OPTIMAL INVESTMENT MODELS WITH STOCHASTIC VOLATILITY AND PORTFOLIO CONSTRAINTS: THE TIME INHOMOGENEOUS CASE.

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ABSTRACT. In a recent paper by Pham [11] a multidimensional model with stochastic volatility and portfolio constraints has been proposed, solving a class of investment problems. One feature which is common with these problems is that the resultant Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) is highly nonlinear. Therefore, a transform is primordial to express the value function in terms of a semilinear PDE with quadratic growth on the derivative term. Some proofs for the existence of smooth solution to this equation have been provided for this equation by Pham [11]. In that paper they illustrated some common stochastic volatility examples in which most of the parameters are time-homogeneous. However, there are cases where time-dependent parameters are needed, such as in the calibrating financial models. Therefore, in this paper we extend the work of Pham[11] to the time-inhomogeneous case.

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1. INTRODUCTION

Pham [11] considered a multidimensional problem with random volatilities which include the degenerate cases and correlation between assets and the stochastic factors. Solving the resultant PDE when finding solution for stochastic volatility models has been a major focus of many researchers (see Benth and Kalsen [1], Kufakunesu [9], Hobson [6], and Heston [5] among others). The HJB equation leads to the characterization of the value function which then gives birth to highly nonlinear PDE [11]. Various authors have devised some transformations which express value functions in terms of a linear or semilinear PDE. One example is the logarithmic transform assuming some constant relative risk aversion utility function introduced by Fleming [2]. In this paper, extending the work of Pham [11], we shall consider the time-dependent utility function (see Karatzas [8] page 34, for this concept). We discuss cases when time-dependent extensions on parameters are possible to lead us to a linear or semilinear PDE. For instance, an example of power transform in Pham [11] suggested by Zariphopoulou [15] is not easily extendible to the time-dependent case of its value function.

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The structure of our paper is similar to that of Pham [11]. In Section 2, we discuss the time-dependent investment and the stochastic control models. In Section 3, a Hamilton-Jacobi-Bellman equation is derived and we apply a time-dependent logarithmic transformation. A verification theorem relating the value function of the solution of the semilinear PDE is stated. In Section 4, the existence proofs of the time dependent PDE are discussed and in Section 5, a series of examples are discussed using the theory from the previous sections.

2. The model

In this section, we describe an extension of the model in Pham [11] to the time-dependent case on parameters. Consider $(\Omega, \mathcal{F}, P)$ to be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,\mathbb{T}]}$ satisfying the usual conditions, where $\mathbb{T} < \infty$ is the time horizon (Protter [12]). In this space, a bond S^0 and n risky assets S_t price processes are defined in the following way: the bond is given as

$$dS_t^0 = rS_t^0 dt, \tag{2.1}$$

where $r \in \mathbb{R}$ is the interest rate. Let B be a d-dimensional and W be an m-dimensional standard Brownian motions, respectively. The price of the n risky assets S are modelled as:

$$dS_t = diag(S_t)\left[b(t, Y_t)dt + \sigma(t, Y_t)dB_t + \beta(t, Y_t)dW_t\right] \ , \quad S(0) > 0, \tag{2.2}$$

where Y_t valued in $\mathbb{R}^d$ is a stochastic volatility process which is defined as

$$dY_t = \eta(t, Y_t)dt + dB_t \, , \quad Y(0) > 0 \, . \tag{2.3}$$

The continuous time-dependent parameter functions which are assumed in Equations 2.2 and 2.3 are as follows: $b : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{(n+1)\times d}$, $\eta : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, and $\beta : [0, \infty) \times \mathbb{R}^{n\times d} \to \mathbb{R}^{(n+1)\times m}$. Here, $diag(S_t)$ is the diagonal $n \times n$ matrix with diagonal elements S_t^i, where S_t^i is the price of asset number i, with $i = 1, \cdots, n$. We assume the following Lipschitz condition **A1** on the $\mathbb{R}^d$ - valued function η :

$$|\eta(y^*) - \eta(z^*)| \le C|y^* - z^*|, \qquad \forall\, y^*, z^* \in [0, \infty) \times \mathbb{R}^d, \tag{2.4}$$

where $y^* = (t, y)$ and $z^* = (t, y)$. From (2.4) the linear growth condition is fulfilled, therefore, by Gronwall's lemma there exists a constant C [11]:

$$|Y_t| \le C\left(1 + \int_0^t |W_u|du + |W_t|\right), \quad t \in [0, \mathbb{T}].$$

Hence, there exists some $\epsilon > 0$ such that

$$\sup_{t\in[0,\mathbb{T}]} \mathbb{E}[\exp(\epsilon|Y_t|^2)] < \infty. \tag{2.5}$$

Denote as in [11] the following:

$$\mu(t, y) = b(t, y) - r\mathbf{e}_{n+1}, \quad y^* \in [0, \mathbb{T}] \times \mathbb{R}^d$$

and this is called the excess rate of return of the risky assets with $\mathbf{e}_{n+1}$ representing the vector of one in $\mathbb{R}^{n+1}$. We also denote

$$\Sigma(t,y) = (\sigma(t,y)\ \beta(t,y))$$

as the $(n+1)\times(d+m)$ matrix-valued volatility function of the risky assets. We assume that for a.e $y^* \in [0,\mathbb{T}]\times\mathbb{R}^d$, $\Sigma(t,y)$ is the full rank equal to $n+1$ so that $\Sigma(t,y)\Sigma(t,y)^T$ is a nonsingular $(n+1)\times(n+1)$ matrix. Here, the symbol T represents the transposition operator. Therefore:

$$\alpha(t,y) = \inf_{\pi\in\mathbb{R}^n,\pi\neq 0} \frac{|\Sigma(t,y)^T\pi|^2}{|\pi|^2}, \quad y^* \in [0,\mathbb{T}]\times\mathbb{R}^d, \tag{2.6}$$

is the smallest eigenvalue of $\Sigma(t,y)\Sigma(t,y)^T$, which is then strictly positive a.e. $y^* \in [0,\mathbb{T}]\times\mathbb{R}^d$. Another assumption is the following (see [11]): there exists some positive constant C such that for a.e. $y^* \in [0,\mathbb{T}]\times\mathbb{R}^d$:

(i) $|\mu(t,y)/\sqrt{\alpha(t,y)} \leq C(1+|y^*|)$.
(ii) $||\sigma(t,y)||/\sqrt{\alpha(t,y)} \leq C$.

