HEFAT2007
5th International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics
Sun City, South Africa
Paper number: MM7

MULTI-FLUID STRATIFIED SHEAR FLOWS IN PIPES. PART 3.

LINEAR STABILITY ANALYSIS

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ABSTRACT

The focus of this paper is on linear stability analysis of steady state solutions developing in horizontal or nearly horizontal pipes. Continuity and momentum equations for incompressible and isothermal flows are derived considering an arbitrary number of fluids. It is shown that the linearization of the governing equations around the steady state solutions and the assumption of a particular form of the perturbations yield an eigen-value problem, the roots of which represent complex wave speeds of the perturbations. The coefficients of the eigen-value problem are derived for the most general case and given in analytical form. The effects of inertia of the fluids, gravity, interfacial tensions, shear stresses and wave length are included explicitly in the stability coefficients. Some general properties of the eigen-value problem are outlined.

INTRODUCTION

Mathematical solutions of fluid dynamical problems are supposed to obey the basic laws of fluid dynamics. However, within the whole set of mathematical solutions satisfying the equations of conservation of mass, momentum and energy for a particular problem, only those which are also stable can actually occur in Nature. It is therefore clear that methods capable of discerning between stable and unstable solutions are fundamental in many practical applications.

The capability of predicting flow pattern regimes and flow pattern transition criteria for given flow conditions represents, for instance, a preliminary and fundamental step to the subsequent calculations of engineering quantities such as pressure gradients, phase fractions, mass and heat transfer rates, pressure oscillations and many others which are reasonably assumed to depend in an intimate way on the flow structure.

A quite common approach to do this in horizontal and nearly horizontal pipes, is based on the hypothesis that stratified flow is the initial configuration from which all other flow pattern regimes depart. The existence of a specific flow pattern regime at given flow conditions is in fact believed not to be affected by the path through which the final state is reached. Indications as to the possibility for transition from stratified flow to other flow patterns may therefore be obtained from a stability analysis of the stratified flow configuration.

Many of these transitions are strictly correlated to the possibility for interfacial wave growth. Clearly, any growing interfacial perturbation induces considerable pressure and shear stress variations, with respect to the undisturbed state, which in turn influence its growth. It is to be noted however that, although these phenomena are undoubtedly three-dimensional, at the present time, one-dimensional models are still the only feasible choice in many practical applications.

In this work linear stability theory of stratified shear flows in pipes inclined from the horizontal will be fully developed. The analysis will be limited to incompressible and isothermal flows. It will be shown that the linearization of the governing equations around the steady state solutions and the assumption of a particular form of the perturbations yield an eigen-value problem, the roots of which represent complex wave speeds of the perturbations. The coefficients of the eigen-value problem will be derived for the most general case and will be given in analytical form. The effects of inertia of the fluids, gravity, interfacial tensions, shear stresses and wave length will be included explicitly in the stability coefficients, making the computation of neutral stability lines and amplification factors of the perturbations consequential.

Different types of Kelvin-Helmholtz stability analyses which is possible to perform will be described and some general properties of the eigen-value problem will be outlined.

NOMENCLATURE

Make reference to Part 1 of “Multi-fluid stratified shear flow in pipes” presented at HEFAT 2007.

GOVERNING EQUATIONS

Consider the pipe geometry shown in **Figure 1**. At any time t the flow of each phase is predominantly along the positive x direction and at any position the pipe is inclined at an angle q from the horizontal. Only incompressible and isothermal flows are here considered, therefore a multi-fluid flow system can be modelled using a combination of one-dimensional continuity and momentum equations in integral form.

