

Integral representation of quaternion elliptical density and its applications

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Abstract: In this paper, an integral representation for the density of a matrix variate quaternion elliptical distribution is proposed. To this end, a weight function is used, based on the inverse Laplace transform of a function of a Hermitian quaternion matrix. Examples of well-known members of the family of quaternion elliptical distributions are given as well as their respective weight functions. It is shown that under some conditions, the proposed formula can be applied for the scale mixture of quaternion normal models. Applications of the proposed method are also given.

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1 Introduction

Quaternions were introduced in the mid-nineteenth century by Hamilton [12] and [13] as an extension of complex numbers and as a tool for manipulating 3-dimensional vectors. Although quaternion field of algebra has many applications in quantum physics, geostatics, the figure and pattern recognition, molecular modeling and the space telemetry (see Liu [17], Wang [21] and Karney [14]), its statistical applications are not considered well yet. It also should be noticed that the statistical theory of quaternions is well proposed (see Anderson [1], Kabe [15], Rautenbach and Roux [20], Dimitriu and Koev [8], Diaz-Garcia and Jaimez [7] and more recently Loots et al. [18]).

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As potential application of matrix quaternion normal distribution in multivariate analysis, Faraway and Coe [9] formulated a setting for modeling head and hand orientation during motion. Since there is an increasing criticism on the inappropriate use of the normal distribution to model, therefore it is often more appropriate to use the elliptical distribution. The elliptical distribution provides a highly impressive list of heavier/lighter tail alternatives to the multivariate normal models. Thus the necessity of formulating quaternion elliptical distribution and the ease of its use highlight here.

In this paper, a unique approach is followed in that we combine the best of both fields (i.e. the matrix variate quaternion normal distribution and the matrix variate elliptical distribution), by expanding the matrix variate quaternion elliptical distribution as an integral of a series of quaternion normal densities, having the same covariance structure to different scales. Therefore, properties of the matrix variate quaternion elliptical distribution can be easily explored due to the fact that it can be described in terms of its quaternion normal components. The different choices for the weight functions for the random variable emphasize the flexibility of this representation of the matrix variate quaternion elliptical distribution. Furthermore, a generalised quaternion distribution in terms of integral series of quaternion Wishart densities is proposed. This result in the derivation of the distributions of the extreme eigenvalues (i.e. maximum and minimum) of this generalised quaternion distribution with special focus on the scenario where the choice of the weight function is that of the matrix variate quaternion t distribution. This methodology provides an uncomplicated approach to the study of the matrix variate quaternion elliptical distribution and its properties.

The paper is organized as follows: In Section 2 a collection of fundamental mathematical results are presented for use in later sections. Section 3 is devoted to the matrix quaternion normal distribution and matrix quaternion elliptical distribution. The key idea is presented in Section 4, that is the integral representation for the density of a matrix variate quaternion elliptical distribution; followed by some properties of the distribution, using this alternative approach. In Section 5 the generalised quaternion distribution formed by integral series of quaternion Wishart densities is proposed, using this alternative methodology, and we conclude by presenting the distributions of the extreme eigenvalues as well as the expression for the density function of Wilks' statistic in the case of this generalised quaternion matrices.

2 Preliminaries and Notation

A number of useful theorems and other general results that are found in the literature will be discussed in this section, and will be referred to frequently in subsequent sections.

Let \mathbb{R} denote the field of real numbers, and \mathbb{Q} the quaternion (Hamiltonion) division algebra over \mathbb{R} , respectively. Hence, every $z \in \mathbb{Q}$ can be expressed as $z = x_1 + ix_2 + jx_3 + kx_4$, where $i, j,$ and k satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$$

and where $x_1, x_2, x_3 \in \mathbb{R}$. The conjugate of a quaternion element is defined in a similar fashion to that of a complex number, and is given by: $\bar{z} = x_1 - ix_2 - jx_3 - kx_4$ and $\|z\| = (z^H z)^{\frac{1}{2}} =$

$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}$. Now let $M_{n \times p}(\mathbb{R})$ and $M_{n \times p}(\mathbb{Q})$ denote the set of all $n \times p$ matrices over \mathbb{R} and \mathbb{Q} , respectively. In the case of square matrices, say $p \times p$, this will be indicated by $M_p(\mathbb{R})$ and $M_p(\mathbb{Q})$ instead. Similar to the scalar form above, any $\mathbf{Z} \in M_{n \times p}(\mathbb{Q})$ may be rewritten as:

$$\mathbf{Z} = [z_{ij}]_{n \times p} = \mathbf{X}_1 + i\mathbf{X}_2 + j\mathbf{X}_3 + k\mathbf{X}_4,$$

where $z_{ij} \in \mathbb{Q}$, and $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, and $\mathbf{X}_4 \in M_{n \times p}(\mathbb{R})$. \mathbf{X}_1 is the real part of \mathbf{Z} , and will be denoted by $\text{Re } \mathbf{Z}$. By setting $n = 1$, this reduces to the vector form in an obvious way.

