ON i-OPERATORS ON REAL NORMED SPACES

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Abstract. We gradually study i-operators on real vector spaces, on real topological vector spaces, and on real normed spaces. Among several things we prove the existence of real topological vector spaces (different from the James' Space) that are free of continuous i-operators. We also prove that every real normed space can be equivalently renormed to be free of norm i-operators. Examples of spaces of continuous functions not admitting norm i-operators and whose unit sphere is free of convex subsets with non-empty interior relative to it are also found. Finally, we also provide some results on a problem posed by Wenzel in [10].

1. i-Operators on real vector spaces

A linear i-operator on a real vector space X is an automorphism $i : X \to X$ of real vector spaces such that $i \circ i = -I_X$. The most basic example of a real vector space with a linear i-operator is a complex vector space seen as a real vector space with the multiplication by i. Conversely, a real vector space X with a linear i-operator $i : X \to X$ can be naturally endowed with a complex structure as follows: $(a + ib)\, x = ax + bi\,(x)$, where $a, b \in \mathbb{R}$ and $x \in X$. In other words, the category of the real vector spaces with an i-operator is isomorphic to the category of the complex vector spaces. The interesting question is to determine whether a certain real vector space admits a linear i-operator. This is what we do next.

Definition 1.1. Let X be a real vector space. We say that X is linearly Cartesian exactly when it can be decomposed as the direct sum of two vector subspaces of the same dimension.

The reader may immeditely notice that a real vector space X is linearly Cartesian if and only if X is not of odd finite dimension.

Theorem 1.2. *Let X be a real vector space. The following conditions are equivalent:*

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(1) *X admits a linear i-operator.*
(2) *X is linearly Cartesian.*

Proof. Assume that X admits a linear i-operator $i : X \to X$. Consider the set $\mathcal{L}$ of all subspaces M of X such that $M \cap i(M) = \{0\}$. Note that $\mathbb{R}x \in \mathcal{L}$ for all $x \in X$. Consider $\mathcal{L}$ ordered by the inclusion. Let $(M_j)_{j \in J}$ be a chain of $\mathcal{L}$. Then $\bigcup_{j \in J} M_j \in \mathcal{L}$. Indeed, if $m, n \in \left(\bigcup_{j \in J} M_j\right) \setminus \{0\}$ are so that $m = i(n)$, then there are $j, k \in J$ such that $m \in M_j$ and $n \in M_k$; since $(M_j)_{j \in J}$ is a chain, we can assume without loss that $M_j \subseteq M_k$, which means that $M_k \cap i(M_k) \neq \{0\}$. By the Zorn's Lemma, we can consider a maximal element M of $\mathcal{L}$. We will show that $X = M + i(M)$. Let $x \in X$ so that $x \notin M$. Then $M + \mathbb{R}x$ strictly contains M, and hence there exist two elements $m + \lambda x, n + \gamma x \in M + \mathbb{R}x$ such that $m + \lambda x = i(n + \gamma x)$. Observe that we cannot have $\lambda = \gamma = 0$. On the one hand, we have that $\lambda x - \gamma i(x) = i(n) - m$. On the other hand, we have that $\lambda i(x) + \gamma x = -n - i(m)$. Then $\lambda^2 x - \lambda \gamma i(x) = \lambda i(n) - \lambda m$ and $\gamma \lambda i(x) + \gamma^2 x = -\gamma n - \gamma i(m)$. Then $(\lambda^2 + \gamma^2) x = -\lambda m - \gamma n + \lambda i(n) - \gamma i(m)$ and

$$x = \frac{-\lambda}{\lambda^2 + \gamma^2} m - \frac{\gamma}{\lambda^2 + \gamma^2} n + i\left(\frac{\lambda}{\lambda^2 + \gamma^2} n - \frac{\gamma}{\lambda^2 + \gamma^2} m\right) \in M + i(M).$$

Conversely, assume that $X = M \oplus N$, where M and N are subspaces of X of the same dimension. Let $\varphi : M \to N$ be an isomorphism. It suffices to define the following linear i-operator:

$$\begin{array}{rrcl} i : & X & \to & X \\ & x = m + n & \mapsto & i(x) = -\varphi^{-1}(n) + \varphi(m). \end{array}$$

□

Corollary 1.3. *A real vector space admits a linear i-operator if and only if it is not of odd finite dimension.*

