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ON THE GROUP SHEAF OF $\mathcal{A}$-SYMPLECTOMORPHISMS

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ABSTRACT. This is a part of a further undertaking to affirm that most of classical module theory may be retrieved in the framework of Abstract Differential Geometry (à la Mallios). More precisely, within this article, we study some defining basic concepts of symplectic geometry on free $\mathcal{A}$-modules by focussing in particular on the group sheaf of $\mathcal{A}$-symplectomorphisms, where $\mathcal{A}$ is assumed to be a torsion-free PID $\mathbb{C}$-algebra sheaf. The main result arising hereby is that $\mathcal{A}$-*symplectomorphisms* locally are products of symplectic transvections, which is a particularly well-behaved counterpart of the classical result.

1. Introduction

The study of symplectic $\mathcal{A}$-transvections parallels that of orthogonal $\mathcal{A}$-symmetries with respect to ($\mathcal{A}$-) hyperplanes. An important result regarding orthogonal $\mathcal{A}$-symmetries is the sheaf-theoretic version of the Cartan-Dieudonné theorem, which stipulates that given a PID $\mathbb{C}$-algebra sheaf and ϕ a non-degenerate $\mathcal{A}$-bilinear form on a convenient $\mathcal{A}$-module $\mathcal{E}$ of rank n, having nowhere-zero (local) isotropic sections, every $\mathcal{A}$-isometry $\sigma \in \mathrm{A}ut_{\mathcal{A}}\,\mathcal{E}$ is a product of at most n orthogonal symmetries with respect to (local) non-isotropic hyperplanes, cf. [11]. But for a Riemannian convenient $\mathcal{A}$-module $\mathcal{E}$ of finite rank, equipped as above with a non-degenerate $\mathcal{A}$-bilinear form ϕ, the condition that $\mathcal{E}$ should have nowhere-zero isotropic sections plays no role, and is therefore needless, cf. [11].

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The Cartan-Dieudonné theorem is part of special features characterizing orthogonal geometry; its counterpart with respect to symplectic geometry is centered around symplectic $\mathcal{A}$-transvections. It states that

> *Given a free symplectic $\mathcal{A}$-module of rank $2n$ on a topological space X, where $\mathcal{A}$ is a PID $\mathbb{C}$-algebra sheaf, and $\sigma \in \mathrm{End}_{\mathcal{A}}\,\mathcal{E}$ an $\mathcal{A}$-symplectomorphism of $\mathcal{E}$, for any open $U \subseteq X$, σ_U is a product of at most $4n-2$ symplectic $\mathcal{A}(U)$-transvections.*

This result is the main result of the paper.

2. Generalities on abstract geometric algebra

Let us review succinctly the basic notions of Abstract Geometric Algebra which we are concerned with in this paper. Most of the concepts in this paper are defined on the basis of the classical ones. These notions may be found in our recent papers such as [8], [9], and [10]. Let $\mathcal{F}$ and $\mathcal{E}$ be $\mathcal{A}$-modules and $\phi\colon \mathcal{F} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ an $\mathcal{A}$-bilinear morphism. Then, we say that the triple $((\mathcal{F}, \mathcal{E}; \phi); \mathcal{A}) \equiv (\mathcal{F}, \mathcal{E}; \phi) \equiv (\mathcal{F}, \mathcal{E}; \mathcal{A})$ forms a *pairing of $\mathcal{A}$-modules* or an *$\mathcal{A}$-pairing.* The sub-$\mathcal{A}$-module $\mathcal{F}^{\perp}$ of $\mathcal{E}$ such that, for every open subset U of X, $\mathcal{F}^{\perp}(U)$ consists of all $r \in \mathcal{E}(U)$ with $\phi_V(\mathcal{F}(V), r|_V) = 0$ for any open $V \subseteq U$, is called the *right kernel* of the pairing $(\mathcal{F}, \mathcal{E}; \mathcal{A})$. In a similar way, one defines the *left kernel* of $(\mathcal{F}, \mathcal{E}; \mathcal{A})$ to be the sub-$\mathcal{A}$-module $\mathcal{E}^{\perp}$ of $\mathcal{F}$ such that, for any open subset U of X, $\mathcal{E}^{\perp}(U)$ is the set of all (local) sections $r \in \mathcal{F}(U)$ such that $\phi_V(r|_V, \mathcal{E}(V)) = 0$ for every open $V \subseteq U$.

If $(\mathcal{E}, \phi)$ is a self $\mathcal{A}$-pairing with ϕ symmetric or skew-symmetric, the kernel $\mathcal{E}^{\perp}$ is called the *radical sheaf* (or *sheaf of $\mathcal{A}$-radicals*, or simply *$\mathcal{A}$-radical*) of $\mathcal{E}$. If $\mathcal{F}$ is a sub-$\mathcal{A}$-module of $\mathcal{E}$, the radical of $\mathcal{F}$ consists of those sections of $\mathcal{F}^{\perp}$ that are also sections of $\mathcal{F}$. In other words, $\mathrm{rad}\,\mathcal{F} = \mathcal{F} \cap \mathcal{F}^{\perp}$. In general, if $(\mathcal{F}, \mathcal{E}; \mathcal{A})$ is a pairing of free $\mathcal{A}$-modules, then $\mathrm{rad}\,\mathcal{E} := \mathcal{E} \cap \mathcal{E}^{\perp}$, and similarly $\mathrm{rad}\,\mathcal{F} := \mathcal{F} \cap \mathcal{F}^{\top}$. An $\mathcal{A}$-module $\mathcal{E}$ such that $\mathrm{rad}\,\mathcal{E} \neq 0$ (resp. $\mathrm{rad}\,\mathcal{E} = 0$) is called *isotropic* (resp. *non-isotropic*); $\mathcal{E}$ is *totally isotropic* if ϕ is identically zero. For any open $U \subseteq X$, a non-zero section $r \in \mathcal{E}(U)$ is called *isotropic* if $\phi_U(r, r) = 0$.

N.B. We assume throughout the paper, unless otherwise mentioned, that the pair $(X, \mathcal{A})$ is an *algebraized space*, where $\mathcal{A}$ is a *unital* $\mathbb{C}$*-algebra sheaf* such that *every nowhere-zero section of* $\mathcal{A}$ is *invertible.* (Consider for example sheaves of continuous, smooth and holomorphic functions.)