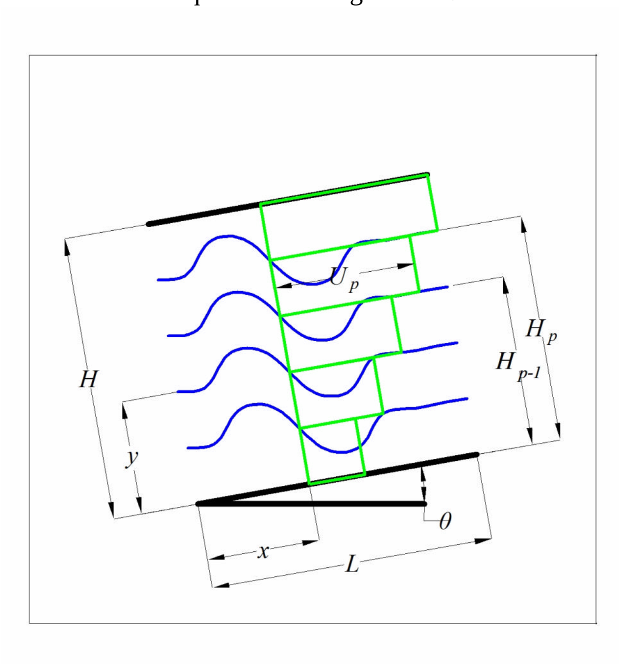

Figure 1. Small interfacial flow perturbations.

The integral form of the continuity equations gives:

$$\frac{\partial}{\partial t}\left(r_p A_p\right)+\frac{\partial}{\partial x}\left(r_p U_p A_p\right)=0 \tag{1}$$

while the integral form of the momentum equations gives:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(r_p U_p A_p\right)+\frac{\partial}{\partial x}\left(r_p y_p U_p^2 A_p\right)=\\ &-\frac{\partial}{\partial x}\left(\int_{A_p} P_p\, dA_p\right)\\ &-P_p^{\inf} P_{p-1}^j \frac{\partial H_{p-1}}{\partial x}+P_p^{\sup} P_p^j \frac{\partial H_p}{\partial x}\\ &-T_p^i P_p^i - T_{p-1}^j P_{p-1}^j + T_p^j P_p^j - r_p A_p g \sin q\end{aligned} \tag{2}$$

Since the flow is incompressible, the continuity equations can be easily expressed in terms of the interfacial heights:

$$\begin{aligned}\frac{\partial U_p}{\partial x} &= \frac{P_{p-1}^j}{A_p}\left(\frac{\partial H_{p-1}}{\partial t}+U_p\frac{\partial H_{p-1}}{\partial x}\right)\\ &-\frac{P_p^j}{A_p}\left(\frac{\partial H_p}{\partial t}+U_p\frac{\partial H_p}{\partial x}\right)\end{aligned} \tag{3}$$

The pressure distribution is assumed to vary hydrostatically along the transverse section and thus, owing to the effects of interfacial tension, the pressure acting on the upper side of each interface differs from the pressure acting on the lower side by a small amount proportional to the interfacial curvature.

By using Leibniz’s theorem for differentiation of an integral:

$$\begin{aligned}\frac{\partial}{\partial x}\left(\int_{A_p} P_p\, dA_p\right) &= \int_{A_p}\frac{\partial P_p}{\partial x}dA_p\\ &+P_p^{\sup} P_p^j \frac{\partial H_p}{\partial x} - P_{p-1}^{\inf} P_{p-1}^j \frac{\partial H_{p-1}}{\partial x}\end{aligned} \tag{4}$$

the momentum equations can be expressed quite straightforwardly as:

$$\begin{aligned}&r_p\frac{\partial U_p}{\partial t}+r_p\left(1-y_p\right)\frac{U_p}{A_p}\frac{\partial A_p}{\partial t}\\ &+r_p y_p U_p\frac{\partial U_p}{\partial x}+r_p U_p^2\frac{\partial y_p}{\partial x}=-\frac{\partial \overline{P}^{\inf}(x,t)}{\partial x}\\ &+\sum_{q=1}^{p-1}\begin{pmatrix}\left(r_q-r_{q+1}\right)g\cos q\,\dfrac{\partial H_q(x,t)}{\partial x}\\ -s_q\dfrac{\partial k_q(x,t)}{\partial x}\end{pmatrix}\\ &-T_p^i\frac{P_p^i}{A_p}-T_{p-1}^j\frac{P_{p-1}^j}{A_p}+T_p^j\frac{P_p^j}{A_p}-r_p g\sin q\end{aligned} \tag{5}$$

