HEFAT2014
10th International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics
14 – 16 July 2014
Orlando, Florida

Modulation Equations for Roll Waves of a liquid film Down an Inclined Plane as a Power-Law Fluid

Zhu K. Boudlal A. and Mompean G.
Mechanics Laboratory of Lille, UMR 8107,
University of Lille 1,
Villeneuve d'Ascq, 59650,
France,
E-mail: kan.zhu@ed.univ-lille1.fr

ABSTRACT

Roll waves of finite amplitude on a thin layer of non-Newtonian fluid modeled as a power-law fluid are considered. In the long wave approximation, the flow is governed by a non-homogeneous hyperbolic system of equations. As the linearized instability analysis of a uniform flow delivers only a diagnosis of instability, the nonlinear stability is investigated and the criterion for roll waves based on the hyperbolicity of the modulation equation is suggested. The main problem in defining the roll wave stability region on a roll wave diagram is due to the singularities of functions for the mean values and their derivatives near the boundaries of roll wave existence. Asymptotic formulae for nonlinear stability of roll waves of small and maximal amplitudes are derived. Numerical calculation reveals that for a Newtonian fluid, as the bottom inclination decreases downwardly the amplitude of admissible waves diminishes, and the stability domain reduces until it disappears. These results remain valid for a slightly non-Newtonian fluid. For highly non-Newtonian fluid, an inversion in the nature of stability is observed.

INTRODUCTION

The thin flow on a wall is of considerable importance for a variety of industrial and natural processes. Over a range of flow parameters, a variety of free surface instabilities may occur. Roll waves are among these flow patterns, which may exhibit quasi-periodic spatial structures of the free boundary. The standard procedure of roll waves consists of a periodic pattern of stable bores separated by continuous profiles increasing monotically from the rear (see an extensive reference quoted in [1]). From the experimental work of Kapitza [2]; Liu, Paul & Gollub [3], among others, have experimentally investigated the linear stability. A detailed study of this work can be found in the book by François Charru [4].

The aim of the paper is to give a nonlinear study on stability of permanent roll waves on a shear thinning fluid in the frame of unsteady, gradually varied, laminar mud flow. Throughout the paper, the averaged equations for one-dimensional flow with the shear stress being evaluated in conventional manner.

Starting from long waves equations averaged over the normal to the bed, periodic roll waves are constructed. As the amplitude and the phase velocity of waves are slowly varying during their propagation and as these variations can give rise to instability, this problem of stability can be solved by deriving modulation equations for wave packets [5,6]. The stability criterion for nonlinear roll waves is formulated in terms of hyperbolicity of modulation equations that need calculation of averaged quantities. The main difficulty to construct the stability domain on the plane of governing parameters is due to the singularities in the hyperbolicity conditions of modulation equations for the waves of the infinitesimal and maximal amplitude.

Using an asymptotic analysis, the stability conditions of roll waves of small and maximal amplitude, as well as the approximate position of boundaries of the hyperbolicity domain are obtained. Note that for a vertically falling film, the system is self-similar, the modulation equations take a rather simple form and the hyperbolicity criterion is reduced to a condition for a function of one variable [5]. Here we present an inclined flow, the stability condition depends on two governing parameters. Numerical calculations of stability diagrams corresponding to an inclined plane are presented. It must be noticed that roll waves without dissipation of energy across the shocks are of small amplitude.

NOMENCLATURE

n	$[-]$	Flow index
β	$[-]$	Momentum flux factor
γ	$[-]$	Energy flux factor
x^*	$[m]$	Cartesian axis direction
y^*	$[m]$	Cartesian axis direction
h^*	$[m]$	Depth of the flow
u^*	$[m/s]$	Means velocity
τ_b^*	[Pa]	Bottom stress
t^*	$[s]$	Time
g	$[m/s^2]$	Gravity acceleration
μ_n	$[kgm^{-1}s^{n-2}]$	Viscosity coefficient
h_1	$[-]$	Depth of the flow before the jump, dimensionless
h_2	$[-]$	Depth of the flow after the jump, dimensionless
h_c	$[-]$	Critical flow depth, dimensionless
z^*	$[-]$	Minimal depth of admissible roll waves, dimensionless

Subscripts

t	Derivative on t
x	Derivative on x
h_c	Derivative on h_c
z	Derivative on z
$\bar{h}$	Derivative on $\bar{h}$
0	Reference

Special characters

m'	m derivative on h_c
D'	D derivative on h_c

GOVERNING EQUATION

Consider a 2D flow of a thin fluid film down a plane inclined with an angle $\varphi, (0 < \varphi < \pi/2)$. The frame of reference is chosen as follows: the axis Ox^* is directed in the flow direction, and the axis Oy^* is perpendicular to it, directed upwards.

