Sibling curves of polynomials

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Abstract. Sibling curves were demonstrated in papers [2, 3] as a novel way to visualize the zeros of complex valued functions. In this paper, we continue the work done in those papers by focusing solely on polynomials. We proceed to prove that the number of sibling curves of a polynomial is the degree of the polynomial.

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1. Introduction. In [1] we followed the historical journey of finding roots of complex functions, both algebraically and graphically. Three ways to visualize the complex roots of polynomial equations were discussed which are all informative but somewhat artificial. Then a fourth under-explored approach by Howard Fehr was considered which led to the idea of sibling curves [2]. This approach turns out to be a rich and useful way of visualizing zeroes of polynomials.

A polynomial f that maps complex numbers onto complex numbers, has n complex roots. This is a result that follows from the Fundamental Theorem of Algebra. For example, if $f(z) = z^2 + 1$, we get two solutions i and $-i$. The roots in this case are imaginary and the question of how to visually represent these zeroes is answered by means of the concept of *sibling curves*.

If we restrict the domain of f to those complex numbers that map onto real values, then the function has real values on this domain and can be represented in three dimensions with the domain in the horizontal plane and the range along the vertical axis. For example, a polynomial f can be written in the form

$$w = f(z) = f(x+iy) = g(x,y) + ih(x,y)$$

for some polynomials g and h. If f maps the complex number $x+iy$ onto a real number w then $h(x,y)=0$. We restrict the domain of the function f to all points in the xy-plane such that $h(x,y)=0$. The condition that $h(x,y)=0$ defines a curve(s) in the Argand plane. The function f with these curves as domain form the sibling curves. We demonstrate this with a few examples.

Example 1.1. Consider the function $f(z)=z^2$. We are interested in finding all z such that $f(z)\in\mathbb{R}$. To do so, let $z=x+iy$ where $x,y\in\mathbb{R}$. Then $f(z)=z^2=(x^2-y^2)+2ixy$ and $f(z)$ is real valued when $2xy=0$, that is $x=0$ or $y=0$. This produces two sibling curves, namely the parabola $f(z)=-y^2$ on $x=0$ and parabola $f(z)=x^2$ on $y=0$. These two sibling curves can be parametrized by $(it,-t^2)$ and (t,t^2), where t is a real number. Note that the root 0 lies on both sibling curves. See Figure 1.

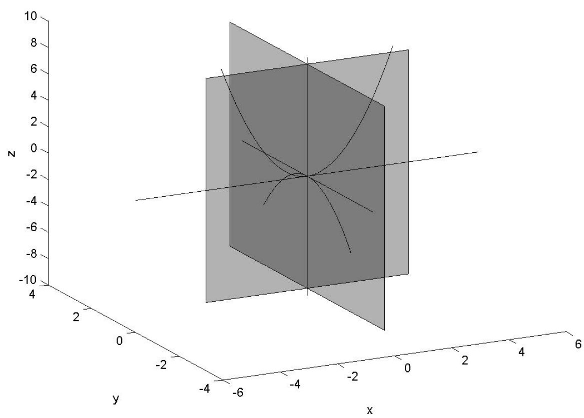

Figure 1 : Sibling curves of $f(z)=z^2$.

Example 1.2. Consider another quadratic function $f(z)=z-z^2$. Suppose $z=x+iy$ where $x,y\in\mathbb{R}$. Then $f(z)=z-z^2=(x-x^2+y^2)+i(y-2xy)$ is real valued when $y(1-2x)=0$. Hence $x=\frac{1}{2}$ or $y=0$. It again produced two sibling curves. One is the parabola in the plane $x=\frac{1}{2}$ and the other is a parabola in the plane $y=0$. These two sibling curves can be parametrized by $(\frac{1}{2}+it,\frac{1}{4}+t^2)$ and $(t,t-t^2)$ with $t\in\mathbb{R}$. Note that the roots 0 and 1 lie on the second sibling curve. See Figure 2.

