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THE TAYLOR VORTEX AND THE DRIVEN CAVITY PROBLEMS BY THE STREAM FUNCTION-VORTICITY FORMULATION.

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ABSTRACT

We present here the Navier-Stokes equations using the stream function-vorticity formulation. In order to solve these equations we have used a simple numerical method based on a fixed point iterative process [1] to solve a nonlinear elliptic system resulting after time discretization. For this kind of problems efficient solvers like Fishpack [2] may be used. Results are presented for two problems: the Taylor vortex problem and the driven cavity problem. We present results, for the driven cavity problem for Reynolds numbers Re=3200 and Re=7500, and for the Taylor vortex problem with the same Reynolds numbers. With respect to the driven cavity problem, results agree very well with those reported in the literature [4 - 7]. For the Taylor vortex problem [3], since the exact solution is known, the relative error was calculated and results were very good. Results are compared with those obtained for the same problems with other formulations [3 - 8].

INTRODUCTION

In this work, the Navier-Stokes equations in stream function-vorticity formulation are numerically solved. Results are obtained using a numerical procedure based on a fixed point iterative process [1] to solve the nonlinear elliptic system that results once a convenient second order time discretization is made. The iterative process leads us to the solution of an uncoupled, well-conditioned, symmetric linear elliptic problem. Fishpack [2] is used to solve this symmetric linear elliptic problem. Fishpack discretizes the elliptic equation which is solved using a generalized cyclic reduction algorithm. Second or fourth order approximations may be used. In particular we used a second order approximation.

MATHEMATICAL MODEL

Let $D = \Omega \times (0, T)$, $T > 0$, $\Omega \subset R^2$, be the region of the flow of an unsteady isothermal incompressible fluid and Γ its boundary. This kind of flow is governed by the non-dimensional system of equations in D, defined by:

$$\boldsymbol{u}_t - \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u} + \nabla \mathrm{p} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \mathrm{f}, \qquad (1)$$

$$\nabla\cdot\boldsymbol{u} = 0. \qquad (2)$$

These are the Navier-Stokes equations in primitive variables, where $\boldsymbol{u}$ is the velocity, p is the pressure and the dimensionless parameter Re is the Reynolds number. This system must be supplemented with appropriate initial and boundary conditions: $\boldsymbol{u}(x, 0) = u_0(x)$ in Ω and $\boldsymbol{u} = g$ on Γ, respectively. In order to avoid the pressure variable and the incompressibility condition (2), the stream function-vorticity formulation is used here.

The stream function ψ is defined by:

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}, \qquad (3)$$

where $\boldsymbol{u} = (u, v)$ with u and v the velocities in x and y-axis, respectively. It is easy to verify that $(\boldsymbol{u}\cdot\nabla)\psi = 0$. The vorticity is defined as the curl of the velocity field, and in 2D it is defined as:

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \quad (4)$$

So, finally, we get the following coupled system of equations:

$$\omega_t - \frac{1}{Re}\nabla^2\omega + \boldsymbol{u}\cdot\nabla\omega = 0\,, \quad (5)$$
$$\nabla^2\,\psi = -\,\omega. \quad (6)$$

This is the system that we are going to solve. For system (5)-(6) whenever the velocity is given on Γ, there are two boundary conditions for ψ and none for *ω*. Glowinski and Pironneau [15] (see also [16]) derived nonhomogeneous Dirichlet boundary conditions for *ω* from the condition on the normal derivative of ψ . We will discuss later how we get the boundary conditions for the two problems shown.

NUMERICAL METHOD

The time derivative is approximated by the second-order scheme

$$\omega_t(x,(n+1)\Delta t) \approx \frac{3\omega^{n+1}-4\omega^{n}+\omega^{n-1}}{2\Delta t}, \quad (7)$$

where $n \geq 1$, $x \in \Omega$ and $\Delta t > 0$ is the time step.

