COMPLEX GROUP ALGEBRAS OF THE DOUBLE COVERS OF THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract. We prove that the double covers of the alternating and symmetric groups are determined by their complex group algebras. To be more precise, let $n \geq 5$ be an integer, G a finite group, and let $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n^{\pm}$ denote the double covers of A_n and S_n, respectively. We prove that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$ if and only if $G \cong \hat{\mathsf{A}}_n$, and $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{+} \cong \mathbb{C}\hat{\mathsf{S}}_n^{-}$ if and only if $G \cong \hat{\mathsf{S}}_n^{+}$ or $\hat{\mathsf{S}}_n^{-}$. This in particular completes the proof of a conjecture proposed by the second and fourth authors that every finite quasi-simple group is determined uniquely up to isomorphism by the structure of its complex group algebra. The known results on prime power degrees and relatively small degrees of irreducible (linear and projective) representations of the symmetric and alternating groups together with the classification of finite simple groups play an essential role in the proofs.

1. Introduction

The complex group algebra of a finite group G, denoted by $\mathbb{C}G$, is the set of formal sums $\{\sum_{g\in G} a_g g \mid a_g \in \mathbb{C}\}$ equipped with natural rules for addition, multiplication, and scalar multiplication. Wedderburn's theorem implies that $\mathbb{C}G$ is isomorphic to the direct sum of matrix algebras over $\mathbb{C}$ whose dimensions are exactly the degrees of the (non-isomorphic) irreducible complex representations of G. Therefore, the study of complex group algebras and the relation to their base groups is important in group representation theory.

In an attempt to understand the connection between the structure of a finite group and its complex group algebra, R. Brauer asked in his landmark paper [3, Question 2]: *when do non-isomorphic groups have isomorphic complex group algebras*? Since this question might be too general to be solved completely, it is more feasible to study

Date: January 12, 2015.

2010 *Mathematics Subject Classification.* Primary 20C30, 20C15; Secondary 20C33.

Key words and phrases. Symmetric groups, alternating groups, complex group algebras, Schur covers, double covers, irreducible representations, character degrees.

H. N. Nguyen was partially supported by the NSA Young Investigator Grant #H98230-14-1-0293 and a Faculty Scholarship Award from Buchtel College of Arts and Sciences, The University of Akron. Tong-Viet's work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number 93408).

more explicit questions/problems whose solutions will provide a partial answer to Brauer's question. For instance, if two finite groups have isomorphic complex group algebras and one of them is solvable, is it true that the other is also solvable? Or, if two finite groups have isomorphic complex group algebras and one of them has a normal Sylow p-subgroup, can we conclude the same with the other group? We refer the reader to Brauer's paper [3] or Section 9 of the survey paper [23] by G. Navarro for more discussions on complex group algebras.

A natural problem that arises from Brauer's question is the following: given a finite group G, determine all finite groups (up to isomorphism) such that their complex group algebra is isomorphic to that of G. This problem is easy for abelian groups but difficult for solvable groups in general. If G is any finite abelian group of order n, then $\mathbb{C}G$ is isomorphic to a direct sum of n copies of $\mathbb{C}$, so that the complex group algebras of any two abelian groups are isomorphic if and only if the two groups have the same order. For solvable groups, the probability that two groups have isomorphic complex group algebra is often fairly 'high'. For instance, B. Huppert pointed out in [10] that among 2328 groups of order 2^7, there are only 30 different structures of complex group algebras. In contrast to solvable groups, simple groups or more generally quasi-simple groups seem to have a stronger connection with their complex group algebras. In [26], two of the four authors have conjectured that every finite quasi-simple group is determined uniquely up to isomorphism by its complex group algebra, and proved it for all quasi-simple groups, except the non-trivial perfect central covers of the alternating groups.

Let A_n denote the alternating group of degree n. (Throughout the paper we always assume that $n \geq 5$ unless otherwise stated.) The *Schur covers* (or *covering groups*) of the alternating groups as well as symmetric groups were first studied and classified by I. Schur [29] in connection with their projective representations. It is known that A_n has one isomorphism class of Schur covers, which is indeed the double cover $\hat{\mathsf{A}}_n$ except when n is 6 or 7, where triple and 6-fold covers also exist. We are able to prove that every double cover of an alternating group of degree at least 5 is determined uniquely by its complex group algebra.

Theorem A. *Let $n \geq 5$. Let G be a finite group and $\hat{\mathsf{A}}_n$ the double cover of A_n. Then $G \cong \hat{\mathsf{A}}_n$ if and only if $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$.*

We prove a similar result for the triple and 6-fold (perfect central) covers of A_6 and A_7, and therefore complete the proof of the aforementioned conjecture.

Theorem B. *Let G be a finite group and H a quasi-simple group. Then $G \cong H$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.*

Proof. This is a consequence of Theorem A, Theorem 6.2, and [26, Corollary 1.4]. □

The symmetric group S_n has two isomorphism classes of Schur double covers, denoted by $\hat{\mathsf{S}}_n^-$ and $\hat{\mathsf{S}}_n^+$. It turns out that these two covers are isoclinic and therefore their complex group algebras $\mathbb{C}\hat{\mathsf{S}}_n^+$ and $\mathbb{C}\hat{\mathsf{S}}_n^-$ are isomorphic [21]. Our next result solves the above problem for the double covers of the symmetric groups.

Theorem C. *Let $n \geq 5$. Let G be a finite group and $\hat{\mathsf{S}}_n^{\pm}$ the double covers of S_n. Then $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^+$ (or equivalently $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^-$) if and only if $G \cong \hat{\mathsf{S}}_n^+$ or $G \cong \hat{\mathsf{S}}_n^-$.*

Let $\mathrm{Irr}(G)$ denote the set of all irreducible representations (or characters) of a group G over the complex field. As mentioned above, two finite groups have isomorphic complex group algebras if and only if they have the same set of degrees (counting multiplicities) of irreducible characters. Therefore, the proofs of our main results as expected depend heavily on the representation theories of the symmetric and alternating groups, their double covers, and quasi-simple groups in general. In particular, we make use of the known results on relatively small degrees and prime power degrees of the irreducible characters of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n^{\pm}$.

The remainder of the paper is organized as follows. In the next section, we give a brief overview of the representation theory of the symmetric and alternating groups as well as their double covers, and then collect some results on prime power character degrees of these groups. The results on minimal degrees are then presented in Section 3. In Section 4, we establish some useful lemmas that will be needed later in the proofs of the main results. The proof of Theorem A is carried out in Section 5 and exceptional covers of A_6 and A_7 are treated in Section 6.

The last four sections are devoted to the proof of Theorem C. Let G be a finite group such that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{\pm}$. We will show that $G' = G''$ and therefore there exists a normal subgroup M of G such that $M \subseteq G'$ and the chief factor G'/M is isomorphic to a direct product of k copies of a non-abelian simple group S. To prove that G is isomorphic to one of $\hat{\mathsf{S}}_n^{\pm}$, one of the key steps is to show that this chief factor is isomorphic to A_n. We will do this by using the classification of finite simple groups to eliminate almost all possibilities for k and S. As we will see, it turns out that the case of simple groups of Lie type in even characteristic is most difficult.

In Section 7, we prove a non-existence result for particular character degrees of $\hat{\mathsf{S}}_n^{\pm}$ and apply it in Section 8 to show that S cannot be a simple group of Lie type in even characteristic. We then eliminate other possibilities for S in Section 9 and complete the proof of Theorem C in Section 10.

We close this introduction with many thanks to the referee for his/her careful reading of the manuscript and several helpful comments.

Notation. Since $\mathbb{C}\hat{\mathsf{S}}_n^+ \cong \mathbb{C}\hat{\mathsf{S}}_n^-$, when working with character degrees of $\hat{\mathsf{S}}_n^{\pm}$, it suffices to consider just one of the two covers. For the sake of convenience, we will write $\hat{\mathsf{S}}_n$ to denote either one of the two double covers of S_n. If X and Y are two multisets, we write $X \subseteq Y$ and say that X is a sub-multiset of Y if the multiplicity of any element in X does not exceed that of the same element in Y. For a finite group G, the number of conjugacy classes of G is denoted by $k(G)$. We write $\mathrm{Irr}_{2'}(G)$ to mean the set of all irreducible characters of G of odd degree. The set and the multiset of character degrees of G are denoted respectively by $\mathrm{cd}(G)$ and $\mathrm{cd}^*(G)$. Finally, we denote by $d_i(G)$ the ith smallest non-trivial character degree of G. Other notation is standard or will be defined when needed.

2. Prime power degrees of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$

In this section, we collect some results on irreducible characters of prime power degree of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$. The irreducible characters of the double covers of the symmetric and alternating groups are divided into two kinds: faithful characters which are also known as spin characters, and non-faithful characters which can be viewed as ordinary characters of S_n or A_n.

2.1. **Characters of S_n and A_n.** To begin with, let us recall some notation and terminology of partitions and Young diagrams in connection with representation theory of the symmetric and alternating groups. A partition λ of n is a finite sequence of natural numbers $(\lambda_1, \lambda_2, ...)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = n$. If $\lambda_1 > \lambda_2 > \cdots$, we say that λ is a *bar partition* of n (also called a *strict partition* of n). The Young diagram associated to λ is an array of n nodes with λ_i nodes on the ith row. At each node (i, j), we define the *hook length* $h(i, j)$ to be the number of nodes to the right and below the node (i, j), including the node (i, j).

It is well known that the irreducible characters of S_n are in one-to-one correspondence with partitions of n. The degree of the character χ_λ corresponding to λ is given by the following *hook-length formula* of Frame, Robinson and Thrall [6]:

$$f_\lambda := \chi_\lambda(1) = \frac{n!}{\prod_{i,j} h(i,j)} .$$

Two partitions of n whose Young diagrams transform into each other when reflected about the line $y = -x$, with the coordinates of the upper left node taken as $(0, 0)$, are called conjugate partitions. The partition conjugate to λ is denoted by $\overline{\lambda}$. If $\lambda = \overline{\lambda}$, we say that λ is self-conjugate. The irreducible characters of A_n can be obtained by restricting those of S_n to A_n. More explicitly, $\chi_\lambda \downarrow_{\mathsf{A}_n} = \chi_{\overline{\lambda}} \downarrow_{\mathsf{A}_n}$ is irreducible if λ is not self-conjugate. Otherwise, $\chi_\lambda \downarrow_{\mathsf{A}_n}$ is the sum of two different irreducible characters of A_n of the same degree. In short, the degrees of irreducible characters of A_n are labeled by partitions of n and are given by

$$\widetilde{f_\lambda} = \begin{cases} f_\lambda & \text{if } \lambda \neq \overline{\lambda}, \\ f_\lambda/2 & \text{if } \lambda = \overline{\lambda}. \end{cases}$$

Irreducible representations of prime-power degree of the symmetric and alternating groups were classified by A. Balog, C. Bessenrodt, J. B. Olsson and K. Ono, see [1]. This result is critical in eliminating simple groups other than A_n involved in the structure of finite groups whose complex group algebras are isomorphic to $\mathbb{C}\hat{\mathsf{A}}_n$ or $\mathbb{C}\hat{\mathsf{S}}_n$.

