

## MACH'S PRINCIPLE IN NEWTONIAN CONTINUUM MECHANICS

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### ABSTRACT

The Principle of Mach has occupied a significant position in mechanics. The present objective is to outline the nature of the Principle in the context of Newtonian continuum mechanics. It is shown that the inertial force can, indeed, be expressed as an integral over the entire universe. The meaning of this integral is explored.

### BACKGROUND

Doubts concerning the concept of absolute space, and upon the nature of the inertial force, arose soon after the first edition of the “*Principia*” by Newton was published (see Newton [1] for the third edition from which quotes are taken). The inertial force or “*inherent force*” — which Newton defined to be “*the power of resisting by which every body, so far as it is able, perseveres in its state either of resting or of moving uniformly straight forward*”. Most vocal were the comments of Mach [2] who insisted that inertial effects are due to the influence of the entire universe on the given body. To quote: “*... the neglect of the rest of the world is impossible*”. Mach was convinced that the entire universe must be included in a discussion of dynamics. As Sciama [3] notes, quoting from Euler: “*very strange and contrary to the dogmas of metaphysics*” referring to the influences of the “*fixed stars*”. The quote from Berkeley [4] “*Therefore if we suppose that everything is annihilated except one globe, it would be impossible to imagine any movement of that globe.*” gives vivid life to the concerns about the existence of absolute space. The present objective is to explore the suggestion of Mach that the entire universe must be included in a proper description of dynamics. This inclusion is most readily carried out using the covariant formulation of Newtonian mechanics which, itself, rests upon the first axiom of Newton and the (post Newtonian concept of) conservation of energy.

It is clear that if the inertial force is to be dependent upon the entire universe then the theory of continuum mechanics must be cast on a global scale

which includes that universe. It then follows that certain properties of the universe, not usually considered in Newtonian mechanics, must then be included. For example: as viewed from earth, the mass distribution in the universe is essentially isotropic (see Raine [5] or Bondi [6] for example). As just noted, the starting point (see Moulden [7], [8]) must be the first axiom of Newton as extended to meet the needs of continuum mechanics. This extension also includes a global view of Newtonian mechanics.

For the motion of a deformable heat conducting continuum body accept the epistemological statement:

**Constraint I.** *Every material body  $\mathcal{B}$  moves as a rigid body unless acted upon by forces, torques or unless thermal energy passes across the surface,  $\partial\mathcal{B}$ , of  $\mathcal{B}$  or between the parts of  $\mathcal{B}$ . Also, no heat sources exist in  $\mathcal{B}$ , nor do electromagnetic fields and thermal radiation act upon  $\mathcal{B}$ .* □

Let  $\mathbb{W}_e$  be the four dimensional space–time that is appropriate for Newtonian mechanics. Let  $\mathcal{W} = \sum(\mathcal{B}_i)$  represent a Newtonian model of the physical universe with material bodies,  $\mathcal{B}_i$ , over which property fields are defined. Attach the coordinate frame,  $\Lambda$ , to the center of mass of the arbitrary body  $\mathcal{B} \prec \mathcal{W}$  then the first axiom of classical continuum thermo–mechanics has two components which define the background to the theory. This is a more useful form of the first axiom for continuum mechanics, than that given in the *Principia* for particle mechanics, and can be stated as:

**Axiom I.** *The axiom of inertia for continuous thermo–mechanical bodies.*

*Accept constraint I then the center of mass of body  $B$  moves relative to the frame  $\Lambda_c$  such that:*

- 1). *With no angular velocity of frame  $\Lambda$  relative to  $\Lambda_c$  there is for linear inertial motion:*
  - a). *the speed,  $|\mathbf{v}|$ , is a bounded constant*
  - b). *the motion is rectilinear**to define the geodesics of space–time.*
- 2). *With no linear displacement of frame  $\Lambda$  relative to  $\Lambda_c$  there is:*
  - c). *the angular speed,  $|\boldsymbol{\omega}| < \infty$ , about a fixed*

axis through the center of mass of  $B$  is constant. The frame,  $\Lambda_c$ , is located at the mass center of  $\mathcal{O}$ .  $\square$

extending, as in Moulden [8], the discussion given in Moulden [7]. No *Neumann body alpha* is required. This formulation of the axiom accepts that the thermally induced components of the body motion cannot be ignored in continuum mechanics. The global motion of the universe  $\mathcal{O}$  is not of interest since there is no matter external to  $\mathcal{O}$  against which such motion can be assessed. *Axiom I* requires the geometry of  $\mathbb{W}_e$  to be Euclidean since the geodesics are defined to be linear. It can be noted that, as far as the dynamics of  $\mathcal{O}$  are concerned, it is not important if the matter in the universe is light or dark. Make the global:

**Assumption I.** *The universe  $\mathcal{O}$  is finite and isolated with no preferred orientation.*  $\square$

which means that there are no agencies outside the universe that have any influence on bodies inside the universe. The theory herein is entirely Newtonian in nature.

