Subjective Bayesian analysis of the elliptical model

by

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Abstract

The multivariate elliptical model is considered, such as to derive subjective Bayesian estimators of the location vector and some functions of the characteristic matrix with the normal-inverse Wishart and the normal-Wishart as prior respectively. The latter was considered by Bekker and Roux (1995) as prior for the normal model. Fang and Li (1999) considered the elliptical model for Bayesian analysis but with an objective prior structure. In addition, the newly developed results is applied to the multivariate normal distribution and the multivariate t-distribution. For the univariate case, a performance study is done to evaluate the normal-gamma and normal-inverse gamma distributions as suitable priors. A practical application for the posterior distributions of the multivariate t-distribution is included by means of Gibbs sampling and a Metropolis-Hastings algorithm.

Keywords: Bayesian, characteristic matrix, elliptically contoured, location vector, normal-inverse Wishart, normal-Wishart.

AMS 2000 subject classification: 62H10; 62E15

1 Introduction

The multivariate normal distribution is a well-known and widely discussed distribution for which a large amount of literature exists. It is however necessary to extend the results that does exist for a more general case such as the elliptical contoured model since there are a lot of natural phenomena, especially in the finance and risk sectors, where the normal model is inadequate as a modelling distribution. The elliptically contoured distribution (Fang and Zhang, 1990) can be viewed as an extension of the multivariate normal distribution and hence possesses some of the same properties such as symmetry. An elliptically contoured distribution is a distribution whose contours of equal density have the same elliptical shape as that of the normal distribution but it can also be long-tailed or short-tailed. This type of model is therefore more flexible than the normal model and hence its increase in popularity. Some examples of an elliptically contoured distribution is the Pearson type VII distributions of which the multivariate t-distribution forms part and the generalized Laplace and Bessel distributions to name but a few. Elliptically contoured distributions has been investigated from as early as 1860 by Maxwell amongst others and have since then received a large amount of interest from modern researchers such as Fang and Zhang (1990) and Gupta and Varga (1993). The contributions made by Díaz-García and Vera (2011), Gómez et al (2003) and Cheung et al (2007) should be acknowledged.

The objective of this paper is to derive Bayesian estimators for the parameters of the elliptically contoured distribution by using *subjective* Bayesian analysis. Fang and Li (1999)

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considered the elliptical model for Bayesian analysis but with an objective prior structure. This study focuses on the univariate and multivariate elliptical model, respectively, as the underlying model. The joint and marginal posterior distributions are derived in this study as well as the Bayesian estimators of the location vector and some functions of the characteristic matrix assuming subjective priors.

Subjective analysis generally produces more admissible results since added information is used than in the case of objective analysis. None the less, very few results and estimators for subjective Bayesian analysis exist and this study attempts to contribute to the literature in terms of more general subjective Bayesian estimation. The Bayesian analysis of the multivariate normal model has been discussed by for example Press (1982) with the assumption of a diffuse prior and also a conjugate prior in the form of the normal model for the location parameter and the inverse Wishart distribution for the covariance matrix of the underlying model. The normal-inverse Wishart prior is known as the natural conjugate prior for the multivariate normal distribution as a special case, Arashi et. al. (2013) developed a conjugate prior structure for the matrix-variate elliptical model. A prior distribution that maximally reflects the prior information is preferred and it is for this reason that a normal-Wishart prior, as discussed by Bekker and Roux (1995), will also be considered in this study to optimally consume the prior information.

For the purpose of this paper we give the definition of a multivariate elliptically contoured distribution as follows:

Definition 1 *(Fang & Zhang,1990). A random vector $\mathbf{X} \in \mathbb{R}^p$ has a multivariate elliptically contoured distribution (ECD) with parameters $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ and g if its density function is*

$$f(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = d_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}} g[\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})], \boldsymbol{\mu} \in \mathbb{R}^p, \boldsymbol{\Sigma} > \mathbf{0} \quad (1)$$

where $d_p = \frac{\Gamma(\frac{1}{2\pi})}{(\frac{1}{2\pi})^{\frac{p}{2}}}(\int_{\mathbb{R}^p} y^{p-1} g(y^2)dy)^{-1}$ with $\Gamma(\cdot)$ the gamma function. The function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is called the density generator and it is a function of the quadratic form $\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$. It is denoted as $\mathbf{X} \sim ECD(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$ with $\boldsymbol{\mu}$ as the location vector and $\boldsymbol{\Sigma}$ as the characteristic matrix.

(The multivariate elliptically contoured distribution will be referred to as the elliptically contoured distribution hereafter.) Note that the density function (1) can also be written as (Chu, 1973):

$$f(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \int_0^\infty w(z) f_{N_{\boldsymbol{\mu}, z^{-1}\boldsymbol{\Sigma}}}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})dz$$

for a scalar function $w(z)$ and with $f_{N_{\boldsymbol{\mu}, z^{-1}\boldsymbol{\Sigma}}}(\cdot)$ as the multivariate normal density function with the same expected value as $\mathbf{X}$ and a covariance matrix $z^{-1}\boldsymbol{\Sigma}$. Note that $\int_0^\infty w(z)dz = 1$ and the difference from the class of scale mixtures of the normal distribution. If a scale mixture of the normal distribution is considered then the weighting function is actually a density function where $0 \leq w(z) \leq 1$ for all values of z. However, in the case of equation the elliptical model, the weighting function does not have the same restriction but $-1 \leq w(z) \leq 1$ for all values of z.

The rest of the paper is organized as follows: In section 2, we derive the posterior distributions, as well as Bayesian estimators of the location vector $\boldsymbol{\mu}$ and characteristic matrix

$\boldsymbol{\Sigma}$ of the elliptically contoured model for the normal-inverse Wishart prior. The normal-Wishart prior forms the base of section 3. Subsequently, the newly developed results of sections 2 and 3 will be applied to the multivariate normal distribution and the multivariate t-distribution as special cases of the elliptically contoured distribution in section 4. In section 5 a real dataset as well as simulation is used to illustrate the methodology developed in this paper.