or as:

$$\begin{aligned}&r_p\frac{\partial U_p}{\partial t}+r_p\left(1-y_p\right)\frac{U_p}{A_p}\frac{\partial A_p}{\partial t}\\ &+r_p y_p U_p\frac{\partial U_p}{\partial x}+r_p U_p^2\frac{\partial y_p}{\partial x}=-\frac{\partial \overline{P}^{\sup}(x,t)}{\partial x}\\ &-\sum_{q=1}^{n-p}\begin{pmatrix}\left(r_{n-q}-r_{n-q+1}\right)g\cos q\,\dfrac{\partial H_{n-q}(x,t)}{\partial x}\\ -s_{n-q}\dfrac{\partial k_{n-q}(x,t)}{\partial x}\end{pmatrix}\\ &-T_p^i\frac{P_p^i}{A_p}-T_{p-1}^j\frac{P_{p-1}^j}{A_p}+T_p^j\frac{P_p^j}{A_p}-r_p g\sin q\end{aligned} \tag{6}$$

depending on whether the hydrostatic pressure distribution is calculated in terms of the pressure condition existing at the bottom or top part of the pipe.

Making use of continuity equations, subtracting the momentum equation of phase $p+1$ from the momentum equation of phase p and rearranging, the following combined momentum equations are obtained:

$$
\begin{aligned}
& r_p \frac{\partial U_p}{\partial t} - r_{p+1} \frac{\partial U_{p+1}}{\partial t} \\
& - r_p \left(1 - y_p\right) U_p \frac{P_{p-1}^j}{A_p} \frac{\partial H_{p-1}}{\partial t} \\
& + \begin{pmatrix} r_p \left(1 - y_p\right) U_p \dfrac{P_p^j}{A_p} + \\ r_{p+1} \left(1 - y_{p+1}\right) U_{p+1} \dfrac{P_p^j}{A_{p+1}} \end{pmatrix} \frac{\partial H_p}{\partial t} \\
& - r_{p+1} \left(1 - y_{p+1}\right) U_{p+1} \frac{P_{p+1}^j}{A_{p+1}} \frac{\partial H_{p+1}}{\partial t} \\
& + r_p y_p U_p \frac{\partial U_p}{\partial x} - r_{p+1} y_{p+1} U_{p+1} \frac{\partial U_{p+1}}{\partial x} \\
& + r_p U_p^2 \frac{\partial y_p}{\partial x} - r_{p+1} U_{p+1}^2 \frac{\partial y_{p+1}}{\partial x} \\
& + \left(r_p - r_{p+1}\right) g \cos q \frac{\partial H_p}{\partial x} - s_p \frac{\partial k_p}{\partial x} = F_p
\end{aligned}
\tag{7}
$$

where the shear stress functions F_p and the wall-fluid and fluid-fluid shear stresses are defined as in Part 1 of "Multi-fluid stratified shear flows in pipes".

LINEAR STABILITY ANALYSIS

The problem of instability of parallel shear flows has been firstly recognised by Helmholtz in 1868 and then posed and solved by Kelvin in 1871. In their honour this kind of instability is now called Kelvin-Helmholtz instability.

The condition of Kelvin-Helmholtz instability mainly represents an imbalance of the destabilising effects of inertia over the stabilising effects of gravity and interfacial tensions. Viscosity, instead, may play either a stabilising or destabilising role on flow stability depending on whether the energy dissipation or the momentum diffusion is predominant. Thus, for certain operational conditions, inviscid flows can be unstable while, for the same conditions, viscous flows can be stable and vice versa [16].

In order to analyse whether stability or instability occurs in a multi-phase flow system, we may consider the state of equilibrium slightly disturbed so that:

$$
H_p = \overline{H_p} + \hat{H}_p \quad U_p = \overline{U_p} + \hat{U}_p \tag{8}
$$

where the superscript bar and the circumflex denote equilibrium and disturbance, respectively.