We will denote the transpose of a matrix \mathbf{Z} as $\mathbf{Z}' = \mathbf{Z}$. The conjugate transpose of \mathbf{Z} is therefore given by

$$\bar{\mathbf{Z}}' = [z'_{ij}]_{p \times n} = \mathbf{X}'_1 - i\mathbf{X}'_2 - j\mathbf{X}'_3 - k\mathbf{X}'_4$$

and we say \mathbf{Z} is Hermitian if $\bar{\mathbf{Z}}' = \mathbf{Z}$, this is will be denoted by \mathbf{Z}^H .

The vec operator is frequently used in expressions involving matrices of quaternions, and is defined as

$$\text{vec } \mathbf{Z} = [\underline{Z}'_1, \dots, \underline{Z}'_p]' \in M_{np \times 1}(\mathbb{Q}),$$

where $\underline{Z}_\alpha \in M_{n \times 1}(\mathbb{Q}), \alpha = 1, \dots, p$ are the columns of \mathbf{Z} .

The trace operator is frequently used in the simplification of expressions, and although the multiplication of quaternions are noncommutative, we may go about it as follows, see Zhang [22].

Let $\text{Retr}(\mathbf{Z}) = \text{tr}(\text{Re } \mathbf{Z})$ for $\mathbf{Z} \in M_p(\mathbb{Q})$, we have

$$\text{Retr}(\mathbf{Z}) = \frac{1}{2} \text{tr}(\mathbf{Z} + \mathbf{Z}^H), \text{Retr}(\mathbf{Z}\mathbf{Y}) = \text{Retr}(\mathbf{Y}\mathbf{Z}) \quad \forall \mathbf{Z}, \mathbf{Y} \in M_p(\mathbb{Q}).$$

Moreover, if $\mathbf{Z} = \mathbf{Z}^H \in M_p(\mathbb{Q})$, i.e. a Hermitian matrix, then $\text{Retr}(\mathbf{Z}) = \text{tr}(\mathbf{Z}) = \sum_{\alpha=1}^p \lambda_\alpha$, where $\lambda_1, \dots, \lambda_p$ are the eigenvalues of \mathbf{Z} .

A number of scattered results that will be used in later sections are briefly presented below. The quaternion multivariate gamma function is defined by (see [11])

$$\begin{aligned} \mathbb{Q}\Gamma_p(a) &= \int_{\mathbf{Z}=\mathbf{Z}^H > \mathbf{0}} |\mathbf{Z}|^{a-2(p-1)-1} \text{etr}(-\mathbf{Z}) d\mathbf{Z} \\ &= \pi^{p(p-1)} \prod_{i=1}^p \Gamma(a - 2(i-1)), \quad \text{Re}(a) > 2(p-1). \end{aligned} \quad (2.1)$$

The quaternion multivariate beta function is given by (see [15])

$$\begin{aligned} \mathbb{Q}B_p(a, b) &= \int_{\mathbf{0} < \mathbf{Z} = \mathbf{Z}^H < \mathbf{I}_p} |\mathbf{Z}|^{a-2(p-1)-1} |\mathbf{I} - \mathbf{Z}|^{b-2(p-1)-1} d\mathbf{Z} \\ &= \frac{\mathbb{Q}\Gamma_p(a)\mathbb{Q}\Gamma_p(b)}{\mathbb{Q}\Gamma_p(a+b)} = \mathbb{Q}B_p(b, a), \quad \text{Re}(a, b) > 2(p-1) \end{aligned} \quad (2.2)$$

The quaternion generalized hypergeometric function of one matrix argument is defined by

$${}_p\mathbb{Q}F_q(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{Z}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_{\kappa} \cdots (a_p)_{\kappa}}{(b_1)_{\kappa} \cdots (b_q)_{\kappa}} \frac{\mathbb{Q}C_{\kappa}(\mathbf{Z})}{k!} \quad (2.3)$$

where $a_i, i = 1, \dots, p, b_j, j = 1, \dots, q$ are arbitrary quaternion numbers, $\mathbf{Z} (p \times p)$ is a quaternion Hermitian matrix, $\mathbb{Q}C_\kappa(\mathbf{Z})$ is the zonal polynomial of $p \times p$ quaternion Hermitian matrix \mathbf{Z} corresponding to the partition $\kappa = (k_1, \dots, k_m), k_1 \geq \dots \geq k_p \geq 0, k_1 + \dots + k_p = k$ and \sum_κ denotes summation over all partitions κ . The quaternion generalized hypergeometric coefficient $(a)_\kappa$ used above is defined by $(a)_\kappa = \prod_{i=1}^m (a - 2(i-1))_{k_i} = \frac{\mathbb{Q}\Gamma_p(a, \kappa)}{\mathbb{Q}\Gamma_p(a)}$ where $(a)_r = a(a+1) \cdots (a-2r+2), r = 1, 2, \dots$ with $(a)_0 = 1$. Conditions for convergence of the series in (2.3) are available in the literature.

The quaternion confluent hypergeometric function with matrix argument, ${}_1\mathbb{Q}F_1(\cdot)$, has the integral representation (see [6])

$${}_1\mathbb{Q}F_1(a; b; \mathbf{X}) = \frac{1}{\mathbb{Q}B_p(a, b-a)} \int_{\mathbf{0} < \mathbf{Z} = \mathbf{Z}^H < \mathbf{I}_p} \exp(-\text{Re tr}(\mathbf{XZ})) |\mathbf{Z}|^{a-2p+1} |\mathbf{I} - \mathbf{Z}|^{b-a-2p+1} d\mathbf{Z}, \quad (2.4)$$

valid for $\text{Re}(b) > \text{Re}(a) + 2(p-1) > 4(p-1) - k_p$.