3. Symplectic Gram-Schmidt theorem

Lemma 1. *Let $(\mathcal{E}, \omega)$ be a symplectic free $\mathcal{A}$-module, U an open subset of X and $(r_1, \ldots, r_n) \subseteq \mathcal{E}(U)$ an arbitrary (local) gauge of $\mathcal{E}$. For any $r \equiv r_i$, $1 \leq i \leq n$, there exists a nowhere-zero section $s \in \mathcal{E}(U)$ such that $\omega_U(r, s)$ is nowhere zero.*

P r o o f. Without loss of generality, assume that $r_1 = r$. On the other hand, since the induced $\mathcal{A}$-morphism $\widetilde{\omega} \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^*)$ is one-to-one and both $\mathcal{E}$ and $\mathcal{E}^*$ have the same finite rank, it follows that the matrix D representing ω_U (see also [1: Theorem 2.21, p. 357; Definition 2.19, p. 356] or [2: Proposition 20.3, p. 343]), with respect to the basis $(r_1, \ldots, r_n)$, has a *nowhere-zero determinant*; so since

$$\det D = \sum_{i=1}^{n} (-1)^{1+i} \omega(r_1, r_i) \det D_{1i} = \omega\left(r_1, \sum_{i=1}^{n} (-1)^{1+i} \det D_{1i} r_i \right),$$

where D_{1i} is the minor of the corresponding $\omega(r_1, r_i)$, and $\det D$ nowhere zero, we thus have a section $s := \sum\limits_{i=1}^{n} (-1)^{1+i} \det D_{1i} r_i \in \mathcal{E}(U)$ such that $\omega(r, s)$ is nowhere zero. □

Theorem 1. *Let $(\mathcal{E}, \omega)$ be a symplectic free $\mathcal{A}$-module of rank $2n$, and I and J two (possibly empty) subsets of $\{1, \ldots, n\}$. Moreover, let $A = \{r_i \in \mathcal{E}(U) : i \in I\}$ and $B = \{s_j \in \mathcal{E}(U) : j \in J\}$ such that r_i, s_j $(i \in I, j \in J)$ are nowhere zero, and*

$$\omega_U(r_i, r_j) = \omega_U(s_i, s_j) = 0, \quad \omega_U(r_i, s_j) = \delta_{ij}, \qquad (i, j) \in I \times J. \qquad (1)$$

Then, there exists a basis $\mathfrak{B}$ *of $(\mathcal{E}(U), \omega_U)$ containing $A \cup B$.*

P r o o f. As in [5: Theorem 1.15, pp. 12, 13], we have three cases. With no loss of generality, we assume that $U = X$.

(1) <u>Case: $I = J = \emptyset$.</u> Since $\mathcal{A}^{2n} \neq 0$ (we already assumed that $\mathbb{C} \equiv \mathbb{C}_X \subseteq \mathcal{A}$), there exists an element

$$0 \neq r_1 \in \mathcal{E}(X) \simeq \mathcal{A}^{2n}(X) \simeq \mathcal{A}(X)^{2n}$$

(take e.g. the image (by the isomorphism $\mathcal{E}(X) \simeq \mathcal{A}^{2n}(X)$) of an element in the canonical basis of (sections) of $\mathcal{A}^{2n}(X)$). By virtue of Lemma 1, there exists a section $\overline{s}_1 \in \mathcal{E}(X)$ such that $\omega_V(r_1|_V, \overline{s}_1|_V) \neq 0$ for any open subset V in X. Thus, based on the hypothesis on $\mathcal{A}$, $\omega_X(r_1, \overline{s}_1)$ is invertible in $\mathcal{A}(X)$. Putting $s_1 := u^{-1}\overline{s}_1$, where $u \equiv \omega_X(r_1, \overline{s}_1) \in \mathcal{A}(X)$, one gets

$$\omega_X(r_1, s_1) = 1.$$

Now, let us consider

$$S_1 := [r_1, s_1],$$

that is, the $\mathcal{A}(X)$-*plane*, spanned by r_1 and s_1 in $\mathcal{E}(X)$, along with *its orthogonal complement in* $\mathcal{E}(X)$, i.e.,

$$S_1^{\perp} \equiv T_1 := \big\{t \in \mathcal{E}(X) : \ \omega_X(t, z) = 0 \ \text{ for all } \ z \in S_1\big\}.$$

The sections r_1 and s_1 are linearly independent, for if $s_1 = ar_1$, with $a \in \mathcal{A}(X)$, then

$$1 = \omega_X(r_1, s_1) = \omega_X(r_1, ar_1) = a\omega_X(r_1, r_1) = 0,$$

a *contradiction.* So, $\{r_1, s_1\}$ is a basis of S_1. Furthermore, we prove that

(i) $S_1 \cap T_1 = 0$,

(ii) $S_1 + T_1 = \mathcal{E}(X)$.

Indeed,
(i) since $\omega_X(r_1, s_1) \neq 0$, we have $S_1 \cap T_1 = 0$.
On the other hand,
(ii) for every $z \in \mathcal{E}(X)$, one has

$$z = \big(-\omega_X(z, r_1)s_1 + \omega_X(z, s_1)r_1\big) + \big(z + \omega_X(z, r_1)s_1 - \omega_X(z, s_1)r_1\big),$$

with

$$-\omega_X(z, r_1)s_1 + \omega_X(z, s_1)r_1 \in S_1,$$

and

$$z + \omega_X(z, r_1)s_1 - \omega_X(z, s_1)r_1 \in T_1.$$

Thus,

$$\mathcal{E}(X) = S_1 \oplus T_1.$$

The restriction $\omega_1 \equiv \omega_{1,X}$ of ω_X to T_1 *is non-degenerate* as $\mathcal{E}$ (in particular, $\mathcal{E}(X)$) is non-isotropic. (T_1, ω_1) *is thus a symplectic free* $\mathcal{A}(X)$*-module of rank* $2(n-1)$. Repeating the construction above $n-1$ times, we obtain a strictly decreasing sequence

$$(\mathcal{E}(X), \omega_X) \supseteq (T_1, \omega_1) \supseteq \cdots \supseteq (T_{n-1}, \omega_{n-1})$$

of symplectic free $\mathcal{A}(X)$-modules with rank $T_k = 2(n-k)$, $k = 1, \ldots, n-1$, and also an increasing sequence

$$\{r_1, s_1\} \subseteq \{r_1, r_2; s_1, s_2\} \subseteq \cdots \subseteq \{r_1, \ldots, r_n; s_1, \ldots, s_n\}$$

of gauges; each satisfying relations (6).