2 Normal-Inverse Wishart prior

Suppose an observed sample $(\mathbf{X}_1, ..., \mathbf{X}_n)$ is available from (1) and one wishes to draw inferences about the mean $\boldsymbol{\mu}$ and the characteristic matrix $\boldsymbol{\Sigma}$ for the subjective prior, the normal-inverse Wishart prior for which known results for this model has not yet been derived, but will be derived in this section. The likelihood function is

$$\begin{aligned} L(\boldsymbol{\mu}, \boldsymbol{\Sigma}|\overline{\mathbf{x}}, \mathbf{V}) &= \int_0^\infty w(z)(2\pi)^{-\frac{p}{2}}|z^{-1}\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp[-\frac{1}{2}\sum_{i=1}^n(\mathbf{x}_i - \boldsymbol{\mu})'z\boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})]dz \\ &\propto \int_0^\infty w(z)z^{\frac{np}{2}}|\boldsymbol{\Sigma}|^{-\frac{n}{2}} etr[-\frac{1}{2}z\boldsymbol{\Sigma}^{-1}(\mathbf{V} + n(\overline{\mathbf{x}} - \boldsymbol{\mu})(\overline{\mathbf{x}} - \boldsymbol{\mu})')]dz \qquad (2) \end{aligned}$$

where

$$\mathbf{V} = \sum_{i=1}^n(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})' \qquad (3)$$

Now let $\boldsymbol{\Psi} = z^{-1}\boldsymbol{\Sigma}$, then the Jacobian is $z^{-\frac{p(p+1)}{2}}$. It is assumed that there is prior knowledge on the parameter space $(\boldsymbol{\mu}, \boldsymbol{\Psi})$ summarized in the following normal-inverse Wishart density functions (see Press (1982)), say

$$\boldsymbol{\mu}|\boldsymbol{\Psi} \sim N(\boldsymbol{\theta}, \frac{1}{a}\boldsymbol{\Psi})$$

i.e.

$$\pi(\boldsymbol{\mu}|\boldsymbol{\Psi}) = (2\pi)^{-\frac{p}{2}}|\frac{1}{a}\boldsymbol{\Psi}|^{-\frac{1}{2}} \exp[-\frac{a}{2}(\boldsymbol{\mu} - \boldsymbol{\theta})'\boldsymbol{\Psi}^{-1}(\boldsymbol{\mu} - \boldsymbol{\theta})]$$

and

$$\boldsymbol{\Psi} \sim W^{-1}(\boldsymbol{\Phi}, p, m)$$

such that

$$\pi(\boldsymbol{\Psi}) = [2^{\frac{p(m-p-1)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^p\Gamma(\frac{m-p-j}{2})]^{-1}|\boldsymbol{\Phi}|^{\frac{1}{2}(m-p-1)}|\boldsymbol{\Psi}|^{-\frac{1}{2}m}etr[-\frac{1}{2}\boldsymbol{\Psi}^{-1}\boldsymbol{\Phi}]$$

The rich parameter structure provided by a, m, $\boldsymbol{\theta}$ and $\boldsymbol{\Phi}$ grant the Bayesian statistician a flexible choice of these parameters so that the assumed prior conforms to the past knowledge available to the experimenter. From the above, the joint prior density function is obtained as follows:

$$\pi(\boldsymbol{\mu}, \boldsymbol{\Psi}|z) = c|\boldsymbol{\Psi}|^{-\frac{1}{2}(m+1)} \exp[-\frac{a}{2}(\boldsymbol{\mu} - \boldsymbol{\theta})'\boldsymbol{\Psi}^{-1}(\boldsymbol{\mu} - \boldsymbol{\theta})]etr[-\frac{1}{2}\boldsymbol{\Psi}^{-1}\boldsymbol{\Phi}]$$

where $c = \dfrac{(2\pi)^{-\frac{p}{2}} a^{\frac{p}{2}}}{2^{\frac{p(m-p-1)}{2}} \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma(\frac{m-p-j}{2})} |\mathbf{\Phi}|^{\frac{1}{2}(m-p-1)}$.

The joint prior density function for $(\boldsymbol{\mu}, \mathbf{\Sigma})$ is now defined as follows:

$$\pi(\boldsymbol{\mu}, \mathbf{\Sigma}) \propto \int_0^\infty z^{-\frac{p(p+1)}{2}} z^{\frac{p(m+1)}{2}} |\mathbf{\Sigma}|^{-\frac{m+1}{2}} etr[-\frac{1}{2} z \mathbf{\Sigma}^{-1}(a(\boldsymbol{\mu}-\boldsymbol{\theta})(\boldsymbol{\mu}-\boldsymbol{\theta})' + \mathbf{\Phi})] dz \tag{4}$$

From equations (2) and (4) the joint posterior distribution is obtained as follows:

$$q(\boldsymbol{\mu}, \mathbf{\Sigma}|\overline{\mathbf{x}}, \mathbf{V}) \propto \int_0^\infty w(z) z^{-\frac{p(p-n-m)}{2}} |\mathbf{\Sigma}|^{-\frac{n+m+1}{2}} etr[-\frac{1}{2} z \mathbf{\Sigma}^{-1}(\mathbf{D} + (n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})')] dz \tag{5}$$

where

$$\mathbf{b} = \frac{n\overline{\mathbf{x}} + a\boldsymbol{\theta}}{n+a} \tag{6}$$

and

$$\mathbf{D} = \mathbf{V} + \frac{an}{n+a}(\overline{\mathbf{x}} - \boldsymbol{\theta})(\overline{\mathbf{x}} - \boldsymbol{\theta})' + \mathbf{\Phi} \tag{7}$$

which is independent of $\boldsymbol{\mu}$, $\mathbf{\Sigma}$ and z and with $\mathbf{V}$ as defined in equation (3).

2.1 Bayesian analysis of $\boldsymbol{\mu}$

The marginal posterior distribution for the location parameter, $\boldsymbol{\mu}$, is derived from the joint posterior distribution (see equation (5)):

$$\begin{aligned} q(\boldsymbol{\mu}|\overline{\mathbf{x}}, \mathbf{V}) &= \int_{\mathbf{\Sigma}>\mathbf{0}} q(\boldsymbol{\mu}, \mathbf{\Sigma}|\overline{\mathbf{x}}, \mathbf{V}) d\mathbf{\Sigma} \\ &\propto \int_0^\infty w(z) z^{-\frac{p(p-n-m)}{2}} \int_{\mathbf{\Sigma}>\mathbf{0}} |\mathbf{\Sigma}|^{-\frac{n+m+1}{2}} \\ &\quad \times etr[-\frac{1}{2} z \mathbf{\Sigma}^{-1}(\mathbf{D} + (n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})')] d\mathbf{\Sigma} dz \end{aligned} \tag{8}$$

Note that the integral with respect to $\mathbf{\Sigma}$ is the integral of the kernel of an inverse Wishart distribution (see Gupta and Nagar (2000) definition 3.4.1.) with parameters $z(\mathbf{D} + (n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})'), p$ and $n+m+1$, hence

$$\begin{aligned} q(\boldsymbol{\mu}|\overline{\mathbf{x}}, \mathbf{V}) &\propto \int_0^\infty w(z) |\mathbf{D} + (n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})'|^{-\frac{n+m-p}{2}} dz \\ &= c|\mathbf{D} + (n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})'|^{-\frac{n+m-p}{2}} \end{aligned} \tag{9}$$

with $c^{-1} = (n+m-2p)^{\frac{p(n+m-p+1)}{2}} |\mathbf{D}|^{-\frac{n+m-p-1}{2}} (n+a)^{\frac{p}{2}}$. Therefore the marginal posterior distribution of the location parameter is a multivariate t-distribution with parameters $\mathbf{b}$ and $\frac{1}{(n+a)(n+m-2p)}\mathbf{D}$ and degrees of freedom $(n+m-2p)$, where $\mathbf{b}$ and $\mathbf{D}$ as defined in equations (6) and (7) respectively.

