

A Simple Method for Computing Equilibria when Asset Markets Are Incomplete

Wei Ma*

Department of economics, University of Pretoria, Pretoria, South Africa

Abstract

The problem of computing equilibria for general equilibrium models with incomplete real asset markets, or GEI models for the sake of brevity, is reconsidered. It is shown here that the rank-dropping behavior of the asset return matrix could be dealt with in rather a simple fashion: We first compute its singular value decomposition, and then, through this decomposition, construct, by the introduction of a homotopy parameter, a new matrix such that it has constant rank before a desired equilibrium is reached. By adjunction of this idea to the homotopy method, a simpler constructive proof is obtained for the generic existence of GEI equilibria. For the purpose of computing these equilibria, from this constructive proof is then derived a path-following algorithm whose performance is finally demonstrated by means of three numerical examples.

Keywords: Incomplete asset markets; General equilibrium theory; Homotopy method
JEL: C68; C63

1 Introduction

After the hard work of a number of researchers over a number of decades, the proposition that the actual market is incomplete has been widely acknowledged to be a fact in the town of economics. This then naturally raises the question of how much of the general equilibrium theory for complete markets could be extended to the case of incomplete markets. With this question in mind, the present paper is concerned with the general equilibrium model with incomplete real asset markets, and studies the existence and computation of its equilibrium.

There are various reasons for the market being incomplete, such as asymmetric information, moral hazard, and transaction costs (see Geanakoplos (1990)), and this incompleteness has both remarkable and far-reaching implications. Two of them, perhaps most notable of all, are that perfect competition need not necessarily lead to market efficiency, and that nominal assets are in marked contrast to real assets, in terms of their impact on the existence of equilibrium. As regards the latter, equilibria have been shown in Werner (1985) and Cass (2006) to exist when the assets marketed are nominal. But, on the other hand, it has been shown in Hart (1975), by way of example, that, when the assets marketed are real, equilibria may fail to exist for GEI models. This unpleasant phenomenon, fortunately, has turned out to be nongeneric, and the generic existence of GEI equilibria has been established in, for instance, Duffie and Shafer (1985); Geanakoplos and Shafer (1990); Hirsch et al. (1990). The basic idea of these papers is to consider the demand function as defined on the cartesian product of the price simplex and a Grassmann manifold, and then make extensive use of results from differential topology. In the same spirit, a procedure for computing GEI equilibria has been devised in Demarzo and Eaves (1996). The efficiency of this procedure however is adversely affected by the high-dimension of the Grassmann manifold.

The first constructive proof that dispenses with the use of Grassmann manifold is to our knowledge presented in Brown et al. (1996). In that paper the ingenious idea of switching homotopies is invented. Relatively speaking, the procedure derived from this idea for computing GEI equilibria is, as remarked in Demarzo and Eaves (1996), not robust and not easy to implement. Motivated presumably by these considerations, Schmedders (1998, 1999) introduce the idea of a penalty function with penalties imposed on transactions on the asset market. On the one hand, this method is intuitively appealing and becomes easy to implement, but, on the other hand, its generic convergence has not yet been established.

*Corresponding author: Department of economics, University of Pretoria, Pretoria 0002, South Africa. Tel: +27-0124204751. Email: maweicityu@gmail.com

Confronted with this state of affairs we propose in the present paper to give another constructive proof for the generic existence of GEI equilibria. Recall that the main difficulty for so doing consists in the fact that the asset return matrix may drop rank as the price varies. To circumvent this our idea is rather simple, and could be intuitively described as follows. Let $R(p)$ be the asset return matrix, where p denotes the price vector. We first find its singular value decomposition, given say by $L_1(p)\Sigma(p)L_2(p)$, where $L_1(p), L_2(p)$ are two orthogonal matrices and $\Sigma(p)$ is a diagonal matrix. From matrix theory it follows that the diagonal entries of $\Sigma(p)$ are nonnegative, and the number of its strictly positive diagonal entries is precisely the rank of $R(p)$. Based on this fact, we then introduce a parameter t in $[0, 1]$, add the term $1 - t$ to every diagonal entry of $\Sigma(p)$, and let the resultant matrix be $\Sigma(p, t)$. Using it we form a new matrix $R(p, t) = L_1(p)\Sigma(p, t)L_2(p)$, which is easily seen to be of full rank for every t in $[0, 1)$. With $R(p, t)$ as a substitute for $R(p)$, we are finally able, by means of the homotopy method, to establish the generic existence of GEI equilibria. This idea is carried out in detail in Section 3. It must be remarked that all the technicalities required to realize the idea have already been available in the literature, especially in Brown et al. (1996); what we have to do is simply put them in order.