Remark 1.4. Let X be a real vector space with a linear i-operator $i : X \to X$. If there exists a vector subspace M of X which is minimal verifying that $M + i(M) = X$, then $M \cap i(M) = \{0\}$. Indeed, assume that there are $m_0, n_0 \in M \setminus \{0\}$ such that $m_0 = i(n_0)$. Consider a maximal subspace N of M such that $n_0 \in N$ and $M = N \oplus \mathbb{R}m_0$. We will reach a contradiction by proving that $X = N + i(N)$. Let $x \in X$. There are $m, n \in M$ such that $x = m + i(n)$. We can find $n_1, n_2 \in N$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $m = n_1 + \lambda_1 m_0$ and $n = n_2 + \lambda_2 m_0$. Then:

$$\begin{array}{rcl} x & = & m + i(n) \\ & = & (n_1 - \lambda_2 n_0) + i(n_2 + \lambda_1 n_0) \\ & \in & N + i(N). \end{array}$$

Remark 1.5. Let X be a real vector space. Note that X^* can be naturally endowed with the ω^* topology with no need for X to have a vector topology. Notice also that a linear map $f : X^* \to X^*$ is ω^*-ω^* continuous if and only if f is a dual map, that is, there exists $g : X \to X$ such that $g^* = f$.

The category of the real vector spaces with a linear i-operator also preserves duality in the following sense: If X is a real vector space with a linear i-operator $i : X \to X$, then the dual map $i^* : X^* \to X^*$ is also a linear i-operator on X^* (which has to be ω^*-ω^* continuous). We will show now the existence of linear i-operators that are not dual.

Lemma 1.6. *Let X be an infinite dimensional real vector space. There are two sequences $(m_k^*)_{k\in\mathbb{N}}, (n_k^*)_{k\in\mathbb{N}} \subseteq X^*$ and two elements $m^*, n^* \in X^*$ such that:*

(1) *$(m_k^*)_{k\in\mathbb{N}}$ and $(n_k^*)_{k\in\mathbb{N}}$ are respectively ω^* convergent to m^* and n^*.*

(2) *The four vector subspaces $\operatorname{span}\{m_k^* : k \in \mathbb{N}\}$, $\operatorname{span}\{n_k^* : k \in \mathbb{N}\}$, $\mathbb{R}m^*$, and $\mathbb{R}n^*$ are pairwise disjoint.*

Proof. Let $(e_i)_{i\in I} \subseteq X$ be a Hamel basis for X. Consider the corresponding orthogonal system $(e_i^*)_{i\in I} \subseteq X^*$ and take any proper subsequence $\left(e_{i_k}^*\right)_{k\in\mathbb{N}}$. Observe that $\omega^* \lim_{k\to\infty} e_{i_k}^* = 0$. Let $m^*, n^* \in X^*$ be linearly independent such that $\operatorname{span}\{m, n\} \cap \operatorname{span}\left\{e_{i_k}^* : k \in \mathbb{N}\right\} = \{0\}$. Finally, take $m_k^* := e_{i_{2k}}^* + m^*$ and $n_k^* := e_{i_{2k+1}}^* + n^*$ for all $k \in \mathbb{N}$. $\square$

Theorem 1.7. *Let X be an infinite dimensional real vector space. There exists a linear i-operator $j : X^* \to X^*$ which is not dual.*

Proof. Consider$(m_k^*)_{k\in\mathbb{N}}, (n_k^*)_{k\in\mathbb{N}} \subseteq X^*$ and $m^*, n^* \in X^*$ as in Lemma 1.6. Now we can find two vector subspaces M and N of X^* such that M and N have the same dimension, $X^* = M \oplus N$, $(m_k^*)_{k\in\mathbb{N}} \subset M$, $m^* \in M$, $(n_k^*)_{k\in\mathbb{N}} \subset N$, and $n^* \in N$. We can define an isomorphism $\phi : M \to N$ such that $\phi(m_k^*) = n_k^*$ for all $k \in \mathbb{N}$ but $\phi(m^*) \neq n^*$. The linear i-operator on X^* defined as

$$\begin{array}{rccc} j: & X^* & \to & X^* \\ & x^* = m^* + n^* & \mapsto & j(x^*) = -\phi^{-1}(n^*) + \phi(m^*) \end{array}$$

cannot be dual because it is not ω^*-ω^* continuous. $\square$

2. i-Operators on real topological vector spaces

The next step is to provide a similar theory for real topological vector spaces. In this sense, a topological i-operator on a real topological vector space X is a continuous linear i-operator $i : X \to X$ (which will actually be an isomorphism since $i^{-1} = -i$). The category of the real topological vector spaces with an i-operator is isomorphic to the category of the complex topological vector spaces.

