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Common Fixed Point of Generalized Weak Contractive Mappings in Partially Ordered G_b-metric Spaces

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Abstract. In this work, using the concepts of G-metric and b-metric we define a new type of metric which we call G_b-metric. We study some basic properties of such metric. We also prove a common fixed point theorem for six mappings satisfying weakly compatible condition in complete partially ordered G_b-metric spaces. A nontrivial example is presented to verify the effectiveness and applicability of our main result.

1. Introduction

Mustafa and Sims [12] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. [15] obtained some fixed point results for mappings satisfying different contractive conditions. Abbas and Rhoades initiated the study of common fixed point theory in generalizes metric spaces. Since then, many authors obtained fixed and common fixed point results in the setup of G-metric spaces [1, 7, 11, 13, 14, 16, 18, 19]. Saadati et al. [18] proved some fixed point results for contractive mappings in partially ordered G-metric spaces (see also, [6]). On the other hand the concept of b-metric space was introduced by Czerwik in [8]. After that, several interesting results for the existence of fixed point for single-valued and multivalued operators in b-metric spaces have been obtained (see [4, 5, 20]). Pacurar [17] proved some results on sequences of almost contractions and fixed points in b-metric spaces. Recently, Hussain and Shah [9] obtained results on KKM mappings in cone b-metric spaces.

The aim of this paper is two fold: We introduce a concept of generalized b-metric spaces, study some basic properties of generalized b-metric and obtain a common fixed point result for six mappings satisfying weakly compatible condition in the framework of complete partially ordered generalized b-metric spaces.

Following is our definition of generalized b-metric spaces.

Definition 1.1. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies :

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,

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(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and pair (X, G) is called a generalized b-metric space or G_b-metric space.

It should be noted that, the class of G_b-metric spaces is effectively larger than that of G-metric spaces given in [12].
Following example shows that a G_b-metric on X need not be a G-metric on X.

Example 1.2. Let (X, G) be a G-metric space, and $G_*(x, y, z) = G(x, y, z)^p$, where $p > 1$ is a real number.

Note that G_* is a G_b-metric with $s = 2^{p-1}$. Obviously, G_* satisfies conditions (G_b1) – (G_b4) of the definition 1.1, so it suffices to show that (G_b5) is hold. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$. Thus for each $x, y, z, a \in X$ we obtain

$$\begin{aligned} G_*(x, y, z) &= G(x, y, z)^p \leq (G(x, a, a) + G(a, y, z))^p \\ &\leq 2^{p-1}(G(x, a, a)^p + G(a, y, z)^p) \\ &= 2^{p-1}(G_*(x, a, a) + G_*(a, y, z)). \end{aligned}$$

So G_* is a G_b-metric with $s = 2^{p-1}$.
Also in the above example, (X, G_*) is not necessarily a G-metric space. For example, let $X = \mathbb{R}$ and G-metric G be defined by

$$G(x, y, z) = \frac{1}{3}(|x - y| + |y - z| + |x - z|),$$

for all $x, y, z \in \mathbb{R}$ (see [12]). Then $G_*(x, y, z) = G(x, y, z)^2 = \frac{1}{9}(|x - y| + |y - z| + |x - z|)^2$ is a G_b-metric on $\mathbb{R}$ with $s = 2^{2-1} = 2$, but it is not a G-metric on $\mathbb{R}$. To see this, let $x = 3$, $y = 5$, $z = 7$, $a = \frac{7}{2}$ we get, $G_*(3, 5, 7) = \frac{64}{9}$, $G_*(3, \frac{7}{2}, \frac{7}{2}) = \frac{1}{9}$, $G_*(\frac{7}{2}, 5, 7) = \frac{49}{9}$, so $G_*(3, 5, 7) = \frac{64}{9} \nleq \frac{50}{9} = G_*(3, \frac{7}{2}, \frac{7}{2}) + G_*(\frac{7}{2}, 5, 7)$.
Now we present some definitions and propositions in G_b-metric space.

Definition 1.3. A G_b-metric G is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 1.4. Let (X, G) be a G_b-metric space then for $x_0 \in X$, $r > 0$, the G_b-ball with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X \mid G(x_0, y, y) < r\}.$$

For example, let $X = \mathbb{R}$ and consider the G_b-metric G defined by

$$G(x, y, z) = \frac{1}{9}(|x - y| + |y - z| + |x - z|)^2$$

for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} B_G(3,4) &= \{y \in X \mid G(3,y,y) < 4\} \\ &= \{y \in X \mid \frac{1}{9}(|y-3| + |y-3|)^2 < 4\} \\ &= \{y \in X \mid |y-3|^2 < 9\} \\ &= (0,6). \end{aligned}$$

By some straight forward calculations, we can establish the following.

Proposition 1.5. Let X be a G_b-metric space, then for each $x, y, z, a \in X$ it follows that:
(1) if $G(x, y, z) = 0$ then $x = y = z$,
(2) $G(x, y, z) \leq s(G(x, x, y) + G(x, x, z))$,
(3) $G(x, y, y) \leq 2sG(y, x, x)$,
(4) $G(x, y, z) \leq s(G(x, a, z) + G(a, y, z))$.

Definition 1.6. Let X be a G_b-metric space, we define $d_G(x, y) = G(x, y, y) + G(x, x, y)$, it is easy to see that d_G defines a b-metric on X, which we call it b-metric associated with G.

Proposition 1.7. Let X be a G_b-metric space, then for any $x_0 \in X$ and $r > 0$, if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.
Proof. Let $y \in B_G(x_0, r)$, if $y = x_0$, then we choose $\delta = r$. Suppose that $0 < G(x_0, y, y) < r$, we consider the set, $A = \{n \in \mathbb{N} \mid \frac{r}{4s^{n+2}} < G(x_0, y, y)\}$. By Archmedean property, A is a nonempty set, then by the well ordering principle, A has a least element m. Since $m - 1 \notin A$, we have $G(x_0, y, y) \leq \frac{r}{4s^{m+1}}$. Now if $G(x_0, y, y) = \frac{r}{4s^{m+1}}$, then we choose $\delta = \frac{r}{4s^{m+1}}$, and if $G(x_0, y, y) < \frac{r}{4s^{m+1}}$ we choose, $\delta = \frac{r}{4s^{m+1}} - G(x_0, y, y)$. □

So from the above proposition the family of all G_b-balls

$$\{B_G(x, r) \mid x \in X,\ r > 0\},$$

is a base of a topology $\tau(G)$ on X, which we call it G_b-metric topology.
Now we generalize proposition 5 in [12] for G_b-metric space as follows:

Proposition 1.8. Let X be a G_b-metric space, then for any $x_0 \in X$ and $r > 0$, we have

$$B_G(x_0, \frac{r}{2s+1}) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r).$$

Thus every G_b-metric space is topologically equivalent to a b-metric space. This allows us to readily transport many concepts and results from b-metric spaces into G_b-metric space setting.

