errors	CPU time to solve u
Fourth Order	0.01	0.1	0.001	3.7467	3.938	0.099
			0.005	3.7467	3.938	0.030
			0.01	3.7467	3.938	0.026
			0.025	3.7467	3.938	0.020
	0.1	0.05	0.005	0.1221	4.922	0.031
			0.00625	0.1248	4.923	0.027
			0.008	0.1291	4.947	0.026
			0.01	0.1351	4.954	0.025
	1	0.05	0.0001	1.4639×10^{-6}	6.238	1.382
			0.000625	2.1045×10^{-6}	5.022	0.122
			0.0008	2.4082×10^{-6}	4.979	0.096
			0.001	2.8468×10^{-6}	4.970	0.084
		0.1	0.0001	1.0875×10^{-6}	4.576	0.683
			0.0008	1.3444×10^{-6}	3.896	0.070
			0.0025	2.3537×10^{-6}	3.910	0.029
			0.004	4.1907×10^{-6}	3.838	0.021
Third Order	0.01	0.1	0.001	1.7159	3.757	0.062
			0.005	1.6560	3.749	0.030
			0.01	1.5843	3.751	0.020
			0.025	1.3859	3.775	0.018
	0.1	0.05	0.005	0.1206	4.892	0.037
			0.00625	0.1213	4.866	0.039
			0.008	0.1223	4.874	0.034
			0.01	0.1234	4.846	0.032
	1	0.05	0.0001	1.8595×10^{-6}	6.336	1.356
			0.000625	1.7792×10^{-6}	4.915	0.122
			0.0008	1.7526×10^{-6}	4.915	0.101
			0.001	1.7224×10^{-6}	4.913	0.084
		0.1	0.0001	1.9841×10^{-6}	4.310	0.653
			0.0008	1.8880×10^{-6}	3.769	0.048
			0.0025	1.6603×10^{-6}	3.771	0.021
			0.004	1.4648×10^{-6}	3.824	0.013
NSFD	0.01	0.1	0.001	0.5743	3.738	0.055
			0.005	0.5869	3.747	0.018
			0.01	0.6034	3.739	0.017
			0.025	0.6563	3.740	0.017
	0.1	0.05	0.005	0.1199	4.864	0.040
			0.00625	0.1196	4.868	0.033
			0.008	0.1190	4.847	0.032
			0.01	0.1184	4.859	0.015
	1	0.05	0.0001	1.8405×10^{-6}	6.058	1.305
			0.000625	1.7916×10^{-6}	4.954	0.116
			0.0008	1.7756×10^{-6}	4.936	0.097
			0.001	1.7573×10^{-6}	4.910	0.077
		0.1	0.0001	1.9179×10^{-6}	4.365	0.638
			0.0008	1.8511×10^{-6}	3.814	0.061
			0.0025	1.6960×10^{-6}	3.768	0.036
			0.004	1.5673×10^{-6}	3.762	0.016

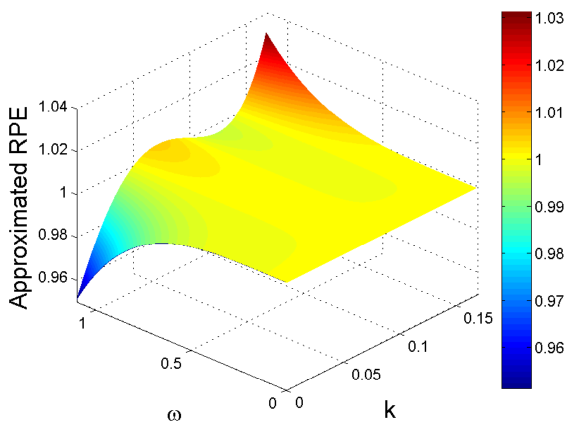
Table 12: Total Variation and CPU time for different values of h and k for Problem 2 at $T = 1$.



(a) Exact RPE versus k vs phase angle for $0 \leq \omega \leq \pi$



(b) Exact RPE versus k vs phase angle for $0 \leq \omega \leq 1.1$



(c) Approximated RPE versus k vs phase angle for $0 \leq \omega \leq 1.1$

Figure 6: Plots of RPE vs k vs ω for the Third Order Upwind scheme for problem 1 with Re= 100 and $h = 0.1$.