Definition 2.1. Let X be a real topological vector space. We say that X is topologically Cartesian exactly when X has a complemented subspace which is isomorphic to its complement.

The version of Theorem 1.2 for real topological vector spaces follows.

Theorem 2.2. *Let X be a real topological vector space. If X is topologically Cartesian, then X admits a topological i-operator.*

The problem of the converse to Theorem 2.2 is somehow in connection with Wenzel's Problem, which we will treat in the next section. The next remark demonstrates that the trivial topology does not serve our purposes of finding a non-topologically Cartesian real topological vector space admitting topological i-operators.

Remark 2.3. Let X be a non-zero real vector space endowed with the trivial topology (which obviously is a vector topology). Observe the following:

(1) Every linear i-operator on X is continous and thus is a topological i-operator.
(2) The only closed vector subspace of X is X.
(3) Every vector subspace of X is topologically complemented (since every linear projection onto that subspace would be continuous).

The next proposition shows that every vector topology can be enlarged to make a certain linear i-operator continuous.

Proposition 2.4. *Let X be a real topological vector space. Let $i : X \to X$ be a linear i-operator. There exists a finer vector topology on X that makes i continuous.*

Proof. Let $\mathcal{U}$ be a local basis of balanced and absorbing neighborhoods of 0. It suffices to consider the vector topology on X generated by the following local basis of balanced and abosorbing neighborhoods of 0:

$$\left\{U \cap i^{-1}(V) : U, V \in \mathcal{U}\right\}$$

$\square$

We will show now the existence of infinite dimensional real topological vector spaces free of topological i-operators.

Theorem 2.5. *Let X be a non-zero real vector space. There exists a non-Hausdorff locally convex vector topology on X that does not admit any topological i-operator.*

Proof. Let $\mathcal{I}$ be the set of all linear i-operators on X. All we have to do is find a vector topology on X that makes none of them continuous. Let $x \in X \setminus \{0\}$ and consider a linear function $f : X \to \mathbb{R}$ such that $f(x) = 1$. Our vector topology on X is the one whose local basis of neighborhoods of 0 is

$$\left\{f^{-1}(-\alpha, \alpha) : \alpha > 0\right\}.$$

Observe that X endowed with this vector topology becomes locally convex but not Hausdorff. Fix an arbitrary $i \in \mathcal{I}$ and consider $f \circ i$. In virtue of [6, Theorem 3.1] it suffices to prove that $(f \circ i)^{-1}(-1, 1)$ has empty interior in order to show that $f \circ i$ is not continuous. So, assume there is $\alpha > 0$ such that $f^{-1}(-\alpha, \alpha) \subseteq (f \circ i)^{-1}(-1, 1)$. Since $X = \ker(f) \oplus \mathbb{R}x$, there must be $m \in \ker(f)$ and $\lambda \in \mathbb{R}$ such that $i(x) = m + \lambda x$. Now, $-x = i(m) + \lambda i(x)$. Therefore, $\lambda x = -\lambda i(m) - \lambda^2 i(x)$. Combining these equations we obtain that

$$i(x) = \frac{1}{1+\lambda^2} m - \frac{\lambda}{1+\lambda^2} i(m).$$

If $f(i(x)) \neq 0$, then we can find $K > 0$ big enough so that

$$|K f(i(x))| > \frac{|\lambda|}{1+\lambda^2}.$$

However, since $Km \in \ker(f) \subset f^{-1}(-\alpha, \alpha) \subseteq (f \circ i)^{-1}(-1, 1)$, we deduce that

$$|K f(i(x))| \leq \frac{|\lambda|}{1+\lambda^2}.$$

This means that $f(i(x)) = 0$, so $i(x) \in \ker(f) \subset (f \circ i)^{-1}(-1, 1)$. Finally, notice that $x = -i(i(x))$ and hence

$$1 = |f(x)| = |(f \circ i)(i(x))| < 1.$$

$\square$

There are other examples of infinite dimensional real topological vector spaces not admitting topological i-operators. For instance, the James' Space (see [7]) is a normable real topological vector space not admitting any topological i-operator.