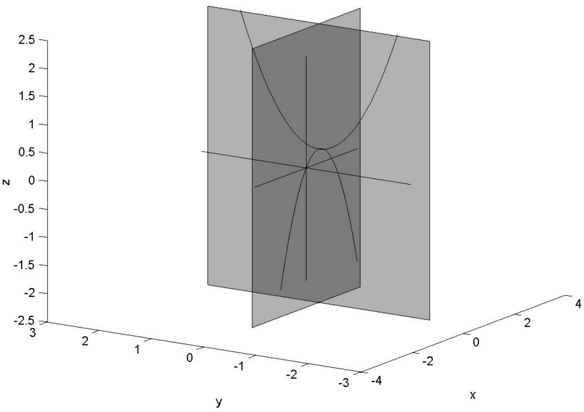

Figure 2 : Sibling curves of $f(z) = z - z^2$.

EXAMPLE 1.3. Now consider a simple cubic function $f(z) = z^3$. Suppose $z = x+iy$ where $x, y \in \mathbb{R}$. Then $f(z) = z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$ is real valued when $y(3x^2 - y^2) = 0$. Solving gives $y = 0$ or $y = \sqrt{3}x$ or $y = -\sqrt{3}x$. This time, we found three sibling curves each living in the planes $y = 0, y = \sqrt{3}x$ and $y = -\sqrt{3}x$ respectively. These sibling curves can be parametrized by $(t, t^3), (t + i\sqrt{3}t, -8t^3)$ and $(t - i\sqrt{3}t, -8t^3)$ with $t \in \mathbb{R}$. Note that the root 0 lies on all the sibling curves. See Figure 3.

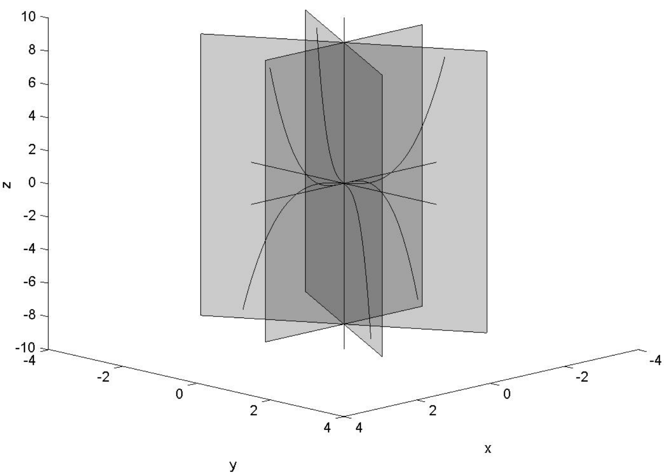

Figure 3 : Sibling curves of $f(z) = z^3$.

From the examples we see that sibling curves are 3D-curves in $\mathbb{C} \times \mathbb{R}$ consisting of the points $(z, f(z))$ in $\mathbb{C} \times \mathbb{R}$ for which $f(z)$ is real. This enables us to define a sibling curve parametrically.

DEFINITION 1.4. If f is a function $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ a function such that $g(t) = (z(t), f(z(t)))$ with $f(z(t)) \in \mathbb{R}$, then the set of points $\{g(t) : t \in \mathbb{R}\}$ is called a *sibling curve* of f.

Armed with the notion of sibling curves, we are ready to tackle the simplest polynomials of degree n.

EXAMPLE 1.5. We will show that the function $f : \mathbb{C} \to \mathbb{C}$ where $f(z) = z^n$ for some positive integer n has n sibling curves. To find the sibling curves of $f(z) = z^n$, we want to find all the values of z such that $f(z) \in \mathbb{R}$ and then form sibling curves using these points. To determine these z, we use De Moivre's Theorem. Assume $z = re^{i\theta}$ for some real numbers r and θ. If $f(z) = z^n = r^n e^{ni\theta} \in \mathbb{R}$, then $n\theta = \pi j$ for some integer j. Hence $\theta = \frac{\pi j}{n}$.

The projection of these points on the Argand plane gives us n straight lines given by $g(t) = te^{\frac{\pi i j}{n}}$ where $t \in \mathbb{R}$ and $j = 0, 1, \ldots, n-1$. The curves defined on these n lines are the n sibling curves. They are parametrized by $g_j : \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ where

$$g_j(t) = (te^{\frac{\pi i j}{n}}, t^n(-1)^j), \quad j = 0, 1, 2, \ldots, n-1; t \in \mathbb{R}.$$

Notice that these n sibling curves contain all points z such that $f(z) \in \mathbb{R}$.