(2) Case $I = J \neq \emptyset$. We may assume without loss of generality that $I = J = \{1, 2, \ldots, k\}$, and let S be the submodule spanned by $\{r_1, \ldots, r_k; s_1, \ldots, s_k\}$. Clearly, $\omega_X|_S$ is non-degenerate; by Adkins-Weintraub [1: Lemma (2.31), p. 360], it follows that $S \cap S^{\perp} = 0$. On the other hand, let $z \in \mathcal{E}(X)$. One has

$$z = \Big(-\sum_{i=1}^{k} \omega_X(z, r_i)s_i + \sum_{i=1}^{k} \omega_X(z, s_i)r_i\Big) + \Big(z + \sum_{i=1}^{k} \omega_X(z, r_i)s_i - \sum_{i=1}^{k} \omega_X(z, s_i)r_i\Big),$$

with

$$-\sum_{i=1}^{k}\omega_X(z,r_i)s_i+\sum_{i=1}^{k}\omega_X(z,s_i)r_i\in S,$$

and

$$z+\sum_{i=1}^{k}\omega_X(z,r_i)s_i-\sum_{i=1}^{k}\omega_X(z,s_i)r_i\in S^{\perp}.$$

Thus,

$$\mathcal{E}(X)=S\oplus S^{\perp}.$$

Based on the hypothesis on S_1 the restriction $\omega_X|_S$ is a symplectic $\mathcal{A}$-bilinear form. (It is also easily seen that the restriction $\omega_X|_{S^\perp}$ is skew-symmetric.) Moreover, since $S\oplus S^\perp=\mathcal{E}(X)$ and $\mathcal{E}(X)^\perp=0$, if there exist $z_1\in S^\perp$ such that $\omega_X(z_1,z)=0$ for all $z\in S^\perp$, then $z_1\in\mathcal{E}(X)^\perp=0$, i.e., $z_1=0$. Thus, $\omega_X|_{S^\perp}$ is non-degenerate and hence a symplectic $\mathcal{A}$-form. Applying Case (1) , we obtain a symplectic basis of $S^\perp$, which we denote as

$$\{r_{k+1},\ldots,r_n;s_{k+1},\ldots,s_n\}.$$

Then,

$$\mathfrak{B}=\{r_1,\ldots,r_n;s_1,\ldots,s_n\}$$

is a symplectic basis of $\mathcal{E}(X)$ with the required property.

(3) *Case $J\setminus I\neq\emptyset$ (or $I\setminus J\neq\emptyset$).* Suppose that $k\in J\setminus I$; since ω_X is non-degenerate there exists $\overline{r}_k\in\mathcal{E}(X)$ such that $\omega_X(\overline{r}_k,s_k)\neq 0$ in the sense that $\omega_V(\overline{r}_k|_V,s_k|_V)\neq 0$ for any open $V\subseteq X$. In other words, the section $v\equiv\omega_X(\overline{r}_k,s_k)\in\mathcal{A}(X)$ is nowhere zero, and is therefore *invertible*. So, if $r_k:=v^{-1}\overline{r}_k$, we have $\omega_X(r_k,s_k)=1$. Next, let us consider the sub-$\mathcal{A}(X)$-module R, spanned by r_k and s_k, viz. $R=[r_k,s_k]$. As in Case (1), we have

$$\mathcal{E}(X)=R\oplus R^{\perp}.$$

Clearly, for every $i\in I$, $r_i\in R^\perp$. To show this, fix i in I, and assume that $r_i=ar_k+bs_k+x$, where $a,b\in\mathcal{A}(X)$ and $x\in R^\perp$. So, one has

$$0=\omega_X(r_i,s_k)=a,\qquad 0=\omega_X(r_i,r_k)=b,$$

which corroborates the claim that $r_i\in R^\perp$ for all $i\in I$. Furthermore, we also clearly have that for every $j\neq k$ in J, $s_j\in R^\perp$. Then $A\cup B\cup\{r_k\}$ is a family of linearly independent sections: the equality

$$a_kr_k+\sum_{i\in I}a_ir_i+\sum_{j\in J}b_js_j=0$$

implies that $a_k=a_i=b_j=0$. Repeating this process as many times as necessary, we are lead back to Case (2), and the proof is finished. $\square$

Referring to Theorem 1, the basis $\mathfrak{B}$ is called a *symplectic $\mathcal{A}(U)$-basis* of $(\mathcal{E}(U),\omega_U)$.

Corollary 1. *Let $(\mathcal{E}, \omega)$ be a symplectic free $\mathcal{A}$-module of finite rank. For any nowhere-zero (local) section $r \in \mathcal{E}(U)$ (U is an open subset of X), there exists a nowhere-zero section $s \in \mathcal{E}(U)$ such that $\omega_U(r, s)$ is nowhere zero.*

P r o o f. Apply Theorem 1 to find a symplectic basis of $(\mathcal{E}(U), \omega_U)$ containing the given nowhere-zero section r, then apply Lemma 1 to find a nowhere-zero section $s \in \mathcal{E}(U)$ such that $\omega_U(r, s)$ is nowhere zero. $\square$

4. Main results

Definition 1. Let $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ be symplectic $\mathcal{A}$-modules. An $\mathcal{A}$-morphism $\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$ is called *symplectic* if

$$\varphi^*\omega' := \omega' \circ (\varphi \times \varphi) = \omega,$$

that is, for any $s, t \in \mathcal{E}(U)$, where U is open in X,

$$\omega'_U(\varphi_U(s), \varphi_U(t)) \equiv \omega'(\varphi(s), \varphi(t)) = \omega(s, t) \equiv \omega_U(s, t).$$

A symplectic $\mathcal{A}$-isomorphism is called an $\mathcal{A}$-*symplectomorphism.* Symplectic $\mathcal{A}$-modules $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ are called symplectomorphic if there is an $\mathcal{A}$-symplectomorphism between them.

Clearly, symplectic $\mathcal{A}$-morphisms are necessarily *injective*; indeed, given a symplectic $\mathcal{A}$-morphism $\varphi\colon (\mathcal{E}, \omega) \longrightarrow (\mathcal{E}', \omega')$, then, for any $s \in \mathcal{E}(U)$, since ω is non-degenerate, $\varphi_U(s) = 0$ implies that $s = 0$.