Definition 1.9. Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:
(1) G_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;
(2) G_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

Using above definitions, we can easily prove the following two propositions.

Proposition 1.10. Let X be a G_b-metric space, Then the following are equivalent:

(1) the sequence $\{x_n\}$ is G_b-Cauchy.
(2) for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq n_0$.

Proposition 1.11. Let X be a G_b-metric space, The following are equivalent:
(1) $\{x_n\}$ is G_b-convergent to x.
(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$.
(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$.

Definition 1.12. A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

Definition 1.13 ([10]). Two self mappings f and g of a set X are said to be weakly compatible if they commute at their coincidence points; i.e., if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

We also need the following definition:

Definition 1.14. Let X be a nonempty set. Then $(X, G, \preceq)$ is called partially ordered G_b-metric space if G is a G_b-metric on a partially ordered set $(X, \preceq)$.

A subset $\mathcal{K}$ of a partially ordered set X is said to be well ordered if every two elements of $\mathcal{K}$ are comparable.

Definition 1.15 ([2, 3]). Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.

Example 1.16 ([2]). Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = \sqrt[3]{x}$. Since $x \preceq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore f is a dominating map.

Definition 1.17 ([2]). Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominated if $fx \preceq x$ for each x in X.

Example 1.18 ([2]). Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \preceq x$ for all $x \in X$. Therefore f is a dominated map.

2. Common Fixed Point Results

The following is the main result of this section.

Theorem 2.1. Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a symmetric G_b-metric G on X such that (X, G) is a complete G_b-metric space. Also let self-mappings f, g, h, S, T and R on X satisfy the following condition

$$\psi(2s^4 G(fx, gy, hz)) \leq \psi(M_s(x, y, z)) - \varphi(M_s(x, y, z)) \tag{1}$$

for all comparable elements $x, y, z \in X$, where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are two mappings such that ψ is a continuous nondecreasing, φ is a lower semi- continuous function with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$, and

$$\begin{aligned} M_s(x, y, z) = \max\{ & G(Rx, Ty, Sz), G(Rx, Ty, gy), G(Ty, Sz, hz), G(Sz, Rx, fx), \\ & \frac{G(fx, Rx, gy) + G(fx, Sz, hz) + G(gy, Ty, hz)}{3s}\}. \end{aligned}$$

If f, g and h are dominated, S, T and R are dominating with $fX \subseteq TX$, $gX \subseteq SX$ and $hX \subseteq RX$, and for a nonincreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$ and

(a) one of fX, gX or hX is a closed subset of X,

(b) The pair $(f, R), (g, T)$ and (h, S) are weakly compatible,

then f, g, h, S, T and R have a common fixed point in X. Moreover, the set of common fixed points of f, g, h, S, T and R is well ordered if and only if f, g, h, S, T and R have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $fX \subseteq TX$, $gX \subseteq SX$ and $hX \subseteq RX$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$\begin{aligned} y_{3n} &= fx_{3n} = Tx_{3n+1}, \\ y_{3n+1} &= gx_{3n+1} = Sx_{3n+2}, \\ y_{3n+2} &= hx_{3n+2} = Rx_{3n+3}, \quad n = 0, 1, 2, \cdots . \end{aligned}$$

By the given assumptions $x_{3n+1} \preceq Tx_{3n+1} = fx_{3n} \preceq x_{3n}$ and $x_{3n+2} \preceq Sx_{3n+2} = gx_{3n+1} \preceq x_{3n+1}$ and $x_{3n+3} \preceq Rx_{3n+3} = hx_{3n+2} \preceq x_{3n+2}$. Thus, for all $n \geq 1$, we have $x_{n+1} \preceq x_n$. Let $G_m = G(y_m, y_{m+1}, y_{m+2})$. We suppose that $y_{3n} \neq y_{3n+1}$ or $y_{3n+1} \neq y_{3n+2}$, for every n. If not then $y_{3n} = y_{3n+1} = y_{3n+2}$, for some n, so $G_{3n} = 0$ and from (1), we obtain

$$\begin{aligned} \psi(G_{3n+1}) &\leq \psi(2s^4 G_{3n+1}) = \psi(2s^4 G(y_{3n+1}, y_{3n+2}, y_{3n+3})) \\ &= \psi(2s^4 G(y_{3n+3}, y_{3n+1}, y_{3n+2},)) = \psi(2s^4 G(fx_{3n+3}, gx_{3n+1}, hx_{3n+2})) \\ &\leq \psi(M_s(x_{3n+3}, x_{3n+1}, x_{3n+2})) - \varphi(M_s(x_{3n+3}, x_{3n+1}, x_{3n+2})), \end{aligned}$$

where

$$\begin{aligned} M_s(x_{3n+3}, x_{3n+1}, x_{3n+2}) &= \max\{G(Rx_{3n+3}, Tx_{3n+1}, Sx_{3n+2}), G(Rx_{3n+3}, Tx_{3n+1}, gx_{3n+1}), G(Tx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ &\quad G(Sx_{3n+2}, Rx_{3n+3}, fx_{3n+3}), \\ &\quad \frac{G(fx_{3n+3}, Rx_{3n+3}, gx_{3n+1}) + G(fx_{3n+3}, Sx_{3n+2}, hx_{3n+2}) + G(gx_{3n+1}, Tx_{3n+1}, hx_{3n+2})}{3s}\} \\ &= \max\{G(y_{3n+2}, y_{3n}, y_{3n+1}), G(y_{3n+2}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \\ &\quad \frac{G(y_{3n+3}, y_{3n+2}, y_{3n+1}) + G(y_{3n+3}, y_{3n+1}, y_{3n+2}) + G(y_{3n+1}, y_{3n}, y_{3n+2})}{3s}\} \\ &= \max\{G_{3n}, G_{3n}, G_{3n}, G_{3n+1}, \frac{G_{3n+1} + G_{3n+1} + G_{3n}}{3s}\} \\ &= \max\{G_{3n}, G_{3n+1}, \frac{G_{3n} + 2G_{3n+1}}{3s}\} \\ &= \max\{0, G_{3n+1}, \frac{0 + 2G_{3n+1}}{3s}\} \\ &= G_{3n+1}. \end{aligned}$$