Note if $n = 2$ in the example above, we must substitute $j = 0$ or $j = 1$ into the formula. This gives the parametrization of two sibling curves $g_0(t) = (t, t^2)$ and $g_1(t) = (it, -t^2)$ as in Example 1.1. If $n = 3$ in the example above we get $g_0(t) = (t, t^3)$ and $g_1(t) = (\frac{1}{2}t + i\frac{\sqrt{3}}{2}t, -t^3)$ and $g_2(t) = (-\frac{1}{2}t + i\frac{\sqrt{3}}{2}t, t^3)$ which look different, but are the same sibling curves as in Example 1.3, although the parametrization is different.

2. An example of a sub-parametrization around a point. Example 1.5 fuels our suspicion that the sibling curves of a polynomial of degree n has n sibling curves. Now to show that the number of sibling curves depends only on the degree of the polynomial, we need to form parametrizations of sibling curves. It suffices to be able to do parametrizations of any portion of a sibling curve. To achieve this goal we use a power series. The next example demonstrates the idea to form a sub-parametrization for a specific polynomial at a specific point.

EXAMPLE 2.1. Consider $f(z) = z - z^2$. We saw earlier that this quadratic has two sibling curves. We focus on the point $z = 0$ and find a sub-parametrization of the sibling curve containing the point $z = 0$.

To find a parametrization for the portion of the sibling curve around $z = 0$, we consider a parametrization of the form $g(t) = (z, f(z))$ with $f(z) \in \mathbb{R}$. We use real number $f(z) = t$ as the parameter, so $g(t) = (h(t), t)$ with $f(h(t)) = t \in \mathbb{R}$. We try $h(t)$ as a power series about 0, that is

$$h(t) = 0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \ldots$$

for t for which it converges. Then

$$(h(t))^2 = a_1^2t^2 + (2a_1a_2)t^3 + (2a_1a_3 + a_2^2)t^4 + (2a_1a_4 + 2a_2a_3)t^5 + (2a_1a_5 + 2a_2a_4 + a_3^2)t^6 + \ldots.$$

Since we want $f(h(t)) = h(t) - (h(t))^2 = t$, we must have

$$\begin{array}{rcl}
a_1 & = & 1 \\
a_2 - a_1^2 & = & 0 \\
a_3 - (2a_1a_2) & = & 0 \\
a_4 - (2a_1a_3 + a_2^2) & = & 0 \\
a_5 - (2a_1a_4 + 2a_2a_3) & = & 0 \\
a_6 - (2a_1a_5 + 2a_2a_4 + a_3^3) & = & 0 \\
 & \cdots &
\end{array}$$

Solving, we obtain $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 5, a_5 = 14, a_6 = 42$, etc. Therefore a portion of one sibling can be parametrized by

$$g(t) = (t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + \ldots, t)$$

if the radius of convergence is greater than 0.

Coincidentally these coefficients are the well-known Catalan numbers [4] named after Eugéne Catalan $(1814 - 1894)$. They appear often in counting problems, for example counting the number of triangulations of a convex polygon. Catalan numbers are formally defined as $C_1 = 1$ and then recursively by

$$C_{n+1} = C_1C_n + C_2C_{n-1} + \ldots + C_{n-1}C_2 + C_nC_1, n \in \mathbb{N}.$$

Applying Stirling approximations to this series, it can be shown that it has a radius of convergence $\frac{1}{4}$. Hence this is indeed a sub-parametrization for the sibling curve containing 0.

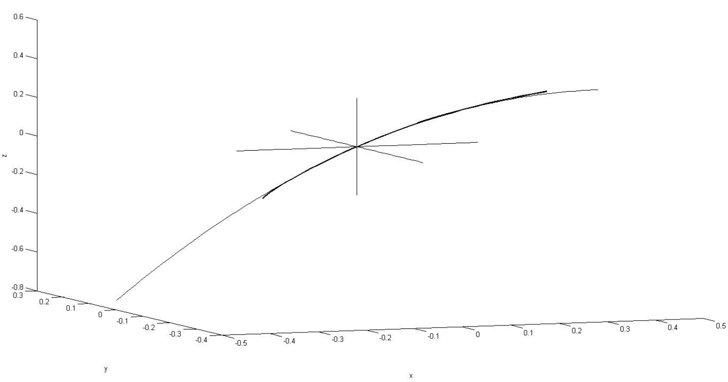

Figure 4 : Partial sibling curves of $f(z) = z - z^2$ around 0.