We notice that from the constructive proof above a path-following algorithm can be derived for computing GEI equilibria. This algorithm is, in comparison with the existing ones (Brown et al. (1996) and Schmedders (1998)), marked by its simplicity and low-dimensional search space. To test its performance we apply it in Section 4 to compute the equilibria for some numerical examples; the results of which turn out to be rather satisfactory, showing that the algorithm derived is both effective and efficient. Before embarking on an elaboration of the preceding results, we have first of all to spell out the model economy with which we shall be concerned; let us do this in the ensuing section.

2 The GEI Model

We shall study the same general equilibrium model with incomplete real asset markets as for instance in Brown et al. (1996); Duffie and Shafer (1985); Schmedders (1998). This model consists of two periods with uncertainty over the states of nature in the second period. In this period we assume there are S number of possible states, which will be indexed by s running from 1 to S ; call the first period state zero. In each state there are L commodities available, which will be indexed by l running from 1 to L . We shall hereafter use the symbol, x_{sl} , to refer to a certain amount of commodity l in state s .

We assume K real assets marketed in the economy, which will be indexed by k running from 1 to K ; to make the market incomplete we further assume $K < S$. By a real asset we shall mean an asset whose payoff in each state is specified by a bundle of commodities in that state. Let $M = L(S + 1)$; we can therefore characterize each asset by a vector of dimension M whose first L components denote its payoff in state zero, the second L components its payoff in state one, and so forth. This in turn enables us to characterize the asset market by a matrix A , of dimension $M \times K$, with its k -column designating the payoff of asset k . Without loss of generality and to simplify the exposition we shall assume that every asset has a zero payoff in state zero.

Let $\Delta = \{p \in \mathbf{R}_+^M \mid \sum_{i=1}^M p_i = 1\}$ be the price simplex and Δ_{++} its relative interior. For each $p \in \Delta_{++}$ it is often convenient to partition it state-wise into $p = (\bar{p}_0, \dots, \bar{p}_S)$ with $\bar{p}_s \in \mathbb{R}^L$ being the price vector for state s , $s = 0, \dots, S$. As in Brown et al. (1996) we define for a particular $p = (\bar{p}_0, \dots, \bar{p}_S) \in \Delta$

$$P = \begin{bmatrix} 0 & \bar{p}_1 & 0 & \cdots & 0 \\ 0 & 0 & \bar{p}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{p}_S \end{bmatrix}_{S \times M} ;$$

the nominal return matrix at the price vector p of the assets is then given by

$$R(p) = PA \in \mathbf{R}^{S \times K}. \quad (1)$$

We assume $H + 1$ households inhabiting in the economy, which will be indexed by h running from 0 to H . Household h is characterized by an endowment vector $e^h \in \mathbf{R}_{++}^M$ and a utility function $U_h : \mathbf{R}_{++}^M \rightarrow \mathbf{R}$ with the following standard properties:

- (i) smoothness: U_h is C^∞ ,
- (ii) strict monotonicity: $DU_h(x) \in \mathbf{R}_{++}^M$ for all x in \mathbf{R}_{++}^M ,
- (iii) differentially strict convexity: $z^\top D^2 U_h(x) z < 0$ for all $z \neq 0$ with $DU_h(x) z = 0$,
- (iv) boundary condition: $\{x \in \mathbf{R}_{++}^M \mid U_h(x) \geq U_h(x_0)\}$ is closed in \mathbf{R}^M for all x_0 in \mathbf{R}_{++}^M .