These two conditions are to be referred as **A2**. If $n=1$, we have

$$\alpha(t,y) = ||\Sigma(t,y)||^2 = ||\sigma(t,y)||^2 + ||\beta(t,y)||^2$$

and its clear that Assumption **A2**(**ii**) is fulfilled. For a general n, $\alpha(t,y)$ is larger than the smallest eigenvalue of $\sigma\sigma^T(t,y)$ that is,

$$\alpha(t,y) \geq \inf_{\pi\in\mathbb{R}^n,\pi\neq 0} \frac{|\Sigma(t,y)^T\pi|^2}{|\pi|^2}, \quad y^* \in [0,\mathbb{T}]\times\mathbb{R}^d$$

and $||\sigma(t,y)||^2$ is bounded by the largest eigenvalue of $\sigma\sigma^T(t,y)$. Assumption **A2**(**i**) can be seen as a condition on the growth of $\mu(t,y)$. As in [11], we consider an investor allocating at any time $t\in[0,\mathbb{T}]$ the amount $\pi_t = (\pi_t^1,\cdots,\pi_t^n)^T$ of her wealth in the risky asset S and $\mathbf{1}-\pi_t^T\mathbf{e}_n$ in the bond. Due to the self-financing hypothesis her wealth process is given by the following differential equation :

$$dX_t = X_t\left[(r+\pi_t^T\mu(t,Y_t))dt + \pi_t^T\sigma(t,Y_t)dB_t + \pi_t^T\beta(t,Y_t)dW_t\right]. \tag{2.7}$$

We let $K\in\mathbb{R}^d$ be fixed closed convex set containing the origin. The portfolio control $\pi=(\pi_t)$ is called *admissible* if it is an $\mathcal{F}_t$ - adapted stochastic process satisfying the following condition:

$$\sup_{t\in[0,\mathbb{T}]} \mathbb{E}[\exp(\epsilon|\Sigma(t,Y_t)^T\pi_t|)] < \infty\,, \tag{2.8}$$

for some $\epsilon>0$, and the constraint $\pi_t\in K,\quad t\in[0,\mathbb{T}]$ a.s. The set of admissible controls is donated by $\mathcal{A}(K)$. Considering Markov controls, means that the investor will allocate the amount $\pi\equiv\pi(t,x,y)\in\mathcal{A}(K)$ into the risky asset when the wealth $X_t=x$ and volatility

$Y_t = y$ (see e.g., [11],[9],[1]). We consider an investor with a time-dependent relative risk aversion utility function $U : [0, \mathbb{T}] \times \mathbb{R}_+ \to \mathbb{R}$

$$U(t,x) = \frac{x^{\delta(t)}}{\delta(t)}, \; x \geq 0,$$

for $\delta(t) < 1$, $\delta(t) \neq 0$. The resulting value in this case is:

$$v(t,x,y) = \sup_{\pi \in \mathcal{A}(K)} \mathbb{E}\left[\frac{X_{\mathbb{T}}^{\delta(\mathbb{T})}}{\delta(\mathbb{T})} \,|\, X_t = x, \; Y_t = y\right], \tag{2.9}$$

where $(t,x,y) \in [0,\mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d$. The task at hand as in [11] is to characterize the value function v as a classical solution of a semilinear partial differential equation and to use this property to provide convergence of numerical schemes for the value function and the optimal portfolio taking into consideration the time-inhomogeneous case of some parameters.

3. Derivation of the HJB Equation

The Hamilton-Jacobi-Bellman (HJB) equation associated to the control problem (2.9) is the following nonlinear PDE [11]:

$$\begin{aligned} &\frac{\partial v}{\partial t} + rx\frac{\partial v}{\partial x} + \eta(t,y)^T D_y v + \frac{1}{2}\Delta_y v \\ &\quad + \max_{\pi \in K}\left[\pi^T \mu(t,y) x \frac{\partial v}{\partial x} + \frac{1}{2}|\Sigma(t,y)^T \pi|^2 x^2 \frac{\partial^2 v}{\partial x^2} + \pi^T \sigma(t,y) x D^2_{xy} v\right] = 0, \end{aligned} \tag{3.1}$$

for $(t,x,y) \in [0,\mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d$. Here we have represented $D_y v$ as the gradient vector of v with respect to y, $\Delta_y v$ as the Laplacian of v with respect to y and $D^2_{xy} v$ as the second order derivative vector v with respect to (t,x,y), [11]. The terminal data is given as follows:

$$v(\mathbb{T},x,y) = \frac{x^{\delta(t)}}{\delta(t)}, \tag{3.2}$$

where $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^d$. We introduce the following logarithmic transform

$$v(t,x,y) = \frac{x^{\delta(t)}}{\delta(t)} \exp(-\phi(t,y)). \tag{3.3}$$

Differentiating we have:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \left[-\frac{\partial \phi}{\partial t} + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)}\right] v, & x\frac{\partial v}{\partial x} &= \delta(t) v, \\ x^2 \frac{\partial^2 v}{\partial x^2} &= \delta(t)(\delta(t)-1) v, & D_y v &= v D\phi, \\ \Delta_y v &= [-\Delta \phi + D\phi^T D\phi] v, & x D^2_{xy} v &= -\delta(t) v D\phi. \end{aligned}$$

This transform simplifies our transform significantly by changing the nonlinear PDE into a semilinear one. We differentiate and substitute in (3.1) we obtain the following:

$$-\phi_t - \frac{1}{2}\Delta\phi_k + F(t,y,D\phi_k) = 0, \quad (t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d \,, \tag{3.4}$$

with terminal condition:

$$\phi(\mathbb{T},y) = 0, \quad y \in \mathbb{R}^d \,, \tag{3.5}$$

and the nonlinear function F is defined on $[0,\mathbb{T}) \times \mathbb{R}^d \times \mathbb{R}^d$ is defined as

$$\begin{aligned} F(t,y,p) &= \frac{1}{2}|p|^2 - p^T\eta(t,y) + \delta(t)r + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)} \\ &\quad + \max_{\pi\in K}\left[\delta(t)\pi^T(\mu(t,y) - \sigma(t,y)p) - \frac{\delta(t)(1-\delta(t))}{2}|\Sigma(t,y)^T\pi|^2\right] . \end{aligned} \tag{3.6}$$

Remark. If we were to substitute Zariphopoulou [15]'s value function endowed with time-dependent parameters:

$$v(t,x,y) = \left(\frac{x^{\delta(t)}}{\delta(t)}\right) j(t,y)^{\beta^*(t)}$$

for $\beta^*(t) \neq 0$ into Equation (3.1), we obtain the following PDE for j:

$$\begin{aligned} &\frac{\partial j}{\partial t} + \frac{\delta(t)}{\beta^*(t)}rj + \eta(t,y)^T Dj + \frac{1}{2}\Delta j + \frac{\beta^*(t)}{2}\frac{|Dj|^2}{j} + \frac{d\beta^*(t)}{dt}\ln(j(t,y)) + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)} \\ &+ \max_{\pi\in K}\left[\frac{\delta(t)}{\beta^*(t)}\pi^T(\mu(t,y)j + \beta^*(t)\sigma(t,y)Dj) - \frac{\delta(t)(1-\delta(t))}{2}|\Sigma(t,y)^T\pi|^2 j\right] = 0. \end{aligned}$$

As noted in [11], this PDE is still highly nonlinear for a general K. For $K = \mathbb{R}^n$ the PDE for j may be given as:

$$\begin{aligned} &\frac{\partial j}{\partial t} + \frac{\delta(t)}{\beta^*(t)} \\ &\left[r + \frac{1}{2(1-\delta(t))}\mu^T(t,y)(\Sigma\Sigma(t,y)^T)^{-1}\mu(t,y) + \frac{d\beta^*(t)}{dt}\ln(j(t,y)) + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)}\right] j \\ &\quad + \left(\eta(t,y) + \frac{\delta(t)}{1-\delta(t)}\sigma^T(t,y)(\Sigma\Sigma(t,y)^T)^{-1}\mu(t,y)\right)^T Dj + \Delta j \\ &\quad + \frac{1}{2j}(Dj)^T\left[(\beta^*(t)-1)\mathbf{I}_d + \frac{\beta^*(t)\delta(t)}{1-\delta(t)}\sigma^T(t,y)(\Sigma\Sigma(t,y)^T)^{-1}\sigma(t,y)\right] Dj = 0. \end{aligned}$$

If we set

$$(\beta^*(t)-1)\mathbf{I}_d + \frac{\beta^*(t)\delta(t)}{1-\delta(t)}\sigma^T(t,y)(\Sigma\Sigma(t,y)^T)^{-1}\sigma(t,y) = 0, \forall (t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d \,,$$

then the PDE does not become linear for j as was the case discussed in [11] for the time-independent case on parameters. A linear PDE in j, in this case, can only be obtained when $\beta^*(t) \equiv \beta^*$ constant parameter. Hence it is possible to write the value function in the following way:

$$v(t,x,y) = \left(\frac{x^{\delta(t)}}{\delta(t)}\right) j(t,y)^{\beta^*}$$

for $\beta^* \neq 0$.

