

# A NOTE ON OPTIMAL INVESTMENT-CONSUMPTION-INSURANCE IN A LÉVY MARKET.

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ABSTRACT. In Shen and Wei (2014) an optimal investment, consumption and life insurance purchase problem for a wage earner with Brownian information has been investigated. This paper discusses the same problem but extend their results to a geometric Itô Lévy jump process. Our modelling framework is very general as it allows random parameters which are unbounded and involves some jumps. It also covers parameters which are both Markovian and non-Markovian functionals. Unlike in Shen and Wei (2014) who considered a diffusion framework, ours solves the problem using a novel approach, which combines the Hamilton-Jacobi-Bellman (HJB) and a backward stochastic differential equation (BSDE) in a Lévy market setup. We illustrate our results by two examples.

## 1. INTRODUCTION

The optimal investment-consumption problem by Merton (1969, 1971) triggered many research extensions. Richard (1975) extended this problem to include life insurance, thereby performing valuations in a combined Financial and Insurance market for the first time. Similar works can be found in literature, (see e.g., Pliska and Ye (2007), Huang and Milevsky (2008), Kwak et al. (2011), Duarte et al. (2012), Shen and Wei (2014)) and references therein. In all these studies the diffusion modelling framework was considered and various approaches have been implemented. These include the stochastic control method introduced by Merton (1969, 1971) where the optimal solution is computed by solving the so called Hamilton-Jacobi-Bellman equation(HJB), the martingale approach introduced by Karatzas et al. (1987) and others. Recently, Shen and Wei (2014) solved the optimal investment-consumption-insurance problem by the stochastic control approach where they applied a combination of the HJB and a backward stochastic differential equation (BSDE). Few works solving this problem in a Jump-diffusion framework appear in the literature.

In this article, we extend this problem to the Jump-diffusion framework. This extension is motivated by the existence in the market of some sudden and rare breaks (jumps) caused by the introduction of external information as prices evolve and these cannot be captured by diffusion models. Our work is closely related to that of Shen and Wei (2014) where the model has random diffusion parameters. Motivated by Delong (2013) where a thorough study has been done on BSDE with jumps in the Financial and Insurance markets, we extend the results in Shen and Wei (2014) to a jump-diffusion model with

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random parameters. Therefore we have managed to extend their results in a more general framework, the Lévy market setup.

This paper is structured as follows. Section 2 provides the modelling dynamics and problem formulation. In Section 3, we provide a verification theorem for the combined HJB equation and the BSDE related to our problem. In Section 4, we provide the main result. In Section 5, we give special examples to illustrate our main result and conclude in Section 6.

## 2. THE FINANCIAL MODEL

We consider a complete filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{0 \leq t \leq T}, \mathbb{P})$ , where  $\{\mathcal{F}(t)\}_{t \in [0, T]}$  is a filtration satisfying the usual conditions (Protter 2000). Let  $\mathcal{B}(E)$  be a Borel  $\sigma$ -field of any set  $E \subset \mathbb{R}$  and  $\mathcal{P}$  the predictable  $\sigma$ -field on  $\Omega \times [0, T]$ . Let  $N(t, E)$  be a Poisson random measure and  $\nu(E)$  the Lévy measure. Moreover, we consider the following spaces:

- $\mathbb{L}^2(\mathbb{R})$ - the space of random variables  $\xi : \Omega \mapsto \mathbb{R}$ , such that  $\mathbb{E}[|\xi|^2] < \infty$ .
- $L^2_{\nu}(\mathbb{R})$ - the space of measurable functions  $v : \mathbb{R} \mapsto \mathbb{R}$  such that

$$\int_{\mathbb{R}} |v(z)|^2 \nu(dz) < \infty,$$

where  $\nu$  is a  $\sigma$ -finite measure.

- $\mathbb{S}^2(\mathbb{R})$ - the space of adapted càdlàg processes  $Y : \Omega \times [0, T] \mapsto \mathbb{R}$  such that

$$\mathbb{E}[\sup |Y(t)|^2] < \infty$$

and

- $\mathbb{H}^2_{\mathcal{N}}$ - the space of predictable processes  $\Upsilon : \Omega \times [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ , such that

$$\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}} |\Upsilon(t, z)|^2 \nu(dz) dt \right] < \infty.$$

We consider a financial market consisting of a risk-free asset  $B := (B(t))_{t \in [0, T]}$  and a risky asset  $S := (S(t))_{t \in [0, T]}$  defined as follows:

$$(2.1) \quad dB(t) = r(t)B(t)dt, \quad B(0) = 1,$$

$$(2.2) \quad dS(t) = S(t) \left[ \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \gamma(t, z) \tilde{N}(dt, dz) \right], \quad S(0) = s > 0,$$

where  $r(t), \alpha(t), \beta(t)$  are  $\mathbb{R}^+$  valued and  $\gamma(t, \cdot) > -1$  are both  $\mathcal{F}(t)$ -adapted and predictable processes.

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$$

is the compensated Poisson random measure and  $W(t)$  is a Brownian motion independent of  $\tilde{N}(dt, dz)$ .