In the long wave approximation, the film thickness depends only on x^* and t^*, and the pressure is hydrostatic: $p^* = \rho g_n (h^* - y^*)$, where $g_n = g\cos\varphi$. The shear stress is modeled by a power law of the form [1,5]:

$$\tau = \mu_n \left(\frac{\partial u}{\partial y}\right)^n (0 < n \le 1) \quad (1)$$

With the following dimensionless quantities defined by:

$$\begin{cases} t^* = \dfrac{l_0}{u_0} t,\ x^* = l_0 x,\ u^* = u_0 u,\ h^* = h_0 h,\ \tau_b^* = \tau_{b0}\tau_b, \\ \alpha = \dfrac{g_n h_0}{u_0^2},\ \tau_{b0} = \mu_n \left[(\dfrac{1+2n}{n}) \dfrac{u_0}{h_0} \right]^n,\ l_0 = \dfrac{u_0^2}{g\sin\varphi}. \end{cases} \quad (2)$$

Here h^* is the depth, u^* is the velocity, t^* is the time, g is the gravity acceleration and τ_b^* is the bottom stress, the subscript 0 stands for the reference quantities. The viscosity coefficient is denoted by μ_n with dimension $\left[ML^{-1}T^{n-2}\right]$ and n is the flow index $(0 < n \le 1)$. The case $n = 1$ corresponds to the Newtonian fluid and μ_1 is the ordinary dynamic viscosity [1, 5].

In dimensionless variables, the governing equations of mass and momentum conservation, averaged in the ordinate direction may take the form [1]:

$$\begin{cases} h_t + (uh)_x = 0 & (3a) \\ (uh)_t + \left(\beta u^2 h + \dfrac{1}{2}\alpha h^2\right)_x = h - \left(\dfrac{u}{h}\right)^n & (3b) \end{cases}$$

The main objective of this work is to investigate the nonlinear stability of roll waves (RW). For this purpose the approach developed in [5] for RW for vertical plane will be applied for thin viscous film flows on an incline.

LINEAR STABILITY

From a temporal linear analysis it is shown in [1] that the stationary solution of (3):

$$u_0 = h_0^{\frac{1+n}{n}}, h_0 = cte \quad (4)$$

is unstable if the following condition is satisfied

$$\alpha < (1+2n)/n^2 \quad (5)$$

The same criterion of transition to unstable flow can be established by making use of the method promoted in [5,6]: there is stability if the velocity of the kinematic waves lies between the velocities of the dynamic wave $\lambda_d^{\pm}$, i.e.

$$\lambda_d^- < \lambda_c < \lambda_d^+ \quad (6)$$

$\lambda_c = h_0^2 + 2u_0, \lambda_d^{\pm} = \beta u_0 \pm u_0\sqrt{(\beta-1)\beta + \alpha},$ where the condition (5) is recovered.

ROLL WAVES

We intend to construct discontinuous periodic-wave solutions which propagate with a constant speed $D\ (0 < u < D)$ (Figure 1). In the frame of reference accompanying the waves the flow is steady and Equations (3) may be expressed in terms of the single variable $\xi = x - Dt$. With this transformation, Equation (3a) may be integrated directly to give

$$m = (D-u)h \tag{7}$$

where the constant m is equal to the apparent discharge rate or progressive discharge according to [7].

Between two successive discontinuities the momentum equation is valid, making use of (7), Equation (3b) leads to an ODE for h which may read:

$$\begin{cases} \dfrac{dh}{d\xi} = \dfrac{F(h)}{\Delta(h)},\ F(h) = h - \left(\dfrac{Dh-m}{h^2}\right)^n \\ G(h) = Dm(1-2\beta) + (\beta-1)D^2h + \beta\dfrac{m^2}{h} + \alpha\dfrac{h^2}{2} \\ \Delta(h) = \dfrac{dG}{dh} = (\beta-1)D^2 - \beta\dfrac{m^2}{h^2} + \alpha h \end{cases} \tag{8}$$

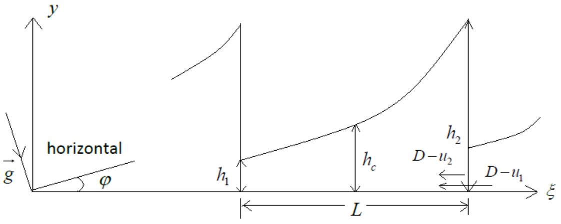

Figure 1 Sketch in the frame of reference accompanying the roll waves

For standard roll waves with only one jump on the period that divides the monotone smooth parts of flow. Therefore, it is necessary for the roll wave existence that the subcritical flow behind the jump $(\Delta > 0)$ transforms into the supercritical flow $(\Delta < 0)$ before the next jump, and there exists the critical depth h_c on the period. For the roll wave existence it is necessary that $F(h)$ and $\Delta(h)$ vanish at the critical depth h_c simultaneously, i.e.

$$F(h_c) = 0,\ \Delta(h_c) = 0 \tag{9}$$

After some algebra manipulations, taking (7) into account, the system (9) leads to

$$m = h_c^{(2n+1)/n}(\theta - 1),\ D = h_c^{(n+1)/n}\theta,$$
$$\theta = \beta + \left\{(\beta-1)\beta + \alpha h_c^{-(n+2)/n}\right\}^{1/2} \tag{10}$$

For given h_c, the periodic solution containing a jump can be constructed if the jump amplitude is known.

To complete the construction of roll waves, we must give the jump conditions.