At each time level the following nonlinear system defined in Ω is obtained:

$$\nabla^2\,\psi = -\,\omega, \qquad \psi|_\Gamma = \psi_{bc}, \quad (8a)$$

$$\alpha\omega - \frac{1}{Re}\nabla^2\omega + \boldsymbol{u}\cdot\nabla\omega = f_\omega\,, \qquad \omega|_\Gamma = \omega_{bc}, \quad (8b)$$

where $\propto = \frac{3}{2\Delta t}$, and $f_\omega = \frac{4\omega^{n}-\omega^{n-1}}{2\Delta t}$. At the first time step, to obtain ω^1 the following discretization is used

$$\omega_t(x,\Delta t) \approx \frac{\omega^1-\omega^0}{\Delta t}$$

with $f_\omega = \frac{\omega^0}{\Delta t}$ and $\propto = \frac{1}{\Delta t}$. Then equation (6) can be used to obtain ψ^1.

The equation (8b) is a transport type equation; a fixed point iterative process is applied to solve it. Denoting

$$R_\omega(\omega,\psi) = \alpha\omega - \frac{1}{Re}\nabla^2\omega + \boldsymbol{u}\cdot\nabla\omega - f_\omega, \text{ in } \Omega, \quad (9)$$

system (8a)-(8b) is equivalent to

$$\nabla^2\,\psi = -\,\omega, \qquad \psi|_\Gamma = \psi_{bc}, \quad (10a)$$

$$R_\omega(\omega,\psi) = 0, \qquad \omega|_\Gamma = \omega_{bc}. \quad (10b)$$

So then (10a)-(10b), at time level n+1, is solved via the following iterative process [1]:

Given $\omega^{n,0} = \omega^n$, and $\psi^{n,0} = \psi^n$, solve "until convergence" in ω:

$$\nabla^2\,\psi^{n,m+1} = -\omega^{n,m}\,, \quad \text{in } \Omega, \qquad \psi^{n,m+1}|_\Gamma = \psi_{bc}^{n+1}, \quad (11a)$$

$$\left(\alpha I - \frac{1}{Re}\nabla^2\right)\omega^{n,m+1} =$$
$$\left(\alpha I - \frac{1}{Re}\nabla^2\right)\omega^{n,m} - \rho R_\omega(\omega^{n,m},\psi^{n,m+1}), \quad \text{in } \Omega, \quad (11b)$$

$$\omega^{n,m+1}|_\Gamma = \omega_{bc}^{n,m}, \ \rho>0;$$

and then take $(\omega^{n+1},\psi^{n+1}) = (\omega^{n,m+1},\psi^{n,m+1})$.

The function ψ_{bc}^{n+1} is a function derived from g at time $(n+1)\Delta t$. By "until convergence" we mean until the following criterion is satisfied:

$$\|\omega^{n,m+1} - \omega^{n,m}\|_\infty < tol.$$

In our experience, the best choice for ρ is ρ=0.9 in order to get faster convergence, since for ρ<0.9 more iterations have to be done.

In order to handle high Reynolds numbers we use Ikeda's upwind scheme [14], which is a second order scheme. To approximate $-\epsilon\nabla^2\omega + \boldsymbol{u}\cdot\nabla\omega$ we use:

$$\max\left(\epsilon,\frac{1}{2}hu\right)\frac{-\omega_{i+1,j}+2\omega_{i,j}-\omega_{i-1,j}}{h^2} + \max\left(\epsilon,\frac{1}{2}hv\right)\frac{-\omega_{i,j+1}+2\omega_{i,j}-\omega_{i,j-1}}{h^2} + u\frac{\omega_{i+1,j}+\omega_{i-1,j}}{2h} + v\frac{\omega_{i,j+1}+\omega_{i,j-1}}{2h},$$

with $\epsilon = \frac{1}{Re}$ and

$$u = \frac{\partial\psi}{\partial y} \approx \frac{\psi_{i,j+1}+\psi_{i,j-1}}{2h}\,, \qquad v = -\frac{\partial\psi}{\partial x} \approx -\frac{\psi_{i+1,j}+\psi_{i-1,j}}{2h}.$$

NUMERICAL EXPERIMENTS AND RESULTS

Two problems are solved in this work, the first one is the un-regularized driven cavity problem, and the second one is the Taylor Vortex problem.

