Abelian Cayley graphs of given degree and diameter 2 and 3

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Abstract

Let $CC_{d,k}$ be the largest possible number of vertices in a cyclic Cayley graph of degree d and diameter k, and let $AC_{d,k}$ be the largest order in an Abelian Cayley graph for given d and k. We show that $CC_{d,2} \geq \frac{13}{36}(d+2)(d-4)$ for any $d = 6p - 2$ where p is a prime such that $p \neq 13$, $p \not\equiv 1 \pmod{13}$, and $AC_{d,3} \geq \frac{9}{128}(d+3)^2(d-5)$ for $d = 8q - 3$ where q is a prime power.

Keywords: Cayley graph, degree, diameter, cyclic group, Abelian group

Let G be a group and let X be a set of generators for this group. Then the vertices of a Cayley graph $C(G, X)$ are the elements of G and there is an edge between two vertices u and v in $C(G, X)$ if and only if there is a generator $a \in X$ such that $v = ua$. In this paper we consider undirected Cayley graphs, hence if $a \in X$, then a^{-1} is also in X. If G is an Abelian group (a cyclic group), then the graph $C(G, X)$ will be called an Abelian Cayley graph (a cyclic Cayley graph).

Let $AC_{d,k}$ be the maximum number of vertices in an Abelian Cayley graph of degree d and diameter k, and let $CC_{d,k}$ be the largest order of a cyclic Cayley graph of degree d and diameter k. The degree-diameter problem for Abelian Cayley graphs and cyclic Cayley graphs is to determine or bound $AC_{d,k}$ and $CC_{d,k}$ for given d and k. It is well-known that the number of vertices in a graph of degree d and diameter k can not exceed the Moore bound $1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$. However there is much better upper bound for Abelian Cayley graphs. For $d \to \infty$ and fixed k we have $CC_{d,k} \leq AC_{d,k} \leq \frac{d^k}{k!} + O(d^{k-1})$; see [7]. Thus $AC_{d,2} \leq \frac{d^2}{2} + O(d)$ and $AC_{d,3} \leq \frac{d^3}{6} + O(d^2)$. On the other hand it is easy to construct Abelian Cayley graphs of order $(\frac{d}{k})^k + O(d^{k-1})$, hence for $d \to \infty$ and fixed k we have $AC_{d,k} \geq (\frac{d}{k})^k + O(d^{k-1})$; see [1]. Let us also mention a work of Garcia and Peyrat [2] who proved that $AC_{d,k} \geq \frac{d^{k-2.17}}{21k!}$ for sufficiently large d and $k \leq d$. Dougherty and Faber [1] presented a number of results on Abelian Cayley graphs for small d and large k. Constructions of Cayley graphs of non-Abelian groups can be found for example in [4] and [5].

We will focus on Abelian Cayley graphs and cyclic Cayley graphs of small diameter. Macbeth, Šiagiová and Širáň [3] showed that $AC_{d,2} \geq \frac{3}{8}(d^2-4)$ for $d = 4q-2$, where q is an odd prime power, and they constructed two families of cyclic Cayley graphs of diameter 2 as well. The first family gives the bound $CC_{d,2} \geq \frac{9}{25}(d+3)(d-2)$ for $d = 5p-3$, where $p \equiv 2 \pmod 3$ is a prime. The other family was also constructed for an infinite set of degrees d and the order $\frac{d^2}{3} + O(d^{\frac{3}{2}})$. These results were generalized in [6], where it is proved that $AC_{d,2} \geq \frac{3}{8}d^2 - 1.45d^{1.525}$ for any sufficiently large d, and $CC_{d,2} \geq 9p(p-1)$ for all $d \geq 12$ and $p \equiv 2 \pmod 3$ such that $\frac{d}{6} \leq p \leq \frac{d+3}{5}$.

We present cyclic Cayley graphs which yield a better bound than lower bounds on $CC_{d,2}$ given in [3] and [6].

Theorem 1. $CC_{d,2} \geq \frac{13}{36}(d+2)(d-4)$ for any $d = 6p-2$ where p is a prime such that $p \neq 13$, $p \not\equiv 1 \pmod{13}$.

Proof. Let F^* be the multiplicative group and let F^+ be the additive group of the Galois field $GF(p)$, where p is a prime such that $p \neq 13$, $p \not\equiv 1$ (mod 13). Let $G = F^* \times F^+ \times Z_{13}$. Since F^*, F^+ and Z_{13} are cyclic groups, and the orders of any two of them have no common divisor greater than 1, the group G is also cyclic. We denote the identity element in F^* by 1, and the identity in F^+ and Z_{13} will be denoted by 0.