Hence, $\psi(G_{3n+1}) \leq \psi(G_{3n+1}) - \varphi(G_{3n+1})$ implies that $G_{3n+1} = 0$, hence $y_{3n+1} = y_{3n+2} = y_{3n+3}$. By a similar argument, we obtain $y_{3n+2} = y_{3n+3} = y_{3n+4}$ and so on. Thus $\{y_n\}$ becomes a constant sequence and by assumption (b), $\{y_{3n}\}$ is the common fixed point of f, g, h, S, T and R. Take $G_n > 0$ for every n. We prove that $\lim_{n\to\infty} G_n = 0$, for this purpose we consider three cases:
If $m = 3n$, then we have

$$\begin{aligned} \psi(G_{3n}) &\leq \psi(2s^4 G_{3n}) = \psi(2s^4 G(y_{3n}, y_{3n+1}, y_{3n+2})) \\ &= \psi(2s^4 G(fx_{3n}, gx_{3n+1}, hx_{3n+2})) \\ &\leq \psi(M_s(x_{3n}, x_{3n+1}, x_{3n+2})) - \varphi(M_s(x_{3n}, x_{3n+1}, x_{3n+2})), \end{aligned} \tag{2}$$

where

$$\begin{aligned}
M_s(x_{3n}, x_{3n+1}, x_{3n+2}) &= \max\{G(Rx_{3n}, Tx_{3n+1}, Sx_{3n+2}), G(Rx_{3n}, Tx_{3n+1}, gx_{3n+1}), G(Tx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\
&\quad G(Sx_{3n+2}, Rx_{3n}, fx_{3n}), \\
&\quad \frac{G(fx_{3n}, Rx_{3n}, gx_{3n+1}) + G(fx_{3n}, Sx_{3n+2}, hx_{3n+2}) + G(gx_{3n+1}, Tx_{3n+1}, hx_{3n+2})}{3s}\} \\
&= \max\{G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n-1}, y_{3n}), \\
&\quad \frac{G(y_{3n}, y_{3n-1}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2}) + G(y_{3n+1}, y_{3n}, y_{3n+2})}{3s}\} \\
&= \max\{G_{3n-1}, G_{3n-1}, G_{3n}, G_{3n-1}, \frac{G_{3n-1} + G_{3n} + G_{3n}}{3s}\} \\
&= \max\{G_{3n-1}, G_{3n}, \frac{G_{3n-1} + 2G_{3n}}{3s}\}.
\end{aligned}$$

We prove that $G_{3n} \leq G_{3n-1}$, for each $n \in \mathbb{N}$. If $G_{3n} > G_{3n-1}$ for some $n \in \mathbb{N}$, then for each $s \geq 1$ we have

$$G_{3n} > G_{3n-1} \geq \frac{1}{3s-2} G_{3n-1} \Longrightarrow G_{3n} > \frac{G_{3n-1} + 2G_{3n}}{3s}.$$

So $M_s(x_{3n}, x_{3n+1}, x_{3n+2}) = G_{3n}$ and from (2) we have

$$\psi(G_{3n}) \leq \psi(G_{3n}) - \varphi(G_{3n}),$$

and so $\varphi(G_{3n}) \leq 0$ which implies that $G_{3n} = 0$, a contradiction to $G_{3n} > 0$. Now, if $m = 3n + 1$, then

$$\begin{aligned}
\psi(G_{3n+1}) &\leq \psi(2s^4 G_{3n+1}) = \psi(2s^4 G(y_{3n+1}, y_{3n+2}, y_{3n+3})) \\
&= \psi(2s^4 G(y_{3n+3}, y_{3n+1}, y_{3n+2},)) = \psi(2s^4 G(fx_{3n+3}, gx_{3n+1}, hx_{3n+2})) \\
&\leq \psi(M_s(x_{3n+3}, x_{3n+1}, x_{3n+2})) - \varphi(M_s(x_{3n+3}, x_{3n+1}, x_{3n+2})), \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}
M_s(x_{3n+3}, x_{3n+1}, x_{3n+2}) &= \max\{G(Rx_{3n+3}, Tx_{3n+1}, Sx_{3n+2}), G(Rx_{3n+3}, Tx_{3n+1}, gx_{3n+1}), G(Tx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\
&\quad G(Sx_{3n+2}, Rx_{3n+3}, fx_{3n+3}), \\
&\quad \frac{G(fx_{3n+3}, Rx_{3n+3}, gx_{3n+1}) + G(fx_{3n+3}, Sx_{3n+2}, hx_{3n+2}) + G(gx_{3n+1}, Tx_{3n+1}, hx_{3n+2})}{3s}\} \\
&= \max\{G(y_{3n+2}, y_{3n}, y_{3n+1}), G(y_{3n+2}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n+2}, y_{3n+3}), \\
&\quad \frac{G(y_{3n+3}, y_{3n+2}, y_{3n+1}) + G(y_{3n+3}, y_{3n+1}, y_{3n+2}) + G(y_{3n+1}, y_{3n}, y_{3n+2})}{3s}\} \\
&= \max\{G_{3n}, G_{3n}, G_{3n}, G_{3n+1}, \frac{G_{3n+1} + G_{3n+1} + G_{3n}}{3s}\} \\
&= \max\{G_{3n}, G_{3n+1}, \frac{G_{3n} + 2G_{3n+1}}{3s}\}.
\end{aligned}$$