With respect to the first one $\Omega = (0,1)\times(0,1)$. The top wall is moving with a nonzero velocity given by $\boldsymbol{u} = (1,0)$ and for the other three walls the velocity is given by $\boldsymbol{u} = (0,0)$, so using (3), ψ is constant on solid and fixed walls; at the moving wall $y = 1$, a constant function for ψ is also obtained. Following Goyon [17] ψ = 0 is chosen on Γ. As already mentioned, ψ is overdetermined on the boundary ($\frac{\partial\psi}{\partial n}|_\Gamma$ is also known) and no boundary condition is given for *ω*. Several alternatives have been proposed, we follow the alternative given by Goyon [17]. A translation of the boundary condition in terms of the velocity (primitive variable) has to be used by Taylor expansion of equation (6). By Taylor expansion of (6) on the boundary for the driven cavity the following boundary conditions for *ω* are obtained:

$$\omega(0,y,t) = -\frac{1}{2h_x^2}[8\psi(h_x,y,t) - \psi(2h_x,y,t)] + O(h_x^2),$$

$$\omega(1,y,t) = -\frac{1}{2h_x^2}[8\psi(1-h_x,y,t) - \psi(1,y,t)] + O(h_x^2),$$

$$\omega(x,0,t) = -\frac{1}{2h_y^2}\left[8\psi(x,h_y,t) - \psi(x,2h_y,t)\right] + O(h_y^2),$$
$$\omega(x,1,t) = -\frac{1}{2h_y^2}\left[8\psi(x,1-h_y,t) - \psi(x,1-h_y,t)\right] - \frac{3}{h_y} + O(h_y^2), \quad (12)$$

with h_x, h_y being space steps.

We report results for Re=3200 and Re=7500. Results are reported through the iso-vorticity contours and streamlines. In Figures 1 and 2, we show the streamlines and the isocontours for the vorticity for Re=3200, respectively. In this case, the steady state is captured. Then, in Figures 3 and 4, we show the streamlines and isocontours for the vorticity with Re=7500. We have here a time-dependent flow.

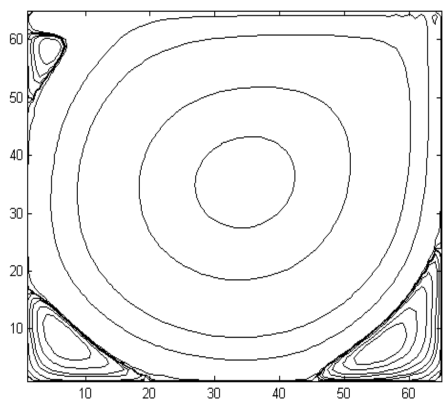

Figure 1 Streamlines for Re=3200, Δt=0.01, t=100 and h=1/64.

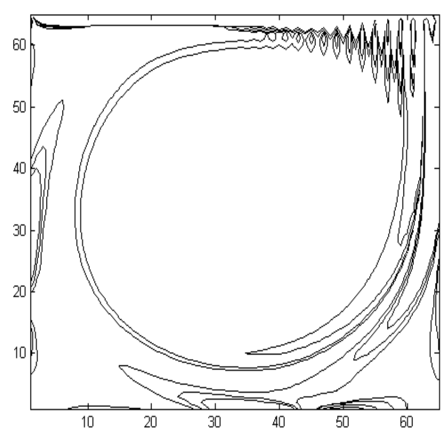

Figure 2 Isocontours for the vorticity with Re=3200, Δt=0.01, t=100 and h=1/64.