## NOMENCLATURE

$\mathbb{B} \equiv \{\mathcal{B}_i\}$	the set of all material bodies
$\mathbb{E}_u, \mathbb{G}_a \subset \mathbb{E}_u$	Euclidean and Galilean groups
$\mathbb{M}_{v,y}^3, \mathbb{M}_{v,k}^3$	skew and symmetric matrices
$\mathbb{G}_z, \mathbb{G}_c$	Axial and composite groups
$\mathbb{SO}_3 \subset \mathbb{O}_3$	orthogonal and special orthogonal groups
$\mathbf{x} \in \mathbb{S}_t \subset \mathbb{R}^3$	coordinate in space of simultaneity
$t \in \mathbb{T} \subset \mathbb{R}$	time and time axis and reals $\mathbb{R}$
$\mathbb{W}_e = \{(\mathbf{x}, t)\}$	the event world
$\mathbf{a}; \boldsymbol{\zeta} \in \mathbb{R}^3$	the acceleration and vorticity vectors
$\mathbf{d}(t) \in \mathbb{R}^3$	arbitrary translation
$\mathbf{D}$ and $\mathbf{W}$	symmetric and skew parts of $\mathbf{L}$
$\mathbf{f}_B; \mathbf{f}_I; \mathbf{f}_S \in \mathbb{R}^3$	body, inertial, surface forces
$\mathbf{f}^m; \mathbf{f}^v$	force per unit mass and per unit volume
$\mathbf{L} = \nabla(\mathbf{v}) \in \mathbb{L}_3$	velocity gradient
$\mathbf{l}_c; \mathbf{m}_c \in \mathbb{R}^3$	body moment and surface couple
$\mathbf{M} \in \mathbb{L}_3$ and $\mathbf{n}$	couple stress tensor and unit normal
$\mathbf{Q} \in \mathbb{SO}_3$ or $\mathbb{O}_3$	rotation operator
$\mathbf{t}$ and $\mathbf{T}$	stress vector and Cauchy stress tensor
$\mathbf{v}, \mathbf{V}_T \in \mathbb{R}^3$	velocity and arbitrary boost velocity
$\mathbf{Z}(t) = \mathbf{Q}\mathbf{Q}^T \in \mathbb{M}_{sk}^3$	Coordinate spin
$\mathbf{Zr} = \mathbf{0} \in \mathbb{R}^3$	axial vector, $\mathbf{r}(t)$ , of $\mathbf{Z}$
$\mathcal{B}; \mathcal{B}^e$	material body and its exterior
$\mathcal{O} = \mathcal{B} \Upsilon \mathcal{B}^e; \emptyset$	universal and empty bodies
$\partial\mathcal{B}$	boundary of $\mathcal{B}$
$\mathcal{D}_t \subset \mathbb{R}^3$	space occupied by $\mathcal{B}$ at time $t$
$M(\mathcal{B}); V(\mathcal{B}) \in \mathbb{R}$	body mass and volume
$\mathcal{A}(\mathcal{B}; t) \in \mathbb{R}$	the working action on $\mathcal{B}$
$E_{tot}(\mathcal{B}) \in \mathbb{R}$	the total energy of $\mathcal{B}$
$P(\mathcal{B}); Q(\mathcal{B}) \in \mathbb{R}$	mechanical and thermal working
$E(\mathcal{B}); K(\mathcal{B}) \in \mathbb{R}$	internal and kinetic energy
$R \in \mathbb{R}$	energy equation residual
$e; \eta \in \mathbb{R}$	specific internal energy and entropy
$\theta, P, \rho \in \mathbb{R}$	pressure, density, Temperature
$d/dt$	material derivative
$\prec, \Upsilon$	part of body, join operator
$\langle \mathbf{a}, \mathbf{b} \rangle = a_i b_i \in \mathbb{R}$	inner product
$ \mathbf{v} ^2 \equiv \langle \mathbf{v}, \mathbf{v} \rangle = v_i v_i \in \mathbb{R}$	vector norm
$\Lambda_c, \Lambda$	standard and arbitrary inertial frames

## SUMMARY OF THE COVARIANT FORMULATION OF NEWTONIAN CONTINUUM MECHANICS

It is understood from the start that a unique time scale,  $t \in \mathbb{T} \subset \mathbb{R}$ , is defined and is universal for all parts  $\mathcal{B} \prec \mathcal{O}$ . Time is thus taken to be absolute, as in Newtonian mechanics (albeit “*the idle metaphysical conception*” as chided by Mach), and not relative as requested by Mach. Making the time scale a subset of the real numbers specifies that time is continuous. In reality if time is measured by a set of standard oscillators (atomic clocks) then these standards are taken to be synchronized and universal but are discrete. The same continuity is imposed upon space by placing  $\mathbf{x} \in \mathbb{S}_t \subset \mathbb{R}^3$ .

*Axiom I* only serves to establish the background space–time and the transformation groups acting on that space–time. The field equations for continuum thermo–mechanics must then be made consistent with these transformation groups on the specified space–time. A motion that satisfies the axiom is said to be *inertial*. This axiom, along with *axiom II* below, identify the theory of Newtonian continuum thermo–mechanics. The term “at rest” here refers to rest relative to the reference frame  $\Lambda_c$ . This reference frame is not fixed in the same sense as the absolute space–time of Newton since the mass center of  $\mathcal{O}$  will change with time as celestial bodies change their relative positions. However, in what follows, it is assumed that this change in the reference frame is sufficiently slow that it may be ignored.

For completeness, the following summary of the definitions and results associated with the covariant formulation is extracted from Moulden [8]. *Axiom I* leads directly to *definition I* since it demands that the geodesics of  $\mathbb{W}_e$  are straight lines and the geometry Euclidean. The general Euclidean transformation group,  $\mathbb{E}_u$ , on space–time is given by:

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t); \quad t^* = t + t_0 \quad (1a)$$

where  $\mathbf{d}(t)$  is an arbitrary (smooth) spatial translation.  $t_0$  is a constant time translation. It then follows that:

$$\begin{aligned} \mathbf{v}^* &= \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{d}} \\ \mathbf{a}^* &= \ddot{\mathbf{Q}}\mathbf{x} + 2\dot{\mathbf{Q}}\mathbf{v} + \mathbf{Q}\mathbf{a} + \ddot{\mathbf{d}} \end{aligned}$$

are the velocity and acceleration transformations for the group  $\mathbb{E}_u$ . Next, introduce the subgroup (consistent with *axiom I(1)*):

**Definition I.** *The Galilean group  $\mathbb{G}_a \subset \mathbb{E}_u$ .*

*The transformations:*

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{V}_T t + \mathbf{x}_0; \quad t^* = t + t_0 \quad (1b)$$

*define the Galilean group  $\mathbb{G}_a \subset \mathbb{E}_u$ .*  $\square$

so that  $\mathbf{v}^* = \mathbf{Q}\mathbf{v} + \mathbf{V}_T$  defines the velocity transformation and  $\mathbf{L}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}$  the corresponding velocity gradient. The acceleration vector is frame indifferent

with  $\mathbf{a}^* = \mathbf{Q}\mathbf{a}$ . The vorticity vector,  $\boldsymbol{\zeta}$ , has a transformation given as:  $\boldsymbol{\zeta}^* = \mathbf{Q}\boldsymbol{\zeta}$ .