3 The Matrix Variate Quaternion Elliptical Distribution

In this section the definitions of the two members that form the basis for the development of the integral representation are given; followed by defining the matrix variate quaternion elliptical distribution from the vector variate quaternion elliptical distribution perspective.

Definition 3.1. The quaternion random matrix \mathbf{X} has a matrix variate quaternion normal distribution (QND) if it has the following density function

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{2^{pn}}{\pi^{2pn} |\Sigma|^{2n} |\Phi|^{2p}} \exp[-2 \text{Re tr}(\Sigma^{-1}(\mathbf{X} - \mathbf{M})^H \Phi^{-1}(\mathbf{X} - \mathbf{M}))]$$

where $\mathbf{M} \in M_{n \times p}(\mathbb{Q}), \Sigma = \mathbf{A}\mathbf{A}^H \in M_p(\mathbb{Q}), \Phi = \mathbf{B}\mathbf{B}^H \in M_n(\mathbb{Q})$, are both invertible. In this case we may use the notation $\mathbf{X} \sim \mathbb{Q}N_{n \times p}(\mathbf{M}, \Phi, \Sigma)$.

Definition 3.2. The quaternion random matrix \mathbf{X} has a matrix-variate quaternion elliptical distribution (QED) if its characteristic function has the form

$$\phi_{\mathbf{X}}(\mathbf{T}) = \exp(\iota \text{Re tr}(\mathbf{T}^H \mathbf{M})) \psi(\text{Re tr}(\Sigma \mathbf{T}^H \Phi \mathbf{T})),$$

where $\mathbf{T}, \mathbf{M} \in M_{n \times p}(\mathbb{Q}), \Sigma = \mathbf{A}\mathbf{A}^H \in M_p(\mathbb{Q}), \Phi = \mathbf{B}\mathbf{B}^H \in M_n(\mathbb{Q})$, are both invertible, ι the usual imaginary complex unit and $\psi: [0, \infty) \rightarrow \mathbb{R}$ is the characteristic generator. In this case we use the notation $\mathbf{X} \sim \mathbb{Q}E_{n \times p}(\mathbf{M}, \Phi, \Sigma, \psi)$.

Then the density function of \mathbf{X} can be expressed as

$$f_{\mathbf{X}}(\mathbf{X}) = g[-2 \text{Re tr}(\Sigma^{-1}(\mathbf{X} - \mathbf{M})^H \Phi^{-1}(\mathbf{X} - \mathbf{M}))],$$

where $g(\cdot)$ is the density generator satisfying the conditions under the real case (see [4]). In this case we may use the notation $\mathbf{X} \sim \mathbb{Q}E_{n,p}(\mathbf{M}, \Phi, \Sigma, g)$.

Subsequently, the vector variate quaternion elliptical distribution can be obtained by taking $p = 1$. In this case we use the notation $\underline{X} \sim \mathbb{Q}E_n(\underline{\mu}, \underline{\Phi}, g)$ ($\underline{\mu} \equiv \mathbf{M} \in M_{p \times 1}(\mathbb{Q})$).

In a similar fashion, using Theorem 9 of Loots et al. [18], we have the following important result:

Theorem 3.3. *Let $\underline{Z}_\alpha \in M_{p \times 1}(\mathbb{Q})$, $\alpha = 1, \dots, n$ be n probability vectors each having a p -variate QED. Now suppose that*

$$\begin{aligned} \mathbf{Z} &= [Z_{\alpha\beta}], \quad \alpha = 1, \dots, n, \beta = 1, \dots, p \\ &= \begin{bmatrix} Z_{11} & \dots & Z_{1p} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \dots & Z_{np} \end{bmatrix} = \begin{bmatrix} \underline{Z}'_1 \\ \vdots \\ \underline{Z}'_n \end{bmatrix} = [\underline{Z}_{(1)} \quad \dots \quad \underline{Z}_{(p)}] \end{aligned}$$

i.e. the rows of \mathbf{Z} are $\mathbb{Q}E_p(\underline{\mu}_\alpha, \underline{\Sigma})$ distributed, $\alpha = 1, \dots, n$ with dependence structure given by $\underline{\Omega} \in M_{n \times n}(\mathbb{Q})$. It may be assumed without loss of generality that $\underline{\Omega}$ is real-valued. Similarly, define $\mathbf{M} = [\mu_{\alpha\beta}]$, $\alpha = 1, \dots, n$, $\beta = 1, \dots, p$. Then

$$\text{vec } \mathbf{Z} = \begin{bmatrix} \underline{Z}_{(1)} \\ \vdots \\ \underline{Z}_{(n)} \end{bmatrix} \sim \mathbb{Q}E_{np}(\text{vec } \mathbf{M}, \underline{\Sigma} \otimes \underline{\Omega})$$

i.e. a matrix-variate QED where

$$\text{vec } \mathbf{M} = \begin{bmatrix} \underline{\mu}_{(1)} \\ \vdots \\ \underline{\mu}_{(p)} \end{bmatrix}.$$

4 Main Results

Chu [5] and Gupta and Varga [10] demonstrated that real elliptical distributions can always be expanded as an integral of a set of normal densities. The next theorem presents the representation of the matrix variate quaternion elliptical distribution as an integral of a series of quaternion normal densities. Due to this form, properties of the matrix variate quaternion elliptical distribution can be described in terms of its quaternion normal components. In addition we demonstrate the flexibility of this alternative approach with some examples.