3. i-Operators on real normed spaces

The final step is to provide a similar theory for real normed spaces. In this sense, a norm i-operator on a real normed space X is a topological i-operator $i : X \to X$ such that $\|ax + bi(x)\|^2 = (a^2 + b^2) \|x\|^2$ for all $a, b \in \mathbb{R}$ and all $x \in X$. Again, the category of the real normed spaces with an i-operator is isomorphic to the category of the complex normed spaces. It is well known that quasi-reflexive spaces (that is, spaces which are of codimension 1 in their bidual) cannot admit an i-operator. A well-known example of a quasi-reflexive space is the James' Space (see [7, 3]). We will show next the existence of infinite dimensional real normed spaces (different from the James' Space) that are free of norm i-operators. Even more, we will prove that every real normed space admits an equivalent norm that makes it free of norm i-operators.

Proposition 3.1. *Let X be a real normed space. Assume that there exists a convex set $C \subset \mathsf{S}_X$ with non-empty interior relative to S_X. Then X is free of norm i-operators.*

Proof. Assume that X admits an i-operator $i : X \to X$. Consider $c \in \operatorname{int}_{\mathsf{S}_X}(C)$. Observe that $\operatorname{span}\{c, i(c)\}$ is a Hilbert space. However, Hilbert spaces are strictly convex but $\operatorname{span}\{c, i(c)\}$ is not. □

Later on in the final section it will be showed that the converse to the previous proposition does not remain true, that is, there are real normed spaces free of norm i-operators whose unit ball is also free of convex subsets with non-empty interior relative to it.

Corollary 3.2. *Let X be a real normed space. There exists an equivalent norm on X that does not allow any norm i-operators.*

Proof. It suffices to consider the equivalent norm on X whose unit ball is

$$\mathsf{B}_X \cap f^{-1}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right),$$

where $f \in \mathsf{S}_{X^*}$ is norm-attaining. The reader may realize that

$$\mathsf{B}_X \cap f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$$

satisfies the hypothesis of Proposition 3.1. □

Our next step is to show that any real normed space admitting a topological i-operator can be equivalently renormed to make the topological i-operator become a norm i-operator. Our first result on this topic is an easy lemma that states that equivalent renormings can be found if we only require the topological i-operator to become an isometry.

Lemma 3.3. *Let X be a real normed space. If $i : X \to X$ is a topological i-operator, then there exists an equivalent norm on X such that i becomes an isometry.*

Proof. Fix $1 \leq p \leq \infty$. Define

$$\lfloor x \rfloor_p := \left\{ \begin{array}{ll} \sqrt[p]{\|x\|^p + \|i(x)\|^p} & 1 \leq p < \infty \\ \max\{\|x\|, \|i(x)\|\} & p = \infty \end{array} \right.$$

for all $x \in X$. $\square$

The following result takes us one step closer to the desired renorming and offers the possibility of controlling the renorming constants, with some exceptions.

Theorem 3.4. *Let X be a real normed space. Let $i : X \to X$ be a topological i-operator on X. Then:*

(1) *If there exists an equivalent renorming on X that makes i become a norm i-operator, then there exists a constant $0 < K \leq 2\|i\|$ such that $K\sqrt{a^2+b^2} \leq \|ax + bi(x)\|$ for all $x \in \mathsf{S}_X$ and all $a, b \in \mathbb{R}$.*
(2) *If there exists a constant $0 < K \leq 2\|i\|$ such that $K\sqrt{a^2+b^2} \leq \|ax + bi(x)\|$ for all $x \in \mathsf{S}_X$ and all $a, b \in \mathbb{R}$, then for every constant $A \geq 2\|i\|$ there exists a pair $(V, |\cdot|_V)$ which is maximal verifying the following:*
   (a) *V is an invariant subspace of X under i.*
   (b) *$K|v|_V \leq \|v\| \leq A|v|_V$ for all $v \in V$.*
   (c) *i is a norm i-operator on V endowed with the norm $|\cdot|_V$.*

Proof.

(1) Assume that $|\cdot|$ is an equivalent norm on X that makes i become a norm i-operator. There exists a constant $K > 0$ such that $K|y| \leq \|y\|$ for all $y \in X$. Fix $x \in \mathsf{S}_X$ and $a, b \in \mathbb{R}$. We have that

$$\begin{array}{rcl} K\sqrt{a^2+b^2} & = & K|ax + bi(x)| \\ & \leq & \|ax + bi(x)\| \\ & \leq & |a| + |b|\,\|i\| \\ & \leq & (|a| + |b|)\,\|i\| \\ & \leq & 2\sqrt{a^2+b^2}\,\|i\|. \end{array}$$

(2) Consider the set $\mathcal{L}$ composed of all pairs $(M, |\cdot|_M)$ such that

(a) M is an invariant subspace of X under i.
(b) $K\,|m|_M \leq \|m\| \leq A\,|m|_M$ for all $m \in M$.
(c) i is a norm i-operator on M endowed with the norm $|\cdot|_M$.