From the system (3), with the help of (7), the relations at jump discontinuity are reduced to

$$G(h_1, h_c) = G(h_2, h_c) \tag{11}$$

where subscripts 1 and 2 denote the right and the left sides of the discontinuity.

Equations (7)-(11) show that the roll waves depend on two parameters. For given values of h_1 and h_c, conditions:

$$\begin{cases} F(h_c, h) < 0 & \text{at } h_1 < h < h_c \\ F(h_c, h) > 0 & \text{at } h_c < h < h_2 \end{cases} \tag{12}$$

are necessary and sufficient for the formation of roll waves. It is shown in [1] that there exist solutions of roll waves only if $z^* < h_1 < h_c$ where z^* and h_c are the roots of the equation $F(h) = 0$ (Figure 2). The value of z^* is obtained only numerically for a given flow parameter.

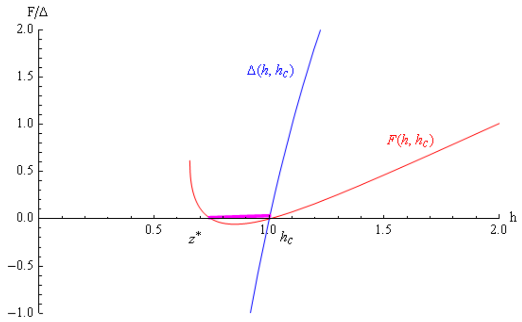

Figure 2 Profiles of Δ and F *vs.* h for $\alpha = 3,\ n = 0.4,\ h_c = 1$

According to the reported observations, between two successive jumps the surface profile must increase, otherwise the slope $dh/d\xi$ must be positive; it is also the condition of irreversibility of hydraulic jump.

Moreover, as $\Delta'(h_c) = \beta m^2/2h_c^3 + \alpha > 0$, therefore a necessary condition: $F'(h_c) > 0$, resulting in:

$$\alpha < [(1+2n)/n^2]h_c^{\frac{n+2}{n}} \tag{13}$$

If the solution of the uniform base flow is critical, the inequality (13) coincides with (5). In this case, the condition of linear instability is necessary for the roll waves formation.

It must be noticed that for $h_c \le 1$, if inequality (13) is verified, the required linear instability criterion (5) is unconditionally satisfied, since the transition from uniform flow to intermittent flow regime is usually tackled by resorting to stability theory according to [7].

MODULATION EQUATIONS

The periodic solution of (3) is defined by two parameters, say h_1 and h_c . The problem on nonlinear stability of periodic wave trains with slowly varying values h_c and h_1 can be solved by analysis of hyperbolicity of the modulation equations for such waves. After averaging (3) over the fixed length scale, which is large enough comparing with the length of roll waves, the following modulation equations:

$$\begin{cases} \overline{h}_t + \left(\overline{uh}\right)_x = 0, \\ \left(\overline{uh}\right)_t + \left(\overline{\beta u^2 h + \frac{1}{2}\alpha h^2}\right)_x = 0 \end{cases} \tag{14}$$

are obtained. All averaged quantities can be expressed as functions of h_1 and h_c as follows:

$$\begin{cases} L = \int_0^L d\xi = \int_z^{w(z,h_c)} A(s,h_c)ds; \\ \overline{h} = L^{-1}\int_0^L h(\xi)d\xi = L^{-1}\int_z^{w(z,h_c)} sA(s,h_c)ds; \\ A = \left\{(\beta-1)D^2 - \beta\frac{m^2}{s^2} + \alpha s\right\}\left\{s - \left(\frac{D}{s} - \frac{m}{s^2}\right)^n\right\}^{-1}; \\ \overline{hu} = D\overline{h} - m;\ \ B = \frac{\beta m^2}{h} + \frac{\alpha h^2}{2} \\ \overline{\beta u^2 h + \frac{1}{2}\alpha h^2} = \beta D^2\overline{h} - 2\beta mD + \overline{B}; \\ \overline{B} = L^{-1}\int_0^L Bd\xi = L^{-1}\int_z^{w(z,h_c)} B(s,h_c)A(s,h_c)ds; \\ s\in[h_1,h_2], z = h_1, w = h_2. \end{cases} \tag{15}$$

In view of (15), the modulation equations take the form:

$$\begin{cases} \overline{h}_t + \left(D\overline{h} - m\right)_x = 0 \\ \left(D\overline{h} - m\right)_t + \left(\beta(D^2\overline{h} - 2mD) + \overline{B}(h_1,h_c)\right)_x = 0 \end{cases} \tag{16}$$

The non stationary evolution of the governing parameters (h_1,h_c) for a periodic wave train is described by equations (16). We say that the roll waves are stable if the modulation equations (16) for corresponding values (h_1,h_c) are hyperbolic. The investigation of hyperbolicity of the modulation equations can be performed more easily for the variables h_c and $\overline{h}$ [8]. It can be done by the transformation $\overline{h} = \overline{h}(h_1,h_c)$ and $\widehat{B}(\overline{h},h_c) = \overline{B}(h_1,h_c)$.