Theorem 4.1. *If $\mathbf{X}_{n \times p}$ is a quaternion random matrix following $\mathbb{Q}E_{n \times p}(\mathbf{M}, \underline{\Phi}, \underline{\Sigma}, g)$, then there exists a scalar weight function $\mathcal{W}(\cdot)$ defined on \mathbb{R}^+ such that*

$$h_{\mathbf{X}}(\mathbf{X}) = \int_{\mathbb{R}^+} \mathcal{W}(t) f_{\mathbf{X}}(\mathbf{X}) dt,$$

where $f_{\mathbf{X}}(\cdot)$ is the density of $\mathbb{Q}N_{n \times p}(\mathbf{M}, \Phi, t^{-1}\Sigma)$ and the weight function $\mathcal{W}(\cdot)$ is given by

$$\mathcal{W}(t) = (\pi/2)^{2np} |\Sigma|^{2n} |\Phi|^{2p} t^{-2np} \mathcal{L}^{-1} \left\{ g \left[-2 \operatorname{Re} \operatorname{tr} \left(\Sigma^{-1} (\mathbf{X} - \mathbf{M})^H \Phi^{-1} (\mathbf{X} - \mathbf{M}) \right) \right] \right\}.$$

where \mathcal{L} is the Laplace transform operator.

Proof: From the density function of \mathbf{X} we have

$$\begin{aligned} h_{\mathbf{X}}(\mathbf{X}) &= g \left[-2 \operatorname{Re} \operatorname{tr} \left(\Sigma^{-1} (\mathbf{X} - \mathbf{M})^H \Phi^{-1} (\mathbf{X} - \mathbf{M}) \right) \right] \\ &= \mathcal{L} \left[\mathcal{W}(t) (2/\pi)^{2np} |\Sigma|^{-2n} |\Phi|^{-2p} t^{2np} \right] \\ &= \int_{\mathbb{R}^+} \mathcal{W}(t) (2/\pi)^{2np} |t^{-1}\Sigma|^{-2n} |\Phi|^{-2p} \\ &\quad \times \exp \left[-2t \operatorname{Re} \operatorname{tr} \left(\Sigma^{-1} (\mathbf{X} - \mathbf{M})^H \Phi^{-1} (\mathbf{X} - \mathbf{M}) \right) \right] dt \\ &= \int_{\mathbb{R}^+} \mathcal{W}(t) f_{\mathbf{X}}(\mathbf{X}) dt \quad \blacksquare \end{aligned}$$

Remark 4.2. (i) In Theorem 4.1 it is stated that f is the density of $\mathbb{Q}N_{n \times p}(\mathbf{M}, \Phi, t^{-1}\Sigma)$. It is realized from the proof that f can be even the density of $\mathbb{Q}N_{n \times p}(\mathbf{M}, t^{-1}\Phi, \Sigma)$ or $\mathbb{Q}N_{n \times p}(\mathbf{M}, t^{-\frac{1}{2}}\Phi, t^{-\frac{1}{2}}\Sigma)$. This fact enables us to adopt each representation whenever is needed.

(ii) Under the assumptions of Theorem 4.1, sufficient conditions to ensure that the inverse Laplace transform exists, is to assume that $g(s^2)$ is differential when s^2 is sufficiently large, and $g(s^2)$ vanishes faster than what s^{-k} , $k > 1$ as $s \rightarrow \infty$.

(iii) From Theorem 4.1 and using Fubini's Theorem, we have

$$1 = \int_{M_{p \times n}(\mathbb{Q})} h_{\mathbf{X}}(\mathbf{X}) d\mathbf{X} = \int_{\mathbb{R}^+} \mathcal{W}(t) \left(\int_{M_{p \times n}(\mathbb{Q})} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{X} \right) dt = \int_{\mathbb{R}^+} \mathcal{W}(t) dt.$$

Thus for a non-negative weight function $\mathcal{W}(\cdot)$, the function $\mathcal{W}(\cdot)$ is a density function of t . In this case Theorem 4.1 can be interpreted as a representation of a scale mixture of quaternion normals.

(iv) Similarly as in Theorem 4.1, one can represent the complex matrix variate elliptical distribution as an integral of series of complex normal densities with possible applications in the communications systems.

Now we give some examples of the weight function in Theorem 4.1 for some well-known quaternion elliptical models. These models are the analogous types proposed in Chu [5].

4.1 Matrix Variate Quaternion Normal Distribution

Based on Definition 3.1 and Theorem 4.1, it can be realized that the weight function $\mathcal{W}(\cdot)$ is given by

$$\mathcal{W}(t) = \delta(t - 1),$$

where $\delta(\cdot)$ is the dirac delta or impulse function having the property $\int_{\mathbb{R}} f(x)\delta(x)dx = f(0)$, for every Borel-measurable function $f(\cdot)$.