Consider on $\mathcal{L}$ the partial order given by $(M, |\cdot|_M) \leq (N, |\cdot|_N)$ exactly when $M \subseteq N$ and $|\cdot|_N\,|_M = |\cdot|_M$. In order to see that $(\mathcal{L}, \leq)$ is an inductive ordered space we need to verify a couple of things:

(a) $\mathcal{L} \neq \varnothing$. Indeed, consider any $x \in \mathsf{S}_X$ and denote $M := \operatorname{span}\{x, i(x)\}$. Define $|\alpha x + \beta i(x)|_M := \sqrt{\alpha^2 + \beta^2}$ for all $\alpha, \beta \in \mathbb{R}$. It is not difficult to see that $(M, |\cdot|_M) \in \mathcal{L}$.
(b) Every chain of $\mathcal{L}$ has an upper bound in $\mathcal{L}$. Indeed, let $\left\{\left(M_i, |\cdot|_{M_i}\right)\right\}_{i\in I}$ be a chain of $\mathcal{L}$. Define $N := \bigcup_{i\in I} M_i$ and $|n|_N := |n|_{M_i}$ if $n \in M_i$. By construction, $(N, |\cdot|_N) \in \mathcal{L}$ and is an upper bound for $\left\{\left(M_i, |\cdot|_{M_i}\right)\right\}_{i\in I}$.

In accordance to the Zorn's Lemma, there exists a maximal element $(V, |\cdot|_V) \in \mathcal{L}$.

□

It is now the time to show that every real normed space admitting a topological i-operator can be equivalently renormed in such a way the topological i-operator becomes a norm i-operator.

Theorem 3.5. *Let X be a real normed space. Assume that X admits a topological i-operator $i : X \to X$. There exists an equivalent renorming on X for which i becomes a norm i-operator.*

Proof. Consider the set

$$\mathcal{B} := \left\{ax + bi(x) : a^2 + b^2 \leq 1, x \in \mathsf{B}_X\right\}.$$

It is not difficult to realize that $\mathcal{B}$ is bounded, closed, absolutely convex and has non-empty interior. Therefore, $\mathcal{B}$ is the unit ball of an equivalent norm on X. In order to see that i becomes a norm i-operator under this equivalent norm it only suffices to realize that the boundary of $\mathcal{B}$ is composed of all elements of the form $ax + bi(x)$ with $a^2 + b^2 = 1$ and $x \in \mathsf{B}_X$. □

The next step is modifying Theorem 2.2 for real normed spaces.

Definition 3.6. Let X be a real normed space. We say that X is norm Cartesian exactly when:

- X has a complemented subspace M which is linearly isometric to its complement N under the surjective linear isometry $\phi : M \to N$.

- For all $m \in M \setminus \{0\}$ we have that $\operatorname{span}\{m, \phi(m)\}$ is a 2-dimensional real Hilbert space having
$$\left\{\frac{m}{\|m\|}, \frac{\phi(m)}{\|m\|}\right\}$$
as a orthonormal basis.

Before modifying Theorem 2.2 for real normed spaces, we would like to make the reader be aware of the following.

Proposition 3.7. *Let X be a real normed space. If X is topologically Cartesian, then X can be equivalently renormed to become norm Cartesian.*

Proof. Let M and N be complemented subspaces of X and let $\varphi : M \to N$ be an isomorphism. Consider the new equivalent norm on N given by $|n| := \|\varphi^{-1}(n)\|$ for all $n \in N$. Observe that φ becomes an isometry with this new norm on N. Finally, consider the new equivalent norm on X given by
$$\lfloor x \rfloor := \sqrt{\|m\|^2 + |n|^2}$$
where $x = m + n$, $m \in M$, and $n \in N$. $\square$

The end of this section is to present, without a proof, the version of Theorem 2.2 for real normed spaces.

Theorem 3.8. *Let X be a real normed space. If X is norm Cartesian, then X admits a norm i-operator.*

4. Wenzel's Question

The problem of the converse to Theorem 3.8 and Theorem 2.2 is basically what Wenzel's Question is all about (see [10]).