We assume that the wage earner is alive at time  $t = 0$ , whose life-time is a non-negative random variable  $\tau$  defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $\lambda(t)$  be an instantaneous force of mortality or the hazard function of the wage earner at time  $t$ . Assume that  $\lambda(t)$  is

$\mathcal{F}(t)$ -adapted process, then the conditional survival probability of the wage earner is given by

$$(2.3) \quad \bar{F}(t) := \mathbb{P}(\tau > t | \mathcal{F}(t)) = \exp\left(-\int_0^t \lambda(s) ds\right)$$

and the conditional survival probability density of the death of the wage earner by

$$(2.4) \quad f(t) := \lambda(t) \exp\left(-\int_0^t \lambda(s) ds\right).$$

We suppose existence of an Insurance market, where the term life insurance is continuously traded. We assume that the wage earner is paying premiums at the rate  $p(t)$ , at time  $t$  for the life insurance contract and the insurance company will pay  $p/\eta(t)$  to his beneficiary for his death, where the  $\mathcal{F}(t)$ -adapted process  $\eta(t) > 0$  is the premium insurance ratio. This parameter  $\eta$  is allowed to be stochastic due to stochastic mortality or safety loading. (See Shen and Wei 2014) and references therein for more discussions. When the wage earner dies, the total legacy to his beneficiary is given by

$$\ell(t) := X(t) + \frac{p(t)}{\eta(t)},$$

where  $X(t)$  is the wealth process of the wage earner at time  $t$  and  $p(t)/\eta(t)$  the insurance benefit paid by the insurance company to the beneficiary if death occurs at time  $t$ .

Let  $c(t)$  be the consumption rate of the wage earner and  $\pi(t)$  the fraction of the wage earner's wealth invested in the risky share. Assuming that the shares are divisible and can be traded continuously and there are no transaction costs, taxes or short-selling constraints in the trading, the wealth process  $X(t)$  is defined by the following stochastic differential equation (SDE):

$$(2.5) \quad \begin{aligned} dX(t) &= [X(t)(r(t) + \pi(t)\mu(t)) - c(t) - p(t)] dt + \pi(t)\beta(t)X(t)dW(t) \\ &\quad + \pi(t)X(t) \int_{\mathbb{R}} \gamma(t, z) \tilde{N}(dt, dz), \quad t \in [0, \tau \wedge T], \\ X(0) &= x > 0, \end{aligned}$$

where  $\mu(t) := \alpha(t) - r(t)$  is the appreciation rate and  $\tau \wedge T := \min\{\tau, T\}$ .

The main problem is to choose the optimal investment-consumption-insurance problem, so that the wage earner maximizes the expected discounted utilities derived from the intertemporal consumption during  $[0, \tau \wedge T]$ , from the legacy if he dies before time  $T$  and from the terminal wealth if he is alive until time  $T$ . We suppose that the discount process rate  $\rho(t)$  is positive and  $\mathcal{F}(t)$ -adapted process.

Given the utility function  $U$  for the intertemporal consumption, legacy and terminal wealth, the wage earner's problem is then to choose an investment consumption insurance strategy so as to optimize the following performance functional:

$$(2.6) \quad J(0, x_0; \pi, c, p) := \sup_{(\pi, c, p) \in \mathcal{A}} \mathbb{E} \left[ \int_0^{\tau \wedge T} e^{-\int_0^s \rho(u) du} U(c(s)) ds \right. \\ \left. + e^{-\int_0^\tau \rho(u) du} U(\ell(\tau)) \mathbf{1}_{\{\tau \leq T\}} + e^{-\int_0^T \rho(u) du} U(X(T)) \mathbf{1}_{\{\tau > T\}} \right],$$

where  $\mathbf{1}_A$  is a characteristic function defined by

$$\mathbf{1}_A(x) := \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise} \end{cases}$$

and  $U$  is the utility function for the consumption, legacy and terminal wealth.

We consider a utility function of the constant relative risk aversion (CRRA) type, given by the following power utility:

$$(2.7) \quad U(x) = \begin{cases} \frac{x^\delta}{\delta}, & \text{if } x \geq 0; \\ -\infty, & \text{if } x < 0, \end{cases}$$

for  $\delta \in (-\infty, 1) \setminus \{0\}$ . From (2.3) and (2.4), the maximum utility (2.6) is equivalent to the following performance functional

$$J(0, x_0, \pi, c, p) = \sup_{(\pi, c, p) \in \mathcal{A}} \mathbb{E} \left[ \int_0^T e^{-\int_0^s \rho(u) du} [\bar{F}(s)U(c(s)) + f(s)U(\ell(s))] ds \right. \\ \left. + e^{-\int_0^T \rho(u) du} \bar{F}(T)U(X(T)) \right].$$

Hence,

$$(2.8) \quad J(0, x_0; \pi, c, p) = \sup_{(\pi, c, p) \in \mathcal{A}} \mathbb{E} \left[ \int_0^T e^{-\int_0^s (\rho(u) + \lambda(u)) du} [U(c(s)) + \lambda(s)U(\ell(s))] ds \right. \\ \left. + e^{-\int_0^T (\rho(u) + \lambda(u)) du} U(X(T)) \right].$$

As we are studying a model with random parameters and jumps, the following assumptions are essential to guarantee the existence and uniqueness of a solution to the BSDE related to our problem in the next section. (See Shen and Wei 2014):

- (A1) the appreciation rate of a share is different from the interest rate;
- (A2) the interest rate, force of mortality, premium insurance ratio and discount rate are bounded away from zero, that is,

$$\exists \epsilon > 0 : |\Lambda(t)| \geq \epsilon, \quad a.e.,$$

where  $\Lambda := r, \lambda, \eta, \rho$ ;

- (A3) the random parameters satisfy the exponential integrability conditions:

$$\mathbb{E} \left[ \exp(\theta \int_0^T |\Lambda(t)| dt) \right] < \infty,$$

for a sufficient large  $\theta$ , where  $\Lambda := r, \lambda, \eta, \rho$ .