The Bayesian estimator of $\boldsymbol{\mu}$ under quadratic loss is given by the expected value of $\boldsymbol{\mu}$ with respect to the posterior distribution. Hence,

$$\widehat{\boldsymbol{\mu}}_B = E_{\boldsymbol{\mu}|\overline{\mathbf{x}}, \mathbf{V}}[\boldsymbol{\mu}|\overline{\mathbf{x}}, \mathbf{V}] = \mathbf{b} = \frac{n\overline{\mathbf{x}} + a\boldsymbol{\theta}}{n+a} \tag{10}$$

It is thus a weighted linear combination of the sample mean and the mean parameter of the prior distribution of $\boldsymbol{\mu}$.

2.2 Bayesian analysis of $\mathbf{\Sigma}$

From equation (5) the marginal posterior distribution of $\mathbf{\Sigma}$ can be derived as follows,

$$\begin{aligned} q(\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V}) &= \int_{\boldsymbol{\mu}\in\mathbb{R}^p} q(\boldsymbol{\mu},\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V})d\boldsymbol{\mu} \\ &= c\int_0^\infty w(z)z^{-\frac{p(p-n-m+1)}{2}}|\mathbf{\Sigma}|^{-\frac{n+m}{2}}etr[-\frac{1}{2}z\mathbf{\Sigma}^{-1}\mathbf{D}]dz \end{aligned} \quad (11)$$

with $c^{-1} = 2^{\frac{p(n+m-p-1)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})|\mathbf{D}|^{-\frac{n+m-1-p}{2}}$ and $\mathbf{D}$ as defined in equation (7). Note that $\int_{\boldsymbol{\mu}\in\mathbb{R}^p} etr[-\frac{1}{2}(\boldsymbol{\mu}-\mathbf{b})'(\frac{1}{z(n+a)}\mathbf{\Sigma})^{-1}(\boldsymbol{\mu}-\mathbf{b}))]d\boldsymbol{\mu} = |\frac{1}{z(n+n_0)}\mathbf{\Sigma}|^{\frac{1}{2}}(2\pi)^{\frac{p}{2}}$.
The Bayesian estimator of $|\mathbf{\Sigma}|$ can be calculated as the expected value of $|\mathbf{\Sigma}|$ with regard to the posterior distribution of $\mathbf{\Sigma}$. Hence from equation (11), we have that the Bayesian estimator of $|\mathbf{\Sigma}|$ is:

$$\begin{aligned} |\widehat{\mathbf{\Sigma}}|_B &= E_{\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V}}[|\mathbf{\Sigma}||\overline{\mathbf{x}},\mathbf{V}] \\ &= \frac{1}{2^{\frac{p(n+m-1-p)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|^{\frac{n+m-1-p}{2}} \\ &\quad \times\int_0^\infty w(z)z^{-\frac{p(p-n-m+1)}{2}}\int_{\mathbf{\Sigma}>\mathbf{0}}|\mathbf{\Sigma}|^{-\frac{n+m-2}{2}}etr[-\frac{1}{2}z\mathbf{\Sigma}^{-1}\mathbf{D}]d\mathbf{\Sigma}dz \\ &= \int_0^\infty zw(z)dz\frac{\prod_{j=1}^{p}\Gamma(\frac{n+m-2-p-j}{2})}{2^p\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}| \end{aligned} \quad (12)$$

where $\mathbf{D}$ is defined in equation (7).

3 Normal-Wishart prior

In this section the prior distribution for the location vector is the normal model and the prior distribution for the characteristic matrix is the Wishart distribution. As in section 2, $\boldsymbol{\mu}|\mathbf{\Psi}\sim N(\boldsymbol{\theta},\frac{1}{a}\mathbf{\Psi})$, but assume that $\mathbf{\Psi}\sim W(\mathbf{\Phi}^{-1},p,m)$ such that

$$\pi(\mathbf{\Psi}) = [2^{\frac{mp}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{m+1-j}{2})]^{-1}|\mathbf{\Phi}|^{\frac{1}{2}m}|\mathbf{\Psi}|^{\frac{1}{2}(m-p-1)}etr[-\frac{1}{2}\mathbf{\Psi}\mathbf{\Phi}]$$

With the joint prior density function for $\boldsymbol{\mu}$ and $\mathbf{\Psi}$:

$$\begin{aligned} \pi(\boldsymbol{\mu},\mathbf{\Psi}|z) &= (2\pi)^{-\frac{p}{2}}|\frac{1}{a}\mathbf{\Psi}|^{-\frac{1}{2}}\exp[-\frac{a}{2}(\boldsymbol{\mu}-\boldsymbol{\theta})'\mathbf{\Psi}^{-1}(\boldsymbol{\mu}-\boldsymbol{\theta})] \\ &\quad \times[2^{\frac{mp}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{m+1-j}{2})]^{-1}|\mathbf{\Phi}|^{\frac{m}{2}}|\mathbf{\Psi}|^{\frac{m-p-1}{2}}etr[-\frac{1}{2}\mathbf{\Psi}\mathbf{\Phi}] \end{aligned}$$

Define now the joint prior density function for the elliptical contoured model as:

$$\pi(\boldsymbol{\mu},\boldsymbol{\Sigma}) \propto \int_0^\infty z^{-\frac{p(p+1)}{2}} z^{-\frac{p(m-1)}{2}} |\boldsymbol{\Sigma}|^{\frac{m-p-2}{2}} etr[-\frac{1}{2}z\boldsymbol{\Sigma}^{-1}a(\boldsymbol{\mu}-\boldsymbol{\theta})(\boldsymbol{\mu}-\boldsymbol{\theta})' - \frac{1}{2}z^{-1}\boldsymbol{\Sigma}\boldsymbol{\Phi})]dz \quad (13)$$

From equations (2) and (13) the joint posterior distribution is obtained as,

$$\begin{aligned} q(\boldsymbol{\mu},\boldsymbol{\Sigma}|\overline{\mathbf{x}},\mathbf{V}) \quad \propto \quad & \int_0^\infty w(z) z^{\frac{p(n-m+1)}{2}} |\boldsymbol{\Sigma}|^{\frac{-n+m-p-2}{2}} etr[-\frac{1}{2}z^{-1}\boldsymbol{\Sigma}\boldsymbol{\Phi}] \\ & \times etr[-\frac{1}{2}z\boldsymbol{\Sigma}^{-1}(\mathbf{V}+(n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})' \\ & +\frac{an}{n+a}(\overline{\mathbf{x}}-\boldsymbol{\theta})(\overline{\mathbf{x}}-\boldsymbol{\theta})')dz \end{aligned} \quad (14)$$

with $\mathbf{b}$ as defined in equation (6).