Given the incomplete market structure the households, in contradistinction to the case of complete markets, will face a sequence of markets. In first period there are spot markets for L current commodities and securities markets for K assets. In the second period the assets will pay off and the spot markets in every state will be reopened for the L commodities. For household h , if we let $\mathbf{x} = (\bar{x}_0, \dots, \bar{x}_S) \in \mathbf{R}_+^M$ be his total consumption vector with \bar{x}_s his consumption vector for state s , $e^h = (\bar{e}_0^h, \dots, \bar{e}_S^h)$, and $\theta = (\theta_1, \dots, \theta_k)$ his portfolio of assets, then at a particular commodity price

vector $p = (\bar{p}_0, \dots, \bar{p}_S) \in \Delta$ and an asset price vector $q \in \mathbf{R}_+^k$, household h , so as to determine his demand, will be confronted with the following problem:

$$\begin{aligned} \max_{(x, \theta)} U_h(x), \quad \text{subject to} \\ \bar{p}_0 \cdot (\bar{x}_0 - \bar{e}_0^h) + q\theta = 0, \\ P(x - e^h) = R(p)\theta. \end{aligned}$$

As is argued in Brown et al. (1996), this problem would have a solution only if the asset price vector q does not admit an arbitrage opportunity. With this consideration the above maximization problem is equivalent to:

$$\begin{aligned} \max_x U_h(x), \quad \text{subject to} \\ p \cdot (x - e^h) = 0, \\ P(x - e^h) = R(p)\theta \text{ for some } \theta. \end{aligned} \quad (2)$$

We now proceed to define the notion of equilibrium. For this let $x_h^*(p, R(p))$ be the solution to problem (2), and define

$$Z^h(p, R(p)) = x_h^*(p, R(p)) - e^h$$

to be the excess demand function for household h . Let $U = (U_0, \dots, U_H)$ and $e = (e^0, \dots, e^H)$; then a GEI economy is completely characterized by the triplet (U, e, A) .

DEFINITION. A GEI equilibrium for the economy (U, e, A) is a price vector $p \in \Delta$ such that $\sum_{h=0}^H Z^h(p, R(p)) = 0$.

Also as is explained in Brown et al. (1996) this definition is not convenient to deal with for certain reasons. In view of this we introduce the Cass trick (see Cass (2006)). Consider household zero; suppose that he faces, instead of problem (2), the following problem with a complete market structure:

$$\max_x U_0(x), \quad \text{subject to } p \cdot (x - e^0) = 0.$$

We call this household an unconstrained one and suppose that $x^u(p)$ is his demand function. Let $Z^u(p) = x^u(p) - e^0$ and $Z(p) = Z^u(p) + \sum_{h=1}^H Z^h(p, R(p))$. Then according to the Cass trick the above definition of equilibrium is equivalent to the following:

DEFINITION. A GEI equilibrium for the economy (U, e, A) is a price vector $p \in \Delta$ such that $Z(p) = 0$.

3 The Equilibrium Existence Theorem

The purpose of this section is to establish the existence of GEI equilibria as defined above. From Hart (1975) we have learned that such an attempt may fail when the assets involved are real; so what we shall do is rather to establish the *generic* existence of GEI equilibria, or, more precisely, the existence of GEI equilibria for almost every $(e, A) \in W$, where $W = \mathbf{R}^{(H+1)M} \times \mathbf{R}^{M \times K}$.

It has become well known, since the work of Hart (1975), that the main difficulty in doing so lies in the fact that the asset return matrix $R(p)$ may drop rank as p varies, leading then to a discontinuous demand function. To circumvent this difficulty we make use of the homotopy method, by first introducing into the picture a parameter $t \in [0, 1]$, and then defining a new asset return matrix $R(p, t)$ as follows. Let the singular value decomposition of $R(p)$ be given by

$$R(p) = L_1(p) \cdot \Sigma(p) \cdot L_2(p),$$

where $L_1(p), L_2(p)$ are two orthogonal matrices and $\Sigma(p)$ is a diagonal matrix. Let $\sigma_1(p), \dots, \sigma_K(p)$ be the diagonal entries of $\Sigma(p)$. From matrix theory it follows that $\sigma_k(p) \geq 0$ for all k , and that the rank of $R(p)$ is equal to the number of $\sigma_k(p)$ that is strictly positive. Using this result we set $\sigma_{k,t}(p) = \sigma_k(p) + (1-t)$, and define

$$R(p, t) = L_1(p) \Sigma_t(p) L_2(p),$$

where $\Sigma_t(p) \in \mathbf{R}^{S \times K}$ is a diagonal matrix with its k -th diagonal entry given by $\sigma_{k,t}(p)$; it is easily seen that $R(p, t)$ has full rank for all $(p, t) \in \mathbf{R}_+^M \times [0, 1]$.