3. The general case. The proof that a polynomial of degree n has n sibling curves needs the existence of sub-parametrizations at any point on a sibling curve. Lemma 3.1 and Lemma 3.2 show that these parametrizations have non-zero radii of convergence. This is needed in Lemma 3.3 to produce the sub-parametrizations when a point has a certain multiplicity.

LEMMA 3.1. *For all positive integers k, n we have*

$$\binom{kn}{n} \leq k^{kn}.$$

Proof. The result is clearly true for $k = 1$. Fix some integer $k \geq 2$. We proceed by induction on n. Note if $n = 1$ then $\binom{k}{1} = k \leq k^k$.

Note $\frac{kn+j}{kn-n+j} \leq k$ if $k \geq 2$ and j any positive integer. This follows from the fact $2kn \leq k^2 n$ and $j \leq kj$ which gives $kn + j \leq k^2 n - kn + kj = k(kn - n + j)$. Using this, we get

$$\begin{aligned}
\binom{k(n+1)}{n+1} &= \binom{kn}{n} \frac{(kn+1)\cdot(kn+2)\cdot\ldots\cdot(kn+k)}{(kn-n+1)\cdot(kn-n+2)\cdot\ldots\cdot(kn-n+k)} \\
&\leq k^{kn}\cdot k\cdot k\cdot\ldots\cdot k \\
&= k^{k(n+1)}.
\end{aligned}$$

This completes our induction. □

LEMMA 3.2. *Assume $a_1, a_2, \ldots, a_n \in \mathbb{C}$ are the first n terms of a sequence. For complex $\alpha_1, \alpha_2, \ldots, \alpha_n$, define the rest of the sequence recursively by*

$$\begin{aligned}
a_{m+1} &= \alpha_1 \sum_{j_p \geq 1} a_{j_1} a_{m+1-j_1} + \alpha_2 \sum_{j_p \geq 1} a_{j_1} a_{j_2} a_{m+1-j_1-j_2} + \ldots \\
&+ \alpha_n \sum_{j_p \geq 1} a_{j_1} a_{j_2} \ldots a_{j_{n-1}} a_{m+1-j_1-i_2\ldots-j_{n-1}}
\end{aligned}$$

where $m \geq n$. Then $\sqrt[m]{|a_m|}$ is a bounded sequence.

Proof. We start by eliminating the coefficients α_i, by defining a new sequence b_i that do not need these coefficients. We then proceed to define another sequence d_i whose recurrence equation is even simpler. With the help of Catalan numbers we then show $\sqrt[m]{d_m}$ is bounded. This then proves $\sqrt[m]{b_m}$ and $\sqrt[m]{|a_m|}$ are both bounded sequences.

We start off by finding a positive real number t such that $|\alpha_j t| \leq t^{j+1}$ for $j = 1, 2, \ldots, n$. Hence by the triangle inequality if $m \geq n$,

$$\begin{aligned}
|ta_{m+1}| &\leq \sum_{j_p \geq 1} |\alpha_1 t a_{j_1} a_{m+1-j_1}| + \sum_{j_p \geq 1} |\alpha_2 t a_{j_1} a_{j_2} a_{m+1-j_1-j_2}| + \ldots \\
&+ \sum_{j_p \geq 1} |\alpha_m t a_{j_1} a_{j_2} \ldots a_{j_{n-1}} a_{m+1-j_1-j_2 \ldots -j_{n-1}}| \\
&\leq \sum_{j_p \geq 1} |(ta_{j_1})(ta_{m+1-j_1})| + \sum_{j_p \geq 1} |(ta_{j_1})(ta_{j_2})(ta_{m+1-j_1-j_2})| + \ldots \\
&+ \sum_{j_p \geq 1} |(ta_{j_1})(ta_{j_2}) \ldots (ta_{j_{n-1}})(ta_{m+1-j_1-j_2 \ldots -j_{n-1}})|.
\end{aligned}$$