3.1 Bayesian analysis for $\boldsymbol{\mu}$

The marginal posterior distribution of $\boldsymbol{\mu}$ can be obtained from equation (14) as,

$$\begin{aligned} q(\boldsymbol{\mu}|\overline{\mathbf{x}},\mathbf{V}) \quad \propto \quad & \int_0^\infty w(z) z^{\frac{p(n-m+1)}{2}} \int_{\boldsymbol{\Sigma}>\mathbf{0}} |\boldsymbol{\Sigma}|^{\frac{-n+m-p-2}{2}} etr[-\frac{1}{2}z^{-1}\boldsymbol{\Sigma}\boldsymbol{\Phi}] \\ & \times etr[-\frac{1}{2}z\boldsymbol{\Sigma}^{-1}\mathbf{F}]d\boldsymbol{\Sigma}dz \end{aligned}$$

with $\mathbf{F}=\mathbf{V}+(n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})'+\frac{an}{n+a}(\overline{\mathbf{x}}-\boldsymbol{\theta})(\overline{\mathbf{x}}-\boldsymbol{\theta})'$, which is independent of $\boldsymbol{\Sigma}$ and z. Using Gupta and Nagar (2000, definition 1.6.5.) it follows that

$$q(\boldsymbol{\mu}|\overline{\mathbf{x}},\mathbf{V}) = cB_{\frac{n+m-2p-3}{2}}(\frac{1}{4}\boldsymbol{\Phi}\mathbf{F}) \quad (15)$$

where $c^{-1}=(2\pi)^{-\frac{p}{2}}(n+a)^{\frac{1}{2}}|\frac{1}{2}\boldsymbol{\Phi}|^{\frac{1}{2}}B_{\frac{n+m-2p-2}{2}}(\frac{1}{4}\boldsymbol{\Phi}(\mathbf{V}+\frac{an}{n+a}(\overline{\mathbf{x}}-\boldsymbol{\theta})(\overline{\mathbf{x}}-\boldsymbol{\theta})'))$ and $B_\nu(\cdot)$ is the Bessel function of the second kind with matrix argument (see Herz (1995)).
Under quadratic loss the Bayesian estimator of $\boldsymbol{\mu}$ is $\widehat{\boldsymbol{\mu}}_B = E_{\boldsymbol{\mu}|\overline{\mathbf{x}},\mathbf{V}}[\boldsymbol{\mu}|\overline{\mathbf{x}},\mathbf{V}]$. Hence,

$$\widehat{\boldsymbol{\mu}}_B = E_{\boldsymbol{\mu}|\overline{\mathbf{x}},\mathbf{V}}[\boldsymbol{\mu}] = \mathbf{b} = \frac{n\overline{\mathbf{x}}+a\boldsymbol{\theta}}{n+a} \quad (16)$$

as before for the normal-inverse Wishart prior (see equation (10)).

Remark 1 *The Bayesian estimator of* $\boldsymbol{\mu}$ *(see (16) and (10) for* $p = 1$ *lifts the value if the sample mean is too small and decreases the value if the sample mean is too large. This is illustrated in Figure 1 for which the true value of the location parameter,* μ*, is 5 and* $a = 5$ *(left) and* $a = 15$ *(right). This lifting and decreasing effect is greatest for small sample sizes, for which Bayesian analysis is most beneficial.*

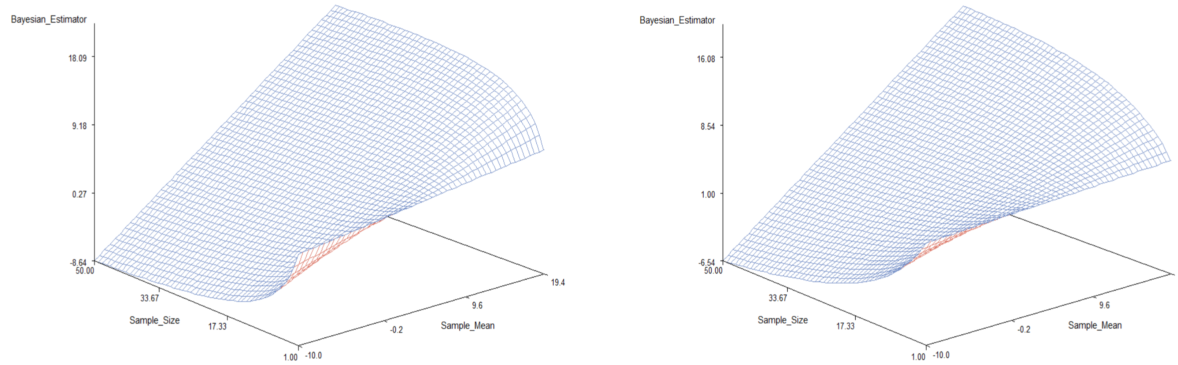

Figure 1. Performance evaluation of the Bayesian estimator of μ

Remark 2 *The tail probabilities of the posterior distribution of* μ *is of interest especially in risk and finance analysis. Equations (9) and (15) are used to calculate the posterior tail probability* $P(\mu > \mu_\alpha) = \int_{\mu_\alpha}^{\infty} q(\mu|\overline{x}, V)d\mu$ *as given in Figure 2. Note that for these calculations,* $\alpha_0 = 1, \beta_0 = 0.5, m_0 = 16, a = 2, n = 10, d = 20$ *and a Bayesian estimate of* $b = 0$ *without loss of generality.*