Let $X = \mathbf{R}_{++}^M \times [0, 1]$, a smooth manifold with boundary of dimension $M+1$, and let

$$E = \{(p, t, w) \in X \times W \mid R(p, t) \text{ has full rank } K\}.$$

Noting that E is an open subset of $X \times W$, it is a smooth submanifold. Replace $R(p)$ with $R(p, t)$ in problem (2) and let $Z^h(p, R(p, t))$ be the counterpart of $Z^h(p, R(p))$ for this new problem. Given a vector v let v_{-1} be the vector formed by deleting the last component of v . Consider the homotopy $H : E \rightarrow \mathbf{R}^M$ defined by

$$H(p, t, w) = \begin{bmatrix} Z_{-1}^u(p) + t \sum_{h=1}^H Z_{-1}^h(p, R(p, t)) \\ \sum_i p_i - 1 \end{bmatrix}; \quad (3)$$

and refer to the first set of equations as $H_{-1}(p, t, w)$. It is by means of this homotopy that we shall establish the existence of GEI equilibria:

THEOREM 1. The economy (U, e, A) has a GEI equilibrium for almost every $(e, A) \in W$. Moreover, if multiple equilibria exist, they are locally unique and odd in number.

PROOF. See the Appendix.

Q.E.D

4 Numerical Test

We notice that, on the basis of the proof of Theorem 1, a path-following algorithm of predictor-corrector type could be devised for the numerical computation of GEI equilibria. The purpose of this section is not to present the details of this algorithm (as it has become a standard practice in the field of applied general equilibrium theory), for which we refer the reader to Allgower and Georg (1993), Eaves and Schmedders (1999); but instead to test its performance through some numerical examples, and compare it with the existing algorithms.

EXAMPLE 1. This example is extracted from Brown et al. (1996). Let there be three households, three future states, two assets, and two commodities in each state. Assume that households 1 and 2 are the same in terms of their characteristics, i.e., they share the same utility function and the same endowment vector. The utility functions take on the following form:

$$U_h(x) = - \sum_{s=0}^S \lambda_s \left(K - \prod_{l=1}^L (x_{sl})^{\alpha_l^h} \right)^2, \quad h = 0, 1,$$

for which $K = 5.7$, $\lambda = [1, 1/3, 1/3, 1/3]$, $\alpha^0 = [1/4, 3/4]$, and $\alpha^1 = [3/4, 1/4]$. The endowment vectors of households 0 and 1 are given respectively by $e^0 = [2, 2; 0.5, 1; 1, 1; 1.5, 1]$ and $e^1 = [1, 1; 2.5, 2; 2, 2; 1.5, 2]$. Assume furthermore that the two assets available are forward contracts for each good; that is, asset 1 delivers one unit of good 1 in each state, and asset 2 delivers one unit of good 2 in each state; or more formally,

$$A^\top = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

In this example of economy, the asset return matrix $R(p)$ drops rank at the price equilibrium vector corresponding to $t = 0.5$. For further analysis of this example consult Brown et al. (1996); here we only present the numerical result of the computation. We implement the algorithm by means of a MATLAB routine. After 22 iterations (by an iteration we mean a combination of the predictor and corrector steps), it terminates with the following price equilibrium vector:

$$[0.3026, 0.2211; 0.0873, 0.0673; 0.0925, 0.0661; 0.0984, 0.0647].$$

It is readily seen that this result is the same, up to normalization, as what is obtained in Brown et al. (1996); it however takes the latter 65 iterations to arrive at this result.