We now state a verification result slightly similar to (see[11] Theorem 3.1) which relates a solution of the semilinear equation to the stochastic control problem (2.9) with time dependent parameters.

Proposition 3.1. *Assume that Equations* (2.4) *and* (2) *given by Assumptions* **A1** *and* **A2** *respectively hold. Suppose there exists a solution* $\phi \in C^{1,2}([0,\mathbb{T}],\mathbb{R}^d) \cap ([0,\mathbb{T}],\mathbb{R}^d)$ *to the semilinear PDE* (3.4) *with terminal condition* (3.5)*. Then the value function of* (2.9) *is given by*

$$v(t,x,y) = \frac{x^{\delta(t)}}{\delta(t)} \exp(-\phi(t,y)) \qquad (t,x,y) \in [0,\mathbb{T}] \times \mathbb{R}_+ \times \mathbb{R}^d. \tag{3.7}$$

Moreover, the Markov optimal control π^* *is given by*

$$\pi^*(t,y) \in \arg\min_{\pi \in K} \left[\frac{1-\delta(t)}{2}|\Sigma(t,y)^T\pi|^2 - \pi^T(\mu(t,y) - \sigma(t,y)D\phi(t,y))\right],$$

for almost every $(t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d$.

Proof. The proof is in essence similar to (Pham [11], Theorem 3.1). We clarify some points concerning the time-dependence of some parameters. Denote as in[11]

$$J(t,x,y,\pi) = \mathbb{E}[U(X_{\mathbb{T}})|X_t = x\,, Y_t = y,$$

the expected utility of terminal wealth for Y and X controlled by $\pi \in \mathcal{A}(K)$. For any $\pi \in \mathcal{A}(K)$, the integrability condition (2.5) ensures that the equivalent martingale measure Q^π, whose Radon-Nikodym derivative is:

$$\begin{aligned} & \frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t \\ = \; & \exp\left(\int_0^t \delta(u)\pi_u^T\sigma(u,y)dB_u + \int_0^t \delta(u)\pi_u^T\delta(u,y)dW_u - \frac{1}{2}\int_0^t |\delta(u)\Sigma(u,Y_u)^T\pi_u|^2 du\right), \\ & t \in [0,\mathbb{T}], \end{aligned}$$

is well defined, (see Pham [11] and Lipster Shiryaev[10]). Applying Itô's formula on $\ln X_t$ we have

$$J(t,x,y,\pi) = \frac{x^{\delta(t)}}{\delta(t)}\mathbb{E}^\pi\left[\exp\left(\int_t^{\mathbb{T}} h(u,Y_u,\pi_u)du\right)|Y_t = y\right], \tag{3.8}$$

where $\mathbb{E}^\pi$ is the expectation with respect to Q^π and

$$h(t,y,\pi) = \delta(t)r + \delta(t)\pi^T\mu(t,y) - \frac{\delta(t)(1-\delta(t))}{2}|\Sigma(t,y)^T\pi|^2 \quad + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)}.$$

Using the Girsanov's theorem, the process of Y under $Q^\pi, \pi \in \mathcal{A}(K)$ we have

$$dY_t = \left(\eta(t,Y_t) + \delta(t)\sigma(t,Y_t)^T\pi_t\right)dt + dB_t^\pi \tag{3.9}$$

where W^π is the d-dimensional Brownian motion under Q^π. Applying to Itô's formula to $\phi(t,Y_t)$ under Q^π, we obtain

$$\begin{aligned}
\phi(t,Y_{\mathbb{T}}) &= \phi(t,Y_t) + \int_t^{\mathbb{T}} \left(\frac{\partial\phi}{\partial u} + (\eta + \delta(u)\sigma^T\pi)^T D\phi + \frac{1}{2}\Delta\phi\right)(u,Y_u)du \\
&\quad + \int_t^{\mathbb{T}} D\phi(u,Y_u)^T dB_u^\pi \\
&\geq \phi(t,Y_t) + \int_t^{\mathbb{T}} h(u,Y_u,\pi_u)du + \int_t^{\mathbb{T}} D\phi(u,Y_u)^T dB_u^\pi \\
&\quad + \frac{1}{2}\int_t^{\mathbb{T}} |D\phi(u,Y_u)|^2 du.
\end{aligned}$$

Since $\phi(\mathbb{T},y) = 0$, we deduce that for all $\pi \in \mathcal{A}(K)$,

$$\mathbb{E}^\pi\left[\exp\left(\int_t^{\mathbb{T}} h(u,Y_u,\pi_u)du\right)|\mathcal{F}_t\right] \leq \exp(-\phi(t,Y_t))\mathbb{E}^\pi\left[\frac{\xi_{\mathbb{T}}^\pi}{\xi_t^\pi}|\mathcal{F}_t\right] \tag{3.10}$$

$$\leq \exp(-\phi(t,Y_t)), \tag{3.11}$$

and this is due to the martingale property of ξ^ϕ under Q^π. Hence, from Equation (3.8) have

$$J(t,x,y,\pi) \leq \frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,Y_t))\,. \tag{3.12}$$

Let

$$F(t,y,\pi) = \frac{1-\delta(t)}{2}|\Sigma(t,y)^T\pi|^2 - \pi^T(\mu(t,y) - \sigma(t,y)D\phi(t,y)).$$

This function F is continuous and convex in $\pi \in \mathbb{R}^n$. We have

$$\forall y \in \mathbb{R}^d, \pi \in \mathbb{R}^n,\ |\Sigma(t,y)^T\pi|^2 \geq \alpha(t,y)|\pi|^2,$$

with $\alpha(t,y) > 0$, a.e. The rest of the proof follows in the lines of the proof of Theorem 3.1 in Pham [11], by adjusting certain parameters into time-dependent ones. □

4. Regularity of the Value function

In this section we analyze the existence of a classical solution to the semilinear PDE (3.4). We follow the idea in Pham [11] by first establishing smoothness of the value function under at least one of the following general conditions:

(G3a). When $K = \mathbb{R}^n$ the no constraint case and

(i) η, $\sigma^T(\Sigma\Sigma^T)^{-1}\sigma$, $\sigma^T(\Sigma\Sigma^T)^{-1}\mu$ are C^1 and Lipschitz;

(ii) $\sigma^T(\Sigma\Sigma^T)^{-1}\mu$ is C^1, bounded and Lipschitz.