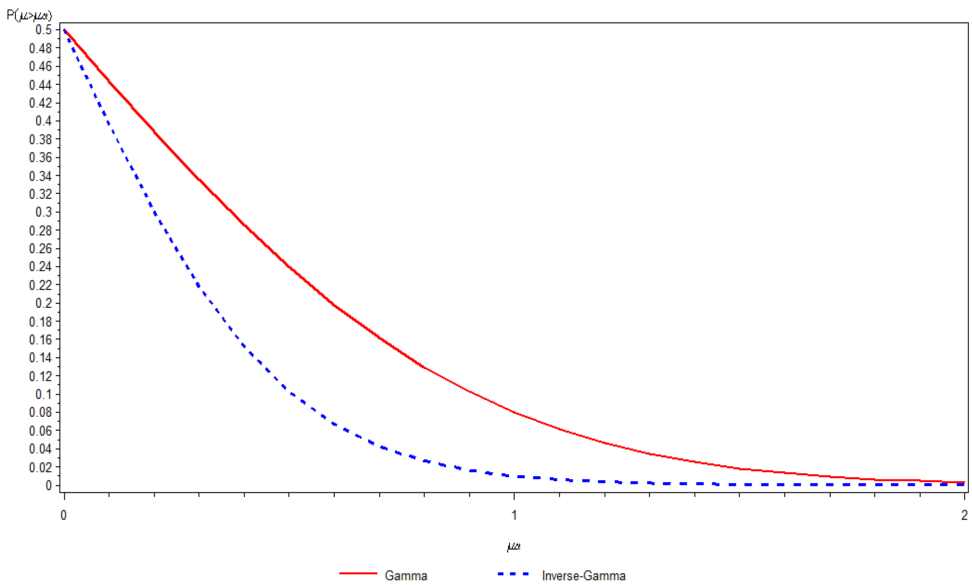

Figure 2. Posterior tail probabilities

The tail probabilities for the gamma prior is higher than for the inverse-gamma prior and the gamma prior is hence preferred in the case of heavy-tailed data.

For different sample sizes:

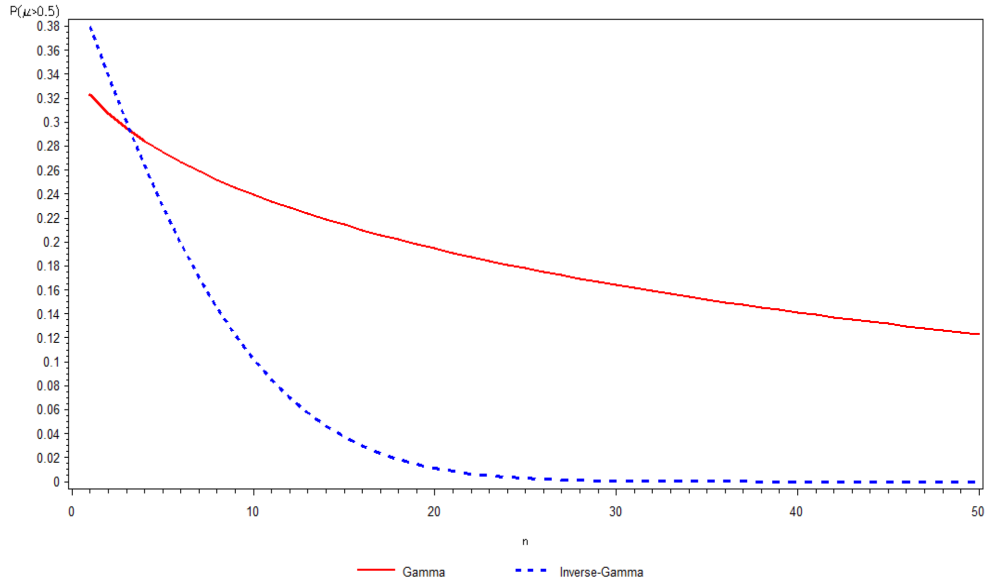

Figure 3. Posterior tail probabilities for different sample sizes

Similar to Figure 2, it is clear from Figure 3 that the gamma prior is again preferred.

3.2 Bayesian analysis for $\mathbf{\Sigma}$

The marginal posterior distribution of $\mathbf{\Sigma}$ can be obtained from equation (14) as is done in this section,

$$\begin{aligned} q(\mathbf{\Sigma}|\bar{\mathbf{x}},\mathbf{V}) &\propto \int_0^\infty w(z)z^{\frac{p(n-m+1)}{2}}|\mathbf{\Sigma}|^{\frac{-n+m-p-2}{2}}etr[-\frac{1}{2}z^{-1}\mathbf{\Sigma\Phi}] \\ &\times\int_{\boldsymbol{\mu}\in\mathbb{R}^p} etr[-\frac{1}{2}z\mathbf{\Sigma}^{-1}(\mathbf{V}+(n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})' \\ &+\frac{na}{n+a}(\bar{\mathbf{x}}-\boldsymbol{\theta})(\bar{\mathbf{x}}-\boldsymbol{\theta})')d\boldsymbol{\mu}dz \end{aligned}$$

where $\int_{\boldsymbol{\mu}\in\mathbb{R}^p} etr[-\frac{1}{2}z\mathbf{\Sigma}^{-1}(n+a)(\boldsymbol{\mu}-\mathbf{b})(\boldsymbol{\mu}-\mathbf{b})']d\boldsymbol{\mu} \propto z^{-\frac{p}{2}}|\mathbf{\Sigma}|^{\frac{1}{2}}$. Then from Gupta and Nagar (2000, definition 1.6.5.) follows that

$$\begin{aligned} q(\mathbf{\Sigma}|\bar{\mathbf{x}},\mathbf{V}) &= c|\mathbf{\Sigma}|^{\frac{-n+m-p-1}{2}}\int_0^\infty w(z)z^{\frac{p(n-m)}{2}}etr[-\frac{1}{2}z^{-1}\mathbf{\Sigma\Phi}] \\ &\times etr[-\frac{1}{2}z\mathbf{\Sigma}^{-1}\mathbf{W}]dz \end{aligned} \tag{17}$$

with $c^{-1} = 2^{-\frac{p(n-m)}{2}}|\mathbf{\Phi}|^{\frac{n-m}{2}}B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi W})$ with

$$\mathbf{W} = \frac{na}{n+a}(\bar{\mathbf{x}}-\boldsymbol{\theta})(\bar{\mathbf{x}}-\boldsymbol{\theta})'+\mathbf{V} \tag{18}$$

From equation (17) and using Gupta and Nagar (2000, definition 1.6.5.) we have that the Bayesian estimator of $|\mathbf{\Sigma}|$ is:

$$\begin{aligned} |\widehat{\mathbf{\Sigma}}|_B &= E_{\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V}}[|\mathbf{\Sigma}||\overline{\mathbf{x}},\mathbf{V}] \\ &= 2^p|\mathbf{\Phi}|^{-1}\frac{B_{\frac{n-m-2}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}\int_0^\infty w(z)z^p dz \end{aligned} \tag{19}$$

Remark 3 *The marginal posterior density function and the Bayesian estimator of* $\mathbf{\Sigma}$ *reduces to the univariate elliptical model for* $p = 1$.

4 Special cases

In this section the newly developed results of section 2 and 3 will be applied to the multivariate normal distribution and the multivariate t-distribution.