EXAMPLE 2. This example is extracted from Schmedders (1998, Section 6.2). It is almost the same as Example 1, except for the following modifications: $K = 57$, $e^0 = [20, 20; 8, 24; 10, 30; 618]$, $e^1 = [10, 10; 25, 20; 20, 20; 15, 20]$, and

$$A^\top = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & -1 & 2 & -1 \end{bmatrix}.$$

In this example of economy, the asset return matrix $R(p)$ drops rank at the starting price equilibrium vector. We run once more our MATLAB routine; after 21 iterations it terminates with the following price equilibrium vector:

$$[0.3034, 0.2323; 0.0898, 0.0584; 0.0966, 0.0518; 0.1106, 0.0571],$$

which is easily seen to be all but identical with what is obtained in Schmedders (1998).

EXAMPLE 3. This is a large-scale example extracted again from Schmedders (1998, Section 6.3.2), and the computation of its equilibrium was claimed there to be the most challenging. This example is given here for the purpose of comparing the performance of the algorithm in this paper with that of Schmedders (1998). For this reason the latter is implemented also by means of a MATLAB routine (instead of the HOMPACk package), and both algorithms are run on a DELL Inspiron 15R laptop.

Let there be three households, five future states, two assets, and five commodities in each state. The utility function of each household is given by the following functional form:

$$U_h(x) = \frac{1}{\gamma^h + 1} \sum_{s=0}^S \lambda_s \sum_{l=1}^L (x_{sl})^{\gamma^h + 1}, \quad h = 0, 1, 2$$

for which $\lambda = [1, 1/5, 1/5, 1/5, 1/5, 1/5]$, $\gamma^0 = -0.5$, $\gamma^1 = -3$, and $\gamma^2 = -8$. The endowment vectors of the households are generated randomly with each e_{sl}^h being a random variable following the uniform distribution on $[0.75, 1.25]$, $h = 0, 1, 2$, $s = 0, \dots, 5$, $l = 1, \dots, 5$. Assume furthermore that the payoffs of the two assets are given by

$$A^\top = \begin{bmatrix} \pi_0 & \pi_0 & \pi_0 & \pi_0 & \pi_0 \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{bmatrix},$$

where $\pi_0 = (1, 0, 0, 0, 0)$, $\pi_s = (0, 0.7 + s, 0, 0, 0)$, $s = 1, \dots, 5$; namely, the payoff of the first asset is one unit of commodity 1 in each state of the second period, and that of the second asset $0.7 + s$ units of commodity 2 in state s , $s = 1, \dots, 5$.

We create 10 samples of this economy by randomly varying the endowment vectors, and use both the algorithm of this paper and that of Schmedders (1998) to compute their equilibria. The performance of these two algorithms is given in Table 1, in which ‘Iteration’ indicates the average number of iterations that it takes for the algorithm to converge, and ‘CPU(s)’ the average time (in seconds); ‘ME’ indicates the maximum error in all samples (see Schmedders (1998, Section 6.3)). From the table we can see that the algorithm of this paper takes less number of iterations but each

	Iteration	CPU(s)	ME $\times 10^{-7}$
Algorithm of this paper	91.5	62.5	8.86
Schmedders (1998)	114.9	64.4	5.80

Table 1: Performance comparison

iteration takes a longer time, and so we may conclude that the two algorithms are on the whole comparable in terms of the running time and maximum error.

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A Appendix: Proof of Theorem 1

We shall proceed in three steps, each of which will be formulated as a lemma. To state the first lemma, we shall need the following notion: Given a $w \in W$ let $H_w(p, t) = H(p, t, w)$; then H_w is said to be *boundary-free* if $\text{cl}\{p \mid (p, t) \in H_w^{-1}(0)\} \subset \mathbf{R}_{++}^M$.

LEMMA 1. The homotopy $H(p, t, w)$ is smooth on E , and, moreover, H_w is boundary-free.

PROOF. To see the smoothness of H , we notice, in view of Proposition 2.7.2 of Mas-Colell (1989), that $Z^u(p)$ is smooth on E , so there remains to prove the smoothness of $Z^h(p, R(p, t))$ for $h = 1, \dots, H$. But this proof, by recalling that $R(p, t)$ has full rank on E , is identical with that of Case 2 of Lemma 2 in Schmedders (1999).