4.2 Matrix Variate ε -Contaminated Quaternion Normal Distribution

We say the quaternion random matrix \mathbf{X} has the matrix variate ε -contaminated quaternion normal distribution with the parameters $\mathbf{M} \in M_{n \times p}(\mathbb{Q})$, $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^H \in M_p(\mathbb{Q})$, $\mathbf{\Phi} = \mathbf{B}\mathbf{B}^H \in M_n(\mathbb{Q})$, are both invertible, $\sigma > 0$ and $0 < \varepsilon < 1$, if it has the following density

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{2^{pn}}{\pi^{2pn} |\mathbf{\Sigma}|^{2n} |\mathbf{\Phi}|^{2p}} \left\{ (1 - \varepsilon) \exp \left[-2 \operatorname{Re} \operatorname{tr} \left(\mathbf{\Sigma}^{-1} (\mathbf{X} - \mathbf{M})^H \mathbf{\Phi}^{-1} (\mathbf{X} - \mathbf{M}) \right) \right] + \frac{\varepsilon}{\sigma^{pn}} \exp \left[-\frac{2}{\sigma^2} \operatorname{Re} \operatorname{tr} \left(\mathbf{\Sigma}^{-1} (\mathbf{X} - \mathbf{M})^H \mathbf{\Phi}^{-1} (\mathbf{X} - \mathbf{M}) \right) \right] \right\}.$$

Then it can be concluded that the weight function is given by

$$\mathcal{W}(t) = (1 - \varepsilon)\delta(t - 1) + \varepsilon\delta(t - \sigma^2).$$

4.3 Matrix Variate Quaternion t -Distribution

We say the quaternion random matrix \mathbf{X} has the matrix variate quaternion t -distribution with the parameters $\mathbf{M} \in M_{n \times p}(\mathbb{Q})$, $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^H \in M_p(\mathbb{Q})$, $\mathbf{\Phi} = \mathbf{B}\mathbf{B}^H \in M_n(\mathbb{Q})$, are both invertible, and $\nu > 0$ denoted by $\mathbf{X} \sim \mathbb{Q}T_{p \times n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{\Phi}, \nu)$, if it has the following density

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{\nu^{2pn} \mathbb{Q}\Gamma(2[pn + \nu])}{\pi^{2pn} \mathbb{Q}_p\Gamma(2\nu)} \left\{ 1 + \frac{1}{\nu} \operatorname{Re} \operatorname{tr} \left(\mathbf{\Sigma}^{-1} (\mathbf{X} - \mathbf{M})^H \mathbf{\Phi}^{-1} (\mathbf{X} - \mathbf{M}) \right) \right\}^{-2(pn + \nu)} \quad (4.1)$$

where $\mathbb{Q}\Gamma(\cdot)$ is the quaternion gamma function.

By applying Theorem 4.1, the weight function $\mathcal{W}(\cdot)$ is then given by

$$\mathcal{W}(t) = \frac{(2t\nu)^{2\nu} e^{-2t\nu}}{t\mathbb{Q}\Gamma(2\nu)} \quad (4.2)$$

where $\mathbb{Q}\Gamma(\cdot)$ is the quaternion gamma function.

The matrix variate quaternion Cauchy distribution is obtained by setting $\nu = 1$ in 4.1.

Theorem 4.3. *Suppose that $\mathbf{X} \sim \mathbb{Q}E_{n \times p}(\mathbf{M}, \mathbf{\Phi}, \mathbf{\Sigma}, g)$, then we have*

- (i) $E(\mathbf{X}) = \mathbf{M}$, and $E(\mathbf{X}^H \mathbf{X}) = \kappa^{(1)} \boldsymbol{\Sigma} \otimes \boldsymbol{\Phi}$,
where

$$\kappa^{(i)} = \int_{\mathbb{R}^+} \left(\frac{1}{t}\right)^i \mathcal{W}(t) dt.$$

- (ii) The characteristic function of \mathbf{X} is isomorph to that of the real case.
(iii) Let $\mathbf{C} \in M_{p \times q}(\mathbb{Q})$, $\mathbf{D} \in M_{n \times m}(\mathbb{Q})$ and $\mathbf{E} \in M_{q \times m}(\mathbb{Q})$ be constant matrices. Further assume that $\mathbf{Y} = \mathbf{C}^H \mathbf{X} \mathbf{D} + \mathbf{E}$. Then

$$\mathbf{Y} \sim \mathbb{Q}E_{n \times p}(\mathbf{C}^H \mathbf{M} \mathbf{D} + \mathbf{E}, \mathbf{D}^H \boldsymbol{\Phi} \mathbf{D}, t^{-1} \mathbf{C}^H \boldsymbol{\Sigma} \mathbf{C}, g)$$

and the density of \mathbf{Y} can be represented as

$$h_{\mathbf{Y}}(\mathbf{Y}) = \int_{\mathbb{R}^+} \mathcal{W}(t) f_{\mathbf{X}}(\mathbf{X}) dt$$

where $f(\cdot)$ is the density of $\mathbb{Q}N_{n \times p}(\mathbf{C}^H \mathbf{M} \mathbf{D} + \mathbf{E}, \mathbf{D}^H \boldsymbol{\Phi} \mathbf{D}, t^{-1} \mathbf{C}^H \boldsymbol{\Sigma} \mathbf{C})$.