Lemma 2.1 ([1], Theorem 2.4)**.** *Let $n \geq 5$. An irreducible character $\chi_\lambda \in \mathrm{Irr}(\mathsf{S}_n)$ corresponding to a partition λ of n has prime power degree $f_\lambda = p^r > 1$ if and only if one of the following occurs:*

(1) *$n = p^r + 1$, $\lambda = (n-1, 1)$ or $(2, 1^{n-2})$, and $f_\lambda = n - 1$;*
(2) *$n = 5$, $\lambda = (2^2, 1)$ or $(3, 2)$, and $f_\lambda = 5$;*
(3) *$n = 6$, $\lambda = (4, 2)$ or $(2^2, 1^2)$, and $f_\lambda = 9$; $\lambda = (3^2)$ or (2^3), and $f_\lambda = 5$; $\lambda = (3, 2, 1)$ and $f_\lambda = 16$;*
(4) *$n = 8$, $\lambda = (5, 2, 1)$ or $(3, 2, 1^3)$, and $f_\lambda = 64$;*
(5) *$n = 9$, $\lambda = (7, 2)$ or $(2^2, 1^5)$, and $f_\lambda = 27$.*

Lemma 2.2 ([1], Theorem 5.1)**.** *Let $n \geq 5$. An irreducible character degree $\widetilde{f_\lambda}$ of A_n corresponding to a partition λ of n is a prime power $p^r > 1$ if and only if one of the following occurs:*

(1) *$n = p^r + 1$, $\lambda = (n-1, 1)$ or $(2, 1^{n-2})$, and $\widetilde{f_\lambda} = n - 1$;*
(2) *$n = 5$, $\lambda = (2^2, 1)$ or $(3, 2)$, and $\widetilde{f_\lambda} = 5$; $\lambda = (3, 1^2)$ and $\widetilde{f_\lambda} = 3$;*
(3) *$n = 6$, $\lambda = (4, 2)$ or $(2^2, 1^2)$ and $\widetilde{f_\lambda} = 9$; $\lambda = (3^2)$ or (2^3) and $\widetilde{f_\lambda} = 5$; $\lambda = (3, 2, 1)$ and $\widetilde{f_\lambda} = 8$;*
(4) *$n = 8$, $\lambda = (5, 2, 1)$ or $(3, 2, 1^3)$, and $\widetilde{f_\lambda} = 64$;*
(5) *$n = 9$, $\lambda = (7, 2)$ or $(2^2, 1^5)$, and $\widetilde{f_\lambda} = 27$.*

2.2. **Spin characters of S_n and A_n.** We now recall the spin representation theory of the symmetric and alternating groups, due to Schur [9, 21, 29, 33]. To each bar partition $\mu = (\mu_1, \mu_2, ..., \mu_m)$ (i.e., $\mu_1 > \mu_2 > \cdots > \mu_m$) of n, there corresponds one or two irreducible characters (depending on whether $n - m$ is even or odd, respectively) of $\hat{\mathsf{S}}_n$ of degree

$$g_\mu = 2^{\lfloor (n-m)/2 \rfloor} \bar{g}_\mu$$

where $\bar{g}_\mu$ denotes the number of shifted standard tableaux of shape μ. This number can be computed by an analogue of the hook-length formula, the *bar formula* [9, Prop. 10.6]. The length $b(i, j)$ of the (i, j)-bar of μ is the length of the $(i, j+1)$-hook in the shift-symmetric diagram of μ (obtained by reflecting the shifted diagram of μ along the diagonal and pasting it onto μ; see [18, p.14] for details). Then

$$\bar{g}_\mu = \frac{n!}{\prod_{i,j} b(i, j)} .$$

The spin character degree may also be computed by the formula

$$g_\mu = 2^{\lfloor (n-m)/2 \rfloor} \frac{n!}{\mu_1! \mu_2! \cdots \mu_m!} \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j}.$$

Again, one can get faithful irreducible characters of $\hat{\mathsf{A}}_n$ by restricting those of $\hat{\mathsf{S}}_n^{\pm}$ to $\hat{\mathsf{A}}_n$ in the following way. If $n-m$ is odd, then the restrictions of the two characters of $\hat{\mathsf{S}}_n^{\pm}$ labeled by μ to $\hat{\mathsf{A}}_n$ are the same and irreducible. Otherwise, the restriction of the one character labeled by μ is the sum of two irreducible characters of the same degree $g_\mu/2$. Let $\widetilde{g}_\mu$ be the degree of the irreducible spin character(s) of $\hat{\mathsf{A}}_n$ labeled by the bar partition μ; we then have

$$\widetilde{g}_\mu = \begin{cases} g_\mu & \text{if } n-m \text{ is odd,} \\ g_\mu/2 & \text{if } n-m \text{ is even.} \end{cases}$$

The classification of spin representations of prime power degree of the symmetric and alternating groups has been done by the first and third authors of this paper in [2].

Lemma 2.3 ([2], Theorem 4.2)**.** *Let $n \geq 5$ and μ be a bar partition of n. The spin irreducible character degree g_μ of $\hat{\mathsf{S}}_n$ corresponding to μ is a prime power if and only if one of the following occurs:*

(1) $\mu = (n)$ *and* $g_\mu = 2^{\lfloor (n-1)/2 \rfloor}$*;*
(2) $n = 2^r + 2$ *for some* $r \in \mathbb{N}$*,* $\mu = (n-1, 1)$*, and* $g_\mu = 2^{2^{r-1}+r}$*;*
(3) $n = 5$*,* $\mu = (3, 2)$*, and* $g_\mu = 4$*;*
(4) $n = 6$*,* $\mu = (3, 2, 1)$*, and* $g_\mu = 4$*;*
(5) $n = 8$*,* $\mu = (5, 2, 1)$*, and* $g_\mu = 64$*.*

Lemma 2.4 ([2], Theorem 4.3)**.** *Let $n \geq 5$ and μ be a bar partition of n. The spin irreducible character degree $\widetilde{g}_\mu$ of $\hat{\mathsf{A}}_n$ corresponding to μ is a prime power if and only if one of the following occurs:*

(1) $\mu = (n)$ *and* $\widetilde{g}_\mu = 2^{\lfloor (n-2)/2 \rfloor}$*;*
(2) $n = 2^r + 2$ *for some* $r \in \mathbb{N}$*,* $\mu = (n-1, 1)$*, and* $\widetilde{g}_\mu = 2^{2^{r-1}+r-1}$*;*
(3) $n = 5$*,* $\mu = (3, 2)$*, and* $\widetilde{g}_\mu = 4$*;*
(4) $n = 6$*,* $\mu = (3, 2, 1)$*, and* $\widetilde{g}_\mu = 4$*;*
(5) $n = 8$*,* $\mu = (5, 2, 1)$*, and* $\widetilde{g}_\mu = 64$*.*

3. Low degrees of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$

We present in this section some results on minimal degrees of both ordinary and spin characters of the symmetric and alternating groups. We start with ordinary characters.

Lemma 3.1 (Rasala [28])**.** *The following hold:*

(1) $d_1(\mathsf{S}_n) = n - 1$ *if* $n \geq 5$*;*
(2) $d_2(\mathsf{S}_n) = n(n-3)/2$ *if* $n \geq 9$*;*
(3) $d_3(\mathsf{S}_n) = (n-1)(n-2)/2$ *if* $n \geq 9$*;*
(4) $d_4(\mathsf{S}_n) = n(n-1)(n-5)/6$ *if* $n \geq 13$*;*

(5) $d_5(\mathsf{S}_n) = (n-1)(n-2)(n-3)/6$ *if* $n \geq 13$*;*
(6) $d_6(\mathsf{S}_n) = n(n-2)(n-4)/3$ *if* $n \geq 15$*;*
(7) $d_7(\mathsf{S}_n) = n(n-1)(n-2)(n-7)/24$ *if* $n \geq 15$.

Lemma 3.2 (Tong-Viet [31])**.** *If* $n \geq 15$ *then* $d_i(\mathsf{A}_n) = d_i(\mathsf{S}_n)$ *for* $1 \leq i \leq 4$ *and, if* $n \geq 22$*, then* $d_i(\mathsf{A}_n) = d_i(\mathsf{S}_n)$ *for* $1 \leq i \leq 7$.

The minimal degrees of spin irreducible representations of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n$ were obtained by A. Kleshchev and P. H. Tiep, see [13, 14]. These minimal degrees are indeed the degrees of the *basic spin* and *second basic spin* representations. Let $\mathfrak{d}_1(\hat{\mathsf{A}}_n)$ and $\mathfrak{d}_1(\hat{\mathsf{S}}_n)$ denote the smallest degrees of irreducible spin characters of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n$, respectively.

Lemma 3.3 ([13], Theorem A)**.** *Let* $n \geq 8$*. The smallest degree of the irreducible spin characters of* $\hat{\mathsf{A}}_n$ *(*$\hat{\mathsf{S}}_n$*, resp.) is* $\mathfrak{d}_1(\hat{\mathsf{A}}_n) = 2^{\lfloor (n-2)/2 \rfloor}$ *(*$\mathfrak{d}_1(\hat{\mathsf{S}}_n) = 2^{\lfloor (n-1)/2 \rfloor}$*, resp.) and there is no degree between* $\mathfrak{d}_1(\hat{\mathsf{A}}_n)$ *(*$\mathfrak{d}_1(\hat{\mathsf{S}}_n)$*, resp.) and* $2\mathfrak{d}_1(\hat{\mathsf{A}}_n)$ *(*$2\mathfrak{d}_1(\hat{\mathsf{S}}_n)$*, resp.).*

Using the above results, we easily deduce the following.

Lemma 3.4. *The following hold:*

(1) *If* $n \geq 31$*, then* $d_i(\hat{\mathsf{S}}_n) = d_i(\mathsf{S}_n)$ *for* $1 \leq i \leq 7$.
(2) *If* $n \geq 34$*, then* $d_i(\hat{\mathsf{A}}_n) = d_i(\mathsf{A}_n)$ *for* $1 \leq i \leq 7$.

Proof. We observe that $\mathfrak{d}_1(\hat{\mathsf{S}}_n) = 2^{\lfloor (n-1)/2 \rfloor} > n(n-1)(n-2)(n-7)/24 = d_7(\mathsf{S}_n)$ if $n \geq 31$. Therefore part (1) follows by Lemmas 3.1 and 3.3. Similarly, we have $\mathfrak{d}_1(\hat{\mathsf{A}}_n) = 2^{\lfloor (n-2)/2 \rfloor} > n(n-1)(n-2)(n-7)/24 = d_7(\mathsf{A}_n)$ if $n \geq 34$ and thus part (2) follows. □

Lemma 3.5. *Let* G *be either* $\hat{\mathsf{A}}_n$ *or* $\hat{\mathsf{S}}_n$*. Then we have:*

(1) *If* $n \geq 8$*, then* $d_1(G) = n-1$.
(2) *If* $n \geq 10$*, then* $d_2(G) = \min\{n(n-3)/2, \mathfrak{d}_1(G)\}$*. Furthermore, if* $n \geq 12$*, then* $d_2(G) > 2n$.
(3) *If* $n \geq 16$*, then* $d_3(G) = (n-1)(n-2)/2$ *and* $d_4(G) = \min\{n(n-1)(n-5)/6, \mathfrak{d}_1(G)\}$.
(4) *If* $n \geq 28$*, then* $d_4(G) = n(n-1)(n-5)/6$*,* $d_5(G) = (n-1)(n-2)(n-3)/6$*,* $d_6(G) = n(n-2)(n-4)/3$*, and* $d_7(G) = \min\{n(n-1)(n-2)(n-7)/24, \mathfrak{d}_1(G)\}$.