Similarly, if $G_{3n+1} > G_{3n}$ for some $n \in \mathbb{N}$, then $M_s(x_{3n+3}, x_{3n+1}, x_{3n+2}) = G_{3n+1}$ and from (3) we have

$$\psi(G_{3n+1}) \leq \psi(G_{3n+1}) - \varphi(G_{3n+1}),$$

and so $\varphi(G_{3n+1}) \leq 0$ which implies that $G_{3n+1} = 0$, a contradiction to $G_{3n+1} > 0$. If $m = 3n + 2$, then

$$\begin{aligned}\psi(G_{3n+2}) &\leq \psi(2s^4 G_{3n+2}) = \psi(2s^4 G(y_{3n+2}, y_{3n+3}, y_{3n+4}))\\ &= \psi(2s^4 G(y_{3n+3}, y_{3n+4}, y_{3n+2},)) = \psi(2s^4 G(fx_{3n+3}, gx_{3n+4}, hx_{3n+2}))\\ &\leq \psi(M_s(x_{3n+3}, x_{3n+4}, x_{3n+2})) - \varphi(M_s(x_{3n+3}, x_{3n+4}, x_{3n+2})), \end{aligned} \tag{4}$$

where

$$\begin{aligned} M_s(x_{3n+3}, x_{3n+4}, x_{3n+2}) &= \max\{G(Rx_{3n+3}, Tx_{3n+4}, Sx_{3n+2}), G(Rx_{3n+3}, Tx_{3n+4}, gx_{3n+4}), G(Tx_{3n+4}, Sx_{3n+2}, hx_{3n+2}),\\ &\quad G(Sx_{3n+2}, Rx_{3n+3}, fx_{3n+3}),\\ &\quad \frac{G(fx_{3n+3}, Rx_{3n+3}, gx_{3n+4}) + G(fx_{3n+3}, Sx_{3n+2}, hx_{3n+2}) + G(gx_{3n+4}, Tx_{3n+4}, hx_{3n+2})}{3s}\}\\ &= \max\{G(y_{3n+2}, y_{3n+3}, y_{3n+1}), G(y_{3n+2}, y_{3n+3}, y_{3n+4}), G(y_{3n+3}, y_{3n+1}, y_{3n+2}),\\ &\quad G(y_{3n+1}, y_{3n+2}, y_{3n+3}),\\ &\quad \frac{G(y_{3n+3}, y_{3n+2}, y_{3n+4}) + G(y_{3n+3}, y_{3n+1}, y_{3n+2}) + G(y_{3n+4}, y_{3n+3}, y_{3n+2})}{3s}\}\\ &= \max\{G_{3n+1}, G_{3n+2}, G_{3n+1}, G_{3n+1}, \frac{G_{3n+2} + G_{3n+1} + G_{3n+2}}{3s}\}\\ &= \max\{G_{3n+1}, G_{3n+2}, \frac{G_{3n+1} + 2G_{3n+2}}{3s}\}. \end{aligned}$$

Similarly, if $G_{3n+2} > G_{3n+1}$ for some $n \in \mathbb{N}$, then $M_s(x_{3n+3}, x_{3n+4}, x_{3n+2}) = G_{3n+2}$ and from (4) we have

$$\psi(G_{3n+2}) \leq \psi(G_{3n+2}) - \varphi(G_{3n+2}),$$

and so $\varphi(G_{3n+2}) \leq 0$ which implies that $G_{3n+2} = 0$, a contradiction to $G_{3n+2} > 0$. Hence for each $n \in \mathbb{N}$ we have $0 < G_n \leq G_{n-1}$. Thus the sequence $\{G_n\}$ is nonincreasing and so there exists $\lim_{n\to\infty} G_n = r \geq 0$. Also we have

$$M_s(p\{x_n, x_{n+1}, x_{n+2}\}) = \max\{G_{n-1}, G_n, \frac{G_{n-1} + 2G_n}{3s}\}.$$

Taking the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} M_s(p\{x_n, x_{n+1}, x_{n+2}\}) = \max\{r, r, \frac{r + 2r}{3s}\} = r.$$

Suppose that $r > 0$. Then

$$\begin{aligned}\psi(G_n) &\leq \psi(2s^4 G_n) = \psi(2s^4 G(y_n, y_{n+1}, y_{n+2})) = \psi(2s^4 G(fx_n, gx_{n+1}, hx_{n+2}))\\ &\leq \psi(M_s(x_n, x_{n+1}, x_{n+2})) - \varphi(M_s(x_n, x_{n+1}, x_{n+2})).\end{aligned}$$

So taking the upper limit as $n \to \infty$ implies that

$$\begin{aligned}\psi(r) &\leq \psi(r) - \liminf_{n\to\infty} \varphi(M_s(x_n, x_{2n+1}, x_{n+2}))\\ &= \psi(r) - \varphi(\liminf_{n\to\infty} M_s(x_n, x_{2n+1}, x_{n+2}))\\ &= \psi(r) - \varphi(r),\end{aligned}$$

a contradiction. Hence

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+2}) = 0. \tag{5}$$

Since $y_n \neq y_{n+1}$ or $y_{n+1} \neq y_{n+2}$ for every n, so by property (G_b3) we obtain

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+2}).$$

Hence

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0. \tag{6}$$

Also, since G is symmetric we have

$$\lim_{n\to\infty} G(y_{n+1}, y_n, y_n) = 0. \tag{7}$$

Now we prove that $\{y_n\}$ is a Cauchy sequence. For this it is sufficient to show that a subsequence $\{y_{3n}\}$ is Cauchy in X. Assume on contrary that $\{y_{3n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{3m_k}\}$ and $\{y_{3n_k}\}$ of $\{y_{3n}\}$ such that m_k is the smallest index for which $3m_k > 3n_k > k$,

$$G(y_{3n_k}, y_{3m_k}, y_{3m_k}) \geq \varepsilon \tag{8}$$

and

$$\begin{aligned} G(y_{3n_k}, y_{3m_k-\alpha}, y_{3m_k-\beta}) &< \varepsilon, \qquad \alpha, \beta \in \{1, 2\}, && (9)\\ G(y_{3n_k-1}, y_{3m_k-\gamma}, y_{3m_k-\delta}) &< \varepsilon. \qquad \gamma, \delta \in \{0, 1, 2, 3\}. && (10)\end{aligned}$$