Remark 4 *The marginal posterior distribution of* $\boldsymbol{\mu}$ *for all elliptically contoured distributions and a normal-inverse Wishart prior is from equation (9) a multivariate t-distribution with parameters* $\mathbf{b}$ *and* $\frac{1}{(n+a)(n+m-2p)}\mathbf{D}$ *and degrees of freedom* $(n+m-2p)$ *and with a normal-Wishart prior from equation (15)*

$$q(\boldsymbol{\mu}|\overline{\mathbf{x}},\mathbf{V}) = cB_{\frac{n+m-2p-3}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{F})$$

with $c^{-1} = (2\pi)^{-\frac{p}{2}}(n+a)^{\frac{1}{2}}|\frac{1}{2}\mathbf{\Phi}|^{\frac{1}{2}}B_{\frac{n+m-2p-2}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})$. *The Bayes estimator of* $\boldsymbol{\mu}$ *is from equation (10) and (16)*

$$\widehat{\boldsymbol{\mu}}_B = \frac{n\overline{\mathbf{x}} + a\boldsymbol{\theta}}{n+a}$$

4.1 Multivariate normal distribution

Let $\mathbf{X}_i$ follow a multivariate normal distribution with parameters $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$. Then from Chu (1973) the associated weight function is:

$$\begin{aligned} w(z) &= \delta(z-1) \\ &= \lim_{d\to 0}\frac{1}{d\sqrt{\pi}}\exp(-\frac{(z-1)^2}{d^2}) \end{aligned} \tag{20}$$

with $\delta(\cdot)$the Dirac delta function.

4.1.1 Normal-Inverse Wishart prior

The marginal posterior distribution of $\mathbf{\Sigma}$ is obtained by using equations (20) and (11),

$$q(\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V}) = \frac{1}{2^{\frac{p(n+m-1-p)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|^{\frac{n+m-1-p}{2}}|\mathbf{\Sigma}|^{-\frac{n+m}{2}}etr[-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{D}] \tag{21}$$

with $\mathbf{D}$ as defined in equation (7). The Bayesian estimator of $|\mathbf{\Sigma}|$ is obtained from equations (12) and (20)

$$\begin{aligned} |\widehat{\mathbf{\Sigma}}|_B &= \int_0^\infty zw(z)dz \frac{\prod_{j=1}^{p} \Gamma(\frac{n+m-2-p-j}{2})}{2^{\frac{p}{2}} \prod_{j=1}^{p} \Gamma(\frac{n+m-p-j}{2})} |\mathbf{D}| \\ &= \frac{\prod_{j=1}^{p} \Gamma(\frac{n+m-2-p-j}{2})}{2^{p} \prod_{j=1}^{p} \Gamma(\frac{n+m-p-j}{2})} |\mathbf{D}| \end{aligned} \tag{22}$$

with $\mathbf{D}$ as defined in equation (7).

Remark 5 *The results obtained for the normal model in section 4.1.1. has been obtained by Press (1982).*

4.1.2 Normal-Wishart prior

From equations (17) and (20) the marginal posterior density function of $\mathbf{\Sigma}$ is

$$q(\mathbf{\Sigma}|\bar{\mathbf{x}}, \mathbf{V}) = \frac{2^{\frac{p(n-m)}{2}}}{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi W})} |\mathbf{\Phi}|^{-\frac{n-m}{2}} |\mathbf{\Sigma}|^{\frac{-n+m-p-1}{2}} etr[-\frac{1}{2}\mathbf{\Sigma\Phi}] etr[-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{W}] \tag{23}$$

and the Bayesian estimator of $|\mathbf{\Sigma}|$ is

$$|\widehat{\mathbf{\Sigma}}|_B = 2^p |\mathbf{\Phi}|^{-1} \frac{B_{\frac{n-m-2}{2}}(\frac{1}{4}\mathbf{\Phi W})}{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi W})} \tag{24}$$

with $\mathbf{V}$ as defined in equation (3).

Remark 6 *These results obtained for the normal model in section 4.1.2. has been obtained by Bekker and Roux (1995) by using the relation given by Herz (1995) as* $B_{-\nu}(\mathbf{D}) = B_\nu(\mathbf{D})|\mathbf{D}|^\nu$.

4.2 Multivariate t-distribution

Let $\mathbf{X}_i$ follow a t-distribution with ν_0 degrees of freedom. Then from Chu (1973) the associated weight function is:

$$w(z) = \frac{(\frac{\nu_0}{2})^{\frac{\nu_0}{2}} z^{\frac{\nu_0}{2}-1} \exp(-\nu_0 z)}{\Gamma(\frac{\nu_0}{2})} \tag{25}$$

4.2.1 Normal-Inverse Wishart prior

The marginal posterior distribution of $\mathbf{\Sigma}$ is from equations (11) and (25):

$$
\begin{aligned}
q(\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V}) &= \frac{1}{2^{\frac{p(n+m-1-p)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|^{\frac{n+m-1-p}{2}}|\mathbf{\Sigma}|^{-\frac{n+m}{2}} \\
&\quad\times\int_0^\infty \frac{(\frac{\nu_0}{2})^{\frac{\nu_0}{2}}}{\Gamma(\frac{\nu_0}{2})}z^{\frac{\nu_0}{2}-\frac{p(p-n-m+1)}{2}-1}\exp(-\nu_0 z-\frac{1}{2}ztr(\mathbf{\Sigma}^{-1}\mathbf{D}))dz \\
&= \frac{1}{2^{\frac{p(n+m-1-p)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|^{\frac{n+m-1-p}{2}}|\mathbf{\Sigma}|^{-\frac{n+m}{2}} \\
&\quad\times\frac{(\frac{\nu_0}{2})^{\frac{\nu_0}{2}}}{\Gamma(\frac{\nu_0}{2})}\left(\frac{\Gamma(\frac{\nu_0}{2}-\frac{p(p-n-m+1)}{2})}{(\nu_0+\frac{1}{2}tr(\mathbf{\Sigma}^{-1}\mathbf{D})])^{\frac{\nu_0}{2}-\frac{p(p-n-m+1)}{2}}}\right)
\end{aligned}
$$

with $\mathbf{D}$ as defined in equation (7). From equations (12) and (25) the Bayesian estimator of $|\mathbf{\Sigma}|$ is

$$
|\widehat{\mathbf{\Sigma}}|_B = \int_0^\infty zw(z)dz\frac{\prod_{j=1}^{p}\Gamma(\frac{n+m-2-p-j}{2})}{2^p\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|
$$

with

$$
\begin{aligned}
\int_0^\infty zw(z)dz &= \int_0^\infty z\frac{(\frac{\nu_0}{2})^{\frac{\nu_0}{2}}z^{\frac{\nu_0}{2}-1}\exp(-\nu_0 z)}{\Gamma(\frac{\nu_0}{2})}dz \\
&= \frac{\Gamma(\frac{\nu_0}{2}+1)}{\Gamma(\frac{\nu_0}{2})}\nu_0 2^{-\frac{\nu_0}{2}}
\end{aligned}
$$