The modulation equations take the following form:

$$\begin{cases} \overline{h}_t + \left(D\overline{h} - m\right)_x = 0 \\ \left(D\overline{h} - m\right)_t + \left(\beta(D^2\overline{h} - 2mD) + \widehat{B}(\overline{h},h_c)\right)_x = 0 \end{cases} \tag{17}$$

The characteristic curves of (17) are

$$\begin{aligned} &\frac{dx^{\pm}}{dt} = \frac{\{2\beta D + (-2m\beta D' + \widehat{B}_{h_c})\delta^{-1}\} \pm \sqrt{Disc}}{2} \\ &Disc = \left\{2(\beta-1)D + (-2m\beta D' + \widehat{B}_{h_c})\delta^{-1}\right\}^2 \\ &\quad + 4\left((\beta-1)D^2 + \widehat{B}_{\overline{h}}\right),\ \delta = D'\overline{h} - m'. \end{aligned} \tag{18}$$

Here the “prime” denotes the full derivation on the variable h_c. The hyperbolicity condition for (18) is

$$\begin{aligned} &Disc = \left\{2(\beta-1)D + (-2m\beta D' + \widehat{B}_{h_c})\delta^{-1}\right\}^2 \\ &\quad + 4\left((\beta-1)D^2 + \widehat{B}_{\overline{h}}\right) > 0 \end{aligned} \tag{19}$$

For the variables (h_c, z) the hyperbolicity condition (17) can be expressed in the form:

$$\begin{aligned} &\left\{2(\beta-1)D + (-2m\beta D' + \overline{B}_{h_c} - \overline{B}_z\,\overline{h}_{h_c}/\overline{h}_z)\delta^{-1}\right\}^2 \\ &+4\left((\beta-1)D^2 + \overline{B}_z/\overline{h}_z\right) > 0. \end{aligned} \tag{20}$$

To check the stability criterion (19) or (20) for given roll wave train, we have to resolve the singularities in (15) for the wave of infinitesimal $(L\to 0)$ and limiting $(L\to\infty)$ amplitude.

ASYMPTOTIC ANALYSIS OF ROLL WAVE

1. Stability of infinitesimal amplitude

Let h_c be the given critical depth, then one has:

$$\begin{aligned} A(h_c,h_c) &= \lim_{z\to h_c} A(z,h_c) \\ &= \lim_{z\to h_c}\frac{F(z,h_c)}{\Delta(z,h_c)} = \frac{F_z(h_c,h_c)}{\Delta_z(h_c,h_c)} > 0 \end{aligned} \tag{21}$$

The conjugate depth $w = w(z,h_c)$ and the corresponding derivatives at $z = h_c$ can be calculated from (11) as follows:

$$\begin{cases} w(h_c,h_c) = h_c,\ w_z(h_c,h_c) = -1, \\ w_{h_c}(h_c,h_c) = 2\left\{(-\beta+1)2DD' + \frac{2\beta mm'}{h_c^2}\right\}\times \\ \times\left\{\frac{2\beta m^2}{h_c^3} + \alpha\right\}^{-1} \\ w_{zz}(h_c,h_c) = -\frac{2}{3}\frac{G_{zzz}(h_c,h_c)}{G_{zz}(h_c,h_c)} \end{cases} \tag{22}$$

For the derivates of the mean values in (15) one obtains

$$\bar{h}_z = \frac{1}{L}\Big(A(w,h_c)(w-\bar{h})w_z - A(z,h_c)(z-\bar{h})\Big)$$
$$= \frac{H}{L}$$
$$\bar{B}_z = \frac{1}{L}\Big(A(w,h_c)(B(w,h_c)-\bar{B})w_z$$
$$-A(z,h_c)(B(z,h_c)-\bar{B})\Big) = \frac{Q}{L} \qquad (23)$$

With the functions $H = H(z,h_c)$ and $Q = Q(z,h_c)$ defined in (21) we have

$$\bar{h}_z\Big|_{z=h_c} = \lim_{z\to h_c}\frac{H(z,h_c)}{L(z,h_c)} = \frac{H_z(h_c,h_c)}{L_z(h_c,h_c)},$$
$$\bar{B}_z\Big|_{z=h_c} = \lim_{z\to h_c}\frac{Q(z,h_c)}{L(z,h_c)} = \frac{Q_z(h_c,h_c)}{L_z(h_c,h_c)}, \qquad (24)$$
$$L_z(h_c,h_c) = -2A(h_c,h_c) < 0$$

It follows from (22)-(24) that

$$\bar{h}_z\Big|_{z=h_c} = -\bar{h}_z\Big|_{z=h_c} = 0, \bar{B}_z\Big|_{z=h_c} = -\bar{B}_z\Big|_{z=h_c} = 0 \qquad (25)$$