Proof. When $n \geq 8$, we see that $\mathfrak{d}_1(\hat{\mathsf{S}}_n) \geq \mathfrak{d}_1(\hat{\mathsf{A}}_n) = 2^{\lfloor (n-2)/2 \rfloor} > n-1$ and $d_1(\mathsf{S}_n) = d_1(\mathsf{A}_n) = n-1$, which imply that $d_1(\hat{\mathsf{A}}_n) = d_1(\hat{\mathsf{S}}_n) = n-1$ and part (1) follows. When $n \geq 10$, we observe that $d_2(\mathsf{A}_n) = d_2(\mathsf{S}_n) = n(n-3)/2$ and part (2) then follows from part (1).

Suppose that $n \geq 16$. It is easy to check that $2^{\lfloor (n-2)/2 \rfloor} > (n-1)(n-2)/2 = d_3(\mathsf{A}_n) = d_3(\mathsf{S}_n)$. It follows that $d_3(\hat{\mathsf{A}}_n) = d_3(\hat{\mathsf{S}}_n) = (n-1)(n-2)/2$ and $d_4(G) = \min\{n(n-1)(n-5)/6, \mathfrak{d}_1(G)\}$, as claimed.

Finally suppose that $n \geq 28$. We check that $2^{\lfloor (n-2)/2 \rfloor} > n(n-2)(n-4)/3 = d_6(\mathsf{A}_n) = d_6(\mathsf{S}_n)$ and part (4) then follows. □

4. Some useful lemmas

We begin with an easy observation.

Lemma 4.1. *There always exists a prime number p with $n < p \leq d_2(G)$ for $G = \hat{\mathsf{A}}_n$ or $\hat{\mathsf{S}}_n$, provided that $n \geq 9$.*

Proof. It is routine to check the statement for $9 \leq n \leq 11$ by using [5]. So we can assume that $n \geq 12$. By Lemma 3.5, it suffices to prove that there exists a prime between n and $2n$. However, this is the well-known Bertrand-Chebyshev theorem [8]. □

The next three lemmas are critical in the proof of Theorem A.

Lemma 4.2. *Let S be a simple group of Lie type of rank l defined over a field of q elements with q even. Then*

$$|\mathrm{Irr}_{2'}(S)| \geq \begin{cases} q^l/(l+1, q-1) & \text{if } S \text{ is of type } A, \\ q^l/(l+1, q+1) & \text{if } S \text{ is of type } {}^2A, \\ q^l/3 & \text{if } S \text{ is of type } E_6, {}^2E_6, \\ q^l & \text{otherwise.} \end{cases}$$

Proof. Let S_{sc} be the finite Lie type group of simply-connected type corresponding to S. By [4, Corollary 3.6], S_{sc} has q^l semisimple conjugacy classes. To each semisimple class s of S_{sc}, Lusztig's classification of complex characters of finite groups of Lie type says that there corresponds a semisimple character of the dual group, say S_{sc}^*, of S_{sc} of degree

$$|S_{sc}|_{2'}/|\mathbf{C}_{S_{sc}}(s)|_{2'} .$$

This means that the dual S_{sc}^* of S_{sc} has at least q^l irreducible characters of odd degree.

If S is of type A, we have $S_{sc}^* = \mathrm{PGL}_{l+1}(q) = S.(l+1, q-1)$ and the lemma follows for linear groups. A similar argument works for unitary groups and E_6 as well as ${}^2E_6$. If S is not of these types, we will have $S = S_{sc} = S_{sc}^*$ and the lemma also follows. □

Lemma 4.3. *Let $n \geq 5$ and let $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$ be the binary expansion of n with $k_1 > k_2 > \cdots > k_t \geq 0$. Then*

$$|\mathrm{Irr}_{2'}(\hat{\mathsf{A}}_n)| \leq |\mathrm{Irr}_{2'}(\hat{\mathsf{S}}_n)| = 2^{k_1+k_2+\cdots+k_t} .$$

Proof. If $\mu = (\mu_1, \mu_2, ..., \mu_m)$ is a bar partition of n, then the 2-part of the spin character degree g_μ of $\hat{\mathsf{S}}_n$ labeled by μ is at least

$$2^{\lfloor (n-m)/2 \rfloor},$$

which is at least 2 as $n \geq 5$ and $n - m \geq 3$. We note that if $n - m = 3$ then $\widetilde{g}_\mu = g_\mu$. Therefore, the 2-part of the degree $\widetilde{g}_\mu$ of $\hat{\mathsf{A}}_n$ is at least 2 as well. In particular, we see that every spin character degree of $\hat{\mathsf{A}}_n$ as well as $\hat{\mathsf{S}}_n$ is even. It follows that

$$|\mathrm{Irr}_{2'}(\hat{\mathsf{A}}_n)| = |\mathrm{Irr}_{2'}(\mathsf{A}_n)| \text{ and } |\mathrm{Irr}_{2'}(\hat{\mathsf{S}}_n)| = |\mathrm{Irr}_{2'}(\mathsf{S}_n)| .$$

As mentioned in [20], the number of odd degree irreducible characters of A_n does not exceed that of S_n. Now the lemma follows from the formula for the number of odd degree characters of S_n given in [17, Corollary 1.3]. □

Lemma 4.4. *Let $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$ be the binary expansion of n with $k_1 > k_2 > \cdots > k_t \geq 0$.*

1. *If $k_1 + k_2 + \cdots + k_t \geq \sqrt{(n-3)/2}$, then $n < 2^{15}$;*
2. *if $k_1 + k_2 + \cdots + k_t \geq \sqrt{n-3} - 3$, then $n < 2^{13}$;*
3. *if $k_1 + k_2 + \cdots + k_t \geq (n-3)/18$, then $n < 2^{10}$;*
4. *if $k_1 + k_2 + \cdots + k_t \geq (n-3)/30$, then $n < 2^{11}$.*

Proof. We only give here a proof for part (2). The other statements are proved similarly. As $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$, we get $k_1 = \lfloor \log_2 n \rfloor$ and hence

$$k_1 + k_2 + \cdots + k_t \geq \lfloor \log_2 n \rfloor (\lfloor \log_2 n \rfloor + 1)/2 .$$

However, it is easy to check that $\sqrt{(n-3)} - 3 > \lfloor \log_2 n \rfloor (\lfloor \log_2 n \rfloor + 1)/2$ if $n \geq 2^{14}$. For $2^{13} \leq n < 2^{14}$, the statement follows by direct computations. □

The following lemma is probably known but we include a short proof for the reader's convenience. It will be needed in the proof of Theorem C.

Lemma 4.5. *Let $k(\hat{\mathsf{A}}_n)$ and $k(\hat{\mathsf{S}}_n)$ denote the number of conjugacy classes of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n$, respectively. Then*

$$k(\hat{\mathsf{S}}_n) < 2k(\hat{\mathsf{A}}_n) .$$

Proof. Let $\mathrm{Irr}_{\text{faithful}}(G)$ and $\mathrm{Irr}_{\text{non-faithful}}(G)$ denote the sets of faithful irreducible characters and non-faithful irreducible characters, respectively, of a group G. Let a and b be the numbers of self-conjugate partitions and of pairs of non-self-conjugate partitions, respectively, of n. Also, let c and d be the numbers of bar partitions of n with $n - m$ even and odd, respectively. We have

$$|\mathrm{Irr}_{\text{non-faithful}}(\hat{\mathsf{S}}_n)| = k(\mathsf{S}_n) = a + 2b \text{ and } |\mathrm{Irr}_{\text{faithful}}(\hat{\mathsf{S}}_n)| = c + 2d$$

and

$$|\mathrm{Irr}_{\text{non-faithful}}(\hat{\mathsf{A}}_n)| = k(\mathsf{A}_n) = 2a + b \text{ and } |\mathrm{Irr}_{\text{faithful}}(\hat{\mathsf{A}}_n)| = 2c + d .$$

Therefore,

$$k(\hat{\mathsf{S}}_n) = a + 2b + c + 2d \text{ and } k(\hat{\mathsf{A}}_n) = 2a + b + 2c + d$$

and the lemma follows. □

5. Complex group algebra of $\hat{\mathsf{A}}_n$ - Theorem A

The aim of this section is to prove Theorem A. Let G be a finite group such that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$. Then G has exactly one linear character, which is the trivial one, so that G is perfect. Let M be a maximal normal subgroup of G. We then have that G/M is non-abelian simple and moreover

$$\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{A}}_n).$$

To prove the theorem, it is clear that we first have to show $G/M \cong \mathsf{A}_n$. We will work towards this aim.

Proposition 5.1. *Let S be a non-abelian simple group such that $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$. Then S is isomorphic to A_n or to a simple group of Lie type in even characteristic.*

Proof. We will eliminate other possibilities for S by using the classification of finite simple groups. If $5 \leq n \leq 9$, then the set of prime divisors of S is contained in that of $\hat{\mathsf{A}}_n$, which in turn is contained in $\{2, 3, 5, 7\}$, hence by using [11, Theorem III] and [5], the result follows easily. From now on we assume that $n \geq 10$.

(i) Alternating groups: Suppose that $S = \mathsf{A}_m$ with $5 \leq m \neq n$. Since $\mathrm{cd}(A_m) \subseteq \mathrm{cd}(\hat{\mathsf{A}}_n)$, we get $d_1(\mathsf{A}_m) \geq d_1(\hat{\mathsf{A}}_n)$. As $d_1(\hat{\mathsf{A}}_n) = n-1 \geq 9$ by Lemma 3.5, it follows that $d_1(\mathsf{A}_m) \geq 9$. Thus $m \geq 10$, and so $m - 1 = d_1(\mathsf{A}_m) \geq d_1(\hat{\mathsf{A}}_n) = n - 1$. In particular, we have $m > n$ as $m \neq n$. It follows that $|S| > 2|\mathsf{A}_n|$ and this violates the hypothesis that $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$.

(ii) Simple groups of Lie type in odd characteristic: Suppose that $S = G(p^k)$, a simple group of Lie type defined over a field of p^k elements with p odd. Since $|S|_p$ is the degree of the Steinberg character of S, we have $|S|_p \in \mathrm{cd}(\hat{\mathsf{A}}_n)$. As $|S|_p$ is an odd prime power, Lemma 2.4 implies that $|S|_p$ must be the degree of a non-faithful character of $\hat{\mathsf{A}}_n$. In other words, $|S|_p \in \mathrm{cd}(\mathsf{A}_n)$. Using Lemma 2.2, we deduce that $|S|_p = n - 1$. Hence, $|S|_p = d_1(\hat{\mathsf{A}}_n)$ is the smallest non-trivial degree of $\hat{\mathsf{A}}_n$ by Lemma 3.5. However, by [32, Lemma 8] we have $d_1(S) < |S|_p = d_1(\hat{\mathsf{A}}_n)$, which is impossible as $\mathrm{cd}(S) \subseteq \mathrm{cd}(\hat{\mathsf{A}}_n)$.