To see that H_w is boundary-free, we take a sequence $(p^r, t^r) \in H_w^{-1}(0)$ and suppose that it converges to (p^*, t^*) . We wish to show that $p^* \in \mathbf{R}_{++}^M$. Suppose not; then if, in the first place, $p_i^* = 0$ for some $1 \leq i \leq M-1$, then $\lim_{p^r \rightarrow p^*} \|Z_{-1}^u(p^r)\| = \infty$, contradicting the fact that $(p^r, t^r) \in H_w^{-1}(0)$. If, in the second place, $p_M^* = 0$, then we have

$$\lim_{p_M^r \rightarrow p_M^*} Z_M^u(p^r) = \infty, \quad (\text{A.1})$$

where Z_M^u denotes the M -th component of Z^u . Recall that the Walras’ law asserts that

$$p_{-1}^r H_{-1}(p^r, t^r, w) + p_M^r [Z_M^u(p^r) + t^r \sum_{h=1}^H Z_M^h(p^r, R(p^r, t^r))] = 0;$$

for which we have, since $(p^r, t^r) \in H_w^{-1}(0)$, that the first term on the left-hand side vanishes and $p_M^r > 0$. It then follows that $Z_M^u(p^r) + t^r \sum_{h=1}^H Z_M^h(p^r, R(p^r, t)) = 0$ for all r . But this, noting that each $Z_M^h(p^r, R(p^r, t))$ is bounded from below, is in contradiction to Eq. (A.1). This proves $p^* \in \mathbf{R}_{++}^M$, and hence H_w is boundary-free. Q.E.D

Let the symbol, ∂ , stand for the operation of taking boundary of a manifold; let $\partial H : \partial X \times W \rightarrow \mathbf{R}^M$ and $\partial H_w(x) = \partial H(x, w)$. For more information on the notions involved from differentiable topology we refer to Mas-Colell (1989) or Milnor (1997).

LEMMA 2. For generic w , the set $H_w^{-1}(0)$ is a smooth, compact one-dimensional submanifold of X , and $\partial H_w^{-1}(0) = H_w^{-1}(0) \cap \partial X$.

PROOF. We first show that $H_w^{-1}(0)$ is a smooth one-dimensional submanifold of X . To see this we claim that 0 is a regular value of both H and ∂H . In fact, it follows from the proof of Proposition 17.D.4 of Mas-Colell et al. (1995) that $D_{e^0}Z_{-1}^u(p) = M - 1$, and hence $D_e H_{-1}(p, t, w) = M - 1$. Let $\mathbf{1} = (1, \dots, 1)$; then we have

$$D_{(e,p)}H(p, t, w) = \begin{bmatrix} D_e H_{-1}(p, t, w) & D_p H_{-1}(p, t, w) \\ 0 & \mathbf{1} \end{bmatrix}.$$

It is a simple matter to show that this matrix has rank M ; so, by its very definition, we obtain that 0 is a regular value of H . Notice that W is a smooth manifold without boundary; it follows from the transversality theorem (see for instance Mas-Colell (1989, p. 45)) that 0 is a regular value of H_w for almost every $w \in W$. Then an application of the above argument to ∂H shows at once that 0 is a regular value of ∂H_w for almost every $w \in W$. Combining the above reasoning we obtain that for generic w , 0 is a regular value of both H_w and ∂H_w ; and therefore, by the implicit function theorem (see for instance Mas-Colell (1989, p. 43)), that $H_w^{-1}(0)$ is a smooth one-dimensional submanifold of X with $\partial H_w^{-1}(0) = H_w^{-1}(0) \cap \partial X$.