Question 4.1 (Wenzel, 1995)**.** *Does there exist a real Banach space X with a norm i-operator $i : X \to X$ such that X is not real-isomorphic to the square of a real Banach space (or equivalently, X is not topologically Cartesian)?*

Similar questions of the same nature as the one above have been posed and solved along the recent years (see, for instance, [5, 2]). In 1986 Bourgain showed the existence of complex Banach spaces which are complex-isomorphic but not real-isomorphic (see [2]). The following remark serves to dismiss possible candidates to solve Question 4.1 in the positive.

Remark 4.2. Bad candidates to solve Question 4.1 in the positive:

- In [8] it is shown an elementary example of complex Banach space Z_α which is not isomorphic to its complex conjugate. Such a space cannot be complex isomorphic to a square. However, in [11] it is proved that Z_α is real-isomorphic to its square $\mathsf{Z}_\alpha \oplus \mathsf{Z}_\alpha$.
- It follows by Maharam's Decomposition Theorem (see [4]) that every infinite dimensional L-space is isomorphic with its Cartesian square (let us recall that a Banach lattice is called an L-space whenever $x \geq 0$ and $y \geq 0$ verify that $\|x+y\| = \|x\| + \|y\|$).

The next remark suggests a possible good candidate to solve Question 4.1 in the positive. We refer the reader to [1, 9] for a wider perspective.

Remark 4.3. For a real or complex Banach space X, consider

$$X_w := \left\{ x^{**} \in X^{**} : \begin{array}{c} \text{for all } (x_n^*)_{n\in\mathbb{N}} \subset X^* \text{ there exists } x \in X \\ \text{such that } x^{**}(x_n^*) = x_n^*(x) \text{ for every } n \in \mathbb{N} \end{array} \right\}.$$

The deficiency of X in X_w is defined as

$$\delta_w(X) := \dim\left(\frac{X_w}{X}\right).$$

It is not difficult to check that

$$\delta_w(X \times Y) = \delta_w(X) + \delta_w(Y)$$

for any Banach spaces X and Y. Pelczynski remarked (see [9, p. 84]) that

$$\delta_w(\mathcal{C}([0,\omega_1], \mathbb{R})) = 1.$$

As a consequence, $\mathcal{C}([0,\omega_1], \mathbb{R})$ cannot be real-isomorphic to a square.

5. i-Operators on spaces of continuous functions

We will begin this final section by proving that the spaces of continuous real-valued functions over connected compact Hausdorff topological spaces are free of norm i-operators.

Theorem 5.1. *Let K be a compact Hausdorff space. If K is connected, then $\mathcal{C}(K,\mathbb{R})$ does not admit any topological i-operator which is an isometry. In particular, $\mathcal{C}(K,\mathbb{R})$ is free of norm i-operators.*

Proof. Let $i : \mathcal{C}(K,\mathbb{R}) \to \mathcal{C}(K,\mathbb{R})$ be a topological i-operator and assume that i is an isometry. The Banach-Stone Theorem allows the existence of $g \in \mathcal{C}(K,\mathbb{R})$ with $|g(k)| = 1$ for all $k \in K$ and a homeomorphism $\phi : K \to K$ such that

$$i(f) = g \cdot (f \circ \phi).$$

Notice that
$$-f = i^2(f) = g \cdot ((g \cdot (f \circ \phi)) \circ \phi).$$
Since K is connected, we deduce that either g is the constant function equal to 1 or to -1. In any case, we have that
$$-f = f \circ \phi^2.$$
The contradiction comes into play when we consider any $f \in \mathcal{C}(K, \mathbb{R})$ such that $\inf f(K) > 0$. $\square$

Next, we will provide a sufficient condition for a space of real-valued continuous functions to admit a topological i-operator.

Definition 5.2. Let K be a Hausdorff compact space. We say that K is almost decomposable if there are two closed subsets L and S of K such that $K = L \cup S$, $L \cap S = \{k_0\}$ is a singleton, and there exists a homeomorphism $\phi : L \to S$ that fixes k_0.

Examples of almost decomposable Hausdorff compact spaces follow.

Example 5.3. The following Hausdorff compact topological spaces are almost decomposable:

- Any non-trivial bounded and closed interval of the real line is an example of an almost decomposable Hausdorff compact metric space.
- A finite discrete space is almost decomposable if and only if its cardinality is odd.
- If K is an infinite discrete topological space, then its one-point compactification $\widehat{K}$ is also almost decomposable.