(iii) Sporadic simple groups and the Tits group: Using [7], we can assume that $n \geq 14$. To eliminate these cases, observe that $n \geq \max\{p(S), 14\}$, where $p(S)$ is the largest prime divisor of $|S|$, and that $d_i(S) \geq d_i(\hat{\mathsf{A}}_n)$ for all $i \geq 1$. With this lower bound on n, we find the lower bounds for $d_i(\hat{\mathsf{A}}_n)$ with $1 \leq i \leq 7$ using Lemmas 3.4 and 3.5. Choose $i \in \{2, 3, \cdots, 7\}$ such that $d_i(\hat{\mathsf{A}}_n) > d_j(S)$ for some $j \geq 1$ such that $|i - j|$ is minimal. If $j \geq i$, then we obtain a contradiction. If $j < i$, then $d_j(S) \in \{d_k(\hat{\mathsf{A}}_n)\}_{k=j}^{i-1}$. Solving these equations for n, we then obtain that either these equations have no solution, or that, for each solution of n, we can find some $k \geq 1$ with $d_k(\hat{\mathsf{A}}_n) > d_k(S)$. As an example, assume that $S = O'N$. Then

$n \geq 31$ since $p(S) = 31$. We have $d_7(\hat{\mathsf{A}}_n) = n(n-1)(n-2)(n-7)/24 \geq 26970$. As $d_4(S) = 26752 < d_7(\hat{\mathsf{A}}_n)$, it follows that $d_4(S) \in \{d_4(\hat{\mathsf{A}}_n), d_5(\hat{\mathsf{A}}_n), d_6(\hat{\mathsf{A}}_n)\}$. However, one can check that these equations have no integer solutions. ☐

Proposition 5.2. *Let S be a non-abelian simple group such that $|S| \mid n!$ and $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$. Then $S \cong \mathsf{A}_n$.*

Proof. In light of Proposition 5.1 and its proof, it remains to assume that $n \geq 9$ and to prove that S cannot be a simple group of Lie type in even characteristic. Assume to the contrary that $S = G_l(2^k)$, a simple group of Lie type of rank l defined over a field of $q = 2^k$ elements. As above, we then have $|S|_2 \in \mathrm{cd}(\hat{\mathsf{A}}_n)$. By Lemmas 2.2 and 2.4, we have that $|S|_2 = n-1$, $|S|_2 = 2^{\lfloor (n-2)/2 \rfloor}$, or $|S|_2 = 2^{n/2+\log_2(n-2)-2}$ when $n = 2^r + 2$. Since the case $|S|_2 = n-1$ can be eliminated as in the proof of the previous proposition, we can assume further that

$$|S|_2 = 2^{\lfloor (n-2)/2 \rfloor} \text{ or } |S|_2 = 2^{n/2+\log_2(n-2)-2} \text{ when } n = 2^r + 2\,.$$

Recalling the hypothesis that $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$, we have

$$|\mathrm{Irr}_{2'}(S)| \leq |\mathrm{Irr}_{2'}(\hat{\mathsf{A}}_n)|. \tag{5.1}$$

(i) $S = B_l(2^k) \cong C_l(2^k)$, $D_l(2^k)$, or ${}^2D_l(2^k)$. Then $|S|_2 = 2^{kl^2}$ or $2^{kl(l-1)}$. In particular,

$$|S|_2 \leq 2^{kl^2}\,.$$

As $|S|_2 \geq 2^{\lfloor (n-2)/2 \rfloor}$, it follows that

$$kl^2 \geq \lfloor (n-2)/2 \rfloor \geq (n-3)/2\,.$$

Therefore

$$kl \geq \sqrt{(n-3)/2}\,.$$

Using Lemma 4.2, we then obtain

$$|\mathrm{Irr}_{2'}(S)| \geq q^l = 2^{kl} \geq 2^{\sqrt{(n-3)/2}}\,.$$

Now Lemma 4.3 and inequality (5.1) imply

$$k_1 + k_2 + \cdots + k_t \geq \sqrt{(n-3)/2},$$

where $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$ is the binary expansion of n. Invoking Lemma 4.4(1), we obtain that $n < 2^{15}$.

(ii) $S = A_l(2^k)$ or ${}^2A_l(2^k)$. Arguing as above, we have

$$k_1 + k_2 + \cdots + k_t \geq \sqrt{n-3} - 3,$$

which forces $n < 2^{13}$ by Lemma 4.4(2).

(iii) S is a simple group of exceptional Lie type. Using Lemma 4.4(3,4), we deduce that $n < 2^{11}$.

For each of the above cases, a computer program has checked that either $|S|$ does not divide $n!$ or S has an irreducible character degree not belonging to $\mathrm{cd}(\hat{\mathsf{A}}_n)$ for "small" n. This contradiction completes the proof. Let us describe the example where $S = \mathrm{Sp}_{2l}(2^k) \cong \Omega_{2l+1}(2^k)$. Then we have

$$kl^2 = \lfloor (n-2)/2 \rfloor \text{ or } kl^2 = n/2 + \log_2(n-2) - 2 \text{ when } n = 2^r + 2.$$

Moreover, the condition $|S| \mid n!$ is equivalent to

$$2^{kl^2} \prod_{i=1}^{l} (2^{2ki} - 1) \mid n!.$$

By computer computations, we can determine all triples (k, l, n) with $n < 2^{15}$ satisfying the above conditions. It turns out that, for each such triple, n is at most 170 and one of the three smallest character degrees of S is not a character degree of $\hat{\mathsf{A}}_n$. The low-degree characters of simple groups of Lie type can be found in [30, 15, 24]. □

We are now ready to prove the first main result.

***Proof of Theorem* A.** Recall the hypothesis that G is a finite group such that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$. Therefore $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{A}}_n)$. In particular, we have $|G| = |\hat{\mathsf{A}}_n|$ and $G = G'$ since $\hat{\mathsf{A}}_n$ has only one linear character. Let M be a maximal normal subgroup of G. Then G/M is a non-abelian simple group, say S. It follows that $\mathrm{cd}^*(S) = \mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(G)$ and hence

$$\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)\,.$$

We also have

$$|S| \mid |G| = |\hat{\mathsf{A}}_n| = n!.$$

Applying Propositions 5.1 and 5.2, we deduce that $S \cong \mathsf{A}_n$.

We have shown that $G/M \cong \mathsf{A}_n$. Since $|G| = |\hat{\mathsf{A}}_n| = 2|\mathsf{A}_n|$, we obtain $|M| = 2$. In particular, M is central in G and therefore $M \subseteq \mathbf{Z}(G) \cap G'$. Thus $G \cong \hat{\mathsf{A}}_n$, as desired. □

6. Triple and 6-fold covers of A_6 and A_7

In this section, we aim to prove that every perfect central cover of A_6 or A_7 is uniquely determined up to isomorphism by the structure of its complex group algebra. To do that, we need the following result from [26, Lemma 2.5]. Here and in what follows, we write $\mathrm{Mult}(S)$ and $\mathrm{Schur}(S)$ to denote the Schur multiplier and the Schur covering group (or the Schur cover for short), respectively, of a simple group S.

Lemma 6.1. *Let S be a non-abelian simple group different from an alternating group of degree greater than* 13*. Assume that S is different from* $\mathrm{PSL}_3(4)$ *and* $\mathrm{PSU}_4(3)$*. Let G be a perfect group and $M \lhd G$ such that $G/M \cong S$, $|M| \le |\mathrm{Mult}(S)|$, and*

$\mathrm{cd}(G) \subseteq \mathrm{cd}(\mathrm{Schur}(S))$. *Then G is uniquely determined up to isomorphism by S and the order of G.*

Now we prove the main result of this section.

Theorem 6.2. *Let G be a finite group and H a perfect central cover of A_6 or A_7. Then $G \cong H$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.*

Proof. First, as $\mathsf{A}_6 \cong \mathrm{PSL}_2(9)$, every perfect central cover of A_6 can be viewed as a quasi-simple classical group, which is already studied in [25, Theorem 1.1]. So it remains to consider the perfect central covers of A_7. Let H be one of those and assume that G is a finite group such that $\mathbb{C}G \cong \mathbb{C}H$.

As before, we see that G is perfect and, if M is a normal maximal subgroup of G, we have that G/M is non-abelian simple and $\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(H)$. In particular,

$$\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(\mathrm{Schur}(\mathsf{A}_7)),$$

where $\mathrm{Schur}(\mathsf{A}_7)$ is the Schur cover (or the 6-fold cover) of A_7. It follows that $|G/M| \leq 6|\mathsf{A}_7| = 7,560$. Inspecting [5], we come up with

$$G/M \cong \mathrm{PSL}_2(q) \text{ with } 5 \leq q \leq 23, \text{ or } \mathrm{PSL}_3(3), \text{ or } \mathrm{PSU}_3(3), \text{ or } \mathsf{A}_7.$$

Since each possibility for G/M except A_7 does not satisfy the inclusion $\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(\mathrm{Schur}(\mathsf{A}_7))$, we deduce that $G/M \cong \mathsf{A}_7$.

On the other hand, as $\mathbb{C}G \cong \mathbb{C}H$, we have $|G| = |H|$. It follows that $|M| = |\mathbf{Z}(H)| \leq 6$. Using Lemma 6.1, we conclude that $G \cong H$. □

7. Excluding critical character degrees of $\hat{\mathsf{S}}_n$

In this section we prove a non-existence result for special character degrees of $\hat{\mathsf{S}}_n$ which will be applied in the next section. Indeed, with the following proposition we prove a little more as only the case of even n will be needed (in fact, the proof shows that also versions with slightly modified 2-powers can be obtained).

Proposition 7.1. *Let $n \in \mathbb{N}$. If $2^{[\frac{n-2}{2}]}(n-1)$ is a character degree of $\hat{\mathsf{S}}_n$, then $n \leq 8$ and the degree is an ordinary degree f_λ for $\lambda \in \{(2), (2,1), (4,2^2)\}$ (or their conjugates), or the spin degree g_μ for $\mu = (4,2)$.*

The strategy for the proof is inspired by the methods used in [1, 2] to classify the irreducible characters of prime power degrees. A main ingredient is a number-theoretic result which is a variation of [1, Theorem 3.1].

First, we define $M(n)$ to be the set of pairs of finite sequences of integers $s_1 < s_2 < \cdots < s_r \leq n$, $t_1 < t_2 < \cdots < t_r \leq n$, with all numbers different from $n-1$, that satisfy

(i) $s_i < t_i$ for all i;
(ii) s_1 and t_1 are primes $> \frac{n}{2}$;

(iii) For $1 \leq i \leq r-1$, s_{i+1} and t_{i+1} contain prime factors exceeding $2n - s_i - t_i$ and not dividing $n-1$.