**Definition 2.1.** The set of strategies  $\mathcal{A} := \{(\pi, c, p) := (\pi(t), c(t), p(t))_{t \in [0, T]}\}$  is said to be admissible if the following conditions hold:

(1) a triple  $(\pi, c, p) \in (0, 1) \times \mathbb{R}^+ \times \mathbb{R}$  is  $\mathcal{F}$ -adapted process such that

$$\int_0^T c(t)dt < \infty, \quad \int_0^T |p(t)|dt < \infty \quad \text{and} \quad \int_0^T \pi^2(t)dt < \infty, \quad \mathbb{P} - a.s.,$$

(2) the SDE (2.5) has a unique strong solution associated with  $(\pi, c, p)$  such that:

$$X(t) \geq 0, \quad \mathbb{P} - a.s.,$$

(3) and finally:

$$\mathbb{E} \left[ \int_0^T e^{-\int_0^s (\rho(u) + \lambda(u))du} [U^-(c(s)) + \lambda(s)U^-(\ell(s))]ds + e^{-\int_0^T (\rho(u) + \lambda(u))du} U^-(X(T)) \right] < \infty,$$

where  $U^- := \max\{-U, 0\}$  is the negative part of  $U$ .

To use the dynamic programming principle, we consider the dynamic version of the performance functional (2.8) given by

$$(2.9) \quad J(t, x, \pi, c, p) = \mathbb{E}_{t,x} \left[ \int_t^T e^{-\int_t^s (\rho(u) + \lambda(u))du} [U(c(s)) + \lambda(s)U(\ell(s))]ds + e^{-\int_t^T (\rho(u) + \lambda(u))du} U(X(T)) \right],$$

where  $\mathbb{E}_{t,x}$  represents the conditional expectation  $\mathbb{E}[\cdot | X(t) = x, \mathcal{F}(t)]$ . The wage earner wishes to maximize the dynamic performance functional (2.9) under the admissible set  $\mathcal{A}$ , subject to the wealth process (2.5). Therefore, the value function of the problem is given by:

$$V(t, x) = \sup_{(\pi, c, p) \in \mathcal{A}} J(t, x, \pi, c, p) = J(t, x, \pi^*, c^*, p^*),$$

where  $(\pi^*, c^*, p^*) \in \mathcal{A}$  is the optimal investment consumption insurance strategy to be determined in the next section. We point out that the value function  $V(t, x)$  is an  $\mathcal{F}(t)$ -measurable random variable, since all model parameters are random in our model, so the value function can not be determined from the partial differential equation as usual. To determine the value function, we will use a combination of the Hamilton-Jacob-Bellman (HJB) equation and the backward stochastic differential equation (BSDE). Although it is not possible to obtain an explicit solution to the optimal  $\pi^*$ , we will show the sufficient conditions to guarantee that the solution  $\pi^* \in (0, 1)$  exists. The optimal  $c^*$  and  $p^*$  are derived explicitly.

## 3. COMBINATION OF HJB EQUATION WITH BSDE WITH JUMP

In this section, we employ a combination of HJB equation with BSDE with jumps to solve our optimal problem. We define the following BSDE with jump:

$$(3.1) \quad \begin{aligned} dY(t) &= -f(t, Y(t), \Upsilon(t))dt + \int_{\mathbb{R}} \Upsilon(t, z) \tilde{N}(dt, dz), \\ Y(T) &= \xi, \end{aligned}$$

where  $\xi$ ,  $f$  and  $\Upsilon$  satisfy the following conditions:

(C1) the terminal value  $\xi \in \mathbb{L}^2(\mathbb{R})$ ;

(C2) the generator  $f : \Omega \times \mathbb{R} \times L^2_{\nu}(\mathbb{R}) \mapsto \mathbb{R}$  is predictable, i.e.  $f \in \mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(L^2_{\nu}(\mathbb{R}))$  and Lipschitz continuous, that is,

$$|f(\omega, t, y, v) - f(\omega, t, y', v')|^2 \leq C(|y - y'|^2 + \int_{\mathbb{R}} |v(z) - v'(z)|^2 \nu(dz)),$$

a.s.,  $(\omega, t) \in \Omega \times [0, T]$  a.e. for all  $(y, v), (y', v') \in \mathbb{R} \times L^2_{\nu}(\mathbb{R})$ ;

(C3)

$$\mathbb{E} \left[ \int_0^T |f(t, 0, 0)|^2 dt \right] < \infty.$$

By the above conditions, we guarantee the existence of a unique solution  $(Y, \Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2_{\nu}(\mathbb{R})$  to the BSDE (3.1) (Theorem 3.1 in Delong 2013). The process  $\Upsilon$  is called control process. It control the process  $Y$  so that  $Y$  satisfies the terminal condition. Let  $\mathcal{S} := [0, T] \times \mathbb{R} \times \mathbb{R}$  be the solvency region. We define  $\Psi : \mathcal{S} \mapsto \mathbb{R}$  such that  $\Psi(\cdot, \cdot, \cdot) \in \mathcal{C}^{1,2,2}(\mathcal{S})$ . We suppose that  $\Psi_t, \Psi_x, \Psi_y, \Psi_{xx}, \Psi_{yy}$  are partial derivatives w.r.t.  $t, x, y$  respectively. We define the following partial differential generator:

$$\begin{aligned} \mathcal{L}^{\pi, c, p}[\Psi(t, x, y)] &= -(\rho(t) + \lambda(t))\Psi(t, x, y) + \Psi_t(t, x, y) \\ &+ [X(t)(r(t) + \pi(t)\mu(t)) - c(t) - p(t)] \Psi_x(t, x, y) \\ &- \Psi_y(t, x, y)f(t, Y(t), \Upsilon(t)) + \frac{1}{2}\pi^2(t)\beta^2(t)X^2(t)\Psi_{xx}(t, x, y) \\ &+ \int_{\mathbb{R}} [\Psi(t, x + \pi x \gamma(t, z), y) - \Psi(t, x, y) - \pi x \gamma(t, z)\Psi_x(t, x, y)] \nu(dz) \\ &+ \int_{\mathbb{R}} [\Psi(t, x, y + \Upsilon(t, z)) - \Psi(t, x, y) - \Upsilon(t, z)\Psi_x(t, x, y)] \nu(dz) \end{aligned}$$

The following theorem is a verification result for the combination of the HJB equation and the BSDE associated with our problem.