Define $b_j = |ta_j|$. If $m \geq n$, then

$$\begin{aligned}
b_{m+1} &\leq \sum_{j_p \geq 1} b_{j_1} b_{m+1-j_1} + \sum_{j_p \geq 1} b_{j_1} b_{j_2} b_{m+1-j_1-j_2} + \ldots \\
&+ \sum_{j_p \geq 1} b_{j_1} b_{j_2} \ldots b_{j_{n-1}} b_{m+1-j_1-j_2 \ldots -j_{n-1}}.
\end{aligned}$$

Define $d_1 = \max\{1, b_1\}$ and $d_j = b_j$ if $j = 2, 3, \ldots, n$. If $m \geq n$, define d_{m+1} as follows

$$\begin{aligned}
d_{m+1} &= \sum_{j_p \geq 1} d_{j_1} d_{m+1-j_1} + \sum_{j_p \geq 1} d_{j_1} d_{j_2} d_{m+1-j_1-j_2} + \ldots \\
&+ \sum_{j_p \geq 1} d_{j_1} d_{j_2} \ldots d_{j_{n-1}} d_{m+1-j_1-j_2 \ldots -j_{n-1}}.
\end{aligned}$$

An easy induction argument shows that $b_j \leq d_j$ for all positive integers j. Now choose a positive real number k such that $n^2 \leq k$ and $d_j \leq k^{j-1} d_1^j C_j$ for $j = 1, 2, \ldots, n$ where C_j are the Catalan numbers defined in Example 2.1. Also for convenience define $d_0 = 1$. Assume for induction that $d_j \leq k^{j-1} d_1^j C_j$ for all $j \leq m$ where $m \geq n$. Consider d_{m+1}.

$$d_{m+1} = \sum_{0 \leq j_p \leq m, j_1 + \ldots + j_n = m+1} d_{j_1} d_{j_2} \ldots d_{j_n}.$$

For each term, one of the d_{j_p} term has to have $j_p > 0$, that is $j_p = 1, 2, \ldots, m$. It can occur at n places, that is $p = 1, 2, \ldots, n$. Thus

$$\begin{aligned}
d_{m+1} \leq n[d_1 &\sum_{0 \leq j_p \leq m, j_1 + \ldots + j_{n-1} = m} d_{j_1} d_{j_2} \ldots d_{j_{n-1}} + \ldots \\
&\ldots + d_m \sum_{0 \leq j_p \leq 1, j_1 + \ldots + j_{n-1} = 1} d_{j_1} d_{j_2} \ldots d_{j_{n-1}}].
\end{aligned}$$

Consider each d_k term. It is possible that one $j_p = m+1-k$. This can occur $(n-1)$ times, giving a total of $(n-1)d_{m+1-k}$. However if none of the $j_p = m+1-k$, then all the j_p is less than $m+1-k$ and

$$\sum_{0 \leq j_p < m+1-k, j_1+\ldots+j_{n-1}=m+1-k} d_{j_1} d_{j_2} \ldots d_{j_{n-1}} \leq d_{m+1-k}$$

which gives

$$d_{m+1} \leq n[d_1(nd_m) + d_2(nd_{m-1}) + \ldots + d_m(nd_1)].$$

Each term now contains n^2 and $n^2 \leq k$. Using the induction hypothesis, we obtain

$$d_{m+1} \leq k[(d_1C_1)(k^{m-1}d_1^mC_m) + (k^1d_1^2C_2)(k^{m-2}d_1^{m-1}C_{m-1}) + \ldots \\ \ldots + (k^{m-1}d_1^mC_m)(d_1C_1)].$$

Simplifying and using the Catalan recursion, we get

$$\begin{aligned} d_{m+1} &\leq k^m d_1^{m+1}(C_1C_m + C_2C_{m-1} + \ldots + C_mC_1) \\ d_{m+1} &\leq k^m d_1^{m+1}C_{m+1}. \end{aligned}$$