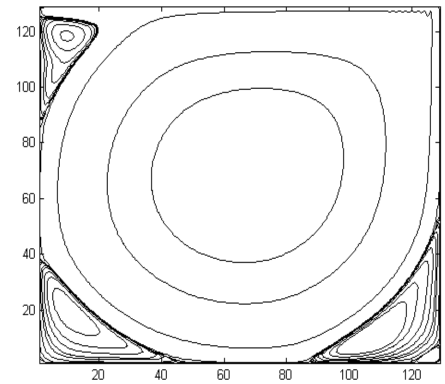

Figure 3 Streamlines for Re=7500, Δt=0.01, t=200 and h=1/128.

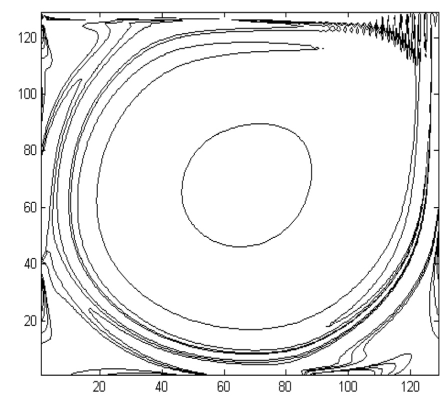

Figure 4 Isocontours for the vorticity with Re=7500, Δt=0.01, t=200 and h=1/128.

For the Taylor Vortex problem we show results for the same values of the Reynolds number, Re=3200 and Re=7500, $h = 1/128$, and t=100. We show the graph in 3D of the stream function and vorticity in order to see the difference in scales at different times and for the two afore mentioned Reynolds numbers. In both cases we show the relative error for the stream function, since we know the exact solution. In this case there is no steady state.

For this problem we use $\Omega = [0,2\pi] \times [0,2\pi]$. The exact solution, in this case, is known, and it is given by the following equations:

$$u\,(x,y,t) = -\,Cos(x)\,Sen(y)\,e^{\frac{-2t}{Re}},$$
$$v\,(x,y,t) = Sen(x)\,Cos(y)\,e^{\frac{-2t}{Re}}. \quad (13)$$

In the primitive variable formulation we have, as initial conditions:

$$u\,(x,y,0) = -\,Cos(x)\,Sen(y),$$
$$v\,(x,y,0) = Sen(x)\,Cos(y). \quad (14)$$

To obtain the initial conditions in the stream function-vorticity formulation we use equations (14):

$$u = \psi_y, \quad v = -\psi_{x,} \quad \omega = u_y - v_x, \qquad (15)$$

$$\begin{aligned} u = \psi_y \ &\Rightarrow \ \psi(x,y,0) = Cos(x)Cos(y) + k_1(x), \\ v = -\psi_x \ &\Rightarrow \ \psi(x,y,0) = Cos(y)Cos(x) + k_2(y), \end{aligned} \qquad (15a)$$

where k_1 and k_2 are taken as zero.

$$\begin{cases} \omega = u_y - v_x, \\ u_y = -Cos(x)\,Cos(y)\,,\ v_x = Cos(x)\,Cos(y), \qquad (15b) \\ \omega(x,y,0) = -2\,Cos(x)\,Cos(y). \end{cases}$$

For the boundary conditions in the primitive variable formulation we have:

$$\begin{cases} u(0,y,t) = -Sin(y)e^{\frac{-2t}{Re}}, \quad u(2\pi,y,t) = -Sin(y)e^{\frac{-2t}{Re}}, \\ v(0,y,t) = 0\,, \quad v(2\pi,y,t) = 0, \end{cases}$$

for $t \geq 0$. We also have:

$$\begin{cases} u(x,0,t) = 0\,, \quad u(x,2\pi,t) = 0, \\ v(x,0,t) = Sin(x)e^{\frac{-2t}{Re}}\,, \quad v(x,2\pi,t) = Sin(x)e^{\frac{-2t}{Re}}, \end{cases}$$

for $t \geq 0$.