Hence,

$$
|\widehat{\mathbf{\Sigma}}|_B = \frac{\nu_0 2^{-\frac{\nu_0}{2}-p}\Gamma(\frac{\nu_0}{2}+1)\prod_{j=1}^{p}\Gamma(\frac{n+m-2-p-j}{2})}{\Gamma(\frac{\nu_0}{2})\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|
$$

4.2.2 Normal-Wishart prior

From equations (17), (25) and Gupta and Nagar (2000, definition 1.6.5.) the marginal posterior density function of $\mathbf{\Sigma}$ is

$$\begin{aligned}
q(\mathbf{\Sigma}|\overline{\mathbf{x}},\mathbf{V}) &= \frac{2^{\frac{p(n-m)}{2}}|\mathbf{\Phi}|^{-\frac{n-m}{2}}}{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}|\mathbf{\Sigma}|^{\frac{-n+m-p-1}{2}}\int_0^\infty w(z)z^{\frac{p(n-m)}{2}} \\
&\quad\times etr[-\frac{1}{2}z^{-1}\mathbf{\Sigma}\mathbf{\Phi}]etr[-\frac{1}{2}z\mathbf{\Sigma}^{-1}\mathbf{W}]dz \\
&= \frac{(\frac{\nu_0}{2})^{\frac{\nu_0}{2}}2^{\frac{p(n-m)}{2}}|\mathbf{\Phi}|^{-\frac{n-m}{2}}}{\Gamma(\frac{\nu_0}{2})B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{W}\mathbf{\Phi})}|\mathbf{\Sigma}|^{\frac{-n+m-p-1}{2}}|\nu_0+\frac{1}{2}tr(\mathbf{\Sigma}^{-1}\mathbf{W})|^{\frac{\nu_0+p(-m+n+1)}{2}} \\
&\quad\times B_{\frac{\nu_0+p(-m+n+1)}{2}}(\frac{1}{2}\mathbf{\Sigma}\mathbf{\Phi}(\nu_0+\frac{1}{2}tr(\mathbf{\Sigma}^{-1}\mathbf{W}))
\end{aligned}$$

with $\mathbf{W} = \frac{na}{n+a}(\overline{\mathbf{x}}-\boldsymbol{\theta})(\overline{\mathbf{x}}-\boldsymbol{\theta})' + \mathbf{V}$. The Bayesian estimator of $|\mathbf{\Sigma}|$ is

$$\begin{aligned}
|\widehat{\mathbf{\Sigma}}|_B &= 2^p|\mathbf{\Phi}|^{-1}\frac{B_{\frac{n-m-2}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}\int_0^\infty \frac{(\frac{\nu_0}{2})^{\frac{\nu_0}{2}}z^{\frac{\nu_0}{2}-1}\exp(-\nu_0 z)}{\Gamma(\frac{\nu_0}{2})}z^p dz \\
&= 2^p|\mathbf{\Phi}|^{-1}\frac{\Gamma(p+\frac{\nu_0}{2})B_{\frac{n-m-2}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}{2^{\frac{\nu_0}{2}}\nu_0^p\Gamma(\frac{\nu_0}{2})B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}
\end{aligned}$$

from equations (19) and (25), with $\mathbf{V}$ as defined in equation (3).

5 Application

5.1 Posterior odds ratio

The posterior odds ratio for $\mathbf{\Sigma}$ for the normal distribution is

$$\begin{aligned}
POR &= \frac{q(\mathbf{\Sigma}_{IW}|\overline{\mathbf{x}},\mathbf{V})}{q(\mathbf{\Sigma}_{W}|\overline{\mathbf{x}},\mathbf{V})} \\
&= \frac{\frac{1}{2^{\frac{p(n+m-1-p)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma(\frac{n+m-p-j}{2})}|\mathbf{D}|^{\frac{n+m-1-p}{2}}|\mathbf{\Sigma}|^{-\frac{n+m}{2}}etr[-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{D}]}{\frac{2^{\frac{p(n-m)}{2}}}{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})}|\mathbf{\Phi}|^{-\frac{n-m}{2}}|\mathbf{\Sigma}|^{\frac{-n+m-p-1}{2}}etr[-\frac{1}{2}\mathbf{\Sigma}\mathbf{\Phi}]etr[-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{W}]} \\
&= \frac{B_{\frac{n-m}{2}}(\frac{1}{4}\mathbf{\Phi}\mathbf{W})|\mathbf{D}|^{\frac{n+m-1-p}{2}}|\mathbf{\Sigma}|^{\frac{-2mp+1}{2}}|\mathbf{\Phi}|^{\frac{n-m}{2}}etr\left[-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{D}\right]}{\Gamma_p(\frac{n+m}{2})2^{\frac{p(n-m)}{2}}etr[-\frac{1}{2}\mathbf{\Sigma}\mathbf{\Phi}]etr[\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{W}]} \qquad (26)
\end{aligned}$$

Although the estimators based on different prior structures are different, note that we are comparing the behaviour of the two posterior density functions (21) and (23). It is not possible to compare the two estimators (22) and (24) by using the posterior odds ratio. For $p=1$ in (26) with $\mathbf{\Phi} = 2\beta_0$ and $m = 2(\alpha_0+1)$ (see van Niekerk, 2012),

$$POR = \frac{K_{\alpha_0-\frac{n}{2}}\left(\sqrt{\frac{2(n+n_0)m_0}{\beta_0}}\right)(n+n_0)^{\frac{\alpha_0}{2}-\frac{n}{4}}d^{\alpha_0+\frac{n}{2}}2^{1-(\frac{3\alpha_0}{2}+\frac{n}{4})}(\sigma^2)^{-\alpha_0-\frac{n+2}{2}}\exp\left[-\frac{d}{2\sigma^2}\right]}{\Gamma(\alpha_0+\frac{n}{2})(m_0\beta_0)^{-\frac{\alpha_0}{2}+\frac{n}{4}}\exp\left[-\frac{\sigma^2}{\beta_0}-\frac{(n+n_0)m_0}{2\sigma^2}\right]}$$

A graphical presentation of the posterior odds ratio for $\sigma^2 = 5$ is given below in Figure 4 and 5 with $\beta_0 = 0.5, m_0 = 16, n_0 = 2, n = 10, d = 20$,