We now show that $H_w^{-1}(0)$ is compact. From the last equation of H we know that $H_w^{-1}(0)$ is bounded, so it remains to show that $H_w^{-1}(0)$ is closed. Suppose by contradiction that there exists a sequence $(p^r, t^r) \in H_w^{-1}(0)$ converging to $(p^*, t^*) \notin H_w^{-1}(0)$. First we have $p^* \in \mathbf{R}_{++}^M$, since H_w is boundary-free. Let $E_w = \{(p, t) \in X \mid (p, t, w) \in E\}$. We claim that $(p^*, t^*) \notin E_w$, for otherwise (p^*, t^*) would belong to $H_w^{-1}(0)$ (as H_w is, by Lemma 1, smooth on E_w). Since $p^* \in \mathbf{R}_{++}^M$ we have $(p^*, t^*) \in X$, so the sole possibility in agreement with the fact of $(p^*, t^*) \notin E_w$ is that $\text{rank } R(p^*, t^*) < K$. Remember that $R(p, t)$ has full rank for every p and every $t \in [0, 1]$; we obtain that $t^* = 1$.

To complete the proof, it suffices now to apply the argument of Theorem V of Brown et al. (1996, p. 17), to show that $(p^*, 1) \notin E_w$ is generically impossible. Since $R(p^r, t^r)$ is of full rank for all r , its span must converge to some K -dimensional subspace of \mathbf{R}^S . Therefore

$$\text{span } R(p^r, t^r) \rightarrow \text{span } \Phi \begin{bmatrix} I \\ V \end{bmatrix}$$

for some permutation matrix Φ and $V \in \mathbf{R}^{(S-K) \times K}$. Then from the continuity of both Z and the sum $\sum_i p_i$, we have

$$Z_{-1}^u(p^*) + \sum_{h=1}^H Z_{-1}^h(p^*, \Phi \begin{bmatrix} I \\ V \end{bmatrix}) = 0 \quad (\text{A.2})$$

$$\sum_i p_i^* - 1 = 0 \quad (\text{A.3})$$

Since $\text{rank } R(p^*, 1) < K$, there must exist a redundant column, say the i -th column. Let $R_{-i}(p^*, 1) \in \mathbf{R}^{S \times (K-1)}$ be the matrix formed by deleting the i -th column of $R(p^*, 1)$. Then we have

$$[-V \ I] \Phi^{-1} R_{-i}(p^*, 1) = 0 \quad (\text{A.4})$$

$$R(p^*, 1) \alpha = 0 \quad (\text{A.5})$$

for some $\alpha \in \mathbf{R}^K$ with α_i nonzero. Let $F(p, V, \alpha, w)$ be the set of functions on the left-hand side of Eqs. (A.2)–(A.5). Direct counting shows that F is a vector of dimension $M + (S - K) \times (K - 1) + S$, with the unknowns (p, V, α) belonging to $\mathbf{R}_{++}^M \times \mathbf{R}^{(S-K) \times K} \times \mathbf{R}^{K-1}$, a manifold of dimension $M + (S - K)K + K - 1$.

Let $F_w(p, V, \alpha) = F(p, V, \alpha, w)$. We claim that $F_w^{-1}(0)$ is empty for almost every w . Indeed, taking the derivative of Eq. (A.2) with respect to e^0 shows that it has rank $M - 1$; taking the derivative of Eq. (A.3) with respect to p shows that it has rank 1; taking the derivative of Eq. (A.4) with respect to A_{-i} shows that it has rank $(S - K)(K - 1)$; and, finally, taking the derivative of Eq. (A.5) with respect to A_i shows that it has rank S . Therefore 0 is a regular value of $F(p, V, \alpha, w)$. From the transversality theorem it then follows that 0 is a regular value of F_w for almost every $w \in W$. By means of the implicit function theorem we have that $F_w^{-1}(0)$ is a manifold of dimension

$$[M + (S - K)K + K - 1] - [M + (S - K) \times (K - 1) + S] = -1.$$

This means that $F_w^{-1}(0)$ is generically empty, hence that $(p^*, 1) \notin E_w$ is generically impossible. From this we conclude that $H_w^{-1}(0)$ is closed, hence compact, for almost every $w \in W$. *Q.E.D*

LEMMA 3. For generic $(e, A) \in W$, the economy (Z^h, e, A) has an equilibrium. Moreover, if multiple equilibria exist, they are locally unique and odd in number.

PROOF. The proof of this lemma is similar to (but simpler than) that of Theorem VI of Brown et al. (1996), so it is omitted here. *Q.E.D*

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