The second derivatives of $\bar{B}$ and $\bar{h}$ can be expressed at $z = h_c$ by formulae

$$\bar{h}_{zz} = \frac{2}{3}\left(\frac{A_z(h_c,h_c)}{A(h_c,h_c)} - \frac{1}{2}\frac{G_{zzz}(h_c,h_c)}{G_{zz}(h_c,h_c)}\right)$$
$$\bar{B}_{zz} = \frac{1}{3}B_{zz}(h_c,h_c) + \frac{2}{3}\frac{B_z(h_c,h_c)A_z(h_c,h_c)}{A(h_c,h_c)}$$
$$+\frac{1}{3}B_z(h_c,h_c)w_{zz}(h_c,h_c) \qquad (26)$$
$$\widehat{B}_{\bar{h}}\Big|_{z=h_c} = \lim_{z\to h_c}\frac{\bar{B}_z(z,h_c)}{\bar{h}_z(z,h_c)} = \frac{\bar{B}_{zz}(z,h_c)}{\bar{h}_{zz}(z,h_c)}$$

To find the sign of (20), we need also the expression of $\bar{h}_{h_c}$ and $\bar{B}_{h_c}$ at $z = h_c$. It follows from (15) that

$$\bar{h}_{h_c}(h_c,h_c) = \frac{1}{2}w_{h_c}(h_c,h_c)$$
$$\bar{B}_{h_c}(h_c,h_c) = \frac{\beta m m'}{h_c} + \frac{1}{2}B_z(h_c,h_c)w_{h_c}(h_c,h_c) \qquad (27)$$

Now the stability of roll wave of infinitesimal amplitude can be checked by (19), (26), (27) using only values of the known function $A(s,h_c), B(s,h_c)$ and their derivatives for $s = h_c$.

2. Stability of limiting roll waves

Suppose that standard roll waves are defined for every critical depth h_c from an interval. It means that there are smooth functions $z^* = z^*(h_c)$ and $w^* = w(z^*(h_c),h_c)$ and the conditions (12) are satisfied for conjugate depths z, w with $G(z) = G(w)$ and $z^* < z < h_c < w < w^*$, $F(z^*,h_c) = 0, F(w^*,h_c) \neq 0$.

When $z \to z^*$, we can use the asymptotic formulae

$$A(s,h_c) = \frac{b(s,h_c)}{s-z^*}, b(z^*,h_c) \neq 0,$$
$$L(z,h_c) = \int_z^w \frac{b(s,h_c)-b(z^*,h_c)}{s-z^*}ds$$
$$+b(z^*,h_c)\ln\frac{w-z^*}{z-z^*}\Big|_{z\to z^*} \to \infty,$$
$$\bar{h}(z,h_c) = \frac{1}{L}\int_z^w (s-z^*)A(s,h_c)ds + z^*\Big|_{z\to z^*} \to z^*, \qquad (28)$$
$$\bar{B}(z,h_c) = \frac{1}{L}\int_z^w (B(s,h_c)-B(z^*,h_c))A(s,h_c)ds$$
$$+B(z^*,h_c)\Big|_{z\to z^*} \to B(z^*,h_c),$$

Excluding the wave length L from (28) we have:

$$\bar{B}(z,h_c) = B(z^*,h_c)$$
$$+\frac{(\bar{h}-z^*)}{\int_z^w b(s,h_c)ds}\int_z^w\frac{(B(s,h_c)-B(z^*,h_c))b(s,h_c)ds}{s-z^*} \qquad (29)$$

The approximate expression for the function $\widehat{B}(\bar{h},h_c) = \bar{B}(z,h_c)$ is given by the following formulae, in which the limits of integrals in (29) are used for $z \to z^*$.

$$\widehat{B}^*(\bar{h},h_c) = B(z^*,h_c) + B'^*(\bar{h}-z^*)$$
$$B'^* = \frac{\displaystyle\int_{z^*}^{w^*}\frac{(B(s,h_c)-B(z^*,h_c))b(s,h_c)ds}{s-z^*}}{\displaystyle\int_{z^*}^{w^*} b(s,h_c)ds} \qquad (30)$$

The approximation (30) can be applied for calculations of the hyperbolicity domain of the modulation equations (17). For that we replace the function $\widehat{B}(\bar{h},h_c)$ by $\widehat{B}^*(\bar{h},h_c)$. The criterion of the hyperbolicity takes the form

$$Disc^* = \left\{2(\beta-1)D + (-2m\beta D' + \widehat{B}^*_{h_c})\delta^{-1}\right\}^2$$
$$+4\left((\beta-1)D^2 + \widehat{B}^*_{\bar{h}}\right) \qquad (31)$$

Due to the linear dependence $\widehat{B}^*$ on $\bar{h}$ in (30) the boundaries $\bar{h}^{\pm} = \bar{h}^{\pm}(h_c)$ of the hyperbolicity domain can be calculated from the quadratic equation $Disc^*(\bar{h},h_c) = 0$ relative to $\bar{h}$ in the explicit form:

$$\begin{aligned}
\bar{h}^{\pm} &= \frac{d_2 \pm m' \sqrt{d_3}}{d_1 \pm D' \sqrt{d_3}} \\
d_1 &= \frac{1}{2} B'^{*}_{h_c} + (\beta - 1) D D' \\
d_2 &= (\beta - 1) D m' + \beta m D' \\
&\quad - \frac{1}{2} (B_{h_c}(z^*, h_c) - B'^{*}_{h_c} z^* - B'^{*} z^*_{h_c}) \\
d_3 &= -\left((\beta - 1) D^2 + B'^{*} \right)
\end{aligned} \tag{32}$$