(G3a'). When $K=\mathbb{R}^n$ the no constraint case,
$\sigma^T(\Sigma\Sigma^T)^{-1}\sigma$ time-dependent correlation and
(i) $\eta,\ \sigma^T(\Sigma\Sigma^T)^{-1}\sigma,\ \sigma^T(\Sigma\Sigma^T)^{-1}\mu$ are C^1 and Lipschitz;
The general constraints framework is given as follows:

(G3b). The uniform elliptic volatility given by $\inf_{(t,y)\in[0,\mathbb{T}]\times\mathbb{R}^d}\alpha(t,y)>0$, $\sigma\equiv 0$ the no correlation case and
(i) $\eta,\ \sigma^T(\Sigma\Sigma^T)^{-1}\sigma,\ \sigma^T(\Sigma\Sigma^T)^{-1}\mu$ are C^1 and Lipschitz;
(ii) the function $\bar{\pi}:=(\frac{1}{1-\delta(t)})(\Sigma\Sigma^T)^{-1}\mu$ is in C^1 and Lipschitz and $(D\bar{\pi})^T(\Sigma\Sigma^T)$ is bounded;
(iii) $\Sigma\Sigma^T$ is C^1, satisfies a polynomial growth condition and

$$||D(\Sigma\Sigma^T)(t,y)||\le C(1+||\Sigma\Sigma^T(t,y)||),$$

for all $(t,y)\in[0,\mathbb{T}]\times\mathbb{R}^d$ is C^1, for $C>0$.

Theorem 4.1. [11] *Under one of the assumptions* **G3a, G3**a' *or* **G3b***, there exists a solution* $\phi\in C^{1,2}([0,\mathbb{T}),\mathbb{R}^d)\cap C^0([0,\mathbb{T}],\mathbb{R}^d)$*, with linear growth condition in* y *on the derivative* $D\phi$*, to the semilinear equation* (3.4) *with terminal condition* (3.5)*.*

As in Pham [11], combining Theorem 3.1 and 4.1, we obtain an extension to Corollary 4.1 in [11].

Corollary 4.2. *Assume* **A2** *and any one of the following assumptions* **G3a, G3**$a\prime$ *or* **G3b** *hold. Then the value function of* (2.5) *is given by*

$$v(t,x,y)=\frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,y))\qquad (t,x,y)\in[0,\mathbb{T}]\times\mathbb{R}_+\times\mathbb{R}^d.$$

and the associated optimal portfolio is given by any Markov control $(\pi_t^*=\pi^*(t,Y_t),\ 0\le t\le\mathbb{T})$ *with*

$$\pi^*(t,y)\in\arg\min_{\pi\in K}\left[\frac{1-\delta(t)}{2}|\Sigma(t,y)^T|^2-\pi^*(\mu(t,y)-\sigma(t,y)D\phi(t,y))\right].$$

a.e. $(t,y)\in[0,\mathbb{T}]\times\mathbb{R}^d$ where ϕ is the unique solution in $\phi\in C^{1,2}([0,\mathbb{T}),\mathbb{R}^d)\cap C^0([0,\mathbb{T}],\mathbb{R}^d)$, with linear growth in y on the derivative $D\phi$, to (3.4)-(3.5).

Remark. When $K=\mathbb{R}^n$, the optimal Markov control is written as follows:

$$\pi^*(t,y)=\frac{1}{1-\delta(t)}(\Sigma(t,y)\Sigma(t,y)^T)^{-1}[\mu(t,y)-\sigma(t,y)D\phi(t,y)]$$

a.e. $(t,y)\in[0,\mathbb{T}]\times\mathbb{R}^d$.

We analyze the function F when $K=\mathbb{R}^n$ we have

$$F(t,y,p)=\frac{1}{2}p^T(\mathbf{I}_d+\rho(t,y))p-p^Tf(t,y)+\lambda(t,y),$$

for $(t, y, p) \in [0, \mathbb{T}] \times \mathbb{R}^d \times \mathbb{R}$, where

$$\rho(t,y) = \frac{\delta(t)}{1-\delta(t)}\sigma(t,y)^T(\Sigma(t,y)\Sigma(t,y)^T)^{-1}\sigma(t,y) \tag{4.1}$$

$$f(t,y) = \eta(t,y) + \frac{\delta(t)}{1-\delta(t)}\sigma(t,y)^T(\Sigma(t,y)\Sigma(t,y)^T)^{-1}\mu(t,y) \tag{4.2}$$

$$\lambda(t,y) = \delta(t)r + \frac{1}{2}\frac{\delta(t)}{1-\delta(t)}\mu(t,y)^T(\Sigma(t,y)\Sigma(t,y)^T)^{-1}\mu(t,y) \tag{4.3}$$

$$+\frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)}. \tag{4.4}$$

Introduce the functions L and L'

$$L, L' : [0, \mathbb{T}) \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

by:

$$\begin{aligned} L(t,y,q) &= \max_{p\in\mathbb{R}^d}\{-q^Tp - F(t,y,p)\} \\ L'(t,y,q) &= \max_{p\in\mathbb{R}^d}\{-(q+f(t,y))^Tp - F(t,y,p)\}\ . \end{aligned}$$

We remark that since $\mathbf{I}_d + \rho(t,y)$ is symmetric definite positive therefore, L and L' are represented by the following forms:

$$L(t,y,q) = \frac{1}{2}(q-f(t,y))^T(\mathbf{I}_d+\rho(t,y))^{-1}(q-f(t,y)) - \lambda(t,y), \tag{4.5}$$

$$L'(t,y,q) = \frac{1}{2}q^T(\mathbf{I}_d+\rho(t,y))^{-1}q - \lambda(t,y). \tag{4.6}$$

Furthermore, since F is a convex continuous function in p the following duality are in order:

$$F(t,y,p) = \max_{p\in\mathbb{R}^d}\{-q^Tp - L(t,y,p)\} \tag{4.7}$$

$$= \max_{p\in\mathbb{R}^d}\{-(q+f(t,y))^Tp - L'(t,y,p)\}\ . \tag{4.8}$$

Consider a compact set $\mathcal{B}_k = \{q \in \mathbb{R}^d : |q| \le k\}$ $k > 0$ where $\mathcal{B}_k \subset \mathbb{R}^d$. We have the following truncated functions:

$$F_k(t,y,p) = \max_{q\in\mathcal{B}_k}\{-q^Tp - L(t,y,p)\} \tag{4.9}$$

$$F'_k(t,y,p) = \max_{q\in\mathcal{B}_k}\{-(q+f(t,y))^Tp - L'(t,y,p)\}\ . \tag{4.10}$$

The functions L, L', $D_{y^*}L$, $D_{y^*}L'$ and C^1 satisfy a polynomial growth condition in y^* on $[0,T)\times\mathbb{R}^d\times\mathcal{B}_k$. Therefore, by Fleming and Rishel [3], there exist unique solutions ϕ_k and $\phi'_k \in C^{1,2}([0,\mathbb{T}),\mathbb{R}^d)\cap C^0([0,\mathbb{T}],\mathbb{R}^d)$ with polynomial growth in y^* for the parabolic partial differential equations:

$$-\frac{\partial \phi_k}{\partial t} - \frac{1}{2}\Delta\phi_k + F_k(t,y,D\phi_k) = 0, \quad (t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d , \tag{4.11}$$

with Cauchy data $\phi_k(\mathbb{T}, y) = 0$ and

$$-\frac{\partial \phi'_k}{\partial t} - \frac{1}{2}\Delta\phi'_k + F_k(t,y,D\phi'_k) = 0, \tag{4.12}$$

on $[0,\mathbb{T}] \times \mathbb{R}^d$ with Cauchy data $\phi'_k(\mathbb{T}, y) = 0$. The estimates for the derivatives of $D\phi_k$ and $D\phi'_k$ are given by a theorem similar to in Pham [11] as follows:

Lemma 4.3. *Under the assumption of* **G3a** *with time-dependent parameters, there exists a positive constant C independent of k such that*

$$|D\phi_k(t,y)| \le C(1+|y|), \quad \forall (t,y) \in [0,\mathbb{T}) \times \mathbb{R}^d.$$

Under the assumption of **G3a′** *with time-dependent parameters, there exists a positive constant C independent of k such that*

$$|D\phi'_k(t,y)| \le C, \quad \forall (t,y) \in [0,\mathbb{T}) \times \mathbb{R}^d.$$

Proof. The proof goes along the lines in Pham [11]. □

As in Pham [11], we note that:

$$\frac{1}{2}|p|^2 - p^T\eta(t,y) = \max_{q\in\mathbb{R}^d}\left[-q^T p - |q-\eta(t,y)|\right],$$

for all $(t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d \times \mathbb{R}^d$. If $\sigma \equiv 0$ then the function F defined in Equation (3.6) can be written in the following forms

$$F(t,y,p) = \max_{q\in\mathbb{R}^d,\, \pi\in K}[-q^T p - \bar{L}(t,y,q,\pi)],$$

where

$$\begin{aligned} \bar{L}(t,y,q,\pi) &= \frac{1}{2}|q-\eta(t,y)|^2 + \frac{\delta(t)(1-\delta(t))}{2}|\Sigma(t,y)^T\pi|^2 \\ &\quad -\delta(t)\pi^T\mu(t,y) - \delta(t)r \\ &= \frac{1}{2}|q-\eta(t,y)|^2 + \frac{\delta(t)(1-\delta(t))}{2}|\Sigma(t,y)^T(\pi-\bar{\pi}(t,y))|^2 - \lambda(t,y), \end{aligned} \tag{4.13}$$

with $\bar{\pi}$ defined in (**G3b**). Consider the truncated function

$$\bar{F}_k(t,y,p) = \max_{q\in\mathcal{B}_k,\pi\in\mathcal{A}_k(K)}\{-q^T p - \bar{L}(t,y,q)\},$$

where $\mathcal{B}_k = \{q \in \mathbb{R}^d : |q| \le k\}$ and $\mathcal{A}_k(K) = \{\pi \in K : |\pi| \le k\}$ are compact sets. By Assumption *G3b* and from [11] and [3] there exists a unique solution $\psi_k \in C^{1,2}([0,\mathbb{T}),\mathbb{R}^d) \cap C^0([0,\mathbb{T}],\mathbb{R}^d)$ with a polynomial growth in y on ψ_k, to the parabolic PDE

$$-\psi_t - \frac{1}{2}\Delta\psi_k + \bar{F}(t,y,D\psi_k) = 0, \quad (t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d , \tag{4.14}$$

with Cauchy data $\psi_k(\mathbb{T}, y) = 0$. The following estimate of $D\psi_k$ is obtained.

Lemma 4.4. *Under the assumption of* **G3b** *with time-dependent parameters there exists a positive constant* C *independent of* k *such that*

$$|D\psi_k(t,y)| \le C(1+|y|), \quad \forall (t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d.$$

Proof. This follows as in (Pham [11], Lemma 4.2 . Here, we clarify some time-dependent parameters in the proof. From the standard verification theorem (see [11],[4]) ψ_k can be represented as the solution of the stochastic control problem:

$$\psi_k(t,y) = \inf_{q\in\mathcal{B}_k\,,\pi\in\mathcal{A}_k(K)} \mathbb{E}^Q\left[\int_t^{\mathbb{T}} L(u, Y^*_u, q_u)\,du \,|\, Y^*_t = y\right], \tag{4.15}$$

where $\mathcal{B}_k = \{q \in \mathbb{R}^d \,:\, |q| \le k\}$ and $\mathcal{A}_k(K) = \{\pi \in K \,:\, |\pi| \le k\}$ are compact sets and the controlled dynamics of Y under Q is given as

$$dY_t = q_t dt + dB_t\,. \tag{4.16}$$

Moreover, an optimal control for (4.15) is Markov with the policy given in feedback form by:

$$(q^*(t,y), \pi^*(t,y)) \in \arg\min_{|q|\le k\,,|\pi|\le k} \{q^T D\psi_k(t,y) + \bar{L}(t,y,q,\pi)\}\,. \tag{4.17}$$

Hence we obtain:

$$\psi_k(t,y) = \mathbb{E}^Q\left[\int_t^{\mathbb{T}} L(u, Y^*_u, q^*(u,Y^*_u))\,du \,|\, Y^*_t = y\right], \tag{4.18}$$

where Y^*_t solves (4.16) with the control $q^*_t = q^*_k(t,Y^*_t)\,, \pi_t = \pi^*(t,Y^*_t)$. From [11] and standard theory (Fleming and Soner [4], Lemma 11.4 on page 209), we have:

$$D\psi_k(t,y) = \mathbb{E}^Q\left[\int_t^{\mathbb{T}} D_y\bar{L}(u, Y^*_u, q^*_k(u,Y^*_u))\,du \,|\, Y^*_t = y\right]. \tag{4.19}$$

From Equation (4.13) and assumption **G3b**, we have $\forall (t,y,q,\pi) \in [0,\mathbb{T}] \times \mathbb{R}^d \times \mathbb{R}^d \times K$,

$$\begin{aligned}
|D_y\bar{L}(t,y,q,\pi)| &\le |D\eta(t,y)||q-\eta(t,y)| + \frac{\delta(t)(1-\delta(t))}{2}||D(\Sigma(t,y)\Sigma(t,y)^T)(t,y)||(\pi-\bar{\pi}(t,y))|^2 \\
&\quad \delta(t)(1-\delta(t))||D\bar{\pi}(t,y)^T\Sigma(t,y)\Sigma(t,y)^T|||(\pi-\bar{\pi}(t,y))| + |D\lambda(t,y)| \\
&\le C\left[|1+|q-\eta(t,y)|^2 + |(\pi-\bar{\pi}(t,y))|^2||\Sigma(t,y)\Sigma(t,y)^T|||(\pi-\bar{\pi}(t,y))|^2\right] \\
&\le C\left[|1+|q-\eta(t,y)|^2 + \Sigma(t,y)^T|(\pi-\bar{\pi}(t,y))|^2\right] \\
&\le C\left[|1+\bar{L}(t,y,q,\pi)|\right],
\end{aligned}$$

where the generic positive constant C changes from line to line in the above estimation process. Here the uniform ellipticity of the matrix function $\Sigma\Sigma^T$ was used in third inequality. ☐

Now we prove Theorem 4.1.

Proof. We analyze the proof as in Pham [11] by the following three cases, **G3**a' **,** **G3a** and **G3b**.

(Case G3a'). From Lemma 4.3, the solution ϕ_k' to Equation (4.12) we have shown that it has a bounded derivative

$$|D\phi_k'(t,y)| \leq C, \quad \forall (t,y) \in [0,\mathbb{T}) \times \mathbb{R}^d.$$

Taking $k \in \mathbb{N}$ sufficiently larger than $C||\mathbf{I_d} + \rho||$ we have:

$$\begin{aligned} F_k'(t,y,D\phi_k'(t,y)) &= \max_{q \in \mathcal{B}_k} \left[-(q + f(t,y))^T D\phi_k'(t,y) - L'(t,y,p) \right] , \\ &= \max_{q \in \mathbb{R}^d} \left[-(q + f(t,y))^T D\phi_k'(t,y) - L'(t,y,p) \right] , \\ &= F(t,y,D\phi_k'(t,y)), \end{aligned}$$

for all $(t,y) \in [0,\mathbb{T}] \times \mathbb{R}^d$. Hence ϕ_k' is a smooth solution with a bounded first derivative.