Retaining only the first order terms in the continuity equations and in the combined momentum equations, differentiating the momentum equations with respect to the space coordinate and substituting from the continuity equations, the final result is:

$$
\begin{aligned}
& H_{1pq} \frac{\partial^2 \hat{H}_q}{\partial t^2} + H_{2pq} \frac{\partial^2 \hat{H}_q}{\partial x \partial t} + H_{3pq} \frac{\partial^2 \hat{H}_q}{\partial x^2} \\
& + H_{4pq} \frac{\partial^4 \hat{H}_q}{\partial x^4} = H_{5pq} \frac{\partial \hat{H}_q}{\partial t} + H_{6pq} \frac{\partial \hat{H}_q}{\partial x}
\end{aligned}
\tag{9}
$$

where the coefficients of the differential equations are evaluated at fully developed flow conditions.

Let us assume that an arbitrary disturbance may be resolved into independent modes of the form:

$$
\hat{H}_p = \hat{H}_p \exp\left(i\left(kx \pm wt\right)\right) \tag{10}
$$

The previous functions represent travelling monochromatic waves propagating in the positive or negative direction of the x axis with wave celerity c:

$$
c = \frac{w}{k} \tag{11}
$$

where the circular frequency w and the wave number k satisfy equations:

$$
w = \frac{2p}{T} \quad k = \frac{2p}{l} \tag{12}
$$

in terms of the period T and the wave length l. The semi-circular hats denote the amplitudes of the perturbations.

Linear stability theory can provide the wave lengths and growth rates of the unstable modes. In temporal stability analysis, the wave number is fixed and real and the frequency is complex while in spatial stability analysis, the frequency is fixed and real and the wave number is complex. The information provided by these two analyses is obviously complementary, however in order to study the growth of a disturbance in time, the usual temporal stability analysis appears more appropriate.

By making the appropriate substitutions and neglecting variations of momentum coefficients, the eventual result is the following linear system:

$$
\left(S_{1pq} c^2 + S_{2pq} c + S_{3pq}\right) \hat{H}_q = 0 \tag{13}
$$

where:

$$
\begin{aligned}
\mathrm{Re}\left(S_{1pq}\right) = & \left(r_p \frac{P_{p-1}^j}{A_p} \right) d\left(p - 1 - q\right) \\
& - \left(r_p \frac{P_p^j}{A_p} + r_{p+1} \frac{P_p^j}{A_{p+1}} \right) d\left(p - q\right) \\
& + \left(r_{p+1} \frac{P_{p+1}^j}{A_{p+1}} \right) d\left(p + 1 - q\right)
\end{aligned}
$$

$$
\mathrm{Im}\left(S_{1pq}\right) = 0
$$

$$\mathrm{Re}\left(S_{2pq}\right) = \pm\left(2\, r_p y_p U_p \frac{P_{p-1}^j}{A_p}\right) d\left(p-1-q\right)$$
$$\mp\left(2\left(r_p y_p U_p \frac{P_p^j}{A_p} + r_{p+1} y_{p+1} U_{p+1} \frac{P_p^j}{A_{p+1}}\right)\right) d\left(p-q\right)$$
$$\pm\left(2\, r_{p+1} y_{p+1} U_{p+1} \frac{P_{p+1}^j}{A_{p+1}}\right) d\left(p+1-q\right)$$

$$\mathrm{Im}\left(S_{2pq}\right) = \frac{1}{k}\left(\mp \frac{P_q^j}{A_q}\frac{\partial F_p}{\partial U_q} \pm \frac{P_q^j}{A_{q+1}}\frac{\partial F_p}{\partial U_{q+1}}\right)$$

$$\mathrm{Re}\left(S_{3pq}\right) = \left(r_p y_p U_p^2 \frac{P_{p-1}^j}{A_p}\right) d\left(p-1-q\right)$$
$$+\left(\begin{array}{l}\left(r_p - r_{p+1}\right) g\cos q + s_p k^2 \\ - r_p y_p U_p^2 \dfrac{P_p^j}{A_p} - r_{p+1} y_{p+1} U_{p+1}^2 \dfrac{P_p^j}{A_{p+1}}\end{array}\right) d\left(p-q\right)$$
$$+\left(r_{p+1} y_{p+1} U_{p+1}^2 \frac{P_{p+1}^j}{A_{p+1}}\right) d\left(p+1-q\right)$$