We then set $t(n) := \max\{t_r \mid ((s_i)_{i=1,\dots,r}, (t_i)_{i=1,\dots,r}) \in M(n)\}$, or $t(n) = 0$ when $M(n) = \emptyset$. Note that for all $n \geq 15$, there are at least two primes $p, q \neq n-1$ with $\frac{n}{2} < p < q \leq n$ (e.g., use [8]); hence for all $n \geq 15$ the set $M(n)$ is not empty.

Theorem 7.2. *Let $n \in \mathbb{N}$. Then $n - t(n) \leq 225$.*
For $15 \leq n \leq 10^9$, we have the tighter bounds

$$n - t(n) \begin{cases} = 7 & \text{for } n \in \{30, 54\}, \\ = 5 & \text{for } n \in \{18, 24, 28, 52, 102, 128, 224\}, \\ \leq 4 & \text{otherwise.} \end{cases}$$

Proof. For $n > 3.9 \cdot 10^8$, the proof follows the lines of the arguments for [1, Theorem 3.1], noticing that in the construction given there the numbers in the sequences are below $n-1$ and that the chosen prime factors do not divide $n-1$; this then gives $n - t(n) \leq 225$.

A computer calculation (with Maple) shows that, for all $n \leq 10^9$, we have the claimed bounds and values for $n - t(n)$. ☐

For any partition λ of n, we denote by $l(\lambda)$ the length of λ, and we let $l_1(\lambda)$ be the multiplicity of 1 in λ. We put $h_i = h(i, 1)$ for $1 \leq i \leq l(\lambda)$; these are the *first-column hook lengths* of λ. We set $\operatorname{fch}(\lambda) = \{h_1, \dots, h_{l(\lambda)}\}$.

First we want to show that Proposition 7.1 holds for ordinary characters. Via computer calculations, the claim is easily checked up to $n = 44$, and in particular, we find the stated exceptions for $n < 9$. Thus we have to show that

$$(*) \qquad f_\lambda = 2^{[\frac{n-2}{2}]}(n-1)$$

cannot hold for $n \geq 9$; if necessary, we may even assume that $n > 44$.

To employ Theorem 7.2, we need some preparations that are similar to corresponding results in [1].

Proposition 7.3. *If q is a prime with $n - l_1(\lambda) \leq q \leq n$ and $q \nmid f_\lambda$, then*

$$q,\ 2q,\ \dots,\ \left[\frac{n}{q}\right] q \in \operatorname{fch}(\lambda)\,.$$

Proof. Put $w = \left[\frac{n}{q}\right]$, $n = wq + r$, $0 \leq r < q$. By assumption, we have $(w-1)q \leq (w-1)q + r = n - q \leq l_1 := l_1(\lambda)$. Thus $q, 2q, \dots, (w-1)q \in \operatorname{fch}(\lambda)$. If $wq \leq l_1$, then we are done. Assume that $l_1 < wq$. At most w hooks in λ are of lengths divisible by q (see e.g. [27, Prop.(3.6)]). If there are only the above $w-1$ hooks in the first column of length divisible by q, then $q | f_\lambda$ since $\prod_{i=1}^{w}(iq) \mid n!$, a contradiction. Let $h(i,j)$ be the additional hook length divisible by q. Since $\lambda \neq (1^n)$, $l_1 \leq h_2$. If $h_2 > l_1$, then $h(i,j) + h(2,1) > q + l_1 \geq n$. By [1, Cor. 2.8] we get $j = 1$. If $h_2 = l_1$, then

$\lambda = (n - l_1, 1^{l_1})$ and since $l_1 < wq$ there has to be a hook of length divisible by q in the first row. Since $n - l_1 \le q$ we must have $h_1 = wq$. $\square$

In analogy to [1, Cor. 2.10], we deduce

Corollary 7.4. *Let $1 \le i < j \le l(\lambda)$. If $h \le n$ has a prime divisor q satisfying $2n - h_i - h_j < q$ and $q \nmid f_\lambda$, then $h \in \mathrm{fch}(\lambda)$.*

We now combine these results with Theorem 7.2, similarly as in [1]; as stated earlier we may assume that $n \ge 15$, and hence there are least two primes $p, q \ne n-1$ with $\frac{n}{2} < p < q \le n$. Assuming $(*)$ for λ, the hook formula implies that there have to be hooks of length p and q in λ. As argued in [1], we then have $p, q \in \mathrm{fch}(\lambda)$ or $p, q \in \mathrm{fch}(\overline{\lambda})$; w.l.o.g., we may assume $p, q \in \mathrm{fch}(\lambda)$. Then the assumption $(*)$ forces *any* prime between $\frac{n}{2}$ and n, except $n-1$ if this is prime, to be in $\mathrm{fch}(\lambda)$. This gives an indication towards the connection with the sequences belonging to the pairs in $M(n)$.

Indeed, we have the following proposition which is proved similarly as the corresponding result in [1].

Proposition 7.5. *Let $n \ge 15$. Let $((s_i)_{i=1..r}, (t_i)_{i=1..r}) \in M(n)$. Let λ be a partition of n such that $(*)$ holds. Then $\{s_1, ..., s_r, t_1, ..., t_r\} \subset \mathrm{fch}(\lambda)$ or $\{s_1, ..., s_r, t_1, ..., t_r\} \subset \mathrm{fch}(\overline{\lambda})$.*

In particular, $n - h_1 \le 225$, and we have tighter bounds for $n - h_1$ when $n \le 10^9$ as given in Theorem 7.2.

Now we can embark on the **first part of the proof of Proposition 7.1**, showing the non-existence of ordinary irreducible characters of the critical degree for $n \ge 9$. As remarked before, we may assume $n \ge 44$.

Set $m = [\frac{n-2}{2}]$, and assume that the partition λ of n satisfies

$$(*) \qquad f_\lambda = 2^m(n-1)\,.$$

Let $c = n - h_1$. By [1, Prop. 4.1] we have the following bound for the 2-part of the degree:

$$(f_\lambda)_2 \le n^2 \cdot ((2c+2)!)_2\,.$$

By Proposition 7.5, we have $c \le 225$, and hence $((2c+2)!)_2 \le (452!)_2 = 2^{448}$. Thus

$$2^m \le 2^{448} n^2 \le 2^{448}(2m+3)^2\,.$$

A short computation gives $m \le 467$, and hence $n \le 937$. By Proposition 7.5, $c \le 5$, unless $n = 54$, where we only get $c \le 7$. But for $n = 54$ we can argue as follows. As λ satisfies $(*)$, w.l.o.g. $43, 47 \in \mathrm{fch}(\lambda)$. Then $l_1(\lambda) \ge 35$ (by [1, Prop. 2.6]), and hence $17, 34 \in \mathrm{fch}(\lambda)$; since $54 > 3 \cdot 17$, by the hook formula there has to be one more hook of length divisible by 17 in λ. As $c \le 7$, this is in the first row or column; if it is not in the first column, we get a contradiction considering this hook and the one of length 43. Thus $51 \in \mathrm{fch}(\lambda)$, and hence $c \le 3$.

Hence for all $n \leq 937$ we have $((2c+2)!)_2 \leq (12!)_2 = 2^{10}$, and

$$2^m \leq 2^{10} n^2 \leq 2^{10}(2m+3)^2 .$$

This implies $m \leq 20$, and hence $n \leq 43$, where the assertion was checked directly. $\square$

Next we deal with the spin characters of $\hat{\mathsf{S}}_n$. Recall that for a bar partition λ of n, $\bar{g}_\lambda$ is the number of shifted standard tableaux of shape λ, and the spin character degree associated to λ is $g_\lambda = 2^{[\frac{n-l(\lambda)}{2}]}\bar{g}_\lambda$. Hence the condition on the spin degree translates into the condition (†) on $\bar{g}_\lambda$ given below.

Proposition 7.6. *Let λ be a bar partition of n. Then*

$$(\dagger) \qquad \bar{g}_\lambda = \begin{cases} 2^{\left\lceil \frac{l(\lambda)-2}{2} \right\rceil}(n-1) & \text{for } n \text{ even} \\ 2^{\left\lceil \frac{l(\lambda)-3}{2} \right\rceil}(n-1) & \text{for } n \text{ odd} \end{cases}$$

only if $n \leq 6$, and $\lambda = (2)$, $\lambda = (4,2)$.

We note that, for $n \leq 34$, the assertion is easily checked by computer calculation (using John Stembridge's Maple package QF), so we may assume that $n > 34$ when needed.

We put $b_i = b(1,i)$ for the *first row bar lengths* of λ, and $\mathrm{frb}(\lambda)$ for the set of first row bar lengths of λ (see [27] for details on the combinatorics of bars).

In analogy to the case of ordinary characters where we have modified the results in [1], we adapt the results in [2] for the case under consideration now. Similarly to Proposition 7.3 before, we have a version of [2, Prop. 2.5] where, instead of the prime power condition for $\bar{g}_\lambda$, the condition $q \nmid \bar{g}_\lambda$ is assumed for the prime q under consideration. For the corresponding variant of [2, Lemma 2.6] that says that any prime q with $\frac{n}{2} < q \leq n$ and $q \nmid \bar{g}_\lambda$ is a first row bar length of λ, we need two primes $p_1, p_2 \neq n-1$ with $p_1 + p_2 - n > \frac{n}{2}$. For $n \geq 33$, $n \neq 42$, we always find two primes $p_1, p_2 \neq n-1$ such that $\frac{3}{4}n < p_1 < p_2 \leq n$. But for $n = 42$, the primes $p_1 = 31$ and $p_2 = 37$ are big enough to have $p_1 + p_2 - n > \frac{n}{2}$.

The largest bar length of λ is $b_1 = b(1,1) = \lambda_1 + \lambda_2$. As before, the preparatory results just described together with our arithmetical Theorem 7.2 show that $n - b_1$ is small for a bar partition λ satisfying (†). More precisely, we obtain:

Proposition 7.7. *Let $n \geq 15$. Let $((s_1, \dots, s_r), (t_1, \dots, t_r)) \in M(n)$. Assume that λ is a bar partition of n that satisfies (†). Then $s_1, \dots, s_r, t_1, \dots, t_r \in \mathrm{frb}(\lambda)$.*

In particular, if λ satisfies (†), then $n - b_1 \leq 225$, and we have tighter bounds for $n - b_1$ when $n \leq 10^9$ as given in Theorem 7.2.

Now we can get into the **second part of the proof of Proposition 7.1**, showing the non-existence of spin irreducible characters of the critical degree for $n \geq 7$. We have already seen that it suffices to prove Proposition 7.6, and that we may assume $n \geq 15$. Set

$$r = \begin{cases} \lceil \frac{l(\lambda)-2}{2} \rceil & \text{for } n \text{ even} \\ \lceil \frac{l(\lambda)-3}{2} \rceil & \text{for } n \text{ odd} \end{cases}$$

and assume that λ is a bar partition of n that satisfies (†).

Let $c = n - b_1$. As seen above, we have $c \leq 225$, and hence $l(\lambda) \leq 23$. Thus $r \leq 11$ in any case, and hence $\bar{g}_\lambda \leq 2^{11}(n-1)$.