**Theorem 3.1.** (Verification theorem) Shen and Wei (2014). *Suppose that  $(\xi, f)$  satisfy the conditions*

(C1) – (C3). *Let  $\bar{\mathcal{S}}$  be the closure of the solvency region  $\mathcal{S}$ . Moreover, let  $\Phi$  denote the following function:*

$$(3.2) \quad \Phi(t, x, Y(t), \pi, c, p) := \mathcal{L}^{\pi, c, p}[\Psi(t, x, y)] + U(c) + \lambda(t)U(x + p/\eta(t)).$$

Suppose there is a function  $\Psi \in \mathcal{C}^2$  and an admissible control  $(\pi^*, c^*, p^*) \in \mathcal{A}$  such that:

- (1)  $\Phi(t, x, Y(t), \pi, c, p) \leq 0$  for all  $(\pi, c, p) \in \mathcal{A}$ ;
- (2)  $\Phi(t, x, Y(t), \pi^*, c^*, p^*) = 0$ ;
- (3) for all  $(\pi, c, p) \in \mathcal{A}$ ,

$$\lim_{t \rightarrow T^-} \Psi(t, x, y) = U(x);$$

- (4) let  $\mathcal{K}$  be the set of stopping times  $\kappa \leq T$ . The family  $\{\Psi(\kappa, X(\kappa), Y(\kappa))\}_{\kappa \in \mathcal{K}}$  is uniformly integrable.

Then

$$\begin{aligned} \Psi(t, x, y) &= \sup_{(\pi, c, p) \in \mathcal{A}} J(t, x; \pi, c, p) \\ &= J(t, x; \pi^*, c^*, p^*) \end{aligned}$$

and  $(\pi^*, c^*, p^*)$  is an optimal control.

*Proof.* Similar to that in Shen and Wei (2014), (Theorem 3.1, page 10) and (Theorem 3.1) in Oksendal and Sulem (2007)  $\square$

From the above theorem, the value function is the solution of the following system of the HJB equation and BSDE with jumps:

$$(3.3) \quad \begin{cases} \sup_{(\pi, c, p) \in \mathcal{A}} \Phi(t, x, Y(t), \pi, c, p) = 0, & \Psi(T, x, \xi) = U(x) \\ dY(t) = -f(t, Y(t), \Upsilon(t))dt + \int_{\mathbb{R}} \Upsilon(t, z) \tilde{N}(dt, dz), & Y(T) = \xi. \end{cases}$$

#### 4. GENERAL SOLUTIONS

In this section, we investigate the solutions to the investment-consumption-insurance problem (3.3) for a wage earner with power utility. To obtain the optimal solution  $(\pi^*, c^*, p^*)$ , from the terminal condition, we try the value function of the form:

$$(4.1) \quad V(t, x) = \Psi(t, x, Y(t)) = \frac{1}{\delta} x^\delta e^{(1-\delta)Y(t)},$$

where  $Y$  is the solution of the BSDE (3.1), for  $\xi = 0$  and  $f$  to be defined later. With this choice, it is clear that  $\Phi$  in equation (3.2) fulfills the standard procedures to solve equation (3.3), i.e.  $\Phi_{\pi\pi} < 0$ ,  $\Phi_{cc} < 0$  and  $\Phi_{pp} < 0$ .

Applying the first order conditions of optimality to  $\Phi$  with respect to  $(\pi, c, p)$  we obtain

$$(4.2) \quad -\Psi_x(t, x, Y(t)) + U'(c) = 0,$$

$$(4.3) \quad -\Psi_x(t, x, Y(t)) + \lambda(t) \frac{\partial U(x + p/\eta(t))}{\partial p} = 0$$

and

$$(4.4) \quad \begin{aligned} &\mu(t)x\Psi_x(t, x, Y(t)) - (1 - \delta)\pi\beta^2(t)x^2\Psi_{xx}(t, x, Y(t)) \\ &+ \int_{\mathbb{R}} [\Psi_\pi(t, x + \pi x\gamma(t, z), Y(t)) - \gamma(t, z)x\Psi_x(t, x, Y(t))] \nu(dz) = 0. \end{aligned}$$

Substituting (2.7) and (4.1) into (4.2)-(4.4) give the following optimal investment consumption insurance strategy

$$(4.5) \quad c^*(t) = xe^{-Y(t)},$$

$$(4.6) \quad p^*(t) = \eta(t)x \left\{ \left[ \frac{\lambda(t)}{\eta(t)} \right]^{\frac{1}{1-\delta}} e^{-Y(t)} - 1 \right\}$$

and  $\pi^*(t)$  is the solution of the following equation

$$(4.7) \quad x^\delta e^{(1-\delta)Y(t)} \left\{ \mu(t) - (1-\delta)\pi(t)\beta^2(t) - \int_{\mathbb{R}} [1 - (1 + \pi(t)\gamma(t, z))^{\delta-1}] \gamma(t, z)\nu(dz) \right\} = 0.$$

Furthermore, from (4.6) the optimal legacy is given by

$$\ell^*(t) = x \left[ \frac{\lambda(t)}{\eta(t)} \right]^{\frac{1}{1-\delta}} e^{-Y(t)}.$$

To ensure the existence of the unique solution  $\pi \in (0, 1)$  in (4.7) we follow the ideas in Benth (2001). To this end, we define a function

$$h(\pi) = \mu(t) - (1-\delta)\pi\beta^2(t) - \int_{\mathbb{R}} [1 - (1 + \pi\gamma(t, z))^{\delta-1}] \gamma(t, z)\nu(dz).$$

If  $\pi = 0$ ,  $h(\pi) = \mu(t) := \alpha(t) - r(t) > 0$  and

$$h'(\pi) = -(1-\delta) \left[ \beta^2(t) + \int_{\mathbb{R}} (1 + \pi\gamma(t, z))^{\delta-2} \gamma^2(t, z)\nu(dz) \right] < 0.$$