NUMERICAL RESULTS

The roll wave diagrams for some significant values of (α, n) are shown in figure 3-6, where the curves $\bar{h} = \bar{h}^{-}$ and $\bar{h} = \bar{h}^{+}$ are the boundaries of hyperbolicity region. In the ellipticity domains Ω_e roll waves are unstable. The curve $\bar{h} = h_c$ corresponds to the roll wave of infinitesimal amplitude. The curve $\Delta E = 0$ corresponds to the roll waves with zero dissipation across the shock [1]. This type of wave is of small amplitude. Obviously the shock is accompanied by the loss of energy $\Delta E < 0$, the calculation of ΔE can be seen in [1].

The computation results for a Newtonian fluid $(n = 1)$ show that as α increase, the amplitude of admissible waves diminishes, and the stability domain reduces until it disappears. These results remain valid for a slightly Non-Newtonian fluid (for example $n = 0.8$). As illustrated in figures 7 to 10 for fixed critical value $\left(h_c = 1\right)$.

For a highly non-Newtonian fluid ($n = 0.1$), we see that as α increase, the amplitude of admissible waves diminishes, and the stability domain of moderate waves increases.

As we can see from the diagrams $\bar{h}^{+} - z^* \ll z^*$, the boundary of the hyperbolicity domain can be found effectively by the above approximation (see (28)-(32)).