Now, by [2, Prop. 2.2] we know that $\bar{g}_\lambda \geq \frac{1}{2}(n-1)(n-4)$ unless we have one of the following situations: $\lambda = (n)$and $\bar{g}_\lambda = 1$, or $\lambda = (n-1,1)$ and $\bar{g}_\lambda = n-2$. None of these exceptional cases is relevant here, and thus we obtain $n - 4 \leq 2^{12}$. But for $n \leq 4100$ we already know that $c \leq 7$. Then $l(\lambda) \leq 6$ and $r \leq 2$, and hence

$$\frac{1}{2}(n-1)(n-4) \leq \bar{g}_\lambda \leq 4(n-1)\,.$$

But then $n - 4 \leq 8$, a contradiction. Thus we have now completed the proof of Proposition 7.1. □

8. Eliminating simple groups of Lie type in even characteristic

Let G be a finite group whose complex group algebra is isomorphic to that of $\hat{\mathsf{S}}_n$. In order to show that G is isomorphic to one of the two double covers of S_n, one has to eliminate the involvement of all non-abelian simple groups other than A_n in the structure of G. The most difficult case turns out to be the simple groups of Lie type in even characteristic.

For the purpose of the next lemma, let $\mathcal{C}$ be the set consisting of the following simple groups:

$$\{{}^2F_4(2)', \mathrm{PSL}_4(2), \mathrm{PSL}_3(4), \mathrm{PSU}_4(2), \mathrm{PSU}_6(2),$$
$$\mathrm{P}\Omega_8^+(2), \mathrm{PSp}_6(2), {}^2B_2(8), G_2(4), {}^2E_6(2)\}.$$

Lemma 8.1. *If S is a simple group of Lie type in characteristic 2 such that $|S|_2 \geq 2^4$ and $S \notin \mathcal{C}$, then $|S|_2 < 2^{(e(S)-1)/2}$, where $e(S)$ is the smallest non-trivial degree of an irreducible projective representation of S.*

Proof. Assume that S is defined over a finite field of size $q = 2^f$. If $|S|_2 = q^{N(S)}$, then the inequality in the lemma is equivalent to

$$e(S) > 2N(S)f + 1. \tag{8.2}$$

The values of $e(S)$ are available in [30, Table II] for classical groups and in [15] for exceptional groups. The arguments for simple classical groups are quite similar. So let us consider the linear groups. Assume that $S = \mathrm{PSL}_m(q)$ with $m \geq 2$. We have $N(S) = m(m-1)/2$. First we assume that $m = 2$. As $|S|_2 = q \geq 2^4$, we deduce that $e(S) = q - 1$ and $f \geq 4$. In this case, we obtain $e(S) - 1 = 2^f - 2$ and $2N(S)f + 1 = 2f + 1$. As $f \geq 4$, we see that (8.2) holds. Next we assume that $m \geq 3$

and $S \neq \mathrm{PSL}_3(4), \mathrm{PSL}_4(2)$. As $|S|_2 \geq 2^4$, we deduce that $S \neq \mathrm{PSL}_3(2)$. We have $e(S) = (q^m - q)/(q-1)$. Now (8.2) is equivalent to

$$\frac{q^m - q}{q-1} > m(m-1)f + 1.$$

It is routine to check that this inequality holds for any $m \geq 3$ and $q \geq 2$.

The arguments for exceptional Lie type groups are also similar. For instance, if $S = {}^2B_2(2^{2m+1})$ with $m \geq 1$ then $|S|_2 = 2^{2(2m+1)}$ and $e(S) = 2^m(2^{2m+1} - 1)$. The inequality can now be easily checked. $\square$

Proposition 8.2. *Let G be a finite group and S a simple group of Lie type in characteristic 2. Suppose that $M \subseteq G'$ is a normal subgroup of G such that $G'/M \cong S$ and $|G : G'| = 2$. Then $\mathrm{cd}^*(G) \neq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ for every integer $n \geq 10$.*

Proof. By way of contradiction, assume that $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ for some $n \geq 10$. Let St_S be the Steinberg character of $G'/M \cong S$. As St_S extends to G/M and $|G/M : G'/M| = 2$, by Gallagher's Theorem (see [12, Corollary 6.17] for instance), G/M has two irreducible characters of degree $\mathrm{St}_S(1) = |S|_2$. As $n \geq 10$, Lemma 3.5(1) yields that $d_1(\hat{\mathsf{S}}_n) = n - 1$.

We claim that $|S|_2 > d_1(\hat{\mathsf{S}}_n) = n - 1$. Suppose for a contradiction that $|S|_2 = d_1(\hat{\mathsf{S}}_n)$. If $G/M \cong S \times C_2$, then $\mathrm{cd}(G/M) = \mathrm{cd}(S) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, which implies that $d_1(S) \geq d_1(\hat{\mathsf{S}}_n) = |S|_2$. However, this is impossible since S always has a non-trivial character degree smaller than $|S|_2$ (see [32, Lemma 8] for instance). Now assume that G/M is almost simple with socle S. If $S \ncong \mathrm{PSL}_2(q)$ with $q \geq 4$, then $d_1(G/M) < |S|_2 = d_1(\hat{\mathsf{S}}_n)$ by [31, Lemma 2.4], which leads to a contradiction as before since $\mathrm{cd}(G/N) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Therefore, assume that $S = \mathrm{PSL}_2(q)$ with $q = 2^f \geq 4$. Then $q = |S|_2 = n - 1 \geq 9$. If $q \equiv -1 \pmod 3$, then $d_1(G) = q - 1 < q = d_1(\hat{\mathsf{S}}_n)$ by [31, Lemma 2.5], which is impossible. Hence, $q \equiv 1 \pmod 3$ and $q + 1 \in \mathrm{cd}(G/M) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. It follows that $q + 1 = (n-1) + 1 = n \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. If $n \geq 12$, then $d_2(\hat{\mathsf{S}}_n) > 2n > n > d_1(\hat{\mathsf{S}}_n)$, hence n is not a degree of $\hat{\mathsf{S}}_n$. Thus $10 \leq n \leq 11$. However, we see that $n - 1$ is not a power of 2 in either case. The claim is proved.

Assume that $n = 2k + 1 \geq 11$ is odd. By Lemmas 2.1 and 2.3, $|S|_2 = 2^k$ is the degree of the basic spin character of $\hat{\mathsf{S}}_n$. However, by [34, Table I] such a degree has multiplicity one, which contradicts the fact proved above that G has at least two irreducible characters of degree $|S|_2$.

Assume $n = 2k \geq 10$ is even. By Lemmas 2.1 and 2.3 and [34, Table I], $\hat{\mathsf{S}}_n$ always has the character degree 2^{k-1} with multiplicity 2; and if $n = 2^r + 2$, then it has the character degree $2^{k-1}(n-2) = 2^{2^{r-1}+r}$ with multiplicity 1. These are in fact the only non-trivial 2-power character degrees of $\hat{\mathsf{S}}_n$. As in the previous case, by comparing the multiplicity, we see that $|S|_2 \neq 2^{2^{r-1}+r}$. Thus $|S|_2 = 2^{k-1}$ is the degree of the basic spin character of $\hat{\mathsf{S}}_n$ with multiplicity 2. Notice that $k \geq 5$ and hence $|S|_2 = 2^{k-1} \geq 2^4$.

Now let $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = n-1$. As $|G : G'| = 2$ and $\psi(1)$ is odd, we deduce that $\phi = \psi\downarrow_{G'} \in \mathrm{Irr}(G')$ and $\phi(1) = n-1$. Let $\theta \in \mathrm{Irr}(M)$ be an irreducible constituent of $\phi\downarrow_M$. Then $\phi\downarrow_M = e(\theta_1 + \cdots + \theta_t)$, where $t = |G' : I_{G'}(\theta)|$, and each θ_i is conjugate to $\theta \in \mathrm{Irr}(M)$. If θ is not G'-invariant, then $\phi(1) = et\theta(1) = n-1 \geq \min(S)$, where $\min(S)$ is the smallest non-trivial index of a maximal subgroup of S. We see that $\min(S) > d_1(S) \geq e(S)$, where $e(S)$ is the minimal degree of a projective irreducible representation of S, and so $n-1 \geq e(S)$. If θ is G'-invariant and $\phi\downarrow_M = e\theta$ with $e > 1$, then e is the degree of a projective irreducible representation of S. It follows that $n-1 \geq e \geq e(S)$. In both cases, we always have

$$k - 1 = \frac{n-2}{2} \geq \frac{e(S)-1}{2}.$$

Therefore,

$$|S|_2 = 2^{k-1} \geq 2^{(e(S)-1)/2}.$$

By Lemma 8.1, we deduce that $S \in \mathcal{C}$. Solving the equation $|S|_2 = 2^{(n-2)/2}$, we get the degree n. However, by using [5] and Lemma 3.5, we can check that $\mathrm{cd}(G/M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ in any of these cases. For example, assume that $S \cong {}^2E_6(2)$. Then $|S|_2 = 2^{36} = 2^{(n-2)/2}$, so $n = 74$. By Lemma 3.5, we have $d_1(\hat{\mathsf{S}}_n) = n-1 = 73$ and $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 = 2627$. Using [5], we know that $\mathrm{cd}(G/M)$ contains the degree 1938. Clearly, $d_1(\hat{\mathsf{S}}_n) < 1938 < d_2(\hat{\mathsf{S}}_n)$, so $1938 \notin \mathrm{cd}(\hat{\mathsf{S}}_n)$, hence $\mathrm{cd}(G/M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, a contradiction.

Finally we assume that $et = 1$. Then θ extends to $\phi \in \mathrm{Irr}(G')$ and to $\psi \in \mathrm{Irr}(G)$. Hence $\phi\downarrow_M = \theta$ and so by Gallagher's Theorem, we have $\psi\tau \in \mathrm{Irr}(G)$ for every $\tau \in \mathrm{Irr}(G/M)$. In particular,

$$2^{k-1}(n-1) = \psi(1)|S|_2 \in \mathrm{cd}(G) = \mathrm{cd}(\hat{\mathsf{S}}_n),$$

which is impossible by Proposition 7.1. $\square$

9. Eliminating simple groups other than A_n

We continue to eliminate the involvement of simple groups other than A_n in the structure of G with $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n$.

Proposition 9.1. *Let G be an almost simple group with non-abelian simple socle S. Suppose that $\mathrm{cd}^*(G) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ for some $n \geq 10$. Then $S \cong \mathsf{A}_n$ or S is isomorphic to a simple group of Lie type in characteristic 2.*

Proof. We make use of the classification of finite simple groups.