Then, the solution exists and is unique if:

$$h(1) = \mu(t) - (1-\delta)\beta^2(t) - \int_{\mathbb{R}} [1 - (1 + \gamma(t, z))^{\delta-1}] \gamma(t, z)\nu(dz) < 0,$$

hence

$$(1-\delta)\beta^2(t) + \int_{\mathbb{R}} [1 - (1 + \gamma(t, z))^{\delta-1}] \gamma(t, z)\nu(dz) > \alpha(t) - r(t).$$

Substituting (4.5), (4.6) and  $\pi^*$  into the HJB equation (3.3) gives the following expression

$$\begin{aligned} & \frac{1-\delta}{\delta} x^\delta e^{(1-\delta)Y} \left\{ -f - \frac{1}{1-\delta}(\rho + \lambda) + \frac{\delta}{1-\delta}(r + \eta) + \right. \\ & \left. \frac{\delta}{1-\delta} \left\{ \pi^* \mu - \frac{1-\delta}{2} (\pi^*)^2 \beta^2 + \frac{1}{\delta} \int_{\mathbb{R}} [(1 + \pi^* \gamma(t, z))^\delta - 1 - \delta \pi^* \gamma(t, z)] \nu(dz) \right\} + \right. \\ & \left. \left[ 1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}} \right] e^{-Y} + \int_{\mathbb{R}} \left\{ \frac{1}{1-\delta} [e^{(1-\delta)\Upsilon} - 1] - \Upsilon \right\} \nu(dz) \right\} = 0. \end{aligned}$$

Then, taking the coefficient of  $x^\delta e^{(1-\delta)Y}$  equal to zero, leads to

$$(4.8) \quad f(t, Y(t), \Upsilon(t)) = K(t) + \left[ 1 + \frac{\lambda^{\frac{1}{1-\delta}}(t)}{\eta^{\frac{\delta}{1-\delta}}(t)} \right] e^{-Y(t)}$$



$$+ \int_{\mathbb{R}} \left\{ \frac{1}{1-\delta} [e^{(1-\delta)\Upsilon(t,z)} - 1] - \Upsilon(t,z) \right\} \nu(dz),$$

where

$$\begin{aligned} K(t) = & -\frac{1}{1-\delta}(\rho(t) + \lambda(t)) + \frac{\delta}{1-\delta}(r(t) + \eta(t)) + \frac{\delta}{1-\delta} \left\{ \pi^*(t)\mu(t) \right. \\ & \left. - \frac{1-\delta}{2}(\pi^*(t))^2\beta^2(t) + \frac{1}{\delta} \int_{\mathbb{R}} [(1 + \pi^*(t)\gamma(t,z))^\delta - 1 - \delta\pi^*(t)\gamma(t,z)] \nu(dz) \right\}. \end{aligned}$$

Under the assumptions **(A1)** – **(A3)**, the BSDE (3.1) with the generator (4.8) and the terminal value  $\xi = 0$ , satisfies the conditions **(C1)** – **(C3)**. Then, by (Theorem 3.1, Delong (2013)), there exists a unique solution  $(Y, \Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ . As in Delong (2013), to obtain this solution, we define the probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  on  $\mathcal{F}(T)$  as follows:

$$(4.9) \quad \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}(T)} = M,$$

where  $M$  is the Radon-Nikodym derivative given by the dynamic:

$$\frac{dM(t)}{M(t)} := \int_{\mathbb{R}} \left[ \frac{e^{(1-\delta)\Upsilon(s,z)} - 1}{(1-\delta)\Upsilon(s,z)} - 1 \right] \tilde{N}(ds, dz).$$

Suppose that

$$(4.10) \quad v := v(t, z) = \frac{e^{(1-\delta)\Upsilon(s,z)} - 1}{(1-\delta)\Upsilon(s,z)} - 1.$$

By Girsanov's theorem,

$$\tilde{N}^{\mathbb{Q}}(dt, dz) := N(dt, dz) - (1 + v(t, z))\nu(dz)dt, \quad 0 \leq t \leq T,$$

is a  $(\mathbb{Q}, \mathcal{F})$ -compensated random measure. Then, under the probability  $\mathbb{Q}$ , the BSDE (3.1) with generator (4.8) becomes:

$$(4.11) \quad dY(t) = - \left\{ K(t) + \left[ 1 + \frac{\lambda^{\frac{1}{1-\delta}}(t)}{\eta^{\frac{\delta}{1-\delta}}(t)} \right] e^{-Y(t)} \right\} dt + \int_{\mathbb{R}} \Upsilon(t, z) \tilde{N}^{\mathbb{Q}}(dt, dz); \quad Y(T) = 0.$$

By (Theorem 11.2.1, Delong (2013)), the solution of (4.11) has the representation

$$(4.12) \quad Y(t) = \mathbb{E}^{\mathbb{Q}} \left\{ \int_t^T \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}} \right) du \right] K(s) ds \right\}$$

and  $\Upsilon$  can be determined by the martingale representation theorem.

As in Shen and Wei (2014),  $Y$  can be interpreted as an intuitive actuarial value process of a consumption rate from current to  $\tau \wedge T$ ,  $\frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}}$  insurance benefit paying at death time  $\tau < T$  and a legacy at terminal  $T$  if the wage earner survives until time  $T$ .