Now, from (1), we have

$$\begin{aligned} \psi(G(y_{3n_k}, y_{3m_k-2}, y_{3m_k-1})) &\leq \psi(2s^4 G(y_{3n_k}, y_{3m_k-2}, y_{3m_k-1})) \\ &= \psi(2s^4 G(fx_{3n_k}, gx_{3m_k-2}, hx_{3m_k-1})) \\ &\leq \psi(M_s(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1})) - \varphi(M_s(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1})), \end{aligned} \tag{11}$$

where

$$\begin{aligned} M_s(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) &= \max\{G(Rx_{3n_k}, Tx_{3m_k-2}, Sx_{3m_k-1}), G(Rx_{3n_k}, Tx_{3m_k-2}, gx_{3m_k-2}), \\ &\quad G(Tx_{3m_k-2}, Sx_{3m_k-1}, hx_{3m_k-1}), G(Sx_{3m_k-1}, Rx_{3n_k}, fx_{3n_k}), \\ &\quad \frac{G(fx_{3n_k}, Rx_{3n_k}, gx_{3m_k-2}) + G(fx_{3n_k}, Sx_{3m_k-1}, hx_{3m_k-1}) + G(gx_{3m_k-2}, Tx_{3m_k-2}, hx_{3m_k-1})}{3s}\} \\ &= \max\{G(y_{3n_k-1}, y_{3m_k-3}, y_{3m_k-2}), G(y_{3n_k-1}, y_{3m_k-3}, y_{3m_k-2}), G(y_{3m_k-3}, y_{3m_k-2}, y_{3m_k-1}), \\ &\quad G(y_{3m_k-2}, y_{3n_k-1}, y_{3n_k}), \\ &\quad \frac{G(y_{3n_k}, y_{3n_k-1}, y_{3m_k-2}) + G(y_{3n_k}, y_{3m_k-2}, y_{3m_k-1}) + G(y_{3m_k-2}, y_{3m_k-3}, y_{3m_k-1})}{3s}\} \\ &= \max\{G(y_{3n_k-1}, y_{3m_k-3}, y_{3m_k-2}), G_{3m_k-1}, G(y_{3m_k-2}, y_{3n_k-1}, y_{3n_k}), \\ &\quad \frac{G(y_{3m_k-2}, y_{3n_k-1}, y_{3n_k}) + G(y_{3n_k}, y_{3m_k-2}, y_{3m_k-1}) + G_{3m_k-3}}{3s}\}. \end{aligned} \tag{12}$$

Using (8) and (G_b4)-(G_b5), we obtain that

$$\begin{aligned}\varepsilon &\le G(y_{3n_k}, y_{3m_k}, y_{3m_k})\\ &\le sG(y_{3n_k}, y_{3m_k-1}, y_{3m_k-1}) + sG(y_{3m_k-1}, y_{3m_k}, y_{3m_k})\\ &\le s^2G(y_{3m_k-1}, y_{3m_k-2}, y_{3m_k-2}) + s^2G(y_{3m_k-2}, y_{3m_k-1}, y_{3n_k}) + sG(y_{3m_k-1}, y_{3m_k}, y_{3m_k}).\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (6), (7) and (9) we obtain

$$\frac{\varepsilon}{s^2} \le \limsup_{k\to\infty} G(y_{3n_k}, y_{3m_k-2}, y_{3m_k-1}) \le \varepsilon. \tag{13}$$

Also

$$\begin{aligned}\varepsilon &\le G(y_{3n_k}, y_{3m_k}, y_{3m_k}) \le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + sG(y_{3n_k-1}, y_{3m_k}, y_{3m_k})\\ &\le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + s^2G(y_{3n_k-1}, y_{3m_k-1}, y_{3m_k-1}) + s^2G(y_{3m_k-1}, y_{3m_k}, y_{3m_k})\\ &\le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + s^3G(y_{3n_k-1}, y_{3m_k-2}, y_{3m_k-2}) + s^3G(y_{3m_k-2}, y_{3m_k-1}, y_{3m_k-1}) + s^2G(y_{3m_k-1}, y_{3m_k}, y_{3m_k})\\ &\le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + s^4G(y_{3m_k-2}, y_{3m_k-3}, y_{3m_k-3}) + s^4G(y_{3m_k-3}, y_{3m_k-2}, y_{3n_k-1})\\ &\quad + s^3G(y_{3m_k-2}, y_{3m_k-1}, y_{3m_k-1}) + s^2G(y_{3m_k-1}, y_{3m_k}, y_{3m_k}).\end{aligned}$$

So from (6), (7) and (10) we get

$$\frac{\varepsilon}{s^4} \le \limsup_{k\to\infty} G(y_{3n_k-1}, y_{3m_k-3}, y_{3m_k-2}) < \epsilon. \tag{14}$$

Moreover, by the symmetrically of G we have

$$\begin{aligned}\varepsilon &\le G(y_{3m_k}, y_{3m_k}, y_{3n_k}) = G(y_{3n_k}, y_{3n_k}, y_{3m_k})\\ &\le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + sG(y_{3n_k-1}, y_{3n_k}, y_{3m_k})\\ &\le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + s^2G(y_{3m_k}, y_{3m_k-2}, y_{3m_k-2}) + s^2G(y_{3n_k-1}, y_{3n_k}, y_{3m_k-2})\\ &\le sG(y_{3n_k}, y_{3n_k-1}, y_{3n_k-1}) + s^3G(y_{3m_k}, y_{3m_k-1}, y_{3m_k-1}) + s^3G(y_{3m_k-1}, y_{3m_k-2}, y_{3m_k-2})\\ &\quad + s^2G(y_{3n_k-1}, y_{3n_k}, y_{3m_k-2}).\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (7) we obtain

$$\frac{\varepsilon}{s^2} \le \lim_{k\to\infty} \sup G(y_{3n_k-1}, y_{3n_k}, y_{3m_k-2}).$$

On the other hand

$$\begin{aligned}G(y_{3n_k-1}, y_{3n_k}, y_{3m_k-2}) &\le sG(y_{3m_k-2}, y_{3m_k-1}, y_{3m_k-1}) + sG(y_{3m_k-1}, y_{3n_k-1}, y_{3n_k})\\ &\le sG(y_{3m_k-2}, y_{3m_k-1}, y_{3m_k-1}) + s^2G(y_{3n_k}, y_{3m_k-1}, y_{3m_k-1})\\ &\quad + s^2G(y_{3n_k-1}, y_{3m_k-1}, y_{3m_k-1}).\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (6), (9) and (10) we get

$$\lim_{k\to\infty} \sup G(y_{3n_k-1}, y_{3n_k}, y_{3m_k-2}) \le 2\epsilon s^2.$$

Consequently,

$$\frac{\varepsilon}{s^2} \le \lim_{k\to\infty} \sup G(y_{3n_k-1}, y_{3n_k}, y_{3m_k-2}) \le 2\epsilon s^2. \tag{15}$$