(i) S is a sporadic simple group or the Tits group. Using [7], we can assume that $n \geq 19$. By Lemma 3.5(2), we have $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 \geq 152$. Since $d_2(G) \geq$

$d_2(\hat{\mathsf{S}}_n) \geq 152$, using [5], we only need to consider the following simple groups:

$$J_3, Suz, McL, Ru, He, Co_1, Co_2, Co_3, Fi_{22}, O'N, HN, Ly, Th, Fi_{23}, J_4, Fi_{24}', B, M.$$

To eliminate these groups, we first observe that $n \geq p(S)$, the largest prime divisor of $|S|$, and $d_i(G) \geq d_i(\hat{\mathsf{S}}_n)$ for all $i \geq 1$. Now with the lower bound $n \geq \max\{19, p(S)\}$, we can find the lower bounds for $d_i(\hat{\mathsf{S}}_n)$ with $1 \leq i \leq 7$ using Lemmas 3.4 and 3.5. Choose $i \in \{2, 3, \cdots, 7\}$ such that $d_i(\hat{\mathsf{S}}_n) > d_j(G)$ for some $j \geq 1$ such that $|i - j|$ is minimal. If $i \geq j$, then we obtain a contradiction. Otherwise, $d_j(G) \in \{d_k(\hat{\mathsf{S}}_n)\}_{k=j}^{i-1}$. Solving these equations for n, we then obtain that either these equations have no solution, or that, for each solution of n, we can find some $k \geq 1$ with $d_k(\hat{\mathsf{S}}_n) > d_k(G)$.

For an example of such a demonstration, assume that $S = O'N$. In this case, we have $|\mathrm{Out}(S)| = 2$, so $G = S$ or $G = S.2$. Since $p(S) = 31$, we have $n \geq 31$. Assume first that $G = S = O'N$. Then $d_4(O'N) = 26752$ and, since $n \geq 31$, by Lemma 3.4, $d_7(\hat{\mathsf{S}}_n) \geq 26970 > d_4(O'N)$. It follows that $d_4(O'N) \in \{d_i(\hat{\mathsf{S}}_n)\}_{i=4}^{6}$. However, we can check that these equations are impossible. Now assume $G = O'N.2$. Then $d_2(G) = 26752 < 26970 \leq d_7(\hat{\mathsf{S}}_n)$ so that $d_2(G) \in \{d_i(\hat{\mathsf{S}}_n)\}_{i=2}^{6}$. As above, these equations cannot hold for any $n \geq 31$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$.

For another example, let $S = M$. Since $|\mathrm{Out}(S)| = 1$, we have $G = S$ so that $p(S) = 71 \in \pi(\hat{\mathsf{S}}_n)$ and hence $n \geq 71$. As $d_1(M) = 196883 < 914480 \leq d_7(\hat{\mathsf{S}}_n)$, we deduce that $d_1(M) \in \{d_i(\hat{\mathsf{S}}_n) \mid i = 1, \cdots, 6\}$. Solving these equations, we obtain $n = 196884$. But then $d_2(\hat{\mathsf{S}}_n) > 21296, 876 = d_2(M)$. Thus $\mathrm{cd}(M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$.

(ii) $S = \mathsf{A}_m$ with $m \geq 7$. Note that we consider $\mathsf{A}_5 \cong \mathrm{PSL}_2(5)$ and $\mathsf{A}_6 \cong \mathrm{PSL}_2(9)$ as groups of Lie type. Let $\lambda = (m-1, 1)$, a partition of m. Since $m \geq 7$, λ is not self-conjugate, hence the irreducible character χ_λ of S_m is still irreducible upon restriction to A_m. Note that $\mathrm{Aut}(\mathsf{A}_m) = \mathsf{S}_m$ as $m \geq 7$. Then $G \in \{\mathsf{A}_m, \mathsf{S}_m\}$ and G has an irreducible character of degree $m - 1$. Since $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, we have $m - 1 \geq d_1(\hat{\mathsf{S}}_n) = n - 1$, so $m \geq n$. If $m = n$ then we are done. On the other hand, if $m > n$ then

$$|G| \geq |S| = |\mathsf{A}_m| > 4|\mathsf{A}_n| = |\hat{\mathsf{S}}_n|$$

and this violates the hypothesis $\mathrm{cd}^*(G) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$.

(iii) S is a simple group of Lie type in odd characteristic. Suppose that $S = G(p^k)$, a simple group of Lie type defined over a field of p^k elements with p odd. Let St_S be the Steinberg character of S. Then, as St extends to G and $\mathrm{St}_S(1) = |S|_p$, we have $|S|_p \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. Using Lemma 2.3, which says that all possible prime power degrees of spin characters of S_n are even, we deduce that $|S|_p \in \mathrm{cd}(\mathsf{S}_n)$. By Lemma 2.1, we then obtain that $|S|_p = n - 1$ since $n \geq 10$. By Lemma 3.5, $n - 1 = d_1(\hat{\mathsf{S}}_n)$ is the smallest non-trivial degree of $\hat{\mathsf{S}}_n$. Assume first that $S \neq \mathrm{PSL}_2(q)$. Then $d_1(G) < |S|_p = d_1(\hat{\mathsf{S}}_n)$ by [31, Lemma 2.4], which is a contradiction as $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Now it remains to consider the case $S = \mathrm{PSL}_2(q)$. We have $q = n-1 \geq 9$. If G has a character degree which is smaller than $|S|_p = q$, then we obtain a contradiction as before. So, by [31, Lemma 2.5], we have $p \neq 3$ and $q \equiv 1 \pmod 3$ or $p = 3$ and $q \equiv 1 \pmod 4$. In both cases, G has an irreducible character of degree $q+1 = n = d_1(\hat{\mathsf{S}}_n)+1$. If $n \geq 12$, then $d_2(\hat{\mathsf{S}}_n) \geq 2n > n > d_1(\hat{\mathsf{S}}_n)$ by Lemma 3.5, so that n is not a character degree of $\hat{\mathsf{S}}_n$. Assume that $10 \leq n \leq 11$. Then $n = 10$ and $q = 9$. However, using [5], we can check that $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ for every almost simple group G with socle $\mathrm{PSL}_2(9) \cong \mathsf{A}_6$. $\square$

Combining Propositions 9.1 and 8.2, we obtain the following results, which will be crucial in the proof of Theorem C.

Proposition 9.2. *Let G be a finite group and let $M \subseteq G'$ be a normal subgroup of G such that G/M is an almost simple group with socle $S \neq \mathsf{A}_n$, where $|G:G'| = 2$ and $G'/M \cong S$. Then $\mathrm{cd}^*(G) \neq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$.*

Proof. If $n \geq 10$, then the result follows from Propositions 9.1 and 8.2. It remains to assume that $5 \leq n \leq 9$ and suppose by contradiction that $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$. Then $|G| = 2n!$ and so $|S| \mid 2n!$, hence $\pi(S) \subseteq \pi(\hat{\mathsf{S}}_n) \subseteq \{2,3,5,7\}$. By [11, Theorem III], one of the following holds:

1. If $\pi(S) = \{2,3,5\}$, then $S \cong \mathsf{A}_5, \mathsf{A}_6$ or $\mathrm{PSp}_4(3)$.
2. If $\pi(S) = \{2,3,7\}$, then $S \cong \mathrm{PSL}_2(7), \mathrm{PSL}_2(8)$ or $\mathrm{PSU}_3(3)$.
3. If $\pi(S) = \{2,3,5,7\}$, then $S \cong \mathsf{A}_k$ with $7 \leq k \leq 10$, J_2, $\mathrm{PSL}_2(49), \mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(5)$, $\mathrm{PSU}_4(3), \mathrm{PSp}_4(7), \mathrm{PSp}_6(2)$ or $\mathrm{P}\Omega_8^+(2)$.

Now it is routine to check that $\mathrm{cd}(G/M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ unless $S \cong \mathsf{A}_n$, where G/M is almost simple with socle S. $\square$

Proposition 9.3. *Let G be a finite group and let $M \subseteq G'$ be a normal subgroup of G. Suppose that $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ and $G/M \cong G'/M \times C_2 \cong S^k \times C_2$ for some positive integer k and some non-abelian simple group S. Then $k = 1$ and $S \cong \mathsf{A}_n$.*

Proof. Since $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(S^k)$, the hypotheses imply that

$$\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)\,.$$

Assume first that $5 \leq n \leq 9$. Since $|S^k| = |S|^k$ divides $|\hat{\mathsf{S}}_n| = 2n!$, we deduce that $\pi(S) \subseteq \pi(\hat{\mathsf{S}}_n)$ and, in particular, $\pi(S) \subseteq \{2,3,5,7\}$. The possibilities for S are listed in the proof of Proposition 9.2 above. Observe that $|S|$ is always divisible by a prime r with $r \geq 5$. Hence, $r^k \mid |\hat{\mathsf{S}}_n|$, which implies that $k = 1$ as $|\hat{\mathsf{S}}_n|$ divides $2 \cdot 9!$. Now the fact that $S \cong \mathsf{A}_n$ follows easily.

From now on we can assume that $n \geq 10$. Using Proposition 9.1, we obtain $S = \mathsf{A}_n$ or S is a simple group of Lie type in even characteristic. It suffices to show that $k = 1$ and then the result follows from Proposition 8.2.

Assume that the latter case holds. Then S is a simple group of Lie type in characteristic 2. By Lemmas 2.1 and 2.3, $\hat{\mathsf{S}}_n$ has at most two distinct non-trivial 2-power character degrees, which are $n-1$ and $2^{\lfloor (n-1)/2 \rfloor}$, or $2^{\lfloor (n-1)/2 \rfloor}$ and $2^{2^{r-1}+r}$ with $n = 2^r + 2$. By way of contradiction, assume that $k \geq 2$. If $k \geq 3$, then $G/M \cong S^k \times C_2$ has irreducible characters of degrees

$$|S|_2^k > |S|_2^{k-1} > |S|_2^{k-2} > 1.$$

Obviously, this is impossible as $\mathrm{cd}(G/M) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Therefore, $k = 2$. In this case, G/M has character degree $|S|_2^2$ with multiplicity at least 2 and $|S|_2$ with multiplicity at least 4. It follows that either $2^{\lfloor (n-1)/2 \rfloor} = |S|_2^2$ and $n-1 = |S|_2$, or $2^{2^{r-1}+r} = |S|_2^2$ and $2^{\lfloor (n-1)/2 \rfloor} = |S|_2$. However, both cases are impossible by comparing the multiplicity.

It remains to eliminate the case $S \cong \mathsf{A}_n$ and $k \geq 2$. By comparing the orders, we see that

$$2(n!/2)^k |M| = 2n!\,.$$

After simplifying, we obtain

$$|M|(n!)^{k-1} = 2^k.$$

Since $n \geq 10$, we see that, if $k \geq 2$, then the left side is divisible by 5 while the right side is not. We conclude that $k = 1$ and the proof is now complete. □

10. Completion of the proof of Theorem C

We need one more result before proving Theorem C.