The philosophical stance adopted in the covariant formulation of classical mechanics is that the field equations of continuum mechanics should be covariant under the transformation groups generated by the first axiom of Newton. There are two such subgroups of  $\mathbb{E}_u$ ;  $\mathbb{G}_a$  as in *definition I* and the axial group,  $\mathbb{G}_z$ , given in *definition II* below. These two transformation groups play very different roles in the theory. First, covariance of the total energy equation under the group  $\mathbb{G}_a$  requires the balance of forces as a compatibility condition. Secondly, covariance under  $\mathbb{G}_z$  gives constraints on the Cauchy stress tensor symmetry. The presentation (see *Moulden [8]* for details) is an extension of the work of *Green and Rivlin [9]*. Taken together these transformation groups lead to a discussion concerning the properties of Mach's Principle as will emerge below.

Next, from the wording of axiom I(2) there arises the definition:

**Definition II.** *The axial group  $\mathbb{G}_z \subset \mathbb{E}_u$ .*

*The transformation:*

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x}; \quad t^* = t + t_0 \quad (1c)$$

with  $\mathbf{Q}(t) = \exp[\mathbf{Z}t] \in \mathbb{S}\mathbb{O}_3$ . Here  $\mathbf{Z} \in \mathbb{M}_{sk}^3$ , an arbitrary global constant, defines the axial group associated with rigid body rotation about the fixed but arbitrary axis,  $\text{span}\{\mathbf{r}\}$ , through the center of mass of  $\mathcal{B}$ .  $\square$

In equation (1c),  $\mathbf{x}$  represents the displacement, in  $\text{span}\{\mathbf{r}\}^\perp$ , from the origin of the subspace  $\text{span}\{\mathbf{r}\}$ . By definition  $\mathbf{Z} \equiv \dot{\mathbf{Q}}\mathbf{Q}^T$  is constant. *Rossmann [10]* shows that the exponential map:

$$\exp : \mathbb{M}_{sk}^3 \rightarrow \mathbb{S}\mathbb{O}_3$$

defines a surjection so that the whole of the group  $\mathbb{S}\mathbb{O}_3$  is included as  $\mathbf{Z}$  varies in  $\mathbb{M}_{sk}^3$ . The velocity vector,  $\mathbf{v}$ , for this group is also in  $\text{span}\{\mathbf{r}\}^\perp$  but with the inner product  $\langle \mathbf{v}, \mathbf{x} \rangle \equiv 0$ . Once the axis,  $\text{span}\{\mathbf{r}\}$ , has been declared the whole transformation group is defined. The velocity transformation related to equation (1c) is given as  $\mathbf{v}^* = \mathbf{Q}(t)[\mathbf{v} + \mathbf{Z}\mathbf{x}]$  and defines the associated velocity field in the subspace  $\text{span}\{\mathbf{r}\}^\perp$ . Then

$$\mathbf{L}^* = \mathbf{Q}(t)[\mathbf{L} + \mathbf{Z}]\mathbf{Q}^T$$

expresses the velocity gradient transformation for this velocity field. The angular velocity,  $\boldsymbol{\omega}$ , and vorticity,  $\boldsymbol{\zeta} \equiv 2\boldsymbol{\omega}$ , transform as:

$$\boldsymbol{\zeta}^* = 2\boldsymbol{\omega}^* \equiv \text{curl}(\mathbf{v}^*) \quad \text{with} \quad \boldsymbol{\omega}^* = \mathbf{Q}(t)[\boldsymbol{\omega} + \mathbf{r}]$$

Also:

$$\mathbf{a}^* = \mathbf{Q}(t)[\mathbf{a} + 2\mathbf{Z}\mathbf{v} + \mathbf{Z}^2\mathbf{x}] \quad (2)$$

defines the acceleration vector transformation under  $\mathbb{G}_z$ . Equation (2) contains both Coriolis and centrifugal acceleration components. Translation in time,  $t^* = t + t_0$ , remains from the Euclidean transformation

group  $\mathbb{E}_u$ .

An important result can be noted:

**Requirement I.** *The material derivative,  $d/dt$ , must commute with  $\int_{\mathcal{B}}(\cdot) dm$*   $\square$

since an integral over  $\mathcal{B}$  with respect to mass is independent of body location.

**Lemma I.**  *$d/dt$  commutes with  $\int_{\mathcal{B}}(\cdot) dm$  iff*  

$$d\rho/dt + \rho \text{div}(\mathbf{v}) = 0$$

PROOF: Follows from the transport theorem.  $\blacksquare$

It can then be shown that:

**Corollary I.** *Requirement I, for an arbitrary body  $\mathcal{B}$ , is equivalent to the condition of mass invariance since:*

PROOF: The proof follows from the definition and the transport theorem, since:

$$M(\mathcal{B}) = \int_{\mathcal{B}} dm \equiv \int_{\mathcal{D}} \rho dV$$

$$\Rightarrow dM(\mathcal{B})/dt = \int_{\mathcal{D}} \left[ \frac{d\rho}{dt} + \text{div}(\rho\mathbf{v}) \right] dV$$

The stated result then follows from the localization theorem when a conservation principle  $dM(\mathcal{B})/dt \equiv 0$  must hold.  $\blacksquare$

Mass is then an invariant of the motion. This mass,  $M(\mathcal{B})$ , is just the rest mass of relativistic theories.