Theorem 5.4. *Let K be a Hausdorff compact space with more than one point. If K is almost decomposable, then $\mathcal{C}(K, \mathbb{R})$ admits a topological i-operator.*

Proof. Consider two closed subsets L and S of K such that $K = L \cup S$, $L \cap S = \{k_0\}$ is a singleton, and there exists a homeomorphism $\phi :$ $L \to S$ that fixes k_0. Observe that $\mathcal{C}(K, \mathbb{R})$ is isomorphic to
$$\mathcal{C}_0 := \{f \in \mathcal{C}(K, \mathbb{R}) : f(k_0) = 0\}$$
via the map
$$f \mapsto f - f(k_0).$$
Therefore, it suffices to construct a topological i-operator on $\mathcal{C}_0$. Let $\phi : L \to S$ be a homeomorphism that fixes k_0. Define the map $i : \mathcal{C}_0 \to \mathcal{C}_0$ as follows: If $f \in \mathcal{C}_0$ and $k \in K$, then
$$i(f)(k) := \begin{cases} f(\phi(k)) & \text{if } k \in L, \\ -f(\phi^{-1}(k)) & \text{if } k \in S. \end{cases}$$

It is not difficult to check that the previous map i is a topological i-operator on $\mathcal{C}_0$. $\square$

Actually, the previous result can be improved in the following way.

Theorem 5.5. *Let K be a Hausdorff compact space with more than one point. If K is almost decomposable, then $\mathcal{C}(K,\mathbb{R})$ is topoloigcally Cartesian.*

Proof. Keeping the same notation as in the proof of Theorem 5.4, it is sufficent to notice that the map

$$\begin{array}{rcl} \mathcal{C}_0 & \to & \mathcal{L}_0 \oplus_\infty \mathcal{S}_0 \\ f & \mapsto & (f|_L, f|_S) \end{array}$$

is a surjective linear isometry, where

$$\mathcal{L}_0 := \{f \in \mathcal{C}(L,\mathbb{R}) : f(k_0) = 0\}$$

and

$$\mathcal{S}_0 := \{f \in \mathcal{C}(S,\mathbb{R}) : f(k_0) = 0\}.$$

$\square$

The final part of this section is aimed at showing an example of a real normed space free of norm i-operators whose unit ball is also free of convex subsets with non-empty interior relative to it.

Lemma 5.6. *Let K be a compact Hausdorff metric space. If $k \in K$, then*

$$\begin{array}{rl} & \operatorname{int}_{\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}}\left(\delta_k^{-1}(\{1\}) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})}\right) \\ = & \left\{f \in \delta_k^{-1}(\{1\}) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})} : \sup|f|(K \setminus \{k\}) < 1\right\}. \end{array}$$

Proof. Let $f \in \mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$ such that $\sup|f|(K \setminus \{k\}) < 1$. If $g \in \mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$ and

$$\|g - f\|_\infty < \frac{1 - \sup|f|(K \setminus \{k\})}{2},$$

then $g(k) = 1$. Conversely, consider

$$f \in \operatorname{int}_{\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}}\left(\delta_k^{-1}(1) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})}\right)$$

and suppose to the contrary that $\sup|f|(K \setminus \{k\}) = 1$. Then there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $K \setminus \{k\}$ such that $(f(k_n))_{n\in\mathbb{N}}$ converges to 1. By passing to a subsequence if necessary we may assume that $(k_n)_{n\in\mathbb{N}}$ is convergent to some element $l \in K$. For every $m \in \mathbb{N}$ we can consider a continuous function

$$\begin{array}{rrcl} g_m : & \{k_n : n \in \mathbb{N}\} \cup \{k, l\} & \to & [-1,1] \\ & z & \mapsto & g_m(z) = \left\{\begin{array}{rl} 1 & \text{if } z = k_m, \\ -1 & \text{otherwise.} \end{array}\right. \end{array}$$

Since $\{k_n : n \in \mathbb{N}\} \cup \{k, l\}$ is closed and K is compact, we can extend g_m continuously to the whole of K. We will keep denoting this extension by g_m. For every $m \in \mathbb{N}$ we define

$$t_m := \begin{cases} 1/m & \text{if } f(k_m) = 1, \\ 1 - f(k_m) & \text{if } f(k_m) < 1. \end{cases}$$