(Case G3a). As in [11], the function $q \to -q^T D\phi_k(t,y) - L(t,y,p)$ attains its maximum on $[0,\mathbb{T}] \times \mathbb{R}^d$ for

$$q_k^*(t,y) = -(\mathbf{I_d} + \rho(t,y))D\phi_k(t,y) + f(t,y).$$

Furthermore, by Lemma 4.3, $D\phi_k$ satisfies a linear growth condition in y uniformly in k. Therefore, for $M > 0$ arbitrary large, there exists a $C > 0$ independent of k:

$$|q_k^*(t,y)| \leq C, \quad \forall t \in [0,\mathbb{T}], \quad |y| \leq M.$$

Hence for $K \geq C$, we obtain

$$\begin{aligned} F_k(t,y,D\phi_k(t,y)) &= \max_{q \in \mathcal{B}_k} \left[-q^T D\phi_k(t,y) - L(t,y,p) \right] , \\ &= \max_{q \in \mathbb{R}^d} \left[-q^T D\phi_k(t,y) - L(t,y,p) \right] , \\ &= F(t,y,D\phi_k(t,y)), \end{aligned}$$

for all $(t,y) \in [0,\mathbb{T}] \times \{|y| \leq M\}$. Letting M go to infinity we have that ϕ_k is a smooth solution with linear growth.

(Case G3b). As in Pham [11], the function $(q,\pi) \to -q^T D\phi_k(t,y) - \bar{L}(t,y,p)$ attains its maximum on $[0,\mathbb{T}] \times \mathbb{R}^d \times K$ for

$$q_k^*(t,y) = -D\psi_k(t,y) + \eta(t,y).$$

and for $\pi^*(t,y)$ satisfying

$$\begin{aligned} \epsilon|\pi^*(t,y)|^2 - \delta(t)|\pi^*(t,y)||\mu(t,y)| &\leq \frac{\delta(t)(1-\delta(t))}{2}|D(\Sigma(t,y)^T|^2(t,y) - \delta(t)\pi^*(t,y)^T\mu(t,y), \\ &\leq 0, \end{aligned}$$

for $\epsilon > 0$. This shows that $|\pi^*(t,y)| \leq \frac{\delta(t)}{\epsilon}|\mu(t,y)|$. The rest of the proof follows as in Pham ([11], page 73). □

5. Examples

In this section, we analyze the examples as they appear in [11], but with time-dependent parameters.

5.1. **Hull-White**[7] **and Scott Models**[13].

5.1.1. *Degeneracy Volatility.* . This example appears in [11] (page 73).

$$\begin{aligned}\frac{dS_t}{S_t} &= (r+\mu(t,Y_t))dt + \rho(t)e^{\gamma(t)Y_t}dB_t + \sqrt{1-\rho(t)^2}e^{\gamma(t)Y_t}dW_t\\ dY_t &= (a(t)-\theta(t)Y_t)dt + dB_t,\end{aligned}$$

where $a(t),\ \gamma(t)\neq 0,\ \rho(t)$ and $\theta(t)$ are time-dependent parameters. Now using the notation given in Section 2 we have: $\eta(t,y) = a(t)-\theta(t)y,\ \sigma(t,y) = \rho(t)e^{\gamma(t)y}$ and $\Sigma\Sigma^T(t,y) = \alpha(t,y) = e^{2\gamma(t)y}$. For the unconstrained case ($K=\mathbb{R}$) the Assumption **G3**a' is satisfied whenever

$$\frac{\mu(t,y)}{e^{\gamma(t)y}},$$

is a bounded, Lipschitz element of C^1. Under this condition, a unique smooth solution exists. In this case the semilinear PDE (3.4) becomes:

$$\begin{aligned}&-\frac{\partial\phi}{\partial t} - \frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2}\left(1+\frac{\delta(t)}{1-\delta(t)}\rho(t)^2\right)\left(\frac{\partial\phi}{\partial y}\right)^2\\ &-\left(a(t)-\theta(t)y+\frac{\delta(t)}{1-\delta(t)}\frac{\rho(t)\mu(t,y)}{e^{\gamma(t)y}}\right)\frac{\partial\phi}{\partial y} + \delta(t)r\\ &+\frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)} + \frac{1}{2}\frac{\delta(t)}{1-\delta(t)}\frac{\mu^2(t,y)}{e^{2\gamma(t)y}} = 0,\end{aligned}$$

with terminal condition $\phi(\mathbb{T},y)=0$. The value function:

$$v(t,x,y) = \frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,y)),$$

is then substituted into the above PDE. Therefore the optimal condition is given by:

$$\pi_t^* = \frac{e^{-\gamma(t)Y_t}}{1-\delta(t)}\left[\frac{\mu(t,Y_t)}{e^{\gamma(t)Y_t}} - \rho(t)\frac{\partial\phi}{\partial y}(t,Y_t)\right],\quad a.s.,\quad t\in[0,\mathbb{T}].$$

5.1.2. *Uniform Elliptic Volatility.* . This example appears in Pham [11] (page 73). We discuss the same problem with time-dependent parameters:

$$\begin{aligned}\frac{dS_t}{S_t} &= (r+\mu(t,Y_t))dt + \rho(t)\sqrt{e^{2\gamma(t)Y_t}+\epsilon}dB_t + \sqrt{1-\rho(t)^2}\sqrt{e^{2\gamma(t)Y_t}+\epsilon}dW_t\\ dY_t &= (a(t)-\theta(t)Y_t)dt + dB_t,\end{aligned}$$

where a constant $\epsilon>0$ is a lower bound on the volatility. In this case, using the notation given in Section 2 we obtain: $\sigma(t,y) = \rho(t)\sqrt{e^{2\gamma(t)Y_t}+\epsilon}$, $\Sigma\Sigma^T(t,y) = \alpha(t,y) = e^{2\gamma(t)Y_t}+\epsilon \geq \epsilon$.

In this case the the semilinear PDE (3.4) becomes:

$$-\frac{\partial\phi}{\partial t} - \frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2}\left(1+\frac{\delta(t)}{1-\delta(t)}\rho(t)^2\right)\left(\frac{\partial\phi}{\partial y}\right)^2$$

$$-\left(a(t)-\theta(t)y+\frac{\delta(t)}{1-\delta(t)}\frac{\rho(t)\mu(t,y)}{\sqrt{e^{2\gamma(t)y}+\epsilon}}\right)\frac{\partial\phi}{\partial y}+\delta(t)r$$
$$+\frac{\delta(t)\ln x-\frac{d\delta(t)}{dt}}{\delta(t)}+\frac{1}{2}\frac{\delta(t)}{1-\delta(t)}\frac{\mu^2(t,y)}{e^{2\gamma(t)Y_t}+\epsilon}=0,$$

with terminal condition $\phi(\mathbb{T},y)=0$. The value function:

$$v(t,x,y)=\frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,y))$$

is then substituted into the above PDE. Therefore, the optimal condition is given by

$$\pi_t^*=\frac{1}{1-\delta(t)}\frac{1}{e^{2\gamma(t)Y_t}+\epsilon}\left[\frac{\mu(t,Y_t)}{\sqrt{e^{2\gamma(t)Y_t}+\epsilon}}-\rho(t)\frac{\partial\phi}{\partial y}(t,Y_t)\right],\quad a.s.,\quad t\in[0,\mathbb{T}].$$

5.2. **The Stein-Stein Model**[14].