Lemma 2. *Let $(\mathcal{E}, \omega)$ be a symplectic $\mathcal{A}$-module. The correspondence*

$$U \longmapsto (\mathrm{Sp}\,\mathcal{E})(U), \tag{2}$$

where U varies over the topology of X, such that $(\mathrm{Sp}\,\mathcal{E})(U)$ is the group (under composition) of all $\mathcal{A}|_U$-symplectomorphisms yields a complete presheaf of groups. The corresponding sheaf, denoted $\mathcal{S}p\,\mathcal{E}$, is called the symplectic group sheaf, *or the* group sheaf of symplectomorphisms *of $\mathcal{E}$ (in fact, of $(\mathcal{E}, \omega)$).*

P r o o f. We first show that for any open set $U \subseteq X$, $(\mathrm{Sp}\,\mathcal{E})(U)$ is a group. In fact, since

$$(\mathrm{Sp}\,\mathcal{E})(U) \subseteq \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$$

and the correspondence

$$U \longmapsto \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \subseteq \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$$

gives rise to a complete presheaf of groups, cf. [7: pp. 285, 286], we need only show that if $\varphi, \psi \in (\mathrm{Sp}\,\mathcal{E})(U)$, then $\varphi \circ \psi$, $\varphi^{-1} \in (\mathrm{Sp}\,\mathcal{E})(U)$. To this end, let V be any open subset of U. Then, we have

$$(\varphi_V \circ \psi_V)^*(\omega_V) = (\psi_V^* \circ \varphi_V^*)(\omega_V) = \psi_V^*(\varphi_V^*(\omega_V)) = \omega_V,$$

from which we deduce that $\varphi \circ \psi \in (\mathrm{Sp}\,\mathcal{E})(U)$.

On the other hand, one also obtains

$$(\varphi_V^{-1})^*(\omega_V) = (\varphi_V^{-1})^*(\varphi_V^*(\omega_V)) = (\varphi_V \circ \varphi_V^{-1})^*(\omega_V) = \omega_V,$$

so that $\varphi^{-1} \in (\mathrm{Sp}\,\mathcal{E})(U)$, as well.

Let us now show that (2) yields a complete presheaf of groups. It is easy to see that Correspondence (2), along with the obvious restriction maps, defines a presheaf of groups on X. Thus, we just prove that the presheaf of groups defined, on X, by (2) is complete.

Indeed, let U be an open subset of X and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of U; let $\varphi, \psi \in (\mathrm{Sp}\,\mathcal{E})(U)$ such that

$$\rho^U_{U_\alpha}(\varphi) \equiv \varphi_{U_\alpha} := \varphi_\alpha = \psi_\alpha =: \psi_{U_\alpha} \equiv \rho^U_{U_\alpha},$$

for all $\alpha \in I$, and where the $\rho^U_{U_\alpha}$ are the restriction maps characterizing the presheaf $((\mathrm{Sp}\,\mathcal{E})(U), \rho^U_V)$. Since

$$(\mathrm{Sp}\,\mathcal{E})(U) \subseteq \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U), \qquad \mathrm{GL}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \subseteq \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U),$$

and the $\{\rho^U_{U_\alpha}\}_{\alpha \in I}$ are also the restriction maps making the diagram

$$U \longmapsto \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \tag{3}$$

into a presheaf, it follows that $\varphi = \psi$. So the presheaf on X, given by (2), satisfies Condition $(S1)$ of presheaves, see [7: p. 46].

For axiom (S2), see [7: p. 47], let

$$(\varphi_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} (\mathrm{Sp}\,\mathcal{E})(U_\alpha) \subseteq \prod_{\alpha \in I} \mathrm{H}om_{\mathcal{A}|_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{E}|_{U_\alpha})$$

be such that

$$\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(\varphi_\alpha) \equiv \varphi_\alpha|_{U_{\alpha\beta}} = \varphi_\beta|_{U_{\alpha\beta}} \equiv \rho^{U_\beta}_{U_\alpha \cap U_\beta}(\varphi_\beta)$$

for any $\alpha, \beta \in I$, with $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$. Hence, since (3) yields a complete presheaf, there exists an element $\varphi \in \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$ such that one has

$$\varphi|_{U_\alpha} = \varphi_\alpha, \qquad \alpha \in I.$$

It only remains to show that

$$\varphi^*\omega = \omega,$$

where $\omega \equiv \omega|_U$ is a symplectic structure on $\mathcal{E}|_U$.

To this end, we first observe that $\varphi^*\omega$, $\omega \in \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{E} \oplus \mathcal{E})|_U, \mathcal{A}|_U)$, with

$$U \longmapsto \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{E} \oplus \mathcal{E})|_U, \mathcal{A}|_U)$$

defining a complete presheaf of $\mathcal{A}$-modules on X, see [7: p. 134]. But,

$$\varphi^*\omega|_{U_\alpha} = (\varphi|_{U_\alpha})^*\omega|_{U_\alpha} = \varphi_\alpha^*\omega|_{U_\alpha} = \omega|_{U_\alpha},$$

therefore

$$\varphi^*\omega = \omega,$$

as desired. □

Symplectic $\mathcal{A}$-transvections, introduced in [12], are a particular case of $\mathcal{A}$-symplectomorphisms. Since we shall need them henceforth, let us recall the definition (due to *Mallios*) of a symplectic $\mathcal{A}$-transvection.

Definition 2. Let $\mathcal{E}$ be an $\mathcal{A}$-module. An element $\varphi \in \mathrm{A}ut\,\mathcal{E}$ is called an *$\mathcal{A}$-transvection* if there exists a sub-$\mathcal{A}$-module $\mathcal{H}$ in $\mathcal{E}$ such that $\mathcal{E}/\mathcal{H} \simeq \mathcal{A}$, and the following conditions are satisfied:

(i) $\varphi|_{\mathcal{H}} = I$.

(ii) $\mathrm{Im}(\varphi - I) \subseteq \mathcal{H}$.

More accurately, we say that φ is an $\mathcal{A}$-transvection with respect to the sub-$\mathcal{A}$-module $\mathcal{H}$.

Purely categorically, condition (i) of Definition 2 means that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, "\tau"] & \mathcal{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{A} \arrow[r] & 0, \end{tikzcd}$$

where the two horizontal rows represent the same short exact sequence, commutes.