$$\mathrm{Im}\left(S_{3pq}\right) = \frac{1}{k}\left(\frac{\partial F_p}{\partial H_q} - U_q \frac{P_q^j}{A_q}\frac{\partial F_p}{\partial U_q} + U_{q+1}\frac{P_q^j}{A_{q+1}}\frac{\partial F_p}{\partial U_{q+1}}\right)$$

d being the Dirac's delta function.

Under the condition that the linear system (13) must admit non-trivial solutions, the dispersion relationship, written in terms of the wave celerity c, follows. Computation of wave speeds and modes satisfying (13) is fundamental for the stability analysis of multi-phase flows.

Once these quantities have been computed, due to the linearity of the system, any arbitrary initial disturbance may be represented as a superposition of the normal solutions except for some multiplicative coefficients w_q which have to be determined in terms of the initial conditions:

$$\hat{H}_p = \sum_{q=1}^{2(n-1)} w_q \hat{H}_{pq} \exp\left(ik\left(x \pm c_q t\right)\right) \qquad (14)$$

Clearly, only the real parts of eigen-vectors represent physical quantities. The dispersion relationship result in a polynomial equation of order $2(n-1)$, the roots of which can be either real or complex. If, for instance, progressive waves are considered, depending on whether $\mathrm{Im}(c) > 0$, $\mathrm{Im}(c) = 0$ or $\mathrm{Im}(c) < 0$ each mode is said to be unstable, neutrally stable or stable respectively. The opposite is true for regressive waves.

As long as the imaginary part of each characteristic root is non positive the flow is deemed stable, conversely the flow is considered unstable when the imaginary part of at least one root is positive because in this case the most unstable mode will emerge and dominate the disturbance field. However, if more than one mode has large growth rates, many modes may appear.

Since in the linear system (13) the effects of inertia, viscosity, actual velocity profiles, interfacial tensions, shear stress modelling and mutual interactions between the interfaces are taken into account, the resulting dispersion equation represents a rather wide-ranging formulation to investigate the stability of multi-phase flow systems. Hence, depending on which of the previous terms are considered and which are neglected, different stability analyses can be performed.

Inviscid versus viscous analysis

Basically the inviscid Kelvin-Helmholtz (IKH) analysis ignores the effects of the shear stresses on the growth of the perturbations [1,5,8,19] while the viscous Kelvin-Helmholtz (VKH) analysis accounts for such effects [2,3,13,14,15,17]. Both theories however are based on steady state solutions which are computed considering wall and interfacial shear stresses.

It is worth noting that, when the fluids are assumed to be inviscid, the coefficients of the dispersion relation become real, thus, in such a case, solutions of the characteristic equation may either be real or occur in pairs of complex conjugate roots. Obviously complex conjugate eigenvalues are associated to complex conjugate eigenmodes. In other words, the inviscid theory implies that for each unstable mode there is a corresponding stable mode, consequently the flow is stable when all the $2(n-1)$ characteristic roots are real.

It is to be noted that for short wave length disturbances ($k \to \infty$) the neutral stability boundaries obtained from the viscous and inviscid theories tend to coincide as it has been observed in two-phase flow stability calculations [10]. It is also interesting to point out that in the general case, if interfacial tensions are ignored, the IKH neutral stability boundaries do not depend on the wave number, while the VKH neutral stability boundaries do.

Complete versus simplified analysis

The complete Kelvin-Helmholtz analysis assumes that perturbations are simultaneously present on all interfaces, so that the stability of each interface is influenced by the presence of a disturbance on the others and vice versa, while the simplified Kelvin-Helmholtz analysis hypothesises that when one interface is perturbed the others remain undisturbed or, in other words, the perturbations do not interfere with each other.