Proof:

- (i) From Theorem 4.1 and Remark 4.2 we get

$$\begin{aligned} E(\mathbf{X}) &= \int_{\mathbb{R}^+} \mathcal{W}(t) E(\mathbf{Z}|t) dt \\ &= \mathbf{M} \int_{\mathbb{R}^+} \mathcal{W}(t) dt = \mathbf{M}, \quad \mathbf{Z}|t \sim \mathbb{Q}N_{n \times p}(\mathbf{M}, \boldsymbol{\Phi}, t^{-1} \boldsymbol{\Sigma}), \text{ also} \\ E(\mathbf{X} \mathbf{X}^H) &= \int_{\mathbb{R}^+} \mathcal{W}(t) E(\mathbf{Z} \mathbf{Z}^H | t) dt \\ &= \int_{\mathbb{R}^+} \mathcal{W}(t) t^{-1} \boldsymbol{\Sigma} \otimes \boldsymbol{\Phi} dt = \kappa^{(1)} \boldsymbol{\Sigma} \otimes \boldsymbol{\Phi}. \end{aligned}$$

- (ii) From Theorem 4.1 and Theorem 10 of Loots et al.[18] we have

$$\begin{aligned} \phi_{\mathbf{X}}(\mathbf{T}) &= E \left[\exp \frac{\iota}{2} \text{tr}(\mathbf{X}^H \mathbf{T} + \mathbf{T}^H \mathbf{X}) \right] \\ &= E \left[\exp \iota \text{Re tr}(\mathbf{X}^H \mathbf{T}) \right] \\ &= \int_{\mathbb{R}^+} \mathcal{W}(t) E \left[\exp \iota \text{Re tr}(\mathbf{X}^H \mathbf{T}) | t \right] dt \\ &= \int_{\mathbb{R}^+} \mathcal{W}(t) \exp \text{Re tr} \left(\iota \mathbf{M} \mathbf{T} - \frac{1}{8t} \boldsymbol{\Sigma} \mathbf{T}^H \boldsymbol{\Phi} \mathbf{T} \right) dt \\ &= \exp \text{Re tr}(\iota \mathbf{M} \mathbf{T}) \psi \left(\text{Re tr}(\boldsymbol{\Sigma} \mathbf{T}^H \boldsymbol{\Phi} \mathbf{T}) \right), \quad (\text{say}). \end{aligned}$$

(iii) follows from (ii) and Theorem 4.1.

5 Applications

Applications of this proposed method in Theorem 4.1 are:

- a definition of a generalised quaternion distribution as an integral series of quaternion Wishart densities;
- expressions for the distributions of the extreme eigenvalues of this generalised quaternion distribution;
- an expression for the density function of Wilks' statistic for matrices from this generalised quaternion distribution.

5.1 Integral Series of Quaternion Wishart Densities Distribution

If $\mathbf{X} \sim \mathbb{Q}N_{n \times p}(\mathbf{0}, \mathbf{I}_n, \mathbf{\Sigma})$ then $\mathbf{W} = \mathbf{X}^H \mathbf{X}$, has the quaternion Wishart distribution denoted by $\mathbb{Q}W_p(n, \mathbf{\Sigma})$ with the following density function

$$f_{\mathbf{W}}(\mathbf{W}) = \frac{2^{np}}{\mathbb{Q}\Gamma_p(2n)|\mathbf{\Sigma}|^{2n}} \exp[-2 \operatorname{Re} \operatorname{tr}(\mathbf{\Sigma}^{-1} \mathbf{W})] |\mathbf{W}|^{2n-2p+1}.$$

where $\mathbb{Q}\Gamma_p(\cdot)$ as defined in (2.1).

In the following result, (see also Díaz- García and Gutiérrez-Jaimes [7]) we rewrite the above result for quaternion elliptical models as a direct consequence of Theorem 4.1, that results from an integral series of quaternion Wishart densities.

Theorem 5.1. *Suppose that $\mathbf{Y} \sim \mathbb{Q}E_{n \times p}(\mathbf{0}, \mathbf{I}_n, \mathbf{\Sigma})$. Then the quadratic form $\mathbf{S} = \mathbf{Y}^H \mathbf{Y}$ has the following density function*

$$f_{\mathbf{S}}(\mathbf{S}) = \frac{2^{np} |\mathbf{S}|^{2n-2p+1} \mathcal{G}(\mathbf{S})}{\mathbb{Q}\Gamma_p(2n) |\mathbf{\Sigma}|^{2n}},$$

where

$$\mathcal{G}(\mathbf{S}) = \int_{\mathbb{R}^+} t^{2n} \exp[-2t \operatorname{Re} \operatorname{tr}(\mathbf{\Sigma}^{-1} \mathbf{S})] \mathcal{W}(t) dt.$$

Under the conditions of Theorem 5.1, \mathbf{S} has the integral series of quaternion Wishart densities distribution, denoted by $\mathbf{S} \sim IS\mathbb{Q}W_p(n, \mathbf{\Sigma}, \mathcal{G})$.