Now we obtain the boundary conditions for the Taylor vortex problem in the stream function-vorticity formulation:

$$\begin{cases} \psi_y = u, \\ \psi_y(0,y,t) = -Sin(y)e^{\frac{-2t}{Re}} \ \Rightarrow \ \psi(0,y,t) = \\ \quad -\int Sin(y)e^{\frac{-2t}{Re}}dy = Cos(y)e^{\frac{-2t}{Re}} + k_1(x), \\ \psi_y(2\pi,y,t) = -Sin(y)e^{\frac{-2t}{Re}} \ \Rightarrow \ \psi(2\pi,y,t) = \\ \quad -\int Sin(y)e^{\frac{-2t}{Re}}dy = Cos(y)e^{\frac{-2t}{Re}} + k_2(x). \end{cases}$$

Where if k_1 and k_2 are taken as zero, then $\psi(0,y,t) = \psi(2\pi,y,t)$, for $t \geq 0$.

$$\begin{cases} \psi_x = -v, \\ \psi_x(x,0,t) = -Sin(x)e^{\frac{-2t}{Re}} \ \Rightarrow \ \psi(x,0,t) \\ \qquad = -\int Sin(x)e^{\frac{-2t}{Re}}dx + k_1(t), \\ \psi(x,0,t) = Cos(x)e^{\frac{-2t}{Re}} + k_1(t), \qquad (16a) \\ \psi_x(x,2\pi,t) = -Sin(x)e^{\frac{-2t}{Re}} \ \Rightarrow \ \psi(x,2\pi,t) \\ \qquad = -\int Sin(x)e^{\frac{-2t}{Re}}dx + k_2(t), \\ \psi(x,2\pi,t) = Cos(x)e^{\frac{-2t}{Re}} + k_2(t). \end{cases}$$

If we take $k_1(t) = k_2(t) = 0$, then $\psi(x,0,t) = \psi(x,2\pi,t)$. ψ is periodic for $t \geq 0$.

The boundary conditions for the vorticity are obtained from equation (14):

$$\begin{cases} \omega(0,y,t) = 2Cos(y)e^{\frac{-2t}{Re}}, \\ \omega(2\pi,y,t) = 2Cos(y)e^{\frac{-2t}{Re}}, \\ \omega(x,0,t) = 2Cos(x)e^{\frac{-2t}{Re}}, \\ \omega(x,2\pi,t) = 2Cos(x)e^{\frac{-2t}{Re}}. \end{cases} \qquad (16b)$$

In Figures 5 and 6 we show the streamlines and the isocontours for the vorticity for Re=3200, h=1/128 and t=100. In Figures 7 and 8 we show the graphs of the stream function and the vorticity for Re=3200 and t=100. As we have said, we show the graphs in 3D in order to see the difference in scales for the two different values of the Reynolds numbers, namely 3200 and 7500.