In the same vein, an $\mathcal{A}$-transvection of a symplectic $\mathcal{A}$-module is called a *sympletic $\mathcal{A}$-transvection.*

Definition 3 (Mallios)**.** Let $\mathcal{E}$ be an $\mathcal{A}$-module. An element $\varphi \in \mathrm{E}nd\,\mathcal{E} \equiv \mathrm{E}nd_{\mathcal{A}}\,\mathcal{E} := \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$ is called a *homothecy* of *ratio* $\alpha \in \mathrm{E}nd_{\mathcal{A}}\,\mathcal{A} =: \mathcal{A}^*(X) \simeq \mathcal{A}(X)$ if

$$\varphi = \alpha \cdot I, \tag{4}$$

where I stands for the identity of the group $\mathrm{E}nd\,\mathcal{E} := \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$.

Section-wise, Equation (4) means that, given any section $s \in \mathcal{E}(U)$, one has

$$\varphi_U(s) = \alpha|_U \cdot s \equiv \alpha \cdot s.$$

Now, suppose we have a *free* $\mathcal{A}$*-module* $\mathcal{E}$ and $\mathcal{H}$ an $\mathcal{A}$*-hyperplane* of $\mathcal{E}$, so that one has

$$\mathcal{E}/\mathcal{H} \simeq \mathcal{A}, \tag{5}$$

cf. [8]. Moreover, let $\varphi \in \mathrm{E}nd\, \mathcal{E}$ such that

$$\varphi(\mathcal{H}) \subseteq \mathcal{H}; \tag{6}$$

then, φ gives rise to an element, say $\widetilde{\varphi}$, of $\mathrm{E}nd(\mathcal{E}/\mathcal{H})$; viz.

$$\widetilde{\varphi} \in \mathrm{E}nd(\mathcal{E}/\mathcal{H}),$$

such that, in view of (6),

$$\widetilde{\varphi} \circ q = q \circ \varphi,$$

where $q \colon \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{H}$ is the *canonical* $\mathcal{A}$-epimorphism. However, due to (5), one has

$$\widetilde{\varphi} \in \mathrm{E}nd(\mathcal{E}/\mathcal{H}) \simeq \mathrm{E}nd\, \mathcal{A} =: \mathcal{A}^*(X) \simeq \mathcal{A}(X),$$

viz. one obtains

$$\widetilde{\varphi} = \alpha \in \mathcal{A}(X) \simeq \mathrm{E}nd\, \mathcal{A},$$

thus

$$\widetilde{\varphi} = \alpha \cdot I,$$

so that α is the *ratio* of $\widetilde{\varphi}$. Hence, φ *induces a homothecy of* $\mathcal{E}/\mathcal{H}(\simeq \mathcal{A})$ *of ratio* α.

Lemma 3. *Let* $\mathcal{E}$ *be a free* $\mathcal{A}$*-module,* $\mathcal{H}$ *an* $\mathcal{A}$*-hyperplane of* $\mathcal{E}$*,* φ *an* $\mathcal{A}$*-endomorphism of* $\mathcal{E}$ *that fixes every section of* $\mathcal{H}$*, and* $\widetilde{\varphi}$ *the* $\mathcal{A}$*-homothecy, of ratio* α*, induced by* φ *on the line* $\mathcal{A}$*-module* $\mathcal{E}/\mathcal{H}$*. Then,*

(1) *If* α *is nowhere* 1*, there exists a unique line* $\mathcal{A}$*-module* $\mathcal{L} \subseteq \mathcal{E}$ *such that* $\mathcal{E} = \mathcal{H} \oplus \mathcal{L}$ *and* $\mathcal{L}$ *is stable by* φ*, i.e.* $\varphi(\mathcal{L}) \simeq \mathcal{L}$*.*

(2) *If* $\alpha = 1$*, then for every* $\mathcal{A}$*-morphism* $\theta \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) = \mathcal{E}^*(X)$ *with* $\ker \theta \simeq \mathcal{H}$*, there exists a unique* $\mathcal{A}$*-morphism* $\psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{H})$ *such that*

$$\varphi = I + \psi. \tag{7}$$

Proof.

(1) <u>Uniqueness.</u> Let $\mathcal{L}$ be a line $\mathcal{A}$-module satisfying the hypotheses of the assertion, and s a nowhere-zero global section of $\mathcal{L}$ (such a section s does exist because $\mathcal{L} \simeq \mathcal{A}$ and $\mathcal{A}$ is unital). Therefore, there exists $b \in \mathcal{A}(X)$ such that $\varphi(s) = \beta s$. Next, assume that q is the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto $\mathcal{E}/\mathcal{H}$. It is clear that $\widetilde{\varphi}_X(q_X(s)) = \beta q_X(s) \equiv \beta q(s)$; thus $\widetilde{\varphi}_X$ is a homothecy of ratio $\alpha = b$, hence, by hypothesis, β is nowhere 1. Now, let u be an element of $\mathcal{E}(X)$ such that $u \notin \mathcal{H}(X)$; then there exists a non-zero $\lambda \in \mathcal{A}(X)$ and an element $t \in \mathcal{H}(X)$ such that

$$u = \lambda s + t.$$

It follows that

$$\varphi(u) = \lambda\beta s + t.$$

Of course, $\varphi(u)$ and u are colinear if and only if $t = 0$. Thus, we have proved that every section $u \in \mathcal{E}(X)$ which is colinear with its image $\varphi(u)$ belongs to $\mathcal{L}(X)$. A similar argument holds should we consider the decomposition $\mathcal{E}(U) = \mathcal{H}(U) \oplus \mathcal{L}(U)$, where U is any other open subset U of X. Hence, $\mathcal{L}$ is the unique complement of $\mathcal{H}$ in $\mathcal{E}$, up to $\mathcal{A}$-isomorphism, and stable by φ.