Now from (12) and using (13), (14) and (15) we get

$$\begin{aligned}
\min\{\frac{\epsilon}{s^4},\frac{\epsilon}{s^2},\frac{\frac{\epsilon}{s^2}+\frac{\epsilon}{s^2}}{3s}\} &\leq \lim_{k\to\infty}\sup M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1})\\
&= \max\{\lim_{k\to\infty}\sup G(y_{3n_k-1},y_{3m_k-3},y_{3m_k-2}),\ \lim_{k\to\infty}\sup G_{3m_k-1},\\
&\lim_{k\to\infty}\sup G(y_{3m_k-2},y_{3n_k-1},y_{3n_k}),\\
&\frac{\lim_{k\to\infty}\sup G(y_{3m_k-2},y_{3n_k-1},y_{3n_k})+\lim_{k\to\infty}\sup G(y_{3n_k},y_{3m_k-2},y_{3m_k-1}) + \lim_{k\to\infty}\sup G_{3m_k-3}}{3s}\}\\
&= \max\{\lim_{k\to\infty}\sup G(y_{3n_k-1},y_{3m_k-3},y_{3m_k-2}),\ 0,\ \lim_{k\to\infty}\sup G(y_{3m_k-2},y_{3n_k-1},y_{3n_k}),\\
&\frac{\lim_{k\to\infty}\sup G(y_{3m_k-2},y_{3n_k-1},y_{3n_k})+\lim_{k\to\infty}\sup G(y_{3n_k},y_{3m_k-2},y_{3m_k-1})+0}{3s}\}\\
&\leq \max\{\epsilon,2\epsilon s^2,\frac{2\epsilon s^2+\epsilon}{3s}\}=2\epsilon s^2.
\end{aligned}$$

So

$$0<\min\{\frac{\epsilon}{s^4},\frac{\epsilon}{s^2},\frac{2\epsilon}{3s^3}\}\leq \lim_{k\to\infty}\sup M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1})\leq 2\epsilon s^2. \tag{16}$$

Similarly we can obtain

$$0<\min\{\frac{\epsilon}{s^4},\frac{\epsilon}{s^2},\frac{2\epsilon}{3s^3}\}\leq \lim_{k\to\infty}\inf M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1})\leq 2\epsilon s^2. \tag{17}$$

Taking the upper limit as $k\to\infty$, in (11) and using (13) and (16) we obtain

$$\begin{aligned}
\psi(2\epsilon s^2) &\leq \psi(2s^4\lim_{k\to\infty}\sup G(y_{3n_k},y_{3m_k-2},y_{3m_k-1}))\\
&= \psi(2s^4\lim_{k\to\infty}\sup G(fx_{3n_k},gx_{3m_k-2},hx_{3m_k-1}))\\
&\leq \psi(\lim_{k\to\infty}\sup M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1}))-\lim_{k\to\infty}\inf\varphi(M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1}))\\
&\leq \psi(2\epsilon s^2)-\varphi(\lim_{k\to\infty}\inf M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1})),
\end{aligned}$$

which implies that

$$\varphi(\lim_{k\to\infty}\inf M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1}))=0,$$

so $\lim\inf M_s(x_{3n_k},x_{3m_k-2},x_{3m_k-1})=0$, a contradiction to (17). It follows that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $y\in X$ such that

$$\begin{aligned}
\lim_{n\to\infty}y_n &= \lim_{n\to\infty}fx_{3n}=\lim_{n\to\infty}gx_{3n+1}=\lim_{n\to\infty}hx_{3n+2}\\
&= \lim_{n\to\infty}Tx_{3n+1}=\lim_{n\to\infty}Rx_{3n+3}=\lim_{n\to\infty}Sx_{3n+2}=y.
\end{aligned}$$

Now, we show that y is a common fixed point of f,g,h,S,T and R.
Let hX be a closed subset X , since $hX\subseteq RX$, so there exist $u\in X$ such that $Ru=y$.
We prove that $fu=y$. since $hx_{3n+2}\preceq x_{3n+2}$ and $hx_{3n+2}\to y$ as $n\to\infty$, $y\preceq x_{3n+2}$ and $u\preceq Ru=y\preceq x_{3n+2}\preceq x_{3n+1}$, so from (1) we obtain

$$\begin{aligned}
\psi(G(fu,gx_{3n+1},hx_{3n+2})) &\leq \psi(2s^4G(fu,gx_{3n+1},hx_{3n+2}))\\
&\leq \psi(M_s(u,x_{3n+1},x_{3n+2}))-\varphi(M_s(u,x_{3n+1},x_{3n+2})),
\end{aligned}\tag{18}$$

where

$$\begin{aligned} M_s(u, x_{3n+1}, x_{3n+2}) &= \max\{G(Ru, Tx_{3n+1}, Sx_{3n+2}), G(Ru, Tx_{3n+1}, gx_{3n+1}), G(Tx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ &\quad G(Sx_{3n+2}, Ru, fu), \\ &\quad \frac{G(fu, Ru, gx_{3n+1}) + G(fu, Sx_{3n+2}, hx_{3n+2}) + G(gx_{3n+1}, Tx_{3n+1}, hx_{3n+2})}{3s}\}. \end{aligned}$$

But

$$\begin{aligned} \lim_{n\to\infty} M_s(u, x_{3n+1}, x_{3n+2}) &= \max\{G(Ru, y, y), G(Ru, y, y), G(y, y, y), G(y, Ru, fu), \\ &\quad \frac{G(fu, Ru, y) + G(fu, y, y) + G(y, y, y)}{3s}\} \\ &= \max\{G(fu, y, y), \frac{2G(fu, y, y)}{3s}\} \\ &= G(fu, y, y). \end{aligned}$$

Now by taking the upper limit as $n \to \infty$ in (18) we get

$$\psi(G(fu, y, y)) \le \psi(G(fu, y, y)) - \varphi(G(fu, y, y)),$$

and $\varphi(G(fu, y, y)) \le 0$ or equivalently $G(fu, y, y) = 0$ and by (1) of proposition 1.5 $fu = y$. Since the pair (R, f) is weakly compatible we have $fRu = Rfu$. Hence $fy = Ry$. We prove that $fy = y$, if $fy \ne y$, then from (1) we have

$$\begin{aligned} \psi(G(fy, gx_{3n+1}, hx_{3n+2})) &\le \psi(2s^4 G(fy, gx_{3n+1}, hx_{3n+2})) \\ &\le \psi(M_s(y, x_{3n+1}, x_{3n+2})) - \varphi(M_s(y, x_{3n+1}, x_{3n+2})), \end{aligned} \tag{19}$$

where

$$\begin{aligned} M_s(y, x_{3n+1}, x_{3n+2}) &= \max\{G(Ry, Tx_{3n+1}, Sx_{3n+2}), G(Ry, Tx_{3n+1}, gx_{3n+1}), G(Tx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ &\quad G(Sx_{3n+2}, Ry, fy), \\ &\quad \frac{G(fy, Ry, gx_{3n+1}) + G(fy, Sx_{3n+2}, hx_{3n+2}) + G(gx_{3n+1}, Tx_{3n+1}, hx_{3n+2})}{3s}\}. \end{aligned}$$