The statistical characteristics of the this distribution can be directly obtained by applying Theorems 4.3 and 5.1. The non-central form can also be defined similarly as in Theorem 4.1. The following result follows naturally from Theorem 4.1 in section 4 and Theorem 6.3 of [6].

Theorem 5.2. Let $\mathbf{S} \sim ISQW_p(n, \Sigma, \mathcal{G})$ and $\Omega \in M_{p \times p}(\mathbb{Q})$, then

$$P(\mathbf{S} < \Omega) = \frac{2^{2np} \mathbb{Q}\Gamma_p(2p-1) |\Omega|^{2n}}{\mathbb{Q}\Gamma_p(2n+2p-1) |\Sigma|^{2n}} \\ \times \int_{\mathbb{R}^+} t^{2np} {}_1\mathbb{Q}F_1(2n; 2n+2p-1; -2t\Omega\Sigma^{-1}) \mathcal{W}(t) dt$$

when $\text{Re}(n) > 4(p-1) - 2k_p$. Also if the constant $2n+2p-1$ is a positive integer, then

$$P(\mathbf{S} > \Omega) = \sum_{k=0}^{p(2n+2p-1)} \sum_{\kappa}^* \frac{1}{k!} \int_{\mathbb{R}^+} t^{2np} \mathbb{Q}C_{\kappa}(2t\Omega\Sigma^{-1}) \text{etr}(2t\Omega\Sigma^{-1}) \mathcal{W}(t) dt,$$

when $\text{Re}(n) > 4(p-1) + 2k_1$ and \sum_{κ}^* denotes summation over those partitions $\kappa = (k_1, \dots, k_p)$ of k with $k_1 \leq 2n+2p-1$.

5.2 Distributions of the Extreme Eigenvalues

Suppose that λ_{max} and λ_{min} are the largest and smallest eigenvalues of \mathbf{S} (where $\mathbf{S} \sim ISQW_p(n, \Sigma, \mathcal{G})$) respectively; then the inequalities $\lambda_{max} < x$ and $\lambda_{min} > y$ are equivalent to $\mathbf{S} < x\mathbf{I}_p$ and $\mathbf{S} > y\mathbf{I}_p$ respectively. and we have the following result:

Corollary 5.3. Let $\mathbf{S} \sim ISQW_p(n, \Sigma, \mathcal{G})$ and $x > 0$, then

$$F_{\lambda_{max}}(x) = P(\lambda_{max} < x) \\ = \frac{2^{2np} \mathbb{Q}\Gamma_p(2p-1) x^{2np}}{\mathbb{Q}\Gamma_p(2n+2p-1) |\Sigma|^{2n}} \\ \times \int_{\mathbb{R}^+} t^{2np} {}_1\mathbb{Q}F_1(2n; 2n+2p-1; -2tx\Sigma^{-1}) \mathcal{W}(t) dt$$

when $\text{Re}(n) > 4(p-1) - 2k_p$. Also if the constant $2n+2p-1$ is a positive integer and $y > 0$, then

$$\bar{F}_{\lambda_{min}}(y) = P(\lambda_{min} > y) \\ = \sum_{k=0}^{p(2n+2p-1)} \sum_{\kappa}^* \frac{1}{k!} \int_{\mathbb{R}^+} t^{2np} \mathbb{Q}C_{\kappa}(2ty\Sigma^{-1}) \text{etr}(2ty\Sigma^{-1}) \mathcal{W}(t) dt,$$

when $\text{Re}(n) > 4(p-1) + 2k_1$.

5.2.1 Example

Specifically, we derive a closed form expression for $P(\lambda_{max} < x)$ of the matrix variate quaternion t-Wishart distribution, i.e. we consider the weight function (4.2). From Corollary 5.3, (4.2), (2.4), ${}_0QF_0(a; A) = \text{etr}(A)$ and equation (3.29) of [5], we have that