Existence. Since α is nowhere 1 on X, there exists a nowhere-zero section $s \in \mathcal{E}(X)$ such that

$$\widetilde{\varphi}_U(q_U(s|_U)) := \widetilde{\varphi}_U(q_U(s_U)) \neq q_U(s_U) =: q_U(s|_U)$$

for any open $U \subseteq X$. As $\widetilde{\varphi} \circ q = q \circ \varphi$, it follows that $r_U := \varphi_U(s_U) - s_U$ does not belong to $\mathcal{H}(U)$, for any open $U \subseteq X$. The line $\mathcal{A}$-module $\mathcal{L} := [r_U]_{X \supseteq U,\,\mathrm{open}}$ clearly complements $\mathcal{H}$. It remains to show that $\mathcal{L}$ is stable by φ: To this end, we first observe that every s_U does not belong to the corresponding $\mathcal{H}(U)$, and $\mathcal{E}(U) \simeq \mathcal{A}(U)s_U \oplus \mathcal{H}(U)$. So, since $r_U \notin \mathcal{H}(U)$ for every open $U \subseteq X$, there exists for every r_U sections $\alpha_U \in \mathcal{A}(U)$ and $t_U \in \mathcal{H}(U)$ such that

$$r_U = \alpha_U s_U + t_U. \tag{8}$$

We deduce from (8) that

$$\varphi_U(r_U) = (\alpha_U + 1)r_U,$$

and the proof is complete.

(2) *Uniqueness* is obvious.

Existence. Let $t \in \mathcal{E}(X)$ be a nowhere-zero section such that $t|_U \notin \mathcal{H}(U)$ for any open $U \subseteq X$. Consider the section

$$r := (\theta(t))^{-1}(\varphi(t) - t). \tag{9}$$

Clearly, $r \in \mathcal{H}(X)$; it is because

$$(q \circ \varphi)(t) - q(t) = (\widetilde{\varphi} \circ q)(t) - q(t) = 0.$$

The $\mathcal{A}(X)$-morphisms

$$s \longmapsto \varphi(s) \tag{10}$$

and

$$s \longmapsto s + \psi(s) := s + \theta(s)r \tag{11}$$

are equal, since $\varphi(t) = t + \psi(t)$, and $\varphi(s) = s + \psi(s)$ for every $s \in \mathcal{H}(X)$. □

From the uniqueness of the $\mathcal{A}$-morphism ψ, we deduce that the global section r in (9) is unique. On another hand, it is clear that φ, as given in (7), is an $\mathcal{A}$-transvection.

In particular, we shall be concerned with symplectic $\mathcal{A}$-transvections of symplectic free $\mathcal{A}$-modules of finite rank. More explicitly, let $(\mathcal{E}, \omega)$ be a symplectic free $\mathcal{A}$-module of finite rank, and φ a symplectic $\mathcal{A}$-transvection of $(\mathcal{E}, \omega)$. By the proof of Lemma 3, there exists a section $r \in \mathcal{E}(X)$ such that

$$\varphi_X(s) = s + \theta_X(s)r, \tag{12}$$

where $\ker \theta$ is an $\mathcal{A}$-hyperplane of $\mathcal{E}$. Clearly, we have

$$\omega_X(\varphi_X(s), \varphi_X(t)) = \omega_X(s, t), \tag{13}$$

for $s, t \in \mathcal{E}(X)$. Using (12), (13) yields

$$\theta_X(s)\omega_X(r, t) = \theta_X(t)\omega_X(r, s).$$

Suppose that r is nowhere zero. By Corollary 1, we let t_0 be a nowhere-zero section such that $\omega_X(r, t_0)$ is nowhere zero. Then

$$\theta_X(s) = \frac{\theta_X(t_0)}{\omega_X(r, t_0)}\omega_X(r, s) \equiv c\omega_X(r, s),$$

where

$$c := \frac{\theta_X(t_0)}{\omega_X(r, t_0)} \in \mathcal{A}(X).$$

Hence,

$$\varphi_X(s) = s + c\omega_X(r, s)r,$$

with $c \in \mathcal{A}(X)$.

Lemma 4. *Let $(\mathcal{E}, \omega)$ be a symplectic orthogonally convenient $\mathcal{A}$-module of finite rank. The correspondence*

$$U \longmapsto (\mathrm{T}v\, \mathcal{E})(U),$$

where U is any open set in X, such that $(\mathrm{T}v\, \mathcal{E})(U)$ is the group of all symplectic $\mathcal{A}|_U$-transvections yields a complete presheaf of groups. The sheaf thus obtained is denoted $\mathcal{T}v\, \mathcal{E}$ and is called the symplectic transvection group sheaf.

P r o o f. Apart from the completeness condition (S2), the rest of the proof is just as straightforward as in the proof of Lemma 2. So, let

$$(\varphi_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} (\mathrm{T}v\, \mathcal{E})(U_\alpha) \subseteq \prod_{\alpha \in I} (\mathrm{S}p\, \mathcal{E})(U_\alpha)$$

be such that

$$\varphi_\alpha|_{U_{\alpha\beta}} = \varphi_\beta|_{U_{\alpha\beta}}$$

for any $\alpha, \beta \in I$, with $U_{\alpha\beta} \neq \emptyset$. Since $((\mathrm{S}p\, \mathcal{E})(U), \rho^U_V)$ is a complete presheaf, there exists $\varphi \in (\mathrm{S}p\, \mathcal{E})(U)$ such that one has

$$\varphi|_{U_\alpha} = \varphi_\alpha, \qquad \alpha \in I.$$

It only remains to show that φ is an $\mathcal{A}|_U$-transvetion. For this purpose, let $s \in \mathcal{E}(U_\alpha \cap U_\beta)$. If $s_0^i \in \mathcal{E}(U_i)$, $i \in I$, is the direction section of the corresponding $\mathcal{A}|_{U_i}$-transvection φ_i, on one hand, we have, for any $s \in \mathcal{E}(U_{\alpha\beta})$,

$$\begin{aligned}\varphi|_{U_{\alpha\beta}}(s) &= \varphi_i|_{U_{\alpha\beta}}(s) \\ &= s + \lambda_{U_i}|_{U_{\alpha\beta}}\omega_{U_{\alpha\beta}}(s_0^{U_i}|_{U_{\alpha\beta}}, s)s_0^{U_i}|_{U_{\alpha\beta}},\end{aligned}$$

where $i = \alpha, \beta$. Whence, we have

$$\lambda_{U_\alpha}|_{U_{\alpha\beta}}\omega_{U_{\alpha\beta}}\big(s_0^{U_\alpha}|_{U_{\alpha\beta}}, s\big)s_0^{U_\alpha} = \lambda_{U_\beta}|_{U_{\alpha\beta}}\omega_{U_{\alpha\beta}}\big(s_0^{U_\beta}|_{U_{\alpha\beta}}, s\big)s_0^{U_\beta}, \tag{14}$$

for every $s \in \mathcal{E}(U_{\alpha\beta})$. Since s is arbitrary, it follows from (14) that

$$s_0^{U_\alpha}|_{U_{\alpha\beta}} = s_0^{U_\beta}|_{U_{\alpha\beta}}, \qquad \lambda_{U_\alpha}|_{U_{\alpha\beta}} = \lambda_{U_\beta}|_{U_{\alpha\beta}}.$$