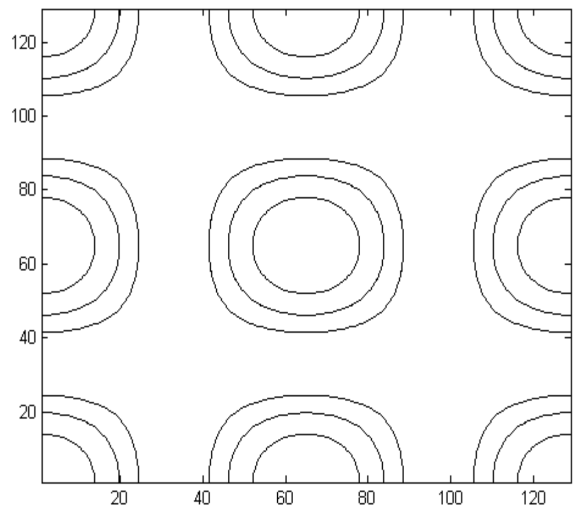

Figure 5 Streamlines for Re=3200, h=1/128 and t=100

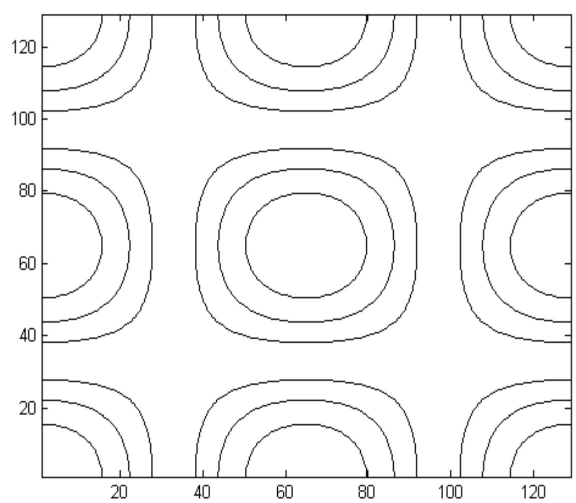

Figure 6 Isocontours for the vorticity for Re=3200, h=1/128 and t=100

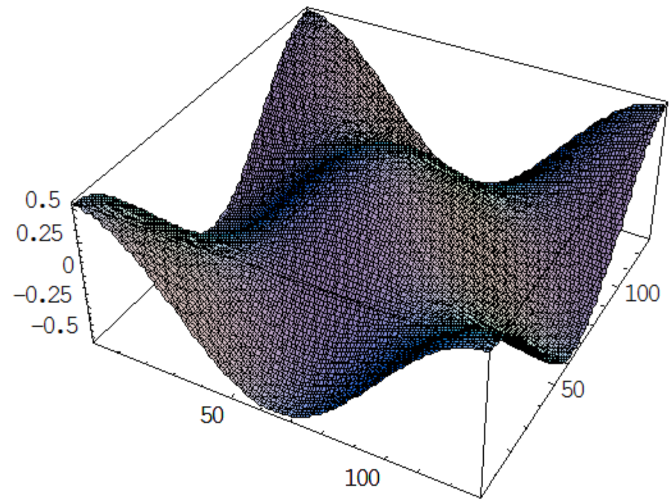

Figure 7 Stream function for Re=3200 and t=100.

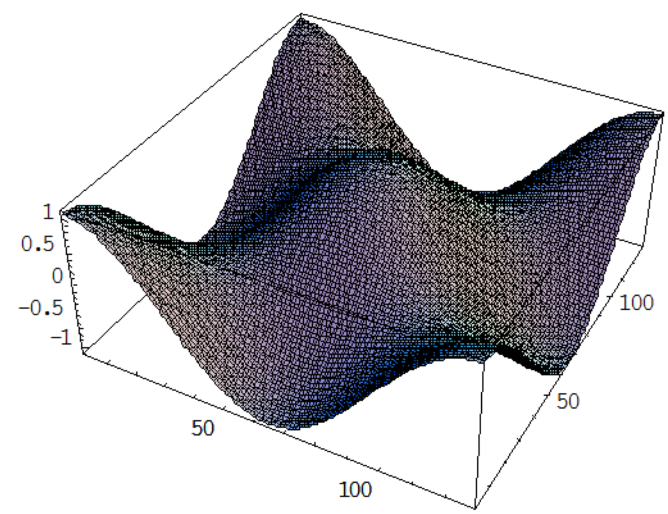

Figure 8 Vorticity function for Re=3200 and t=100.