We then summarize the above analysis in the following theorem:

**Theorem 4.1.** *Under the assumptions (A1) – (A3), the optimal investment-consumption-insurance strategy of the problem is:*

$$c^* = xe^{-Y(t)}, \quad p^* = \eta(t)x \left\{ \left[ \frac{\lambda(t)}{\eta(t)} \right]^{\frac{1}{1-\delta}} e^{-Y(t)} - 1 \right\},$$

and  $\pi^*$  is the solution of the equation

$$\mu(t) - (1 - \delta)\pi\beta^2(t) - \int_{\mathbb{R}} [1 - (1 + \pi\gamma(t, z))^{\delta-1}] \gamma(t, z)\nu(dz) = 0,$$

where  $Y \in \mathbb{S}^2(\mathbb{R})$  is given by (4.12). Moreover, the value function is given by:

$$V(t, x) = \Psi(t, x, Y(t)) = \frac{1}{\delta} x^\delta e^{(1-\delta)Y(t)}, \quad \delta \in (-\infty, 1) \setminus \{0\}.$$

## 5. SPECIAL EXAMPLES

In this section we present two examples. In each case we consider a model with one random parameter and all others deterministic functions of  $t$ . We derive the explicit solution  $Y$  and consequently the Theorem 4.1 is verified.

In the first example, we consider a stochastic mortality with jumps. Jumps in a mortality process might occur for a variety of reasons: sudden changes in environmental conditions or a radical medical changes Cairns et al. (2008). At the second example we consider the case of stochastic appreciation rate with jumps. This case is motivated by the cointegrated model in Chiu and Wong (2011), where the log-prices depend on the diffusion appreciation rate.

**Example 5.1.** We consider the force of mortality given by the following geometric jump diffusion model:

$$(5.1) \quad d\lambda(s) = \lambda(s)[ads + bdW(s) + \int_{z>-1} z\tilde{N}(ds, dz)], \quad \lambda(t) = \lambda^+, \quad 0 \leq t \leq s \leq T,$$

where  $a, b \in \mathbb{R}$  are constants and  $z > -1$ .

Then,  $Y(t)$  in (4.12) is represented as follows:

$$(5.2) \quad Y(t) = \int_t^T \mathbb{E}^{\mathbb{Q}} \left\{ \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}} \right) du \right] \right\} L(s) ds \\ - \frac{1}{1-\delta} \mathbb{E}^{\mathbb{Q}} \left\{ \int_t^T \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}} \right) du \right] \lambda(s) ds \right\},$$

where

$$L(s) := -\frac{\rho(s)}{1-\delta} + \frac{\delta}{1-\delta}(r(s) + \eta(s)) + \frac{\delta}{1-\delta} \left\{ \pi^* \mu(s) - \frac{1-\delta}{2} (\pi^*)^2 \beta^2(s) \right. \\ \left. + \frac{1}{\delta} \int_{\mathbb{R}} [(1 + \pi^* \gamma(s, z))^{\delta} - 1 - \delta \pi^* \gamma(s, z)] \nu(dz) \right\}$$

and  $\mathbb{E}^{\mathbb{Q}}[\cdot]$  is the conditional expectation under  $\mathbb{Q}$ , given  $\mathcal{F}(t)$ .

Under the probability  $\mathbb{Q}$ , the dynamics of the mortality process is given by:

$$d\lambda(s) = \lambda(s) \left[ \left( a + \int_{z>-1} z v(s, z) \nu(dz) \right) ds + b dW^{\mathbb{Q}}(s) + \int_{z>-1} z \tilde{N}^{\mathbb{Q}}(ds, dz) \right].$$

Then, the conditional expectation  $\mathbb{E}^{\mathbb{Q}}[\lambda(s)]$ , given  $\mathcal{F}(t)$  is given by:

$$(5.3) \quad \mathbb{E}^{\mathbb{Q}}[\lambda(s)] = \lambda \exp \left\{ \int_t^s \left[ a + \int_{z>-1} z v(s, z) \nu(dz) \right] ds \right\}.$$

We define a new probability measure  $\tilde{\mathbb{Q}}$  equivalent to  $\mathbb{Q}$  as follows:

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} := \frac{\lambda(s)}{\mathbb{E}^{\mathbb{Q}}[\lambda(s)]} = \exp \left\{ \int_0^s \left[ -\frac{1}{2} b^2 + \int_{z>-1} (\ln(1+z) - z) \nu(dz) \right] du \right\}.$$

By change of measures, (5.2) becomes:

$$(5.4) \quad Y(t) = \int_t^T \mathbb{E}^{\tilde{\mathbb{Q}}} \left\{ \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}} \right) du \right] \right\} L(s) ds \\ - \frac{1}{1-\delta} \int_t^T \mathbb{E}^{\tilde{\mathbb{Q}}} \left\{ \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}}{\eta^{\frac{\delta}{1-\delta}}} \right) du \right] \right\} \cdot \mathbb{E}^{\mathbb{Q}}[\lambda(s)] ds.$$

Applying the Itô's formula to  $\lambda^{\frac{1}{1-\delta}}$  under  $\tilde{\mathbb{Q}}$ , we obtain:

$$d\lambda^{\frac{1}{1-\delta}}(s) = \lambda^{\frac{1}{1-\delta}}(s) \left\{ \frac{1}{1-\delta} \left\{ a + \frac{2-\delta}{2(1-\delta)} b^2 + \int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] \right. \right. \right. \\ \left. \left. \left. + z v(s, z) - \delta z \right\} \nu(dz) \right\} ds + \frac{1}{1-\delta} b dW^{\tilde{\mathbb{Q}}}(s) \right. \\ \left. + \int_{z>-1} \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] \tilde{N}^{\tilde{\mathbb{Q}}}(ds, dz) \right\}.$$

Hence, taking expectation  $\mathbb{E}[\cdot]$ , under  $\tilde{\mathbb{Q}}$ , gives:

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[\lambda^{\frac{1}{1-\delta}}(s)] = \lambda^{\frac{1}{1-\delta}} \exp \left\{ \int_t^s \frac{1}{1-\delta} \left\{ a + \frac{2-\delta}{2(1-\delta)} b^2 \right. \right. \\ \left. \left. + \int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] + z v(u, z) - \delta z \right\} \nu(dz) \right\} du \right\},$$

since

$$\int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] + z v(u, z) - \delta z \right\} \nu(dz) < \infty.$$