Thus, there exist $s_0^U \in \mathcal{E}(U)$ and $\lambda_U \in \mathcal{A}(U)$ such that

$$s_0^U|_{U_\alpha} = s_0^{U_\alpha}, \qquad \lambda_U|_{U_\alpha} = \lambda_{U_\alpha}.$$

Hence,

$$\varphi_U(s) = s + \lambda_U \omega_U(s_0^U, s)s_0^U,$$

for every $s \in \mathcal{E}(U)$. □

We shall now show $\mathcal{A}$-symplectomorphisms are generated by symplectic $\mathcal{A}$-transvections, the counterpart of a classical result that may be found in [3: pp. 18–20] or [4: pp. 422–424] or [6: p. 372–374].

Lemma 5. *If $s, t \in \mathcal{E}(U)$ are sections of $\mathcal{E}$, over an open subset U of X, such that $\omega_U(s,t)|_V = \omega_V(s|_V, t|_V) \neq 0$, for any open $V \subseteq U$, then, there exists a symplectic transvection τ on $\mathcal{E}(U)$ such that $\tau(s) = t$. On the other hand, if $\omega_U(s,t)|_V = 0$ and $s|_V \neq t|_V$ for some open $V \subseteq U$, there is no symplectic transvection on $\mathcal{E}(U)$ carrying s onto t.*

P r o o f. Since $\omega_U(s,t)$ is nowhere zero, and every nowhere-zero section of $\mathcal{A}$ is invertible, it suffices to take

$$\tau(u) = u - \frac{1}{\omega_U(s,t)}\omega_U(t-s, u)(t-s).$$

As for the second part of the lemma, suppose there exists a symplectic transvection τ, determined by a section $a \in \mathcal{E}(U)$, mapping s onto t. Then, on V, we have

$$c|_V \omega_V(a|_V, s|_V)^2 = 0;$$

if $c|_V = 0$ or $\omega_V(a|_V, s|_V) = 0$, it follows that

$$\tau_V(s|_V) = s|_V = t|_V,$$

which contradicts the hypothesis. □

Lemma 6. *Let τ be a symplectomorphism of $\mathcal{E}(U)$ and F_τ the free sub-$\mathcal{A}(U)$-module of $\mathcal{E}(U)$ consisting of all fixed sections in $\mathcal{E}(U)$ (by τ). Then, there exists a transvection α such that the free sub-$\mathcal{A}(U)$-module, F_ϕ, of fixed sections of $\phi \equiv \alpha \circ \tau$, satisfies the property that rank $F_\phi > \operatorname{rank} F_\tau$, if and only if there exists a section s_1 such that $\omega_U(\tau(s_1), s_1)$ is nowhere zero, (which implies that $\tau(s_1)|_V \neq s_1|_V$ for any open $V \subseteq U$, which, in turn, implies that $s_1 \notin F_\tau$).*

Proof. Suppose there exists s_1 with $\omega_U(\tau(s_1), s_1)$ nowhere zero; it follows that

$$\omega_U(\tau(s_1) - s_1, s_1)|_V \neq 0$$

for any open $V \subseteq U$. Based on Theorem 1, there exists a symplectic basis of $(\mathcal{E}(U), \omega_U)$ containing $\tau(s_1) - s_1$; so it is clear that $(\tau(s_1) - s_1)^\perp$ is an $\mathcal{A}(U)$-hyperplane. Furthermore $(\tau(s_1) - s_1)^\perp$ contains F_τ, for, if $\tau(s) = s$,

$$\omega_U(\tau(s_1) - s_1, s) = \omega_U(\tau(s_1), s) - \omega_U(s_1, s) = \omega_U(\tau(s_1), \tau(s)) - \omega_U(s_1, s) = 0.$$

Considering the transvection

$$\alpha(s) = s - \frac{1}{\omega_U(\tau(s_1), s_1)} \omega_U\big(s_1 - \tau(s_1), s\big)\big(s_1 - \tau(s_1)\big),$$

one has

$$\alpha(\tau(s_1)) = s_1.$$

Its fixed sections yield the $\mathcal{A}(U)$-hyperplane $(s_1 - \tau(s_1))^\perp$, which contains F_τ. Since

$$\phi(s_1) = (\alpha \circ \tau)(s_1) = s_1$$

and s_1 is nowhere zero and is contained in F_ϕ, so rank $F_\phi > \operatorname{rank} F_\tau$.

Conversely, if $\phi = \alpha \circ \tau$ for some transvection α such that rank $F_\phi > \operatorname{rank} F_\tau$, it follows that there exists a nowhere-zero section $s_1 \in \mathcal{E}(U)$ such that $\phi(s_1) = (\alpha \circ \tau)(s_1) = s_1$ and $\tau(s_1)|_V \neq s_1|_V$ for any open $V \subseteq U$. Assume that the transvection α is given by

$$\alpha(s) = s + c\omega_U(a, s)a,$$

where $c \in \mathcal{A}(U)$ and $a \in \mathcal{E}(U)$. Then,

$$(\alpha \circ \tau)(s_1) = \tau(s_1) + c\omega_U(a, \tau(s_1))a = s_1,$$

which implies that

$$\tau(s_1) - s_1 = c\omega_U(a, \tau(s_1))a.$$

Since $(\tau(s_1) - s_1)|_V \neq 0$ for any open $V \subseteq U$, $\omega_U(a, \tau(s_1))|_V \neq 0$ and $c|_V \neq 0$ for any open $V \subseteq U$. But, for any open $V \subseteq U$,

$$\omega_U(a, \tau(s_1))|_V = \omega_U(\alpha(a), \alpha(\tau(s_1)))|_V = \omega_U(a, s_1)|_V \neq 0$$