Place  $\mathcal{B} \Upsilon \mathcal{B}^e = \mathcal{O}$  to define the exterior  $\mathcal{B}^e$  of the body  $\mathcal{B}$  (relative to the universe  $\mathcal{O}$ ). Allow that the body  $\mathcal{B}$ , like every part of  $\mathcal{O}$ , has an energy content specified as the sum:

**Definition III.** *Total energy  $E(\mathcal{B})$ .*

*For any  $\mathcal{B} \in \mathbb{B}$  there is a total energy given, in frame  $\Lambda_c$ , as an integral over  $\mathcal{B}$ :*

$$E(\mathcal{B}) = \int_{\mathcal{B}} [e(\mathbf{x}, t) + \langle \mathbf{v}, \mathbf{v} \rangle / 2] dm \quad (3)$$

*Then, let  $\mathcal{A}(\mathcal{B}; t)$  denote the material derivative of  $E$ :  $\mathcal{A}(\mathcal{B}; t) \equiv dE(\mathcal{B})/dt$ .*  $\square$

Here  $e(\mathbf{x}, t)$  is the Euclidean invariant specific internal energy as a field over  $\mathcal{O}$ . The magnitude of the total energy  $E(\mathcal{O})$  has no direct physical significance in the Newtonian theory since that quantity is not Euclidean invariant. The material derivative,  $\mathcal{A}(\mathcal{O}; t)$ , of  $E(\mathcal{O})$  does, however, have invariance under time translation since the inductive statement of Clausius requires that:

**Axiom II.** *(Clausius: Total energy invariance of the universe). In the given inertial frame,  $\Lambda_c$ , the total energy,  $E(\mathcal{O})$ , is invariant under time translation. That is  $\mathcal{A}(\mathcal{O}; t) \equiv 0 \forall t$ .*  $\square$

*Axiom II* has made reference, for any material body in  $\mathbb{B}$ , to the scalar quantity  $\mathcal{A}(\mathcal{B}; t)$ :

$$\mathcal{A}(\mathcal{B}; t) = \frac{d}{dt} E(\mathcal{B}) = \int_{\mathcal{B}} \left[ \frac{de}{dt} + \langle \mathbf{v}, \mathbf{a} \rangle \right] dm \quad (4)$$

However  $\mathcal{A}(\mathcal{B}; t)$  for  $\mathcal{B} \prec \mathcal{O}$  does not vanish, in general, and represents the action of  $\mathcal{B}^e$  on  $\mathcal{B}$  as contained in the first principle of thermo–mechanics (stated in *axiom III* below). The integral form for  $\mathcal{A}(\mathcal{B}; t)$  does, of course, rely upon *Lemma I*.

**Notation I.** For simplicity the following notation is adopted to denote the sum over all separate bodies in  $\mathcal{O}$ . That is:

$$\int_{\mathcal{O}}(\cdot)dm = \sum \int_{\mathcal{B}_i}(\cdot)dm$$

over all bodies in  $\mathcal{O}$ . Then:

$$\int_{\mathcal{B}^e}(\cdot)dm = \int_{\mathcal{O}}(\cdot)dm - \int_{\mathcal{B}}(\cdot)dm$$

for any  $\mathcal{B} \prec \mathcal{O}$ .  $\square$

*Axiom II* is the only conservation principle that has been explicitly stated in the present formulation. The physical principle of matter conservation is well known from early work in chemistry (see *Lavoisier [11]*) but this has been modified by the findings of modern physics. Thus *Feynman [12]* notes that energy, but not mass, is one of the great conservation principles of the universe. The conservation of mass in a moving body is only a Newtonian approximation (valid for low speed, relative to the speed of light, motion). *Lemma I* shows that mass invariance is a compatibility condition for a certain class of integrals that arise in continuum mechanics. It is not known to what extent, if at all, “dark energy” is included in *axiom II*. Such considerations are not Newtonian.

*Axiom II* is not part of the classical Newtonian mechanical theory but is an essential component of the thermo–mechanics for continuous bodies. Its origins (in the form of a mechanical energy conservation principle) are, however, Newtonian. The extension to include thermal (and other forms of energy) arose in the mid nineteenth century with the work of Mayer and Joule, among many others (thus extending the experimental observations of Count Rumford). These concepts were then distilled into an energy conservation equation by Helmholtz in 1847. As has been standard since that contribution of Helmholtz, introduce the energy conservation equation (as in *axiom III* below). The term “energy” was introduced by *Thomas Young (1773–1829)* in 1807 but the word “force” was in common use in the literature for this concept until the 1870’s.

**Axiom III.** The first thermo–mechanical principle. The rate of change of the total energy of body  $\mathcal{B}$  is determined by a linear combination of mechanical  $\mathbf{P}(\mathcal{B})$  and thermal  $\mathbf{Q}(\mathcal{B})$  workings:

$$\mathcal{A}(\mathcal{B}; t) \equiv \int_{\mathcal{B}}[de/dt + \langle \mathbf{a}, \mathbf{v} \rangle] dm = \mathbf{P}(\mathcal{B}) + \mathbf{Q}(\mathcal{B}) \quad (5)$$

where  $\mathbf{P}(\mathcal{B})$  and  $\mathbf{Q}(\mathcal{B})$  are made specific in definition IV below.  $\square$

Then  $\mathcal{A}(\mathcal{B}; t) \equiv 0$  iff  $\mathbf{P}(\mathcal{B}) + \mathbf{Q}(\mathcal{B}) = \mathbf{0}$ . The mechan-

ical  $\mathbf{P}(\mathcal{B})$  and thermal  $\mathbf{Q}(\mathcal{B})$  workings represent the influence of the exterior,  $\mathcal{B}^e$ , on the given body  $\mathcal{B}$  and, for the present discussion, are assumed to include the following:

**Definition IV.** Mechanical and thermal workings: The mechanical working is given as:

$$\begin{aligned} \mathbf{P}(\mathcal{B}) = \int_{\mathcal{D}} \rho[\langle \mathbf{f}_B^m, \mathbf{v} \rangle + \langle \boldsymbol{\zeta}, \boldsymbol{\ell}_c \rangle] dV \\ + \int_{\partial \mathcal{D}} [\langle \mathbf{t}, \mathbf{v} \rangle + \langle \boldsymbol{\zeta}, \mathbf{m}_c \rangle] dA \end{aligned}$$

due to the body and surface forces and couples. The body moment  $\boldsymbol{\ell}_c$  and surface couple  $\mathbf{m}_c$  are assumed to be frame indifferent under  $\mathbb{E}_a$ . For simplicity, restrict the thermal working to:

$$\mathbf{Q}(\mathcal{B}) = - \int_{\partial \mathcal{D}} \langle \mathbf{q}, \mathbf{n} \rangle dA$$