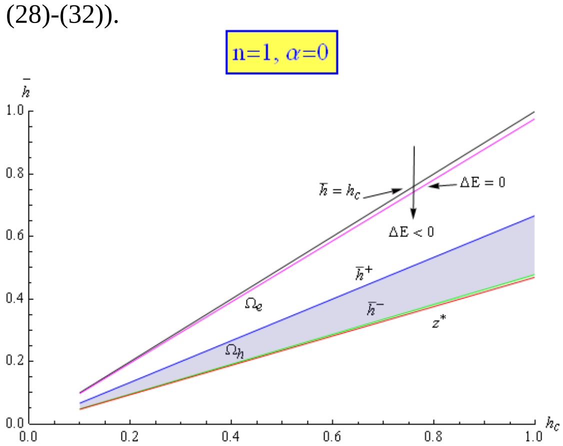

Figure 3 the diagram of stability of roll wave for $n = 1, \alpha = 0$

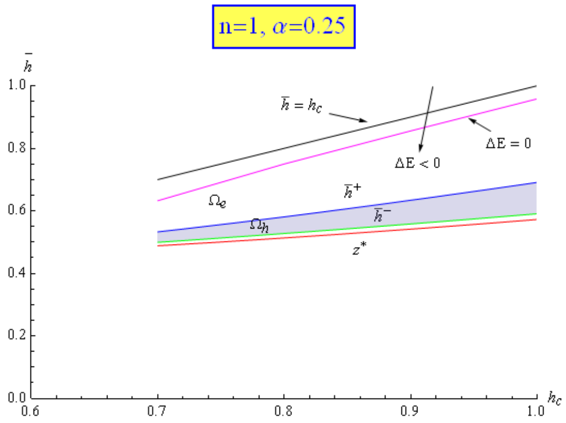

Figure 4 the diagram of stability of roll wave for $n = 1, \alpha = 0.25$

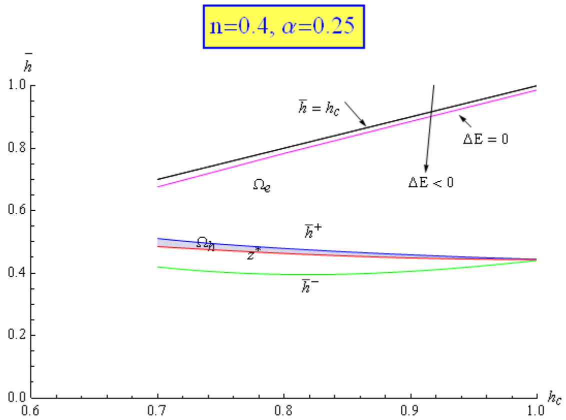

Figure 5 the diagram of stability of roll wave for $n = 0.4, \alpha = 0.25$

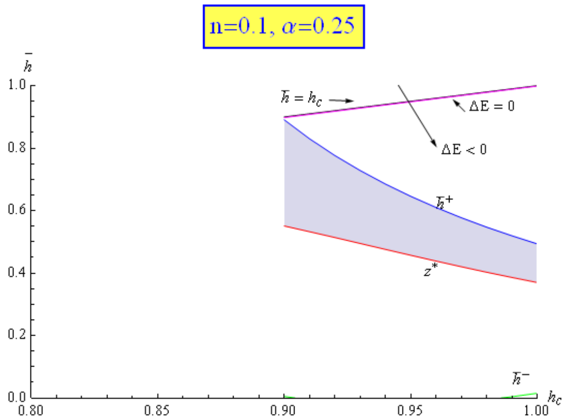

Figure 6 the diagram of stability of roll wave for $n = 0.1, \alpha = 0.25$

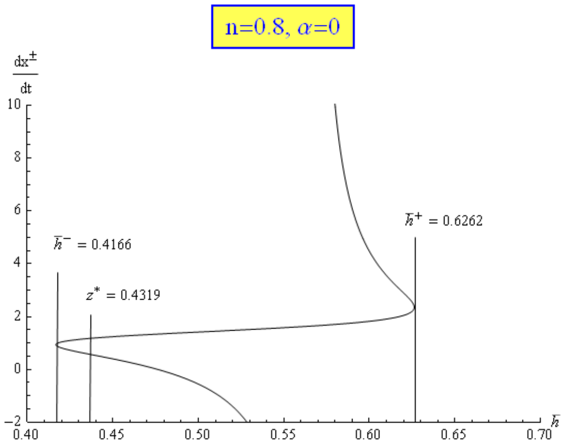

Figure 7 the diagram of stability of roll wave for $n = 0.8, \alpha = 0, h_c = 1$

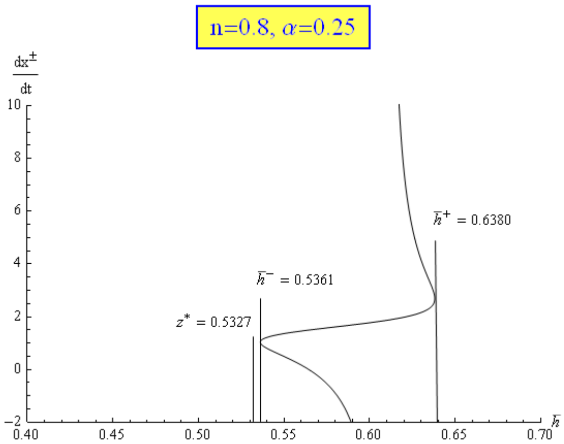

Figure 8 the diagram of stability of roll wave for $n = 0.8, \alpha = 0.25, h_c = 1$

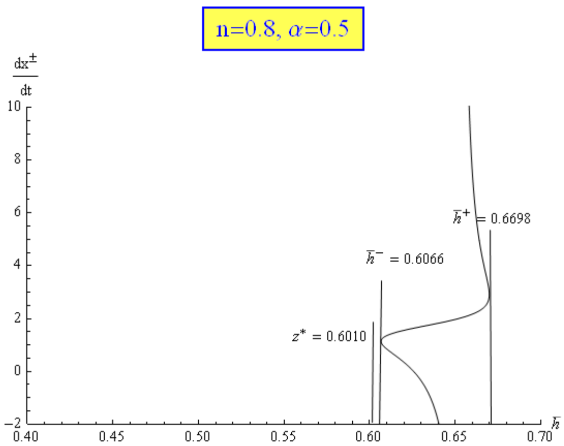

Figure 9 the diagram of stability of roll wave for $n = 0.8, \alpha = 0.5, h_c = 1$

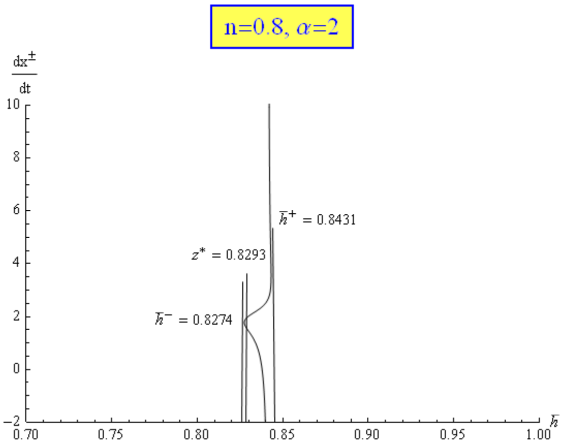

Figure 10 the diagram of stability of roll wave for $n = 0.8, \alpha = 2, h_c = 1$

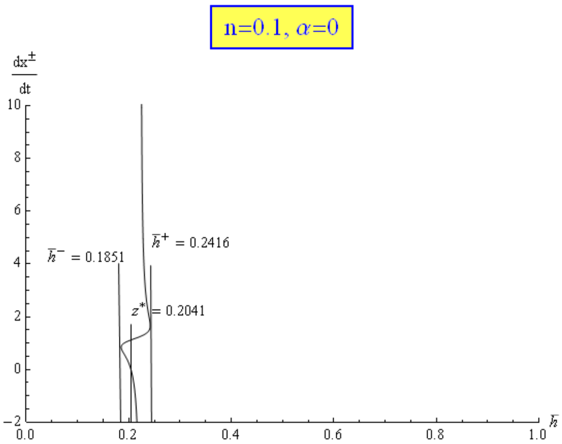

Figure 11 the diagram of stability of roll wave for $n = 0.1, \alpha = 0, h_c = 1$