This completes the induction and therefore $\sqrt[m]{d_m} \leq k^{1-1/m} d_1 \sqrt[m]{C_m}$. Using $C_m = \frac{\binom{2m}{m}}{m+1} \leq \frac{4^m}{m+1}$ by Lemma 3.1, we see that $\sqrt[m]{d_m}$ is a bounded sequence. Since $0 \leq b_m \leq d_m$, it follows that $\sqrt[m]{b_m}$ is bounded and consequently $\sqrt[m]{|a_m|}$ is also a bounded sequence. □

Lemma 3.3. *If f is a polynomial with a root of multiplicity m at the origin, then there are m distinct sub-parametrizations of sibling curves around $z = 0$.*

Proof. Since f is a polynomial with a root of multiplicity m at the origin $f(z) = c_nz^n + c_{n-1}z^{n-1} + \ldots c_mz^m$ for some complex coefficients $c_m, \ldots, c_n$, $c_m \neq 0$ and $m \leq n$.

Suppose $h(t) = a_1t + a_2t^2 + a_3t^3 + \ldots$ for some coefficients a_j. The aim here is to find $h(t)$ such that $f(h(t)) = t^m$ and then to manipulate it into forming m sub-parametrizations.

Looking at the t^m term of $f(h(t))$ we want $c_ma_1^m = 1$. Select any of the m values for a_1 to satisfy this equation. To proceed, consider the t^k term where $k \geq m+1$. We want

$$\begin{aligned} 0 &= c_n \sum_{j_p \geq 1} a_{j_1}a_{j_2} \ldots a_{j_{n-1}} a_{k-j_1-j_2-\ldots-j_{n-1}} + \ldots \\ &+ c_m \sum_{j_p \geq 1} a_{j_1}a_{j_2} \ldots a_{j_{m-1}} a_{k-j_1-j_2-\ldots-j_{m-1}}. \end{aligned}$$

Using this equation, we can solve for a_k uniquely by induction. Continuing in this manner, we form a series that satisfies the equation $f(h(t)) = t^m$ and by Lemma

3.2, we know that $\limsup |a_n|^{1/n}$ exists, thus the power series has a non-zero radius of convergence.

Using this power series, we can now form m sub-parametrizations of sibling curves around 0. In the formula below $j = 0, 1, 2, \ldots, m-1$ and

$$g_j(t) = \begin{cases} (h(\sqrt[m]{t}e^{\frac{\pi i j}{m}}), t(-1)^j) & \text{if } t \geq 0 \\ (h(\sqrt[m]{-t}e^{\frac{\pi i j}{m}+\pi i}), t(-1)^{j+m+1}) & \text{if } t < 0. \end{cases}$$

It should be noted that each g_j is continous and differentiable on the interval of convergence. Note if m is even, then the sub-parametrization produces curves with either non-negative or non-positive z values. And if m is odd, then the sub-parametrization produces curves with both negative and positive z values. □

Before we prove the general result, let us demonstrate this theorem with an example.

Example 3.4. Consider $f(z) = z^2 - z^3$. To find two sub-parametrizations around 0, we take

$$h(t) = a_1 t + a_2 t^2 + a_3 t^3 + \ldots.$$

Then

$$\begin{aligned} (h(t))^2 &= a_1^2 t^2 + 2a_1 a_2 t^3 + (a_2^2 + 2a_1 a_3)t^4 + \ldots \\ (h(t))^3 &= a_1^3 t^3 + (3a_1^2 a_2)t^4 + \ldots. \end{aligned}$$

Thus if $f(h(t)) = t^2$, then

$$\begin{aligned} a_1^2 &= 1 \\ 2a_1 a_2 - a_1^3 &= 0 \\ a_2^2 + 2a_1 a_3 - 3a_1^2 a_2 &= 0 \\ \ldots &. \end{aligned}$$

Selecting $a_1 = 1$, we can solve the other coefficients uniquely, as $a_2 = \frac{1}{2}, a_3 = \frac{5}{8}, \ldots$. Thus $h(t) = t + \frac{1}{2}t^2 + \frac{5}{8}t^3 + \ldots$. By Lemma 3.2 we know this series has a radius of convergence bigger than 0. Using $h(t)$ we now form two sub-parametrizations around 0.