5.2.1. *Degenerate Volatility.* . We analyze the Stein-Stein stochastic volatility model presented in Pham [11]. We extend this model by adding time-dependent parameters:

$$\begin{aligned}\frac{dS_t}{S_t} &= (r+\mu(t,Y_t))dt+\rho(t)|\gamma(t)Y_t|dB_t+\sqrt{1-\rho(t)^2}|\gamma(t)Y_t|dW_t\\ dY_t &= (a(t)-\theta(t)Y_t)dt+dB_t,\end{aligned}$$

where $a(t)$, $\gamma(t)\neq 0$, $\rho(t)$ and $\theta(t)$ are time-dependent parameters. Here the parameter $\rho(t)$ is the correlation between asset and volatility. Now using the notation given in Section 2 we have: $\eta(t,y)=a(t)-\theta(t)y$, $\sigma(t,y)=\rho(t)|\gamma(t)y|$ and $\Sigma\Sigma^T(t,y)=\alpha(t,y)=\gamma^2(t)y^2$. For the unconstrained case ($K=\mathbb{R}$) the Assumption H3a' is satisfied whenever

$$\frac{\mu(t,y)}{|y|},$$

is a bounded, Lipschitz element of C^1. Under this condition, a unique smooth solution exists. In this case the semilinear PDE (3.4) becomes:

$$-\frac{\partial\phi}{\partial t}-\frac{1}{2}\frac{\partial^2\phi}{\partial y^2}+\frac{1}{2}\left(1+\frac{\delta(t)}{1-\delta(t)}\rho(t)^2\right)\left(\frac{\partial\phi}{\partial y}\right)^2$$
$$-\left(a(t)-\theta(t)y+\frac{\delta(t)}{1-\delta(t)}\frac{\rho(t)\mu(t,y)}{|\gamma(t)y|}\right)\frac{\partial\phi}{\partial y}+\delta(t)r$$
$$+\frac{\delta(t)\ln x-\frac{d\delta(t)}{dt}}{\delta(t)}+\frac{1}{2}\frac{\delta(t)}{1-\delta(t)}\frac{\mu^2(t,y)}{\gamma^2(t)y^2}=0,$$

with terminal condition $\phi(\mathbb{T},y)=0$. The value function:

$$v(t,x,y)=\frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,y)),$$

is then substituted into the above PDE. Therefore, the optimal condition is given by

$$\pi_t^* = \frac{1}{1-\delta(t)}\frac{1}{|\gamma(t)Y_t|}\left[\frac{\mu(t,Y_t)}{|\gamma(t)Y_t|} - \rho(t)\frac{\partial\phi}{\partial y}(t,Y_t)\right], \quad a.s., \quad t\in[0,\mathbb{T}].$$

5.2.2. *Uniform Elliptic Volatility.* . This example appears in [11] (page 74). In this case a uniform ellipticity condition is assumed. We discuss this problem with time-dependent parameters:

$$\begin{aligned}\frac{dS_t}{S_t} &= (r+\mu(t,Y_t))dt + \rho(t)\sqrt{\gamma^2(t)Y_t^2+\epsilon}dB_t + \sqrt{1-\rho(t)^2}\sqrt{\gamma^2(t)Y_t^2+\epsilon}dW_t\\ dY_t &= (a(t)-\theta(t)Y_t)dt + dB_t,\end{aligned}$$

where a constant $\epsilon > 0$ is a lower bound on the volatility. In this case, using the notation given in Section 2 we have: $\sigma(t,y) = \rho(t)\sqrt{\gamma^2(t)Y_t^2+\epsilon}$, $\Sigma\Sigma^T(t,y) = \alpha(t,y) = \gamma^2(t)Y_t^2+\epsilon \geq \epsilon$.

In this case the the semilinear PDE (3.4) becomes:

$$\begin{aligned}&-\frac{\partial\phi}{\partial t} - \frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2}\left(1+\frac{\delta(t)}{1-\delta(t)}\rho(t)^2\right)\left(\frac{\partial\phi}{\partial y}\right)^2\\ &-\left(a(t)-\theta(t)y + \frac{\delta(t)}{1-\delta(t)}\frac{\rho(t)\mu(t,y)}{\sqrt{\gamma^2(t)y^2+\epsilon}}\right)\frac{\partial\phi}{\partial y} + \delta(t)r\\ &+\frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)} + \frac{1}{2}\frac{\delta(t)}{1-\delta(t)}\frac{\mu^2(t,y)}{\gamma^2(t)y^2+\epsilon} = 0,\end{aligned}$$

with terminal condition $\phi(\mathbb{T},y)=0$. The value function:

$$v(t,x,y) = \frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,y)),$$

is then substituted into the above PDE. Therefore, the optimal condition is given by:

$$\pi_t^* = \frac{1}{1-\delta(t)}\frac{1}{\sqrt{\gamma^2(t)y^2+\epsilon}}\left[\frac{\mu(t,Y_t)}{\sqrt{\gamma^2(t)y^2+\epsilon}} - \rho(t)\frac{\partial\phi}{\partial y}(t,Y_t)\right], \quad a.s., \quad t\in[0,\mathbb{T}].$$

As in [11], when $\rho = 0$ the the semilinear PDE (3.4) becomes:

$$\begin{aligned}&-\frac{\partial\phi}{\partial t} - \frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 - (a(t)-\theta(t)y)\frac{\partial\phi}{\partial y}\\ &+\delta(t)r + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)}\\ &+\min_{\pi\in K}\left[\frac{1}{2}\delta(t)(1-\delta(t))\pi^2(\sqrt{\gamma^2(t)y^2+\epsilon}) - \delta(t)\pi\mu(t,y)\right] = 0,\end{aligned}$$

with terminal condition $\phi(\mathbb{T}, y) = 0$. Hence the optimal portfolio is independent of ϕ since the correlation $\rho(t) = 0$ which leads to:

$$\pi_t^* \in \arg\min_{\pi\in K}\left[\frac{1-\delta(t)}{2}\pi^2\left(\gamma^2(t)y^2+\epsilon\right)-\pi\mu(t,Y_t)\right], \quad a.s., \quad t\in[0,\mathbb{T}].$$

5.3. **A Two-Asset Stochastic Volatility Model.** . Lastly, we consider the two risky assets with nondegenerate and bounded volatility with time-dependent parameters. This model has been adapted in Pham [11] (on page 76).