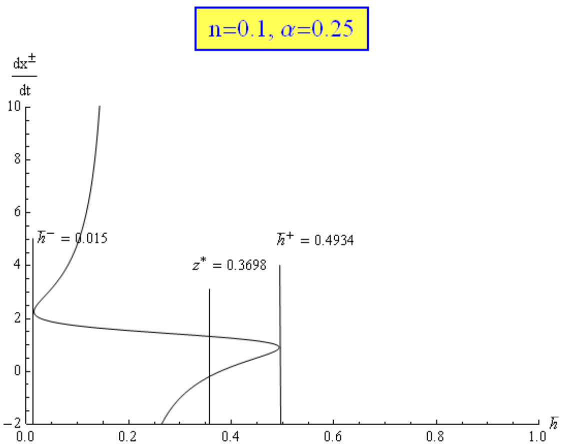

Figure 12 the diagram of stability of roll wave for $n = 0.1, \alpha = 0.25, h_c = 1$

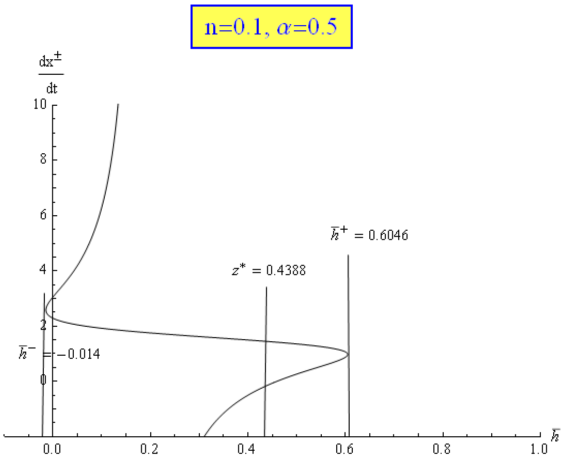

Figure 13 the diagram of stability of roll wave for $n = 0.1, \alpha = 0.5, h_c = 1$

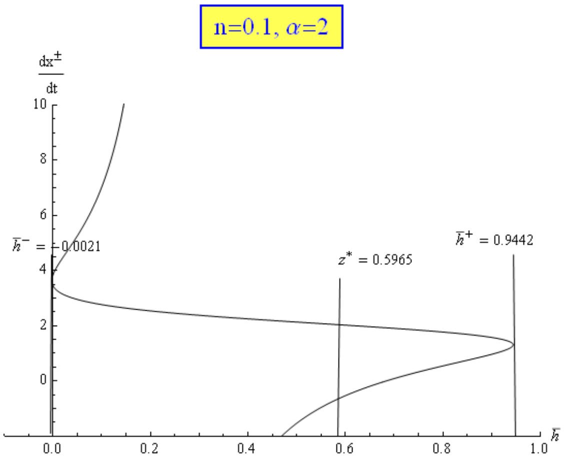

Figure 14 the diagram of stability of roll wave for $n = 0.1, \alpha = 2, h_c = 1$

CONCLUSION

We have investigated the roll waves generation on laminar flow of the thin layer down an inclined plane by nonlinear hyperbolic system [1], in which the rheological behavior is modeled by a power law. It has been shown that the roll waves solution can be described by two parameters analogously to roll waves in open channel flow. The linear stability criterion is unconditionally satisfied together with a roll wave required condition if the inequality (11) is verified, while dimensionless critical depth is less than 1 $(h_c \leq 1)$. A stability criterion based on hyperbolicity of modulated equations has been presented. The asymptotic analysis has been performed and the stability criterion for roll waves of small and maximal amplitude as well as the approximate position of boundaries of the stability region has been derived.

Numerical calculations have been performed for some significant flow parameter. They have revealed that for a Newtonian fluid, as the bottom inclination decreases downwardly the amplitude of admissible waves diminishes, and the stability domain has been reduced until the disappearance. These results have remained valid for a slightly Non-Newtonian fluid. For highly non-Newtonian fluid, an inversion in the nature of stability has been observed.

REFERENCES

[1] C.O.Ng., C.C.Mei, Roll waves on shallow layer of mud modelled as a power-law fluid. Journal of Fluid Mechanics, Vol.263, 1994, pp.151-183.
[2] Kapitza, P.L. 1948 Wave flow of thin layers of a viscous fluid. Zh.Eksp.Teor.Fiz.18,3-28.
[3] Liu,J.,Paul, J.D.&Gollub, J.P. Mesurements of the primary instabilities of film flows. Journal of Fluid Mechanics, Vol.250,1993, pp.69-101.
[4] François Charru, Instabilités hydrodynamiques, 2007.
[5] Boudlal A. Liapidevskii, Modulation Equation for Roll Waves on Vertically Falling Films of a Power-Law Fluid. W.J.M., 2012, 2, pp.1-8
[6] G.B.Whitham, Linear and Nonlinear Waves, John Wiley and Sons, New York, 1974.
[7] F.Dressler, Mathematical Solution of the Problem of Roll Waves in Inclines Open Channels. Communications on pure and Applied Mathematics, Vol.49, No.6,1949, pp.149-194.
[8] Boudlal A. Liapidevskii, Stability of roll waves in open channel flows, C.R.Mecanique 330(2002) 291-295.