with  $\mathbf{q}$  the (frame indifferent) heat flux vector.  $\square$

*Axiom III* is an extension of the mechanical energy equation, which for point masses is the statement  $M(\mathcal{B})\langle \mathbf{a}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle$  and is consistent with *axiom I*. The physical nature of the body force  $\mathbf{f}_B(\mathcal{B}, \mathcal{B}^e)$  is specified from outside of continuum mechanics and may include the gravitational force. The vector  $\mathbf{f}_B(\mathcal{B}, \mathcal{B}^e)$  is, however, constrained by *corollary I* if the context requires. The vector  $\mathbf{q}$  vanishes on the surface  $\partial \mathcal{O}$ . The constituents  $\mathbf{P}(\phi)$  and  $\mathbf{Q}(\phi)$  vanish identically for the void, or empty body,  $\phi$ . *Green and Rivlin [9]* observed that the first principle of thermo–mechanics, equation (5), is covariant under the Galilean group iff all the forces acting on  $\mathcal{B}$  are balanced (the second axiom of Newton). That is if the statement:

$$\mathbf{f}_I(\mathcal{B}) + \mathbf{f}_B(\mathcal{B}) + \mathbf{f}_S(\mathcal{B}) = \mathbf{0}$$

holds true. This covariance makes the first principle of thermo–mechanics consistent with the first axiom of Newtonian continuum mechanics and also provides the second axiom of Newton as a compatibility condition. As in *Moulden [8]* put in place:

**Definition V.** An ancillary vector  $\mathbf{g}(\mathcal{B})$

Introduce the vector valued kinematic quantity,  $\mathbf{g}(\mathcal{B})$ , for any  $\mathcal{B} \prec \mathcal{O}$ , in the inertial frame  $\Lambda_c$ :

$$\begin{aligned} \mathbf{g}(\mathcal{B}) = - \frac{d}{dt} \int_{\mathcal{B}} \mathbf{v}(\mathbf{x}, t) dm = - \frac{d}{dt} \mathbf{M}_{om}(\mathcal{B}) \\ \equiv - \int_{\mathcal{B}} \mathbf{a}(\mathbf{x}, t) dm \end{aligned} \quad (6)$$

from *Lemma I* since  $M(\mathcal{B})$  is an invariant of the motion (from *corollary I*).  $\square$

Below, physical significance will be attached to the vector  $\mathbf{g}(\mathcal{B})$ . In fact it will play a central role in the discussion of Mach’s Principle. As a first property:

**Proposition I.**  $\mathbf{g}(\mathcal{B}) \mapsto \mathbf{Q} \mathbf{g}(\mathcal{B})$  under  $\mathbb{G}_a$ .  $\blacksquare$

which follows since the acceleration vector,  $\mathbf{a}(\mathbf{x}, t)$ , is frame indifferent under  $\mathbb{G}_a$ . It is assumed in the above

that  $\mathbf{v}(\mathbf{x}, t)$  is a smooth function of space and time so that the acceleration field,  $\mathbf{a}(\mathbf{x}, t)$ , is well defined. This regularity assumption, along with uniqueness of the field equations, must be checked for consistency *a posteriori* but that aspect of the theory is not complete (particularly in fluid mechanics — see Lions [13], for example, for details in the case of constant density fluid motion).

The requirement of Green and Rivlin [9] can be stated formally as:

**Axiom IV.** *As given in Axiom III, the first principle of thermo–mechanics must be covariant under the Galilean group for any  $\mathcal{B} \prec \mathcal{O}$ .*  $\square$

in order for *axiom III* to be consistent with *axiom I*.

As shown below, axial group covariance of equation (5) provides the principle of angular momentum. Introducing the residue  $\mathbf{R}(\mathbf{Z}, \mathbf{V}_T)$  defined by the need for equation (5) to be covariant under the groups  $\mathbb{G}_a$  and  $\mathbb{G}_z$ . That is, place:

$$\begin{aligned} \mathcal{A}^*(\mathcal{B}; t) - [\mathbf{P}^*(\mathcal{B}) + \mathbf{Q}^*(\mathcal{B})] - \mathcal{A}(\mathcal{B}; t) \\ + [\mathbf{P}(\mathcal{B}) + \mathbf{Q}(\mathcal{B})] \equiv \mathbf{R}(\mathbf{Z}, \mathbf{V}_T) = 0 \end{aligned} \quad (7)$$

The invariance properties of the residue  $\mathbf{R}(\mathbf{Z}, \mathbf{V}_T)$  then provide the covariance structure of classical thermo–mechanics. First establish covariance under Galilean rotations:

**Result I.** *Covariance under Galilean rotations. Equation (5) is covariant under Galilean rotations.*  $\blacksquare$

since Galilean rotations are independent of time. As discussed in Moulden [8] there is:

**Lemma II.** *Covariance under Galilean boosts*

*PROOF:* Under the boost component of  $\mathbb{G}_a$ , that is when  $\mathbf{Z} = \mathbf{O}$  and  $\mathbf{Q} = \mathbf{I}$ , there is:

$$\begin{aligned} \mathbf{R}(\mathbf{O}, \mathbf{V}_T) = -\langle \mathbf{g}, \mathbf{V}_T \rangle - \int_{\mathcal{D}} \langle \rho \mathbf{f}_B^m, \mathbf{V}_T \rangle dV \\ - \int_{\partial \mathcal{D}} \langle \mathbf{t}, \mathbf{V}_T \rangle dA \equiv 0 \end{aligned}$$

from equation (5). Hence the following equality must hold:

$$\langle \mathbf{V}_T, [\mathbf{g} + \int_{\mathcal{D}} \rho \mathbf{f}_B^m dV + \int_{\partial \mathcal{D}} \mathbf{t} dA] \rangle = 0 \quad (8)$$

as  $\mathbf{V}_T$  is an arbitrary constant vector.  $\blacksquare$

It follows directly that the result of Green and Rivlin [9] is recovered:

**Theorem I.** *Axiom IV implies the existence of a force balance.*

*PROOF:* The inner product identified in equation (8) vanishes when:

$$\begin{aligned} \mathbf{g} + \int_{\mathcal{D}} \rho \mathbf{f}_B^m dV + \int_{\partial \mathcal{D}} \mathbf{t} dA = \mathbf{0} \\ \equiv \mathbf{f}_I(\mathcal{B}, \mathcal{B}^e) + \mathbf{f}_B(\mathcal{B}, \mathcal{B}^e) + \mathbf{f}_S(\mathcal{B}, \mathcal{B}^e) \end{aligned} \quad (9)$$

since the boost velocity,  $\mathbf{V}_T$  was defined to be an arbitrary constant and bounded vector.  $\blacksquare$

It now follows, on comparison with the balance of forces specified by Newton in the *Principia*, that:

**Result II.** *Equation (9) represents a global force balance equation that identifies the vector  $\mathbf{g}$  with the inertial force  $\mathbf{f}_I$ .*  $\blacksquare$

By construction, equation (9) is covariant under  $\mathbb{G}_a$  but it is not covariant under the group  $\mathbb{G}_z$ . Equation (8) represents the virtual working of the vector sum of forces acting on body  $\mathcal{B}$  due to the arbitrary constant Galilean boost velocity  $\mathbf{V}_T$ .

Equation (9) is not, in general, covariant under the transformation  $t \mapsto -t$  a result which points to the concept of universal dissipation as treated in the second principle of thermo–mechanics. Not mentioned in the above is the principle of universal dissipation and its characteristic function, the entropy,  $\eta(\mathbf{x}, t)$ . Then, in general, entropy is not a conserved quantity. Rather,  $d\eta/dt \geq 0$ .

In the context of the present development (Moulden [8]) allow that:

**Axiom V.** *Equation (5) is covariant under both  $\mathbb{G}_a$  and  $\mathbb{G}_z$  when  $\mathbf{R}(\mathbf{Z}, \mathbf{O}) = 0$  in equation (7) and when equation (9) holds locally for any  $\mathbf{Z} \in \mathbb{M}_{sk}^3$  with  $\mathbf{V}_T = \mathbf{O}$ .*  $\square$

To determine the content of *axiom V*, decompose the Cauchy stress tensor as  $\mathbf{T} = \mathbf{T}^+ + \mathbf{T}^-$  with  $\mathbf{T}^+ \in \mathbb{M}_{sy}^3$  the symmetric part of  $\mathbf{T}$ . Let the axial vector of the skew part  $\mathbf{T}^- \in \mathbb{M}_{sk}^3$  of  $\mathbf{T}$  be written as  $\mathbf{s} \in \mathbb{R}^3$  with  $s_i = \epsilon_{ijk} T_{kj} / 2$ . Now the second order tensor  $\mathbf{T}$  must be frame indifferent under  $\mathbb{E}_u$  and so both its skew and symmetric parts are also frame indifferent. From the definition it follows that:

$$\text{trace}(\mathbf{T}\mathbf{Z}) \equiv \text{trace}(\mathbf{T}^-\mathbf{Z}) = -2\langle \mathbf{s}, \mathbf{r} \rangle$$

As shown in Moulden [8] *axiom V* leads to the angular momentum equation. This equation has the local form (when the localization theorem, Gurtin [14], holds):

$$\text{div}[\mathbf{M}^T(\mathbf{x}, t)] + \rho(\mathbf{x}, t) \boldsymbol{\ell}_c(\mathbf{x}, t) = \mathbf{s}(\mathbf{x}, t) \quad (10)$$

This axial vector determines the skew component,  $\mathbf{T}^-$ , of  $\mathbf{T}$ . Else:

$$\mathbf{s}(\mathbf{x}, t) = \mathbf{0} \quad \Rightarrow \quad \mathbf{T}^- \equiv \mathbf{O} \text{ and } \mathbf{T}(\mathbf{x}, t) \in \mathbb{M}_{sy}^3$$

Development of the field equations is now complete and a specific example can be considered:

**Example I** *The linear viscous fluid*

Here the mass invariance constraint has the form:

$$\partial \rho / \partial t + \text{div}(\rho \mathbf{v}) = 0 \quad (11a)$$

while the Cauchy stress tensor involves the fluid pressure  $P$  and the stretching tensor  $\mathbf{D}$ . That is:

$$\mathbf{T} = 2\mu \mathbf{D} - P^* \mathbf{I}; \quad P^* = P - \mu^* \text{trace}(\mathbf{D}) \quad (11b)$$

if  $\mu$ ,  $\mu^*$  are the first and second coefficients of viscosity and  $\mathbf{T}$  is symmetric — no body moments or couple stresses exist for this model of fluid motion. This completes the linear momentum equation which has the form:

$$\rho d\mathbf{v}/dt = \text{div}(\mathbf{T}) + \mathbf{f}_B^v$$

The energy equation is, with these restrictions:

$$\rho de/dt = \text{trace}(\mathbf{TL}) + \kappa \nabla^2(\theta)$$

while the entropy changes according to the equation:

$$\rho \theta d\eta/dt = \rho \langle \mathbf{v}, \mathbf{f}^v \rangle + \text{trace}(\mathbf{TL}) + P \text{div}(\mathbf{v}) + \text{div}(\mathbf{q})$$

to accommodate the second principle of thermodynamics. Here  $\kappa$  represents the (Fourier) thermal conductivity.  $\triangle$

## CONSISTENCY OF THE THEORY

It is standard in cosmology to treat, at least parts of the universe, with the classical Newtonian continuum equations for a compressible fluid. This is done, for example by *Lin and Roberts [15]* for a discussion of spiral galaxy development. These structures are, of course, small scale relative to the entire universe. It has also been shown, using classical inviscid compressible flow theory that *Bondi accretion*, see *Chakrabarti [16]*, includes a transonic flow region.

*Axiom I* has the implication that Euclidean geometry is appropriate to discuss the background structure for Newtonian continuum mechanics. Such a comment would not have been needed before the advent of relativistic mechanics. Thus *Thomson and Tait [17]* make no mention of the space–time structure required to discuss Newtonian mechanics; adopting Euclidean geometry without comment. Similarly the second part of *axiom I* was just mentioned by *Tait [18]* without any suggestion of its implications. The adoption of the *transformation group* terminology, to discuss such matters, did not arise until much later; where the work of Kline was clearly influential.