$$\begin{aligned}
\frac{dS_t^1}{S_t^1} &= (r+\mu_1(t,Y_t))dt+\sqrt{\vartheta_1(t,Y_t)}\left[\rho_1(t)dB_t^1+\sqrt{1-\rho_1(t)^2}dW_t^1\right]\\
\frac{dS_t^2}{S_t^2} &= (r+\mu_2(t,Y_t))dt+\sqrt{\vartheta_2(t,Z_t)}\left[\rho_1(t)dB_t^1+\sqrt{1-\rho_1(t)^2}dW_t^1\right]\\
&\quad +\sqrt{1-\rho(t)^2}\sqrt{\vartheta_2(t,Z_t)}\left[\rho_2(t)dB_t^2+\sqrt{1-\rho_2(t)^2}dW_t^2\right]\\
dY_t &= \eta_1(t,Y_t)dt+dB_t^1\\
dY_t &= \eta_2(t,Y_t)dt+dB_t^2,
\end{aligned}$$

where $\vartheta_i(t,y)$ are bounded in $[0,\mathbb{T}]\times C^1$ functions. Here, for $i=1,2$, η_i, and μ_i are time-dependent parameters, $\rho_i(t)$ represents the time-dependent parameters between asset S^i and its volatility, and $-1<\rho(t)<1$ is the correlation between assets S^i. We shall represent $\eta(t,y,z)=(\eta_1(t,y),\eta_2(t,z))^T$, $\mu(t)=(\mu_1(t),\mu_2(t))^T$,

$$\sigma(t,y)=\begin{pmatrix}\rho_1(t)\sqrt{\vartheta_1(t,y)} & 0\\ \rho(t)\rho_1(t)\sqrt{\vartheta_2(t,z)} & \sqrt{1-\rho^2(t)}\rho_2(t)\sqrt{\vartheta_2(t,z)}\end{pmatrix}$$

and

$$\Sigma\Sigma^T(t,y,z)=\begin{pmatrix}\vartheta_1(t,y) & \rho(t)\sqrt{\vartheta_1(t,y)\vartheta_2(t,z)}\\ \rho(t)\sqrt{\vartheta_1(t,y)\vartheta_2(t,z)} & \vartheta_2(t,z)\end{pmatrix}.$$

The smallest eigenvalue of the matrix $\Sigma\Sigma^T(t,y,z)$ [11] is:

$$\alpha(t,y,z)=\frac{1}{2}\left[\vartheta_1(t,y)+\vartheta_2(t,z)-\sqrt{(\vartheta_1(t,y)-\vartheta_2(t,z))^2+4\rho^2(t)\vartheta_1(t,y)\vartheta_2(t,z)}\right]$$

$$\begin{aligned}
\alpha(t,y,z) &= \frac{2(1-\rho^2(t))\vartheta_1(t,y)\vartheta_2(t,z)}{\left[\vartheta_1(t,y)+\vartheta_2(t,z)-\sqrt{(\vartheta_1(t,y)-\vartheta_2(t,z))^2+4\rho^2(t)\vartheta_1(t,y)\vartheta_2(t,z)}\right]}\\
&\geq \frac{(1-\rho^2(t))\vartheta_1(t,y)\vartheta_2(t,z)}{\vartheta_1(t,y)+\vartheta_2(t,z)}\\
&\geq \frac{(1-\rho^2(t))}{2}\min(\epsilon_1,\epsilon_2).
\end{aligned}$$

Here $\vartheta_1(t,y)$ and $\vartheta_2(t,z)$ are lower bounded uniformly by $\min(\epsilon_1,\epsilon_2)$. Also

$$\mu^T(t)(\Sigma\Sigma^T(t,y,z))^{-1}\mu(t)=\frac{1}{1-\rho^2(t)}\left[\frac{\mu_1^2}{\vartheta_1(t,y)}-2\rho(t)\frac{\mu_1(t)\mu_2(t)}{\sqrt{\vartheta_1(t,y)\vartheta_2(t,z)}}+\frac{\mu_2(t)^2}{\vartheta_2(t,z)}\right]$$

and

$$\sigma(t,y)^T(\Sigma\Sigma^T(t,y,z))^{-1}\mu(t) = \begin{pmatrix} \frac{\rho_1(t)\mu_1(t)}{\sqrt{\vartheta_1(t,y)}} \\ \frac{\rho_2 t)\mu_2(t)}{\sqrt{1-\rho^2(t)}\sqrt{\vartheta_2(t,z)}} - \frac{\rho(t)\rho_2(t)\mu_1(t)}{\sqrt{1-\rho^2(t)}\sqrt{\vartheta_1(t,y)}} \end{pmatrix},$$

of which both are $[0,\mathbb{T}]\times C^1$. Hence the Assumption $\mathbf{H3}a'$ is satisfied for the case $K=\mathbb{R}^2$. There exist a unique solution for the following PDE:

$$\begin{aligned}
&-\frac{\partial\phi}{\partial t} - \frac{1}{2}\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \\
&+\frac{1}{2}\left[\left(1+\frac{\delta(t)}{1-\delta(t)}\rho_1(t)^2\right)\left(\frac{\partial\phi}{\partial y}\right)^2 + \left(1+\frac{\delta(t)}{1-\delta(t)}\rho_2(t)^2\right)\left(\frac{\partial\phi}{\partial z}\right)^2\right] \\
&-\left[a_1(t)-\theta_1(t)y+\frac{\delta(t)}{1-\delta(t)}\frac{\rho_1(t)\mu_1(t)}{\sqrt{\vartheta_1(t,y)}}\right]\frac{\partial\phi}{\partial y} \\
&-\left[a_2(t)-\theta_2(t)y+\frac{\delta(t)}{1-\delta(t)}\left(\frac{\rho_2 t)\mu_2(t)}{\sqrt{1-\rho^2(t)}\sqrt{\vartheta_2(t,z)}} - \frac{\rho(t)\rho_2(t)\mu_1(t)}{\sqrt{1-\rho^2(t)}\sqrt{\vartheta_1(t,y)}}\right)\right]\frac{\partial\phi}{\partial z} \\
&+\delta(t)r + \frac{\delta(t)\ln x - \frac{d\delta(t)}{dt}}{\delta(t)} \\
&+\frac{1}{2}\frac{\delta(t)}{1-\delta(t)}\frac{1}{1-\rho^2(t)}\left[\frac{\mu_1^2(t)}{\vartheta_1(t,y)} - 2\rho(t)\frac{\mu_1(t)\mu_2(t)}{\sqrt{\vartheta_1(t,y)\vartheta_2(t,z)}} + \frac{\mu_2^2(t)}{\vartheta_2(t,z)}\right] = 0,
\end{aligned}$$

with terminal condition $\phi(\mathbb{T},y,z)=0$. The value function:

$$v(t,x,y,z) = \frac{x^{\delta(t)}}{\delta(t)}\exp(-\phi(t,y,z)).$$

is then substituted into the above PDE. We obtain the optimal condition given by:

$$\begin{aligned}
\pi_t^* &= \frac{1}{(1-\delta(t))(1-\rho(t))} \\
&\times \begin{pmatrix} \frac{\mu_1(t)}{\vartheta_1} - \frac{\rho(t)\mu_2(t)}{\sqrt{\vartheta_1\vartheta_2}} - \frac{\rho_1(t)(1-\rho^2(t))}{\vartheta_1}\frac{\partial\phi}{\partial y} + \frac{\rho(t)\rho_2(t)(1-\rho^2(t))}{\vartheta_1}\frac{\partial\phi}{\partial z} \\ \frac{\mu_1(t)}{\vartheta_1} - \frac{\rho(t)\mu_1(t)}{\sqrt{\vartheta_1\vartheta_2}} - \frac{\rho_2(t)(1-\rho^2(t))}{\vartheta_2}\frac{\partial\phi}{\partial z} \end{pmatrix}(t,Y_t,Z_t),
\end{aligned}$$

a.s., for all $t\in[0,\mathbb{T}]$.

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