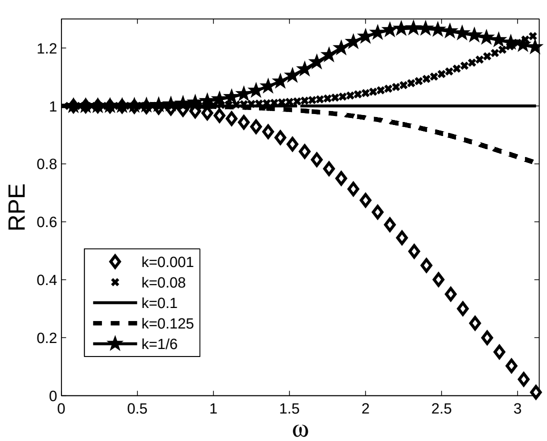
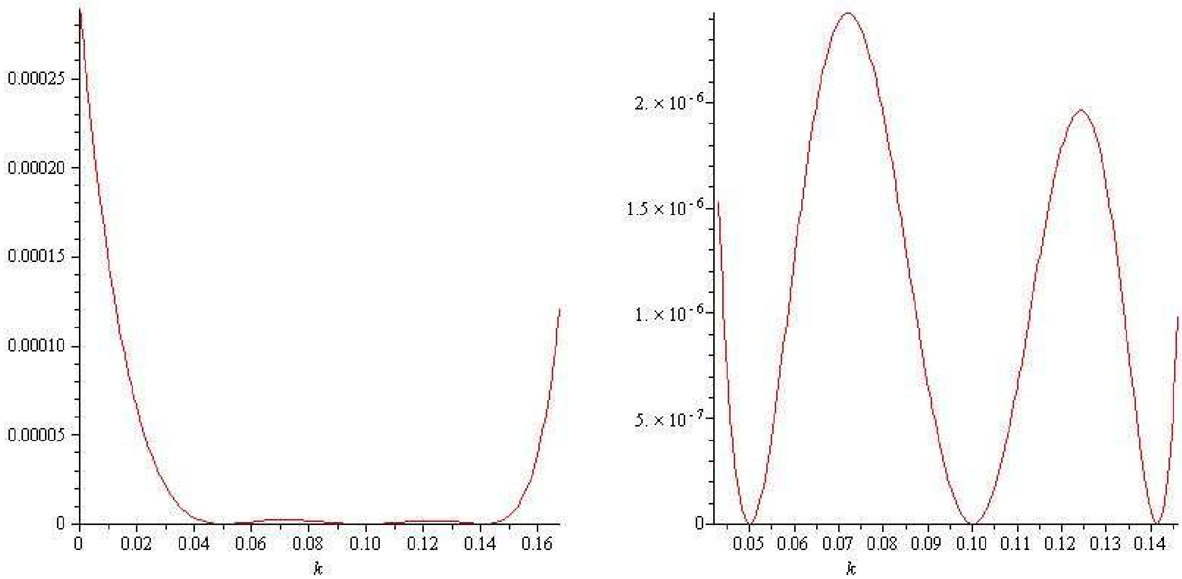
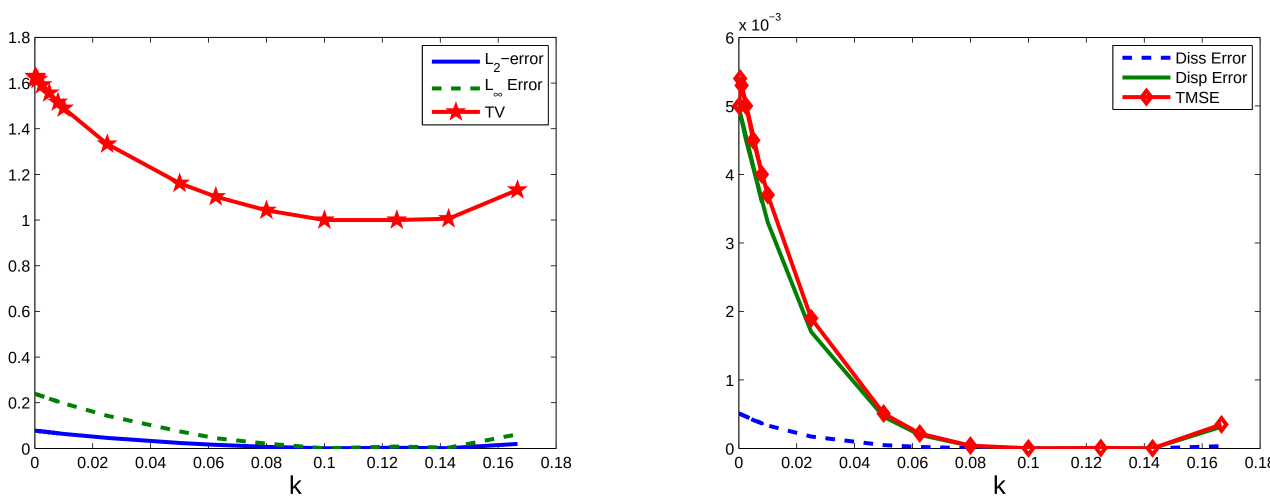


Figure 7: Plots of Exact RPE vs $\omega \in [0, \pi]$ when $k = 0.001, 0.08, 0.1, 0.125$ and $\frac{1}{6}$ for the Third Order Upwind scheme.



(a) Integrated error versus k (b) Integrated error versus k zoomed plot of (8a)

Figure 8: Integrated error versus k for Third Order Upwind scheme when Re=100 and $h = 0.1$.



(a) L_2, L_∞ errors and TV versus k (b) Dissipation, dispersion errors and TMSE versus k

Figure 9: Computed errors versus k for Third Order Upwind scheme when Re=100 and $h = 0.1$ at $T = 1$.