$$\begin{aligned}
& F_{\lambda_{max}}(x) \\
&= \frac{2^{np} \mathbb{Q}\Gamma_p(2p-1)x^{2np}}{\mathbb{Q}\Gamma_p(2n+2p-1)|\Sigma|^{2n}} \int_{\mathbb{R}^+} t^{2np} {}_1QF_1(2n; 2n+2p-1; -2tx\Sigma^{-1}) W(t) dt \\
&= \frac{2^{np} \mathbb{Q}\Gamma_p(2p-1)x^{2np}}{\mathbb{Q}\Gamma_p(2n+2p-1)\mathbb{Q}B_p(2n, 2p-1)|\Sigma|^{2n}} \\
&\quad \times \int_{\mathbb{R}^+} t^{2np} \left[\int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}} \exp(-2tx \text{Re tr}(\Sigma^{-1}\mathbf{Y})) |\mathbf{Y}|^{2n-2p+1} d\mathbf{Y} \right] \frac{(2t\nu)^{2\nu} e^{-2t\nu}}{t\mathbb{Q}\Gamma(2\nu)} dt \\
&= \frac{2^{np}(2\nu)^{2\nu} x^{2np}}{\mathbb{Q}\Gamma(2\nu)\mathbb{Q}\Gamma_p(2n)|\Sigma|^{2n}} \int_{\mathbb{R}^+} t^{2(np+\nu)-1} e^{-2t\nu} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}} \mathbb{Q}C_{\kappa}(-2tx\Sigma^{-1}\mathbf{Y}) |\mathbf{Y}|^{2n-2p+1} d\mathbf{Y} dt \\
&= \frac{2^{2(np+\nu)} \nu^{2\nu} x^{2np} \mathbb{Q}\Gamma_p(2p-1)}{\mathbb{Q}\Gamma(2\nu) \mathbb{Q}\Gamma_p(2n+2p-1) |\Sigma|^{2n}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(2n)_{\kappa} \mathbb{Q}C_{\kappa}(-2x\Sigma^{-1})}{k! (2n+2p-1)_{\kappa}} \int_{\mathbb{R}^+} t^{2np+2\nu+k-1} e^{-2t\nu} dt \\
&= \frac{\nu^{-2np} \mathbb{Q}\Gamma_p(2p-1) x^{2np}}{\mathbb{Q}\Gamma(2\nu) \mathbb{Q}\Gamma_p(2n+2p-1) |\Sigma|^{2n}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \mathbb{Q}C_{\kappa}(-x\Sigma^{-1}) \frac{(2n)_{\kappa} \mathbb{Q}\Gamma(2np+2\nu+k)}{(2n+2p-1)_{\kappa} \nu^k}
\end{aligned}$$

5.3 Wilks' Statistic for ISQW Matrices

Another application arises in the context of hypothesis testing of a mean matrix. (The reader is referred to the valuable contribution of Bhavsar [3] where asymptotic distributions of likelihood ratio criteria for two testing problems are considered.)

Let the rows of $\mathbf{X} \in M_{n_1 \times p}(\mathbb{Q})$ and $\mathbf{Y} \in M_{n_2 \times p}(\mathbb{Q})$ be independently $\mathbb{Q}E_p(\mathbf{0}, \Sigma)$ and $\mathbb{Q}E_p(\underline{\mu}, \Sigma)$ distributed, respectively. Then from Theorem 3.3, it is clear that $\mathbf{X} \sim \mathbb{Q}E_{n_1 \times p}(\mathbf{0}, \Sigma \otimes \mathbf{I}_{n_1})$ and $\mathbf{Y} \sim \mathbb{Q}E_{n_2 \times p}(\mathbf{M}, \Sigma \otimes \mathbf{I}_{n_2})$. Now from Theorem 5.1, we get $\mathbf{A} = \mathbf{X}^H \mathbf{X} \sim ISQW_p(n_1, \Sigma, \mathcal{G})$. Further as an extension to the result of Kabe [15], $\mathbf{B} = \mathbf{Y}^H \mathbf{Y} \sim ISQW_p(n_2, \Sigma, \mathcal{G})$ (non-central ISQW distribution), where $\mathbf{\Xi} = \Sigma^{-1} \mathbf{M} \mathbf{M}^H$. As discussed in Bekker et al. [2], the Wilks' statistic

$$\Lambda = \frac{|\mathbf{X}^H \mathbf{X}|}{|\mathbf{X}^H \mathbf{X} + \mathbf{Y}^H \mathbf{Y}|} = \frac{|\mathbf{A}|}{|\mathbf{A} + \mathbf{B}|}$$

can be used as a likelihood ratio criterion for testing whether the matrix mean \mathbf{M} is equal to zero or not. The solution of this problem depends on the derivation of the distribution of Λ . Using Theorem 4.1 and the result of Loots et al. [18], similar to Bekker et al. [2], one can directly conclude that

$$\begin{aligned}
f_{\Lambda}(\lambda) &= \frac{\exp[\text{Retr}(-2\mathbf{\Xi})]}{\mathbb{Q}\Gamma_p(2n_1)} \\
&\quad \times \sum_{k=0}^{\infty} \sum_{\kappa} \mathbb{Q}C_{\kappa}(2\mathbf{\Xi}) \frac{\mathbb{Q}\Gamma_p(2(n_1+n_2), \kappa)}{k!} G_{p,p}^{p,0} \left(\lambda \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_p \end{array} \right. \right)
\end{aligned}$$

where $a_l = 2(n_1 + n_2 - l) + k_l + 1$, and $b_l = 2n_1 - 2l + 1$, $l = 1, \dots, p$, and $G(\cdot)$ is Meijer's G-function (see [19], pp 60).

6 Concluding Remarks

We have derived an integral representation for the density function of the matrix variate quaternion elliptical distribution. This representation eases theoretical derivations of the properties of the matrix variate quaternion elliptical distribution, as well as the development of the generalised quaternion distribution in terms of integral series of quaternion Wishart densities. It also accommodates different distribution models for different choices of the weight function. We would like to conclude that the presented model in Theorem 4.1 and in particular the expressions for the cumulative distribution functions of the extreme eigenvalues, pave the way for transforming the theory into practice for the user. In this regard, the result of Li et al. [16] can be extended under a more general setup considering non-normality and hyper-complex systems.

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