The equations governing the complete case are the ones here reported. The equations governing the simplified case, instead, can be deduced from the general case simply neglecting the extra-diagonal terms. In such a case, the characteristic polynomial equation of order $2(n-1)$ reduces to $(n-1)$ quadratic polynomial equations, each representing an eigenvalue relation for the single interface. Quite obviously the complete analysis requires much more theoretical and computational work than the simplified one.

The physical implications of neutral stability boundaries obtained from both complete IKH and VKH analyses and simplified IKH and VKH analyses follow immediately from previous definitions.

Physical considerations

Kelvin-Helmholtz theory states the conditions under which infinitesimal waves appearing in stratified flows can be expected to grow or decay. Over the past few decades many authors [3,5,6,7,8,11,12,19] have dedicated large attention primarily to the stability analysis of two-phase flows and have recognised that neutral stability boundaries can be used to model flow pattern transitions in a variety situations.

Some of them [5,6,7,8], in particular, have given a physical interpretation of both the IKH and VKH analyses in gas-liquid systems, showing that although the neutral stability boundaries can delimitate quite different regions on a flow pattern map, the amplification factors predicted by both analyses are very similar.

More precisely, in the region where both the IKH and the VKH analyses predict stable flow, the IKH amplication factor is zero while the VKH amplification factor is slightly negative so that stratified smooth flow regimes are expected to exist.

In the region where both the IKH and the VKH analyses predict unstable flow, both the IKH and the VKH amplification factors are positive and rapidly growing, thus either slug or annular flow is expected to exist depending on whether the pipe filling is greater or lower than 50%. This condition follows from simple physical observations. Consider a finite amplitude sinusoidal wave appearing on a liquid layer. If this wave is supposed to grow, slug flow can take place only if the liquid level is large enough to provide the liquid needed by the wave to block the pipe section, otherwise, if the level is inadequate, the wave is swept up around the pipe wall and annular flow occurs.

Lastly, in the region where the IKH analysis predicts stable flow and the VKH analysis predicts unstable flow, the IKH amplification factor is zero while the VKH amplification factor is slightly positive so that either slug or stratified wavy flow is expected to exist depending on whether the pipe filling is greater or lower than 50%.

Comparison of predictions with available experimental data have then clearly shown that the IKH and VKH neutral stability boundaries and the half-pipe filling curve can be used to predict mechanistically the transition from stratified flow to either stratified wavy, slug or annular flow.

Stability analysis of stratified three-phase flows, despite its practical importance, has been much less investigated and the few obtained results are far from being exhaustive [9,18]. It is conviction of the writers that much work is still to be done on this subject.

DISPERSION EQUATION AND THE EQUIVALENT EIGEN-VALUE PROBLEM

The dispersion equation worked out in the previous paragraph appears in matrix form as:

$$\left(S_1 c^2 + S_2 c + S_3\right)\hat{\boldsymbol{H}} = \boldsymbol{0} \tag{15}$$

By making the substitutions:

$$\boldsymbol{V_1} = \hat{\boldsymbol{H}} \tag{16}$$

$$\boldsymbol{V_2} = c\hat{\boldsymbol{H}} \tag{17}$$

the problem may be recast as:

$$c\boldsymbol{V_1} = \boldsymbol{V_2} \tag{18}$$

$$c\boldsymbol{V_2} = -S_1^{-1}S_3\boldsymbol{V_1} - S_1^{-1}S_2\boldsymbol{V_2} \tag{19}$$

Therefore we have:

$$c\begin{bmatrix}\boldsymbol{V_1}\\ \boldsymbol{V_2}\end{bmatrix} = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & -S_1^{-1}S_2 \end{bmatrix}\begin{bmatrix}\boldsymbol{V_1}\\ \boldsymbol{V_2}\end{bmatrix} \tag{20}$$

or in other form:

$$c\begin{bmatrix}\hat{\boldsymbol{H}}\\ c\hat{\boldsymbol{H}}\end{bmatrix} = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & -S_1^{-1}S_2 \end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{H}}\\ c\hat{\boldsymbol{H}}\end{bmatrix} \tag{21}$$

which is the well known form of an eigenvalue problem with Z and I being respectively the zero and identity matrixes of order $n-1$.