and

$$\omega_U(s_1, \tau(s_1)) = \omega_U(\alpha(s_1), \alpha(\tau(s_1))) = \omega_U(s_1 + c\omega_U(a, s_1)a, s_1) = c\omega_U(a, s_1)^2,$$

therefore

$$\omega_U(s_1, \tau(s_1))|_V \neq 0$$

for any open $V \subseteq U$. □

Lemma 7. *Let τ be a symplectomorphism of $\mathcal{E}(U)$ such that $\omega_U(\tau(s), s) \equiv \omega(\tau(s), s) = 0$ for all $s \in \mathcal{E}(U)$. Then, if τ is not the identity, there exists a transvection α of $\mathcal{E}(U)$ such that $F_{\alpha\circ\tau} \equiv F_\phi = F_\tau$, where F_ϕ and F_τ are free sub-$\mathcal{A}(U)$-modules of $\mathcal{E}(U)$ consisting of all fixed sections of ϕ and τ, respectively. Furthermore, there exists $s_1 \in \mathcal{E}(U)$ such that $\omega(\phi(s_1), s_1)$ is nowhere zero.*

P r o o f. By polarization, one has

$$0 = \omega(\tau(s+t), s+t) = \omega(\tau(s), t) + \omega(\tau(t), s),$$

so that

$$\omega(\tau(s), t) = \omega(s, \tau(t))$$

for all $s, t \in \mathcal{E}(U)$. In other words, τ is a symmetric $\mathcal{A}(U)$-endomorphism ($\tau^* = \tau$); as τ is symplectic, one has

$$\tau^* \circ \tau = \mathrm{Id}_{\mathcal{E}(U)} \equiv 1 = \tau^2.$$

It follows that τ is a symplectic involution, which splits the $\mathcal{A}(U)$-module $\mathcal{E}(U)$ into two sub-$\mathcal{A}(U)$-modules: $\mathcal{E}(U)_+$ and $\mathcal{E}(U)_-$, corresponding to eigen-value sections $+1$ and -1, respectively, i.e.

$$\mathcal{E}(U) = \mathcal{E}(U)_+ \oplus \mathcal{E}(U)_-.$$

Observe that if $s \in \mathcal{E}(U)_+$ and $t \in \mathcal{E}(U)_-$:

$$\omega(s, t) = \omega(\tau(s), \tau(t)) = \omega(s, -t) = -\omega(s, t) = 0$$

(*assuming that the characteristic of* $\mathcal{A}(U)$ *is not* 2). Thus, $\mathcal{E}(U)_+$ and $\mathcal{E}(U)_-$ are orthogonal, hence non-isotropic. Clearly, $\mathcal{E}(U)_+$ is the free sub-$\mathcal{A}(U)$-module F_τ of fixed sections of τ.

If $\tau \neq 1$, $\mathcal{E}(U)_-$ is non-void and non-isotropic. Take in $\mathcal{E}(U)_-$ a nowhere-zero section s_0, so that by Corollary 1, there is a nowhere-zero section s_1 such that $\omega(s_0, s_1)$ is nowhere zero. Let α be the transvection determined by s_0 and a nowhere-zero coefficient $\lambda \in \mathcal{A}(U)$. We contend that $\phi := \alpha \circ \tau$ is not an involution. Indeed,

$$\omega(\phi(s_1), s_1) = \omega(-s_1 - \lambda\omega(s_0, s_1)s_0, s_1) = -\lambda\omega(s_0, s_1)^2$$

is nowhere zero.

Since $s_0 \in \mathcal{E}(U)_-$, s_0 is orthogonal to $\mathcal{E}(U)_+$, that is, $\mathcal{E}(U)_+ \subseteq s_0^{\perp}$, which is the free sub-$\mathcal{A}(U)$-module consisting of all fixed sections of α. Thus, $F_\tau \subseteq F_\phi$, but, by virtue of Lemma 6, we necessarily have that $F_\phi = F_\tau$. □

Putting Lemmas 5, 6 and 7, we have

THEOREM 2. *Let $(\mathcal{E}, \omega)$ be a free sympletcic $\mathcal{A}$-module of rank $2n$ and $\sigma \in \mathrm{E}nd_{\mathcal{A}}\,\mathcal{E}$ an $\mathcal{A}$-symplectomorphism of $\mathcal{E}$. Then, for any open subset U of X, $\sigma \equiv \sigma_U$ is a product of at most $4n-2$ symplectic $\mathcal{A}(U)$-transvections.*

Proof. We use induction on the rank of the free sub-$\mathcal{A}(U)$-module $F_\sigma \equiv F_{\sigma_U}$ of fixed sections of σ, showing that if $\sigma \neq I$, there exist at most two symplectic transvections, α and β, of $\mathcal{E}(U)$ such that, if $\phi := \alpha\beta\sigma$, rank $F_\phi >$ rank F_σ.

Besides, if rank $F_\sigma = 2n-1$, then σ is a symplectic transvection. Indeed, in such a case, there are at most two transvections α and β such that

$$2n = \operatorname{rank} F_\phi > \operatorname{rank} F_\sigma,$$

where $\phi := \alpha\beta\sigma = I_{\mathcal{E}(U)}$. Thus,

$$\sigma = \beta^{-1}\alpha^{-1}.$$

Now, suppose that rank $F_\sigma < 2n-1$; so there exist m symplectic $\mathcal{A}(U)$-transvections $\alpha_1, \ldots, \alpha_m$ of $\mathcal{E}(U)$, where $m \leq 4n-2$, such that

$$\alpha_m \alpha_{m-1} \cdots \alpha_1 \sigma = I.$$

Hence,

$$\sigma = \alpha_1^{-1}\alpha_2^{-1} \cdots \alpha_m^{-1},$$

which is a product of symplectic $\mathcal{A}(U)$-transvections. □

We thus obtain our major result, which is

THEOREM 3. *Let $(\mathcal{E}, \omega)$ be a free symplectic $\mathcal{A}$-module of finite rank. Then, the symplectic group sheaf $\mathcal{S}p\,\mathcal{E}$ is generated by the symplectic transvection group sheaf $\mathcal{T}v\,\mathcal{E}$.*

Proof. Suppose that $\sigma \in (\mathcal{S}p\,\mathcal{E})(U) \subseteq \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E})(U) = \mathrm{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)$, where U is an open subset of X. By Theorem 2, every σ_V, where V is open in U, is a product of symplectic $\mathcal{A}(V)$-transvections. □

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