Consequently

$$\begin{aligned} \lim_{n\to\infty} M_s(y, x_{3n+1}, x_{3n+2}) &= \max\{G(Ry, y, y), G(Ry, y, y), G(y, y, y), G(y, Ry, fy), \\ &\quad \frac{G(fy, Ry, y) + G(fy, y, y) + G(y, y, y)}{3s}\}, \\ &= \max\{G(fy, y, y),\ G(y, fy, fy), \frac{G(fy, fy, y) + G(fy, y, y)}{3s}\} \\ &= \max\{G(fy, y, y),\ \frac{2G(fy, y, y)}{3s}\} \\ &= G(fy, y, y). \end{aligned}$$

As $n \to \infty$ in (19) we obtain

$$\psi(G(fy,y,y)) \leq \psi(G(fy,y,y)) - \varphi(G(fy,y,y)),$$

a contradiction. Therefore $Ry = fy = y$, that is, y is a common fixed point of R, f. Since $y = fy \in fX \subseteq TX$, hence there exists $v \in X$ such that $Tv = y$. Now we show that $gv = y$. Since $v \preceq Tv = y \preceq x_{3n+2}$, hence from (1) we have

$$\begin{aligned} \psi(G(y,gv,hx_{3n+2})) &= \psi(G(fy,gv,hx_{3n+2})) \leq \psi(2s^4G(fy,gv,hx_{3n+2})) \\ &\leq \psi(M_s(y,v,x_{3n+2})) - \varphi(M_s(y,v,x_{3n+2})), \end{aligned} \tag{20}$$

where

$$\begin{aligned} M_s(y,v,x_{3n+2}) &= \max\{G(Ry,Tv,Sx_{3n+2}), G(Ry,Tv,gv), G(Tv,Sx_{3n+2},hx_{3n+2}), \\ &\quad G(Sx_{3n+2},Ry,fy), \\ &\quad \frac{G(fy,Ry,gv) + G(fy,Sx_{3n+2},hx_{3n+2}) + G(gv,Tv,hx_{3n+2})}{3s}\} \\ &= \max\{G(y,y,Sx_{3n+2}), G(y,y,gv), G(y,Sx_{3n+2},hx_{3n+2}), G(Sx_{3n+2},y,y), \\ &\quad \frac{G(y,y,gv) + G(y,Sx_{3n+2},hx_{3n+2}) + G(gv,y,hx_{3n+2})}{3s}\}, \end{aligned}$$

and

$$\lim_{n\to\infty} M_s(y,v,x_{3n+2}) = G(y,y,gv).$$

Taking the upper limit as $n \to \infty$ in (20) we obtain

$$\psi(G(y,y,gv)) \leq \psi(G(y,y,gv)) - \varphi(G(y,y,gv)).$$

Thus $gv = y$. By the weakly compatibility of the pair (g,T) we have $Tgv = gTv$.
Hence $Ty = gy$. We prove that $gy = y$, if $gy \neq y$, then from (1) we have

$$\begin{aligned} \psi(G(fy,gy,hx_{3n+2})) &\leq \psi(2s^4G(fy,gy,hx_{3n+2})) \\ &\leq \psi(M_s(y,y,x_{3n+2})) - \varphi(M_s(y,y,x_{3n+2})), \end{aligned} \tag{21}$$

where

$$\begin{aligned} M_s(y,y,x_{3n+2}) &= \max\{G(Ry,Ty,Sx_{3n+2}), G(Ry,Ty,gy), G(Ty,Sx_{3n+2},hx_{3n+2}), \\ &\quad G(Sx_{3n+2},Ry,fy), \\ &\quad \frac{G(fy,Ry,gy) + G(fy,Sx_{3n+2},hx_{3n+2}) + G(gy,Ty,hx_{3n+2})}{3s}\} \\ &= \max\{G(y,gy,Sx_{3n+2}), G(y,gy,gy), G(gy,Sx_{3n+2},hx_{3n+2}), G(Sx_{3n+2},y,y), \\ &\quad \frac{G(y,y,gy) + G(y,Sx_{3n+2},hx_{3n+2}) + G(gy,gy,hx_{3n+2})}{3s}\}. \end{aligned}$$

Taking the limit as $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} M_s(y,y,x_{3n+2}) &= \max\{G(y,gy,y),G(y,gy,gy),G(gy,y,y),G(y,y,y),\\ &\quad \frac{G(y,y,gy)+G(y,y,y)+G(gy,gy,y)}{3s}\}\\ &= \max\{G(y,y,gy),\frac{2G(y,y,gy)}{3s}\}\\ &= G(y,y,gy).\end{aligned}$$

Taking the upper limit as $n\to\infty$ in (21) we obtain

$$\psi(G(y,gy,y)) \le \psi(G(y,y,gy)) - \varphi(G(y,y,gy)),$$

a contradiction. Therefore, $gy = Ty = y$, that is, y is a common fixed point of g, T.
Similarly, since $y = gy \in gX \subseteq SX$, hence there exists $w \in X$ such that $Sw = y$. We prove that $hw = y$. Since $w \preceq Sw = y$, so from (1) we get

$$\begin{aligned}\psi(G(y,y,hw)) &= \psi(G(fy,gy,hw)) \le \psi(2s^4G(fy,gy,hw))\\ &\le \psi(M_s(y,y,w)) - \varphi(M_s(y,y,w))\\ &= \psi(G(y,y,hw)) - \varphi(G(y,y,hw)).\end{aligned}$$

Thus $hw = y$. Since the pair (h,S) is weakly compatible we have $hSw = Shw$, hence $hy = Sy$. We prove that $hy = y$, if $hy \neq y$, then

$$\begin{aligned}\psi(G(y,y,hy)) &= \psi(G(fy,gy,hy)) \le \psi(2s^4G(fy,gy,hy))\\ &\le \psi(M_s(y,y,y)) - \varphi(M_s(y,y,y))\\ &= \psi(G(y,y,hy)) - \varphi(G(y,y,hy)).\end{aligned}$$