PROPERTIES OF THE EIGEN-VALUE PROBLEM

Key properties of the eigenvalue problem are here outlined through simple and well known results of which proof is not given.

Statement 1

Let complex matrices S_1, S_2 and S_3 be associated to the eigen-value problems:

$$c^+\begin{bmatrix}\hat{\boldsymbol{H}}^+\\ c^+\hat{\boldsymbol{H}}^+\end{bmatrix} = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & -S_1^{-1}S_2 \end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{H}}^+\\ c^+\hat{\boldsymbol{H}}^+\end{bmatrix} \tag{22}$$

$$c^-\begin{bmatrix}\hat{\boldsymbol{H}}^-\\ c^-\hat{\boldsymbol{H}}^-\end{bmatrix} = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & +S_1^{-1}S_2 \end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{H}}^-\\ c^-\hat{\boldsymbol{H}}^-\end{bmatrix} \tag{23}$$

If c^+ is an eigen-value and $\hat{\boldsymbol{H}}^+$ the corresponding eigen-vector for problem (22) then $c^- = -c^+$ is an eigen-value and $\hat{\boldsymbol{H}}^- = \pm\hat{\boldsymbol{H}}^+$ the corresponding eigen-vector for problem (23).

Statement 2

Let real matrices S_1, S_2 and S_3 be associated to the eigen-value problems:

$$c^+\begin{bmatrix}\hat{\boldsymbol{H}}^+\\ c^+\hat{\boldsymbol{H}}^+\end{bmatrix} = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & -S_1^{-1}S_2 \end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{H}}^+\\ c^+\hat{\boldsymbol{H}}^+\end{bmatrix} \tag{24}$$

$$c^-\begin{bmatrix}\hat{\boldsymbol{H}}^-\\ c^-\hat{\boldsymbol{H}}^-\end{bmatrix} = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & +S_1^{-1}S_2 \end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{H}}^-\\ c^-\hat{\boldsymbol{H}}^-\end{bmatrix} \tag{25}$$

If c^+ is an eigen-value and $\hat{\boldsymbol{H}}^+$ the corresponding eigen-vector for problem (24) then also its complex conjugate $\overline{c^+}$ is an eigen-value and its corresponding eigen-vector is $\overline{\hat{\boldsymbol{H}}^+}$.

If c^- is an eigen-value and $\hat{\boldsymbol{H}}^-$ the corresponding eigen-vector for problem (25) then also its complex conjugate $\overline{c^-}$ is an eigen-value and its corresponding eigen-vector is $\overline{\hat{\boldsymbol{H}}^-}$.

Statement 3

Let complex matrices S^+ and S^- be the stability matrices:

$$S^+ = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & -S_1^{-1}S_2 \end{bmatrix} \qquad (26)$$

$$S^- = \begin{bmatrix} Z & I \\ -S_1^{-1}S_3 & +S_1^{-1}S_2 \end{bmatrix} \qquad (27)$$

Since the block matrices, which the stability matrices are made of, are commutative, it follows that:

$$\det(S^+) = \det(S^-) = \det(S_1^{-1}S_3) = \frac{\det(S_3)}{\det(S_1)} \qquad (28)$$

CONCLUSIONS

In this paper a general mathematical model aiming at definition and computation of stability characteristics of stratified flows in horizontal or nearly horizontal pipes has been developed. The model stems from 1-D continuity and momentum equations in integral form treating an arbitrary number of fluids. It has been shown that the linearization of the governing equations around the steady state solutions and the assumption of a particular form of the perturbations yield an eigen-value problem, the roots of which represent complex wave speeds of the perturbations. The coefficients of the eigen-value problem have been derived for the most general case and given in analytical form. The effects of inertia of the fluids, gravity, interfacial tensions, shear stresses and wave length have been included explicitly in the stability coefficients. Some general properties of the eigen-value problem have been outlined.

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