Proposition 10.1. *Let G be a finite group and let S be a non-abelian simple group. Suppose that $|G : G'| = 2$ and $G' \cong S^2$ is the unique minimal normal subgroup of G. Then $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$.*

Proof. Assume, to the contrary, that $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Let $\alpha \in \mathrm{Irr}(S)$ with $\alpha(1) > 1$ and put $\theta = \alpha \otimes 1 \in \mathrm{Irr}(G')$. Observe that θ is not G-invariant, so that $I_G(\theta) = G'$; hence $\theta^G \in \mathrm{Irr}(G)$ and so $\theta^G(1) = 2\alpha(1) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. On the other hand, if $\varphi = \alpha \otimes \alpha \in \mathrm{Irr}(G')$, then φ is G-invariant and, since G/G' is cyclic, we deduce that φ extends to $\psi \in \mathrm{Irr}(G)$, so $\psi(1) = \alpha(1)^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. Thus, we conclude that

$$\text{if } a \in \mathrm{cd}(S)\backslash\{1\} \text{ then } 2a, a^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n). \tag{10.3}$$

Let r be an odd prime divisor of $|S|$. The Ito-Michler theorem then implies that r divides some character degree, say a, of S. Since $a^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$ by (10.3), we have $r^2 \mid 2n!$ and hence $n \geq 2r$ as $r > 2$. Thus, we have shown that

$$\text{if } r \in \pi(S) - \{2\}, \text{ then } r^2 \mid 2n! \text{ and } n \geq 2r. \tag{10.4}$$

Using the classification of finite simple groups, we consider the following cases.

(i) $S = \mathsf{A}_m$, with $m \geq 7$. As $7 \in \pi(S)$, it follows from (10.4) that $n \geq 14$. Since $m-1 \in \mathrm{cd}(S)$, both $2(m-1)$ and $(m-1)^2$ are in $\mathrm{cd}(\hat{\mathsf{S}}_n)$ by (10.3). As $m \geq 7$, we also have that $m(m-3)/2, (m-1)(m-2)/2 \in \mathrm{cd}(S)$ and so $m(m-3), (m-1)(m-2) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. We claim that $m < n$. Suppose for a contradiction that $m \geq n$. As $n \geq 14$, by Lemma 4.1, there exists a prime r such that $n/2 < r \leq n$. Hence, the r-part of $2n!$ is just r. However, as $r \leq n \leq m$, r divides $|\mathsf{A}_m|$ and so $r^2 \mid 2n!$ by (10.4), a contradiction. Thus $m < n$ as claimed.

Since $m \geq 7$, we obtain that

$$1 < 2(m-1) < m(m-3) < (m-1)(m-2) < (m-1)^2. \tag{10.5}$$

By Lemma 3.5(1), we have $d_1(\hat{\mathsf{S}}_n) = n-1$, so $2(m-1) \geq n-1$ and thus $n \leq 2m-1$. As $n \geq 14$, we deduce that $m \geq 8$.

Assume first that $m \in \{8, 9, 10\}$. Then $(m-1)^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$ is a prime power. As $3^3 \neq (m-1)^2 > d_1(\hat{\mathsf{S}}_n)$, Lemmas 2.1 and 2.3 yield that $(m-1)^2$ is a power of 2 and thus $m = 9$. Since $\{2^3, 3^3\} \subseteq \mathrm{cd}(\mathsf{A}_9)$, we have $\{2^4, 2^6, 3^6\} \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ by (10.4). As $n \geq 14$, we have $2^{\lfloor (n-1)/2 \rfloor} > 2^4$ and $n(n-3)/2 > 2^4$, so $d_2(\hat{\mathsf{S}}_n) > 2^4$ by Lemma 3.5(2). This forces $2^4 = d_1(\hat{\mathsf{S}}_n) = n-1$ or, equivalently, $n = 17$. But then Lemmas 2.1 and 2.3 yield $3^6 = d_1(\hat{\mathsf{S}}_n)$, which is impossible.

Assume next that $m \geq 11$. Then $n \geq 22$ by (10.4). By Lemma 3.5(2), we have $d_2(\hat{\mathsf{S}}_n) > 2n > 2m$. In particular, $2(m-1) < d_2(\hat{\mathsf{S}}_n)$. By (10.3), we have $2(m-1) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$, hence $2(m-1) = d_1(\hat{\mathsf{S}}_n) = n-1$, which implies that $n = 2m-1$. By Lemma 3.5(3), we have $d_3(\hat{\mathsf{S}}_n) = (n-1)(n-2)/2$ and thus by (10.5), we obtain that

$$(m-1)(m-2) \geq d_3(\hat{\mathsf{S}}_n) = (n-1)(n-2)/2 = (m-1)(2m-3).$$

After simplifying, we have $m-2 \geq 2m-3$ or, equivalently, $m \leq 1$, a contradiction.

(ii) S is a finite simple group of Lie type in characteristic p, with $S \neq {}^2F_4(2)'$. As $|S|$ is always divisible by an odd prime $r \geq 5$, we have $n \geq 2r \geq 10$ by (10.4). Let St_S denote the Steinberg character of S. We can check that $\mathrm{St}_S(1) = |S|_p \geq 4$. Since $\mathrm{St}_S(1) \in \mathrm{cd}(S)$, $2\mathrm{St}_S(1)$ and $\mathrm{St}_S(1)^2$ are character degrees of $\hat{\mathsf{S}}_n$ by (10.3). As $2\mathrm{St}_S(1) < \mathrm{St}_S(1)^2$, we have $\mathrm{St}(1)^2 > d_1(\hat{\mathsf{S}}_n) = n-1$. Since $n \geq 10$, Lemmas 2.1 and 2.3 yield that $\mathrm{St}_S(1)^2$ is a 2-power. Hence, $2\mathrm{St}(1)$ is also a 2-power. By [32, Lemma 8], there exists a non-trivial character degree x of S such that $1 < x < \mathrm{St}_S(1)$. It follows that $2x < 2\mathrm{St}_S(1)$ is also a character degree of $\hat{\mathsf{S}}_n$. Therefore, $2\mathrm{St}_S(1) > d_1(\hat{\mathsf{S}}_n) = n-1$. Hence, $\hat{\mathsf{S}}_n$ has two distinct non-trivial 2-power character degrees, neither of which is $n-1$. It follows that $n = 2^r + 2 \geq 10$ and furthermore

$$2\mathrm{St}_S(1) = 2^{\lfloor (n-1)/2 \rfloor} \text{ and } \mathrm{St}_S(1)^2 = 2^{2^{r-1}+r}$$

by Lemma 2.3. We write $\mathrm{St}_S(1) = 2^N$. Then $2^{r-1} + r = 2N$ and $2^{r-1} = N+1$ since $\lfloor (n-1)/2 \rfloor = 2^{r-1}$. Solving these equations, we have $r = N-1$ and $2^{r-1} = r+2$. As $n \geq 10$, we deduce that $r \geq 3$. In this case, it is easy to check that the equation $2^{r-1} = r+2$ has no integer solution.

(iii) S is a sporadic simple group or the Tits group. Since the arguments are fairly similar, we consider just the case $S = J_3$ as an example. Recall that $p(S)$ is the largest prime divisor of $|S|$. By (10.4), we have $n \geq 2p(S)$. Since $n \geq 2p(S) \geq 22$, we have $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 \geq p(S)(2p(S)-3)$ by applying Lemma 3.5(3). For $i = 1, 2$, we have $2d_i(S) \in \mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ with $1 < 2d_1(S) < 2d_2(S)$. For each possibility for S, we can check using [5] that $p(S)(2p(S)-3) > 2d_2(S)$, hence $d_2(\hat{\mathsf{S}}_n) > 2d_2(S) > 2d_1(S)$, which is a contradiction.

□

We are now ready to prove the main Theorem C, which we restate below for the reader's convenience.

Theorem C. *Let $n \geq 5$. Let G be a finite group and $\hat{\mathsf{S}}_n^{\pm}$ the double covers of S_n. Then $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{+}$ (or equivalently $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{-}$) if and only if $G \cong \hat{\mathsf{S}}_n^{+}$ or $G \cong \hat{\mathsf{S}}_n^{-}$.*

Proof. By the hypothesis that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n$, we have $|G| = 2n!$ and as $\hat{\mathsf{S}}_n$ has two linear characters, we also have $|G : G'| = 2$.

First we claim that $G' = G''$. Assume not. Then $H := G/G'$ is a group whose commutator subgroup H' is non-trivial abelian of index 2. Now the induction of any non-principal (linear) character of H' to H must be irreducible and 2-dimensional. This is not possible since $\hat{\mathsf{S}}_n$ with $n \geq 5$ does not have any irreducible character of degree 2. Thus $G' = G''$.

As $G' = G''$ and G' is non-trivial, one can choose a normal subgroup M of G such that $M < G'$ and

$$G'/M \cong S^k,$$

where S is a non-abelian simple group and S^k is a chief factor of G. Let

$$C/M := \mathbf{C}_{G/M}(G'/M)\,.$$

(A) First we consider the case $C = M$. Then G'/M is the unique minimal normal subgroup of G/M. Therefore G/M permutes the direct factors of G'/M (which is isomorphic to S^k). It then follows that $k \leq 2$ as $|G : G'| = 2$. Invoking Proposition 10.1, we deduce that $k = 1$ and thus $G'/M \cong S$. Therefore, G'/M is the socle of G/M. As $\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$, Proposition 9.2 then implies that $G'/M \cong \mathsf{A}_n$. Thus $G/M \cong \mathsf{S}_n$ and also $|M| = 2$. In particular, M is central in G and therefore $M \subseteq \mathbf{Z}(G) \cap G'$. We conclude that G is one of the two double covers of S_n, as desired.

(B) It remains to consider the case $C > M$. Since $C/M \lhd G/M$ and $\mathbf{Z}(G'/M) = 1$, it follows that G' is a proper subgroup of $G'C$. As $|G : G'| = 2$, we then deduce that $G = G'C$ and hence

$$G/M = G'/M \times C/M \text{ where } C/M \cong C_2\,.$$

Applying Proposition 9.3, we obtain that $k = 1$ and S is isomorphic to A_n. In other words, $G'/M \cong \mathsf{A}_n$. So $G/M \cong \mathsf{A}_n \times C_2$. Comparing the orders, we get $|M| = 2$ and so $M \subseteq \mathbf{Z}(G)$. As $M \leq G' = G''$, it follows that $M \leq \mathbf{Z}(G') \cap G''$, which in turn implies that G' is the double cover of A_n. We have proved that

$$G' \cong \hat{\mathsf{A}}_n. \tag{10.6}$$

Moreover, as $C/M \cong C_2$ and $|M| = 2$, we have

$$C \cong C_4 \text{ or } C_2 \times C_2. \tag{10.7}$$

Now we claim that G is an (internal) central product of G' and C with amalgamated central subgroup M. To see this, let $x, y \in G'$ and $c \in C$. Then the facts $C/M = \mathbf{C}_{G/M}(G'/M)$ and $M \leq \mathbf{Z}(G)$ imply

$$[x,y]^c = [x^c, y^c] = [xm_1, ym_2] = [x,y],$$

for some $m_1, m_2 \in M$. Therefore, C centralizes $G' = G''$ and the claim follows. This claim together with (10.6) and (10.7) yield

$$4k(\hat{\mathsf{A}}_n) = 4k(G') = k(G' \times C) \leq k(M)k(G) = 2k(G),$$

where the inequality comes from the well-known result that $k(X) \leq k(N)k(X/N)$ for N a normal subgroup of X (see [22] for instance). Since $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$, we have $k(G) = k(\hat{\mathsf{S}}_n)$. It follows that $2k(\hat{\mathsf{A}}_n) \leq k(\hat{\mathsf{S}}_n)$. This however contradicts Lemma 4.5 and the theorem is now completely proved. $\square$

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