The theory of continuum thermo–mechanics as outlined above is only fully consistent if it can be shown that the equations possess a unique solution for all time. Of course, the initial and boundary conditions are part of the specification of an existence and uniqueness theorem. The regularity assumptions made in developing the theory also need to be justified. In particular both  $\partial\mathbf{v}/\partial t$  and  $\partial\mathbf{v}/\partial\mathbf{x}$  are assumed to be defined for all time. Existence theory is a large subject and is by no means complete in the case of the linear viscous fluid. The situation is more complete for the classical theory of linear elasticity (see *Gurtin [19]* for example). A suitable reference for mathematical problems related to fluid mechanics is the four volume review edited by *Friedlander and Serre Eds. [20]*. For the linear viscous fluid (even in the ideal fluid limit,  $\mu \rightarrow 0$ ,  $\mu^* \rightarrow 0$ ,  $\kappa \rightarrow 0$ ) there is no complete regularity result available. However, given the required regu-

larity that the operator  $\mathbf{D}$  have bounded eigenvalues, then the theorem of *Gurtin [14]*, provides a uniqueness result for the constant density equations. This fails in the inviscid limit where shock waves are represented by discontinuities. *Serrin [21]* gives a result for the compressible Navier Stokes equations.

## THE LINEAR PRINCIPLE OF MACH

Introduce a definition of the Mach Principle that will be addresses herein in the context of Newtonian mechanics. Of course, relativistic theories and modern cosmology may contrive different circumstances.

**Definition VI.** *The Principle of Mach for Newtonian mechanics.*

*There are two distinct parts:*

a). *The inertia of body  $\mathcal{B} \prec \mathcal{O}$  is due to that part,  $\mathcal{B}^e$ , of the universe that is exterior to  $\mathcal{B}$ .*

b). *The Galilean group is the appropriate group for the mechanics.*  $\square$

It is appropriate to consider some properties of the functions already introduced.

As noted at the start, time is taken to be absolute and not relative. Thus, some invariance properties of the function  $\mathcal{A}(\mathcal{O}; t)$  can be written down.  $\mathcal{A}(\mathcal{O}; t)$  is, by definition, the material rate of change of the total energy in  $\mathcal{O}$  and so the equality  $\mathcal{A}(\mathcal{O}; t) \equiv 0$  is assumed to hold. First, consider the transformation of  $\mathcal{A}(\mathcal{O}; t)$  under the axial group.

$$\mathcal{A}(\mathcal{O}; t) = \int_{\mathcal{O}} \left[ \frac{de}{dt} + \langle \mathbf{a}, \mathbf{v} \rangle \right] dm \mapsto \mathcal{A}(\mathcal{O}; t) + I_a(\mathcal{O}) \quad (12a)$$

where:

$$I_a(\mathcal{O}) = \int_{\mathcal{O}} \langle \mathbf{Zx}, \mathbf{a} + \mathbf{Zv} \rangle dm \quad (12b)$$

from equation (5). Next, consider the invariance of  $\mathcal{A}(\mathcal{O}; t)$  under the Galilean group when equation (5) shows that:

$$\mathcal{A}(\mathcal{O}; t) \mapsto \mathcal{A}(\mathcal{O}; t) + I_g(\mathcal{O}) \quad (12c)$$

where:

$$I_g(\mathcal{O}) = \int_{\mathcal{O}} \langle \mathbf{a}, \mathbf{Q}^T \mathbf{V}_T \rangle dm \quad (12d)$$

*Assumption I* implies that there must be covariance for  $\mathcal{A}(\mathcal{O}; t)$  so that both  $I_a(\mathcal{O})$  and  $I_g(\mathcal{O})$  must vanish. Considering these separately:

a).  $I_a(\mathcal{O})$ : this integral vanishes identically from the geometry of rotation on the subspace  $\text{span}\{\mathbf{r}\}^\perp$ .

b).  $I_g(\mathcal{O})$ : This condition,  $I_g(\mathcal{O}) = 0$ , holds iff  $\int_{\mathcal{O}} \langle \mathbf{a}, \mathbf{Q}^T \mathbf{V}_T \rangle dm \equiv 0$  from equation (12d). Since both the vector  $\mathbf{V}_T$  and the rotation  $\mathbf{Q}$  are arbitrary constants there is:  $\langle \mathbf{Q}^T \mathbf{V}_T, \mathbf{g}(\mathcal{O}) \rangle = 0$  on using the definition in equation (6) applied to the entire universe  $\mathcal{O}$ . Hence it is concluded from equation (12d) that for Galilean covariance:

**Result III.** *Vanishing of the vector  $\mathbf{g}(\mathcal{W})$ .*

$$I_g(\mathcal{W}) = 0 \quad \Rightarrow \quad \mathbf{g}(\mathcal{W}) \equiv - \int_{\mathcal{W}} \mathbf{a} \, dm = \mathbf{0} \quad (13)$$

and  $d[\mathbf{M}_{om}(\mathcal{W})]/dt = \mathbf{0}$ . ■

Hence the momentum of the universe,  $\mathbf{M}_{om}(\mathcal{W})$ , is time invariant as a constraint on possible motions of the entire universe.

In the words of *Bondi [6]*, the principle of Mach may be summarized as: “*The local inertial frame is determined by some average of the motion of the distant astronomical objects*”. This can be shown from equation (13) where the statement  $\mathbf{g}(\mathcal{W}) \equiv \mathbf{0}$  followed from Galilean covariance of the action  $\mathcal{A}(\mathcal{W})$ . Hence the Newtonian statement:

**Lemma III** *Since  $\mathcal{A}(\mathcal{W})$  is Galilean invariant the Principle of Mach, in Definition VI(a) holds true.*

*Proof:* There is from the definition of the vector  $\mathbf{g}(\mathcal{B})$  and result III:

$$\begin{aligned} \mathbf{g}(\mathcal{W}) \equiv \mathbf{0} &= \mathbf{g}(\mathcal{B}) + \mathbf{g}(\mathcal{B}^e) \\ \Rightarrow \quad \mathbf{f}_I(\mathcal{B}, \mathcal{B}^e) - \frac{d}{dt} \int_{\mathcal{B}^e} \mathbf{v} \, dm &\equiv \mathbf{0} \end{aligned} \quad (14)$$

as linear momentum is additive over  $\mathcal{W}$ . ■

*Lemma III* thus provides the continuum thermo-mechanics statement of Mach’s Principle for Newtonian mechanics in the model universe considered.