To obtain the expectation  $\mathbb{E}^{\tilde{\mathbb{Q}}}[\cdot]$  in (5.4), we consider:

$$Z(s) = \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du.$$

By linearity of conditional expectation, we see that:

$$(5.5) \quad \mathbb{E}^{\tilde{\mathbb{Q}}}[Z] = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du \right] = \int_t^s \left( 1 + \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \right]}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du.$$

Then, from Teh et al. (2006), we know that, for a random variable  $X$  such that  $\mathbb{E}[X] \gg 0$ , the third derivative is small and the Taylor approximation series to  $\mathbb{E}[\ln(1 + X)]$  is very accurate, that is:

$$\mathbb{E}[\ln(1 + X)] = \ln(1 + \mathbb{E}[X]) - \frac{\mathbb{V}(X)}{2(1 + \mathbb{E}[X])^2} + \mathcal{O},$$

where  $\mathcal{O} \rightarrow 0$ . Applying this techniques in (5.4) we have:

$$(5.6) \quad \mathbb{E}^{\tilde{\mathbb{Q}}} \left\{ \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du \right] \right\} = \ln \left[ 1 + \int_t^s \left( 1 + \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \right]}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du \right] - \Pi + \mathcal{O},$$

where  $\mathcal{O} \rightarrow 0$  and

$$\Pi = \frac{\mathbb{V} \left[ \int_t^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du \right]}{2 \left( 1 + \int_t^s \left( 1 + \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \right]}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du \right)^2}.$$

We then need to obtain the variance:

$$\mathbb{V}(Z) := \mathbb{E}^{\tilde{\mathbb{Q}}}[Z^2] - \left( \mathbb{E}^{\tilde{\mathbb{Q}}}[Z] \right)^2.$$

Provided that the integrand of  $Z$  is absolutely convergent, by Fubini's theorem we can change the order of integration and applying the linearity of conditional expectation, leads to

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{Q}}}[Z^2] &= \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \int_{v=t}^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(v)}{\eta^{\frac{\delta}{1-\delta}}(v)} \right) dv \cdot \int_{u=t}^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) du \right] \\ &= \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \int_{v=t}^s \int_{u=t}^s \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(v)}{\eta^{\frac{\delta}{1-\delta}}(v)} \right) \cdot \left( 1 + \frac{\lambda^{\frac{1}{1-\delta}}(u)}{\eta^{\frac{\delta}{1-\delta}}(u)} \right) dudv \right] \\ &= \int_{v=t}^s \int_{u=t}^s \left\{ 1 + \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \right]}{\eta^{\frac{\delta}{1-\delta}}(u)} + \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(v) \right]}{\eta^{\frac{\delta}{1-\delta}}(v)} + \frac{\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \cdot \lambda^{\frac{1}{1-\delta}}(v) \right]}{\eta^{\frac{\delta}{1-\delta}}(u) \cdot \eta^{\frac{\delta}{1-\delta}}(v)} \right\} dudv. \end{aligned}$$

It remains to get the expectation  $\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \cdot \lambda^{\frac{1}{1-\delta}}(v) \right]$  in the above expression. By Itô's formula, we know that

$$\begin{aligned}
d \left[ \lambda^{\frac{1}{1-\delta}}(u) \cdot \lambda^{\frac{1}{1-\delta}}(v) \right] &= \lambda^{\frac{1}{1-\delta}}(u) \cdot \lambda^{\frac{1}{1-\delta}}(v) \left\{ \frac{1}{1-\delta} \left\{ a + \frac{2-\delta}{2(1-\delta)} b^2 \right. \right. \\
&\quad \left. \left. + \int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] + zv(u, z) - \delta z \right\} \nu(dz) \right\} du \\
&\quad + \frac{1}{1-\delta} b dW^{\tilde{\mathbb{Q}}}(u) + \int_{z>-1} \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] \tilde{N}^{\tilde{\mathbb{Q}}}(du, dz) \\
&\quad + \frac{1}{1-\delta} \left\{ a + \frac{2-\delta}{2(1-\delta)} b^2 + \int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] \right. \right. \\
&\quad \left. \left. + zv(v, z) - \delta z \right\} \nu(dz) \right\} dv + \frac{1}{1-\delta} b dW^{\tilde{\mathbb{Q}}}(v) \\
&\quad + \int_{z>-1} \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] \tilde{N}^{\tilde{\mathbb{Q}}}(dv, dz) \\
&\quad + \left\{ \frac{1}{(1-\delta)^2} b^2 + \int_{z>-1} \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right]^2 \nu(dz) \right\} du \wedge dv \\
&\quad + \int_{z>-1} \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right]^2 \tilde{N}^{\tilde{\mathbb{Q}}}(du \wedge dv, dz) \left. \right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}} \left[ \lambda^{\frac{1}{1-\delta}}(u) \cdot \lambda^{\frac{1}{1-\delta}}(v) \right] &= \lambda^{\frac{2}{1-\delta}} \exp \left\{ \frac{1}{1-\delta} \int_t^u \left\{ a + \frac{2-\delta}{2(1-\delta)} b^2 \right. \right. \\
&\quad \left. \left. + \int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] + zv(x, z) - \delta z \right\} \nu(dz) \right\} dx \right. \\
&\quad \left. + \frac{1}{1-\delta} \int_t^v \left\{ a + \frac{2-\delta}{2(1-\delta)} b^2 + \int_{z>-1} \left\{ (1-\delta) \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right] \right. \right. \right. \\
&\quad \left. \left. + zv(x, z) - \delta z \right\} \nu(dz) \right\} dx \\
&\quad \left. + \int_t^{u \wedge v} \left\{ \frac{1}{(1-\delta)^2} b^2 + \int_{z>-1} \left[ (1+z)^{\frac{1}{1-\delta}} - 1 \right]^2 \nu(dz) \right\} dx \right\}.
\end{aligned}$$

Then, substituting (5.3) and (5.6) into (5.4) we obtain  $Y$ . Taking  $Y$  into Theorem 4.1 we obtain the optimal investment consumption insurance strategy and the value function.