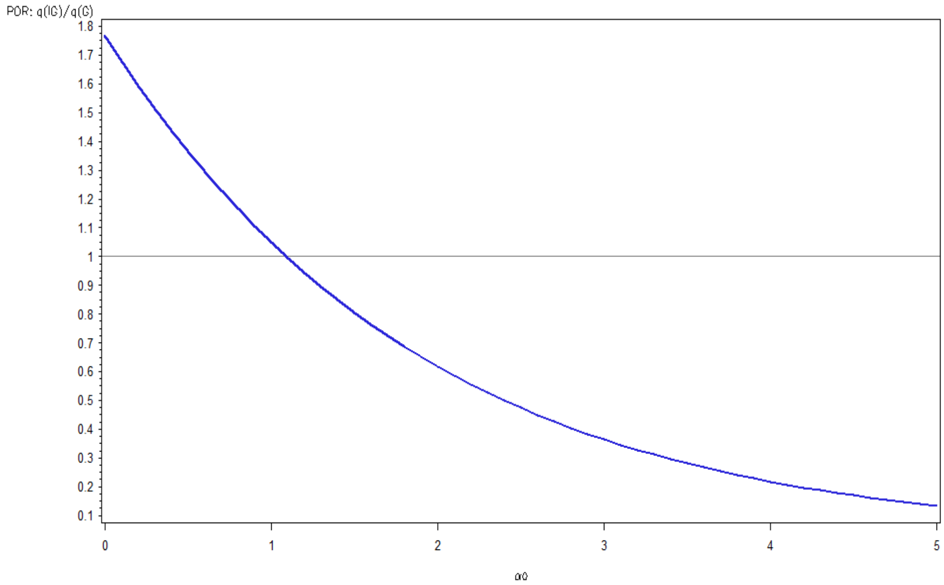

Figure 4. Posterior odds ratio for the univariate normal model

Note that when the posterior odds ratio is lower than 1 then the gamma prior is more appropriate than the inverse gamma prior. In the case for $\beta_0 = 0.5$, the gamma prior provides a better fit that the inverse gamma prior for $\alpha_0 > 1$. We thus suggest to use the normal-gamma prior.

5.2 Real data

5.2.1 Linear regression

In this section the dataset presented in Miller (1980) is used which consists of simultaneous pairs of measurements of serum kanamycin level in blood samples drawn from twenty babies. One of the measurements were obtained by a heelstick method (X) and the other by using an umbilical catheter (Y). A simple linear regression model is fit to the data to test whether there is a systematic difference in the two methods. Hence

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

with the assumption $\varepsilon_i \sim t(0, \sigma_\varepsilon^2, \nu_\varepsilon)$. The prior distributions for β_0 and β_1 are taken to be standard normal distributions (similar to Bolfarine and Arellano-Valle, 2011) and from the distribution on the error it follows that $Y_i \sim t(\beta_0 + \beta_1 X_i, \sigma_\varepsilon^2, \nu_\varepsilon)$. The prior distribution for the error variance is then either an inverse gamma or gamma distribution and these results are then compared. Firstly the inverse-gamma prior, $\sigma_\varepsilon^2 \sim IG(\alpha_1, \beta_1)$ and secondly the gamma prior, $\sigma_\varepsilon^2 \sim G(\alpha_2, \beta_2)$. The hyperparameters are chosen as $\alpha_1 = 4$ and $\beta_1 = 3$ (Bolfarine and Arellano-Valle, 2011) and accordingly $\alpha_2 = 4$ and $\beta_2 = 0.333$. The error degrees of freedom is assumed to be 7. A Gibbs algorithm was used and the convergence was verified graphically (see Gelman and Rubin, 1992), the posterior results are summarized in Tables 1 and 2 below.

Parameter	Mean	MC error	95% HPD	Length
β_0	0.7059	0.004239	(-1.237,2.641)	3.878
β_1	1.761	0.0006637	(1.488,2.04)	0.552
σ_ε^2	22.67	0.03401	(10.47,43.1)	32.63

Table 1. Results for $\sigma_\varepsilon^2 \sim IG(4,3)$

Parameter	Mean	MC error	95% HPD	Length
β_0	0.701	0.003878	(-1.232,2.628)	3.86
β_1	1.762	0.0006222	(1.487,2.04)	0.553
σ_ε^2	23.03	0.03581	(10.74,43.7)	32.96

Table 2. Results for $\sigma_\varepsilon^2 \sim G(4, \frac{1}{3})$

The length of the intervals are very close for both priors. The interval for β_0 is shorter for the gamma prior than the inverse gamma prior. It is shown in this example that the inverse gamma prior is not superior to the gamma prior and in future a gamma prior might be considered in applications.

5.2.2 Simulation

A univariate Students' t dataset was simulated with location parameter, $\mu = 10$, scale parameter, $\gamma^2 = 5$ and degrees of freedom, $\nu = 5$. The prior for the degrees of freedom were taken to be a uniform(1,10) distribution. The two priors, normal-inverse gamma ($\mu|\gamma \sim N(12, \frac{\gamma}{2})$ and $\gamma^2 \sim IG(4,3)$) and normal-gamma ($\mu|\gamma \sim N(12, \frac{\gamma}{2})$ and $\gamma^2 \sim G(4, \frac{1}{3})$) were then applied in a Gibbs sampler that converged graphically and the posterior results are as follows:

Parameter	Mean	MC error	95% HPD	Length
μ	11.13	0.001766	(10.14,12.11)	1.97
γ^2	5.718	0.00791	(2.758,11.21)	8.452
ν	5.003	0.01964	(1.355,9.717)	8.362

Table 3. Results for $\gamma^2 \sim IG(4,3)$

Parameter	Mean	MC error	95% HPD	Length
μ	11.12	0.001734	(10.22,12.02)	1.8
γ^2	5.029	0.00402	(2.84,8.824)	5.984
ν	5.428	0.01108	(3.094,9.685)	6.591

Table 4. Results for $\gamma^2 \sim G(4, \frac{1}{3})$

Notice the similar performance as with the real dataset between the two priors where the confidence intervals for the gamma prior is shorter than for the inverse-gamma prior.

6 Conclusion

This paper focused on subjective aspects regarding the Bayesian paradigm with the elliptical contoured model as the underlying distribution. The normal-inverse Wishart joint prior distribution and normal-Wishart joint prior distribution were assumed respectively, for the location vector and characteristic matrix of the underlying model. The joint posterior density functions, marginal posterior density functions and Bayesian estimators of the parameters were derived. The multivariate normal distribution and the multivariate

t-distribution were considered as special cases in section 4. The Bayesian estimator of the location parameter is robust in the sense that it is independent of the prior distribution of the characteristic matrix and the form of the density function of the specific elliptical model in consideration. This is a very good result since this estimator can then be easily applied to all elliptical models.
In the univariate case it was demonstrated that the normal-gamma prior is a suitable competitor to the normal-inverse gamma prior.
This paper makes a substantial contribution to the field of modern multivariate analysis with the implementation of the elliptically contoured models form a subjective prior view.

7 Acknowledgements

The authors would like to hereby thank the reviewers for their constructive comments that led to adding additional details in the paper and improved the presentation and acknowledge the support of the StatDisT group. This work is based upon research supported by the National Research foundation, Grant (Re:2007043000003) and The Research and Development Programme, WC Post-doctoral fellowship of the University of Pretoria.

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