**Result IV** *The integral in equation (14) is invariant under Galilean transformations (since, by proposition I,  $\mathbf{g}(\mathcal{B})$  is frame indifferent under  $\mathbb{G}_a$  for any  $\mathcal{B}$ ) but not under general space–time transformations. □*

this result responds to *Definition VI(b)* and goes some way towards resolving the concerns of Mach. That is the integral over  $\mathcal{B}^e$  in equation (14) defines the transformation group for the inertial force on  $\mathcal{B}$  as well as the inertial force itself.

More can be said about *Definition VI(a)*. Equation (14) shows directly that the inertial force vanishes if the integral  $I_u = \int_{\mathcal{B}^e} \mathbf{v} \, dm$  is a constant. In that case the given body  $\mathcal{B}$  would continue to move with its initial velocity.

A couple of examples, one from antiquity and the other more current, can be given:

$\alpha$ ). For the universe of Aristotle, wherein the fixed stars are taken to have zero velocity relative to the coordinate system defined at the mass center of  $\mathcal{W}$ , the integral  $I_u \equiv 0$ . This statement has noticed the above finding that  $I_a(\mathcal{W}) \equiv 0$ : the motion is transparent to constant angular velocity coordinate rotations. Such findings reflect the views of Aristotle.

$\beta$ ). For the isotropic expanding universe, relative to the mass centered frame, it can be expected that  $I_u$  is essentially zero. In this case the earth is again taken at the coordinate origin.

In both cases, it is the universe as a whole that dictates that a body will move at constant speed if devoid of the action of an external force.

As noted before, dark matter is included in the integral of equation (14). The inertial force vector,  $\mathbf{f}_I(\mathcal{B}, \mathcal{B}^e)$ , depends only on the rate of change of linear momentum of the exterior body  $\mathcal{B}^e$ . It can be noted that *Noll [22]* made a similar observation from a different perspective. The result in *lemma III* can also be construed as a special case of the third axiom of Newton.

A more complete discussion of the Mach Principle, along with a historical perspective, is contained in *Barbour [23]*. See also *Sciama [3]* and *Mach [2]*. In addition, *Barbour [23]* also suggests that Kepler, and to some extent Aristotle, had thoughts about the workings of the universe that were a prequel to those of Mach.

## THE ANGULAR PRINCIPLE OF MACH

As observed by *Tait [18]*, Newton did, in the *Principia*, note that a spinning hoop continues to rotate until slowed by resistance of the air. This comment of Newton has received very little recognition but does point to an angular inertial effect that needs discussion in the spirit of Mach. It is evident that neither the universe of Aristotle, with its fixed stars, nor the isotropic universe can create a torque that would change the angular momentum of Newton’s hoop; as reflected by the vanishing of the integral  $I_a(\mathcal{W})$ . This topic will be addressed elsewhere.

## IMPLICATIONS

Now it has been shown that the distant stars define coordinate frames in the sense that the material derivative of the integral  $I_u = \int_{\mathcal{B}^e} \mathbf{v} \, dm$  remains constant under a transformation  $\mathbf{v} \mapsto \mathbf{v} + \mathbf{V}_T$  for any constant boost velocity  $\mathbf{V}_T$ . Also, since  $\mathbf{g}(\mathcal{W})$  is frame indifferent, there is invariance in equation (14) under the Galilean group. The statement does not give any other information about admissible frames: it only delivers an equivalence class of coordinate frames.

The formal application of the theory of classical Newtonian mechanics is not affected by the validity, or otherwise, of the Principle of Mach. However, the above would suggest that Newtonian mechanics is a local theory appropriate only to parts of the universe wherein the integral  $I_u$  is constant. For a body  $\mathcal{B}$  near the boundary  $\partial\mathcal{W}$  of the universe, it is evident that the integral  $I_u$  is not zero and the inertial effects for such a body would then be very different.

The importance of body mass in the formulation of Mach’s Principle is made evident in the above since the integrals  $I_a(\mathcal{W})$ ;  $I_g(\mathcal{W})$  and  $I_u$  are all written with respect to mass.

## FINAL COMMENTS

A Newtonian version of the Principle of Mach has been established and adds insight into the structure of Newtonian mechanics.

It has been found that this Newtonian continuum mechanics version of the Mach Principle depends upon assumptions about the nature of the universe. These assumptions include:

*i).* The Clausius inductive statement of energy invariance for the universal body. The cosmological findings of dark matter and dark energy in the universe certainly call this into question in the classical form of *axioms II and III*.

*ii).* No agency exists outside of  $\mathcal{O}$  to change the properties of that universe. Associated with this is the requirement that the universe be bounded. Some of the assumptions made herein may need revision as more is understood about the universe at large.

*iii).* The mass of the universe is time invariant. This is related to the comment on dark matter made in *(i)*. With the mass~energy duality taken into account, the present deliberations would need considerable extension and revision.

*iv).* It appears that cosmology must make appeal to quantum mechanics (and so call in question the physical nature of space and time at small scales — see Huggett *et al.* (Eds.) [24] for more information. These effects are important at length scales which are on the order of the Planck length, we are told). Such deliberations are far removed from Newtonian mechanics. The consideration of quantum effects has often brought anthropic concerns to the fore (see Barrow and Tipler [25] for a discussion of this suggestion) but these concepts are not part of the classical theory.

The above Newtonian findings are contrary to current cosmological views. Thus, while a Newtonian version of the Principle of Mach has been constructed, it need not reflect the reality of the universe around us.

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