**Example 5.2.** Similar to the cointegrated asset price model in Chiu and Wong (2011), we consider an appreciation rate with jump  $\mu(s)$  governed by the following dynamic

$$d\mu(s) = (a(s) - b\mu(s))ds + \int_{\mathbb{R}} \psi(s, z) \tilde{N}(ds, dz), \quad \mu(t) = \mu^+; \quad 0 \leq t \leq s \leq T,$$

where  $a(t) \in \mathbb{R}$  is deterministic and uniformly bounded,  $b \in \mathbb{R}$  is constant.

Under the probability  $\mathbb{Q}$ , the appreciation rate  $\mu(s)$  follows the dynamics

$$(5.7) \quad d\mu(s) = \left[ a(s) - b\mu(s) + \int_{\mathbb{R}} \psi(s, z)v(s, z)\nu(dz) \right] ds + \int_{\mathbb{R}} \psi(s, z)\tilde{N}^{\mathbb{Q}}(ds, dz).$$

Then, taking the conditional expectation  $\mathbb{E}[\cdot]$  under  $\mathbb{Q}$  we obtain

$$(5.8) \quad \mathbb{E}^{\mathbb{Q}}[\mu(s)] = e^{-b(s-t)}\mu_0 + \int_t^s \left[ a(u) + \int_{\mathbb{R}} \psi(u, z)v(u, z)\nu(dz) \right] e^{-b(u-s)} du,$$

since

$$\int_{\mathbb{R}} \psi(u, z)v(u, z)\nu(dz) < \infty.$$

The solution (4.12) becomes

$$(5.9) \quad Y(t) = \int_t^T \ln \left[ 1 + \int_t^s \left( 1 + \frac{\lambda \frac{1}{1-\delta}}{\eta \frac{\delta}{1-\delta}} \right) du \right] \cdot \left( M(s) + \frac{\delta}{1-\delta} \pi^* \mathbb{E}^{\mathbb{Q}}[\mu(s)] \right) ds,$$

where

$$\begin{aligned} M(s) = & -\frac{\rho(s) + \lambda(s)}{1-\delta} + \frac{\delta}{1-\delta}(r(s) + \eta(s)) + \frac{\delta}{1-\delta} \left\{ -\frac{1-\delta}{2} (\pi^*)^2 \beta^2(s) \right. \\ & \left. + \frac{1}{\delta} \int_{\mathbb{R}} [(1 + \pi^* \gamma(s, z))^\delta - 1 - \delta \pi^* \gamma(s, z)] \nu(dz) \right\}. \end{aligned}$$

Taking  $Y$  in (5.9) into Theorem 4.1 we obtain the optimal investment consumption insurance strategy and the value function.

## 6. CONCLUSION

The paper focused on an optimal investment-consumption-insurance with random parameters which include jumps. These random parameters need not to be bounded. Therefore, the application of a combination of the HJB equation and BSDE made it possible to characterize our optimal problem and the value function in terms of a unique solution of a BSDE. We have dealt with our problem using one riskless and one risk asset in our financial and insurance markets. Finally, we provided special examples to illustrate our results.

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## APPENDIX

Given the geometric mortality rate with jump studied in this paper,

$$d\lambda(s) = \lambda(s)[ads + bdW(s) + dN(t)], \quad \lambda(t) = \lambda > 0, \quad 0 \leq t \leq s \leq T$$

and a geometric diffusion mortality in Shen and Wei (2014)

$$d\lambda(s) = \lambda(s)[ads + bdW(s)], \quad \lambda(t) = \lambda > 0, \quad 0 \leq t \leq s \leq T.$$

To illustrate the effect of the jump in the mortality rate we consider the following parameters  $a = -0.035$ ,  $b = 0.1$ ,  $\lambda(0) = 0.05$  and  $T = 40$  years. Graphically, we see that the jump-diffusion mortality can capture the high rates of mortality (solid curve) in the figure given below. This can happen in the cases of radical change in environmental conditions such as war, earthquakes, sudden pandemics etc. Clearly, in this case a stochastic mortality considered in Shen and Wei (2014) (dashed curve) fails.

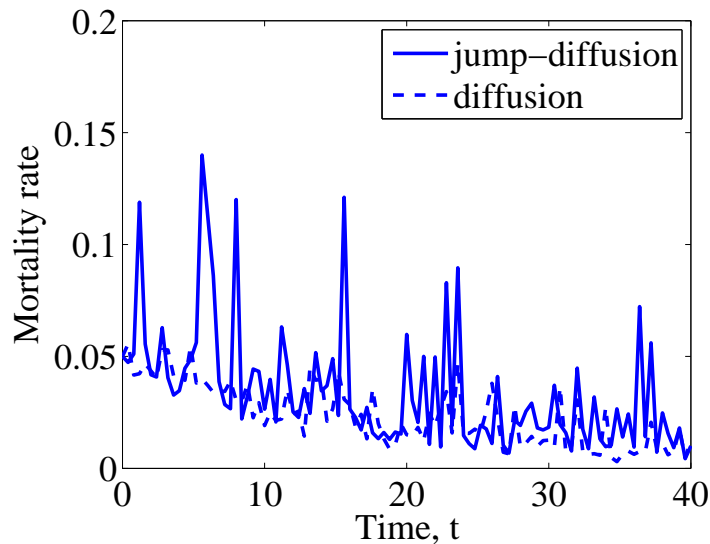


FIGURE 1. Mortalities with diffusion and jump-diffusion processes.

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