$$g_0(t) = \begin{cases} (\sqrt{t} + \frac{1}{2}t + \frac{5}{8}t\sqrt{t} + \ldots, t) & \text{if } t \geq 0 \\ (-\sqrt{-t} - \frac{1}{2}t + \frac{5}{8}t\sqrt{-t} + \ldots, -t) & \text{if } t < 0 \end{cases}$$

and

$$g_1(t) = \begin{cases} (i\sqrt{t} - \frac{1}{2}t - \frac{5}{8}it\sqrt{t} + \ldots, -t) & \text{if } t \geq 0 \\ (-i\sqrt{-t} + \frac{1}{2}t - \frac{5}{8}it\sqrt{-t} + \ldots, t) & \text{if } t < 0. \end{cases}$$

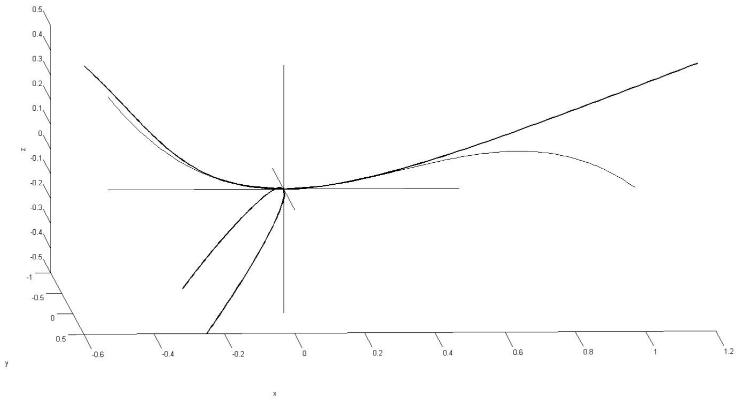

Figure 5 : Partial sibling curves of $f(z) = z^2 - z^3$ around 0.

With the help of Lemma 3.3, we are ready to prove the main result of this paper.

THEOREM 3.5. *If $f(z)$ is a complex polynomial of degree n, then f has n sibling curves.*

Proof. We will show for each real value of w that there are always n portions of sibling curves containing the solutions of $f(z) = w$. If we glue these subcurves together, we get the desired n sibling curves.

For some fixed $w \in \mathbb{R}$, we know $f(z) = w$ has n solutions by the fundamental theorem of algebra. Some may have multiplicity higher than 1. Take solution z_1 with multiplicity m. Then $f(z) - w = q(z - z_1)$ where $q(z) = c_m z^m + \ldots + c_n z^n$. Hence, if $q(z) = 0$ then $f(z + z_1) = w$. So we only need to show that we get m sibling subcurves containing the solutions $q(z) = 0$. Note there is at most $n - 1$ real values w such that $g(z) = f(z) - w$ has a root with multiplicity higher than one.

Noting $q(z) = f(z_1) - w = 0$, the solution now lies in using Lemma 3.3. There we proved that it is possible to define m sibling subcurves of q around 0. By [5] they contain all the solutions in that neighbourhood. Furthermore each of them has the same non-zero radius of convergence. Thus each value of w produces n sub-parametrizations.

Now for any real value w, we have n sub-parametrizations. Suppose R is the smallest radius of convergence for these sub-parametrizations. That is, each sub-parametrization is valid on the interval $(w - R, w + R)$. Now consider the real values $w - \frac{R}{2}$ and $w + \frac{R}{2}$. They each have n sub-parametrizations. By gluing two sub-parametrizations that overlap together, we form n piece-wise functions on the interval $[w - \frac{R}{2}, w + \frac{R}{2}]$. Continuing in this manner we produce the n sibling curves. $\square$

It should be noted that the proof above is not only true for real polynomials like $z^4 + 2z^2 + z + 2$ or $z^6 - 1$, but that it holds for any polynomial, including those with complex coefficients, like $z^8 + iz + 3$ or $z^7 - (i + 2)z^3 + z^2 + i$.

This result gives us a richer and more visual understanding of the roots of a polynomial. The fundamental theorem of algebra merely tells us that a polynomial of degree n has n roots. Now with the aid of this theorem we see that each polynomial of degree n has n special curves associated with it and they contain the zeroes of the polynomial. Moreover, we get a better visual understanding of the four dimensional graph of a complex function of complex variables by looking at the three dimensional cut of the graph on which the function values are real.

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