A contradiction to $hy \neq y$. therefore, $hy = Sy = y$, that is, y is a common fixed point of h, S. Thus $fy = gy = hy = Sy = Ty = Ry = y$. Similarly, if fX or gX is closed then result follows.
Now suppose that the set of common fixed points of f, g, h, S, T and R is well ordered. We show that common fixed point of f, g, h, S, T and R is unique. Assume on contrary that z is another fixed point of f, g, h, S, T and R i.e., $fz = gz = hz = Sz = Tz = Rz = z$ such that $y \neq z$. Then by our assumption, we apply (1) to obtain

$$\begin{aligned}\psi(G(y,y,z)) &= \psi(G(fy,gy,hz))\\ &\le \psi(2s^4G(fy,gy,hz)) \le \psi(M_s(y,y,z)) - \varphi(M_s(y,y,z)),\end{aligned}$$

where

$$\begin{aligned}M_s(y,y,z) &= \max\{G(Ry,Ty,Sz),G(Ry,Ty,gy),G(Ty,Sz,hz),G(Sz,Ry,fy),\\ &\quad \frac{G(fy,Ry,gy)+G(fy,Sz,hz)+G(gy,Ty,hz)}{3s}\}\\ &= \max\{G(y,y,z),G(y,z,z),\frac{G(y,z,z)+G(y,y,z)}{3s}\}\\ &= \max\{G(y,y,z),\frac{2G(y,z,z)}{3s}\}\\ &= G(y,y,z).\end{aligned}$$

Hence

$$\psi(G(y,y,z)) \le \psi(G(y,y,z)) - \varphi(G(y,y,z)),$$

which implies that $G(y,y,z) = 0$, a contradiction to $G(y,y,z) > 0$. Therefore $y = z$. Converse is obvious. Now we give an example to support our result.

Example 2.2. Let $X = [0,\infty)$ be endowed with the usual ordering on $\mathbb{R}$ and Gb-metric on X be given by $G(x,y,z) = \frac{1}{9}(|x-y| + |y-z| + |x-z|)^2$, where $s = 2$. Since $G(x,y,y) = G(y,x,x) = \frac{4}{9}|x-y|^2$, for all $x,y \in X$, so G is symmetric. Define self-maps f,g,h,S,T and R on X by

$$\begin{aligned} f(x) &= \ln(x+1), & S(x) &= e^{2x} - 1,\\ g(x) &= \ln(\tfrac{x}{2}+1), & T(x) &= e^{3x} - 1,\\ h(x) &= \ln(\tfrac{x}{3}+1), & R(x) &= e^{6x} - 1. \end{aligned}$$

For each $x \in X$, we have $1 + x \le e^x$, and $1 + \frac{x}{2} \le e^x$ and $1 + \frac{x}{3} \le e^x$, hence $f(x) = \ln(x+1) \le x$ and $g(x) = \ln(\frac{x}{2}+1) \le x$ and $h(x) = \ln(\frac{x}{3}+1) \le x$. So f,g,h are dominated maps.
Also for each $x \in X$, we have $x \le e^{2x} - 1 = S(x)$ and $x \le e^{3x} - 1 = T(x)$ and $x \le e^{6x} - 1 = R(x)$, so S,T and R are dominating maps. Furthermore $fX = TX = gX = SX = hX = RX = [0,\infty)$ and the pair $(f,R),(g,T)$ and (h,S) are weakly compatible.
Define control functions as $\psi,\varphi : [0,\infty) \to [0,\infty)$, $\psi(t) = bt$ and $\varphi(t) = (b-1)t$, $1 < b \le \frac{9}{8}$ for all $t \in [0,\infty)$.
Now we show that f,g,h,S,T and R satisfy (1). Using the mean value theorem we have

$$\begin{aligned} \psi(2s^4G(fx,gy,hz)) &= \frac{32b}{9}(|f(x)-g(y)| + |f(x)-h(z)| + |g(y)-h(z)|)^2\\ &= \frac{32b}{9}\left(\left|\ln(x+1) - \ln(\tfrac{y}{2}+1)\right| + \left|\ln(x+1) - \ln(\tfrac{z}{3}+1)\right| + \left|\ln(\tfrac{y}{2}+1) - \ln(\tfrac{z}{3}+1)\right|\right)^2\\ &\le \frac{32b}{9}\left(\frac{1}{2}|2x-y| + \frac{1}{3}|3x-z| + \frac{1}{6}|3y-2z|\right)^2\\ &= \frac{32b}{9}\,\frac{(|6x-3y| + |6x-2z| + |3y-2z|)^2}{36}\\ &\le \frac{8b}{81}(|e^{6x}-e^{3y}| + |e^{6x}-e^{2z}| + |e^{3y}-e^{2z}|)^2\\ &\le \frac{1}{9}(|R(x)-T(y)| + |R(x)-S(z)| + |T(y)-S(z)|)^2\\ &= G(Rx,Ty,Sz) \le M_s(x,y,z)\\ &= \psi(M_s(x,y,z)) - \varphi(M_s(x,y,z)). \end{aligned}$$

Thus (1) is satisfied for all $x,y,z \in X$. Therefore all condition of Theorem 2.1 are satisfied. Moreover, 0 is a unique common fixed point of f,g,h,S,T and R. □

Corollary 2.3. Let $(X,\preceq)$ be a partially ordered set. Suppose that there exists a symmetric G_b-metric G on X such that (X,G) is a complete G_b-metric space. Also S,T and R be surjective self-maps on X satisfy the following condition

$$\psi(2s^4G(x,y,z)) \le \psi(M_s(x,y,z)) - \varphi(M_s(x,y,z)),$$

for all comparable elements $x, y, z \in X$, where $\varphi, \psi : [0,\infty) \to [0,\infty)$ are two mappings such that ψ is a continuous nondecreasing, φ is a lower semi- continuous function with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$, and

$$\begin{aligned} M_s(x,y,z) &= \max\{G(Rx,Ty,Sz), G(Rx,Ty,y), G(Ty,Sz,z), G(Sz,Rx,x), \\ &\quad \frac{G(x,Rx,y) + G(x,Sz,z) + G(y,Ty,z)}{3s}\}. \end{aligned}$$

If S, T and R are dominating and for a nonincreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x$, then S, T and R have a common fixed point in X. Moreover, the set of common fixed points of S, T and R is well ordered if and only if S, T and R have one and only one common fixed point.

Proof. Taking f, g and h as identity maps on X, the result follows from Theorem 2.1. □

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