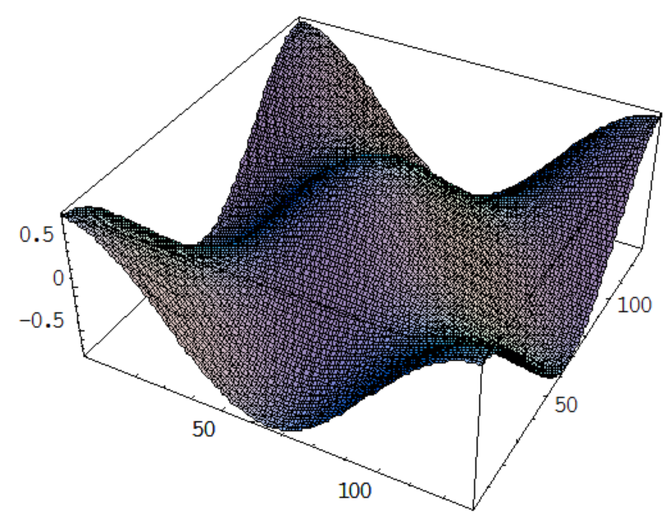

Figure 9 Stream function for Re=7500 and t=100

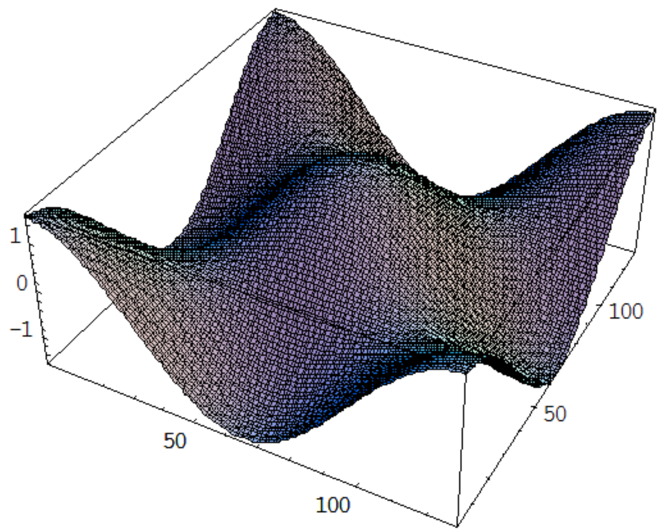

Figure 10 Vorticity function for Re=7500 and t=100

Next we show the minimum and the maximum of the stream function for the Taylor Vortex problem. In the first table, we show the minimum and the maximum for the exact solution for t=100 with Re=3200 and Re = 7500, and in the second table, the minimum and the maximum for the numerical solution obtained for the same values of Re and t.

In Table 3 we show the relative error obtained for the two values of Re mentioned and t=100. This error is obtained as the maximum of the differences between the exact solution and the numerical solution, divided by the value of the exact solution.

Re.	**Time**	**Stream function**	
		Min.	**Max.**
3200	1000	-0.5352614285045	0.5352614285189
7500	1000	-0.7659283383439	0.7659283383646

Table 1 Minimum and maximum values of the stream function (exact solution).

Re.	**Time**	**Stream function**	
		Min.	**Max.**
3200	1000	-0.5352614285101	0.5352614331597
7500	1000	-0.7659283383473	0.7659283450006

Table 2 Minimum and maximum values of the stream function (numerical solution).

Re.	**Time**	**Relative error**
3200	100	3.167780632146758e-008
7500	100	3.167314295814883e-008

Table 3 Relative error.

CONCLUSIONS

We are presenting an efficient numerical scheme for solving the non-steady Navier-Stokes equations in the stream function-vorticity formulation. Two problems were solved: the driven cavity problem and the Taylor vortex problem. For the driven cavity problem results were compared with those reported in the literature [4 - 7], [9 - 11], where other formulations of the Navier-Stokes equations were used. Results agree very well with those reported. As the Reynolds number increases, more primary and secondary vortexes appear and smaller values of h have to be used, numerically by stability matters and physically to capture the fast dynamics of the flow [12]. In Figures 2 and 4 oscillations occur for the vorticity and in order to avoid this problem smaller values of h should be used.

For the Taylor Vortex problem in [8] the velocity-vorticity formulation was used and results agree very well. In this case we were able to compare with the exact solution and the error, as shown in Table 3, was very small.

We are working now on reducing computing time. In the fixed point iterative process the coefficient matrix for each system solved is matrix A (symmetric and positive definite) resulting from the discretization of the laplacian. In a future work we will work with both matrixes, A and B, where matrix B results from the discretization of the advective term [13].

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