Notice that the sequence

$$\left(\frac{f + t_m g_m}{\|f + t_m g_m\|} \right)_{m \in \mathbb{N}}$$

converges to f and is contained in $\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$, therefore we can find $m_0 \in \mathbb{N}$ such that

$$\frac{f + t_{m_0} g_{m_0}}{\|f + t_{m_0} g_{m_0}\|} \in \delta_k^{-1}(\{1\}) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})}.$$

As a consequence,

$$1 - t_{m_0} = \|f + t_{m_0} g_{m_0}\| \geq f(k_{m_0}) + t_{m_0},$$

which is impossible by construction of $(t_m)_{m \in \mathbb{N}}$. □

Theorem 5.7. *Let K be a compact Hausdorff metric space. Let $k \in K$. The following assertions are equivalent:*

1. *k is an isolated point of K.*
2. *There exists $f \in \mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$ such that $\sup|f|(K \setminus \{k\}) < 1$.*

Proof. If k is an isolated point of K, then the function $f : K \to \mathbb{R}$ such that $f(k) = 1$ and $f(K \setminus \{k\}) = \{0\}$ is continuous and verifies the desired properties. Conversely, assume that there exists $f \in \mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$ such that $\sup|f|(K \setminus \{k\}) < 1$ and suppose to the contrary that there exists a sequence $(k_n)_{n \in \mathbb{N}}$ in $K \setminus \{k\}$ converging to k. Since $(f(k_n))_{n \in \mathbb{N}}$ converges to $f(k) = 1$, by passing to a subsequence if necessary we may assume that $f(k_n) > 0$ for all $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ we can consider a continuous function

$$\begin{array}{rrcl} g_m : & \{k_n : n \in \mathbb{N}\} \cup \{k\} & \to & [0,1] \\ & k_n & \mapsto & g_m(k_n) = \delta_{nm}, \\ & k & \mapsto & g_m(k) = 0. \end{array}$$

Since $\{k_n : n \in \mathbb{N}\} \cup \{k\}$ is closed and K is compact, we can extend g_m continuously to the whole of K. We will keep denoting this extension by g_m. Notice that the sequence

$$\left(\frac{f + 2(1 - f(k_m)) g_m}{\|f + 2(1 - f(k_m)) g_m\|} \right)_{m \in \mathbb{N}}$$

converges to f and is contained in $\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$, therefore we can find $m_0 \in \mathbb{N}$ such that

$$\frac{f + 2\left(1 - f\left(k_{m_0}\right)\right) g_{m_0}}{\left\|f + 2\left(1 - f\left(k_{m_0}\right)\right) g_{m_0}\right\|} \in \delta_k^{-1}\left(\{1\}\right) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})}.$$

As a consequence,

$$\begin{aligned} 1 &= \left\|f + 2\left(1 - f\left(k_{m_0}\right)\right) g_{m_0}\right\| \\ &\geq f\left(k_{m_0}\right) + 2\left(1 - f\left(k_{m_0}\right)\right) \\ &= 2 - f\left(k_{m_0}\right) \\ &\geq 2 - \sup |f|\left(K \setminus \{k\}\right) \\ &> 1, \end{aligned}$$

which is impossible. □

Corollary 5.8. *Let K be a compact Hausdorff metric space. The unit sphere of $\mathcal{C}(K,\mathbb{R})$ has a convex set with non-empty interior relative to it if and only if K has an isolated point.*

Proof. If $k \in K$ is an isolated point, then the previous two results assure that $\delta_k^{-1}\left(\{1\}\right) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})}$ has non-empty interior relative to $\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$. Conversely, let C be a convex subset of $\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$ with non-empty interior relative to $\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$. It is well known that C must be supported on $\mathsf{B}_{\mathcal{C}(K,\mathbb{R})}$ by an extreme point of the unit ball of $\mathcal{C}(K,\mathbb{R})^*$. That extreme point is of the form δ_k for some $k \in K$. Then $\delta_k^{-1}\left(\{1\}\right) \cap \mathsf{B}_{\mathcal{C}(K,\mathbb{R})}$ has non-empty interior relative to $\mathsf{S}_{\mathcal{C}(K,\mathbb{R})}$, so k is an isolated point of K in accordance to the previous two results. □

Example 5.9. If $b < c$ are real numbers, then $\mathcal{C}([b,c],\mathbb{R})$ verifies the following:

- It admits a topological i-operator in virtue of Theorem 5.4.
- It does not admit a norm i-operator in accordance with Theorem 5.1.
- Its unit sphere is free of convex subsets with non-empty interior relative to it according to Corollary 5.8.

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