Let $a_0 = (1,0,1)$, $a(x) = (x,x,1)$, $b(x_1) = (x_1,0,3)$ and $c(x_2) = (1,x_2,4)$, where $x, x_1 \in F^*$ and $x_2 \in F^+$. Then $a_0^{-1} = (1,0,-1)$, $a(x)^{-1} = (x^{-1},-x,-1)$, $b(x_1)^{-1} = (x_1^{-1},0,-3)$ and $c(x_2)^{-1} = (1,-x_2,-4)$. We use the generating set $X = \{a_0, a_0^{-1}, a(x), a(x)^{-1}, b(x_1), b(x_1)^{-1}, c(x_2), c(x_2)^{-1} \mid \text{for any } x, x_1 \in F^*$ and $x_2 \in F^+\}$. The Cayley graph $C(G,X)$ is of degree $d = |X| = 6p-2$ and order $|G| = 13p(p-1) = \frac{13}{36}(d+2)(d-4)$.

In order to prove that the diameter of $C(G,X)$ is 2, it suffices to show that any non-identity element of G which is not in X can be obtained as a product of 2 generators of X. It follows from [3] that if $x_1 \neq 1$ and $x_2 \neq 0$, then

$$(x_1,x_2,0) = a(x_1x_2u)a(x_2u)^{-1} = (x_1x_2u, x_1x_2u, 1)((x_2u)^{-1}, -x_2u, -1),$$

where $u = (x_1-1)^{-1}$. It is easy to see that

$(x_1,0,0) = b(x_1)b(1)^{-1} = (x_1,0,3)(1,0,-3)$ for any $x_1 \in F^*$ and

$(1,x_2,0) = c(x_2)c(0)^{-1} = (1,x_2,4)(1,0,-4)$ for any $x_2 \in F^+$.

We consider all the other elements of G. For any $x_1 \in F^*$, $x_2 \in F^+$ and $i,j \in \{-1,1\}$, we have

$(x_1,x_2,3i+4j) = b(x_1^i)^i c(jx_2)^j = (x_1,0,3i)(1,x_2,4j)$,

$(x_1,x_2,i+4j) = a(x_1^i)^i c(jx_2 - ijx_1^i)^j = (x_1,ix_1^i,i)(1,x_2-ix_1^i,4j)$,

$(x_1,0,i+3j) = a_0^i b(x_1^j)^j = (1,0,i)(x_1,0,3j)$,

Finally, if $x_2 \neq 0$,

$(x_1,x_2,i+3j) = a(ix_2)^i b(ix_1^j x_2^{-ij})^j = (ix_2^i,x_2,i)(ix_1x_2^{-i},0,3j)$.

Hence any element of G can be expressed as a product of at most 2 generators in X. The proof is complete. $\Box$

Much less is known about large Abelian Cayley graphs of diameter 3. We mentioned above that for $d \to \infty$ and fixed k, $\frac{d^3}{27} + O(d^2) \leq AC_{d,3} \leq \frac{d^3}{6} + O(d^2)$. The following result improves the lower bound on $AC_{d,3}$ considerably.

Theorem 2. $AC_{d,3} \geq \frac{9}{128}(d+3)^2(d-5)$ for any $d = 8q-3$ where q is a prime power.

Proof. Let $G = F^* \times F^+ \times F^+ \times Z_{36}$, where F^* is the multiplicative group and F^+ is the additive group of the Galois field $GF(q)$; q is a prime power. Again, the identity element in F^* is 1, and the identity in F^+ and Z_{36} is denoted by 0. Let

$a_0 = (1,0,0,1)$ and $b_0 = (1,0,0,3)$.

For $x, \bar{x}, x_1, x_2 \in F^*$ and $x_3 \in F^+$ we define

$a(x) = (x,x,0,1)$, $b(\bar{x}) = (\bar{x},0,\bar{x},3)$, $c(x_1) = (x_1,0,0,9)$,

$d(x_2) = (1,x_2,0,0)$ and $e(x_3) = (1,0,x_3,18)$.

Then

$a_0^{-1} = (1,0,0,-1)$, $b_0^{-1} = (1,0,0,-3)$,

$a(x)^{-1} = (x^{-1},-x,0,-1)$, $b(\bar{x})^{-1} = (\bar{x}^{-1},0,-\bar{x},-3)$,

$c(x_1)^{-1} = (x_1^{-1},0,0,-9)$, $d(x_2)^{-1} = d(-x_2)$ and $e(x_3)^{-1} = e(-x_3)$.

The generating set $X = \{a_0,\ a_0^{-1},\ b_0,\ b_0^{-1},\ a(x),\ a(x)^{-1},\ b(\bar{x}),\ b(\bar{x})^{-1},\ c(x_1),\ c(x_1)^{-1},\ d(x_2),\ e(x_3) \mid$ for any $x, \bar{x}, x_1, x_2 \in F^*$ and $x_3 \in F^+\}$. The Cayley graph $C(G,X)$ is of degree $d = |X| = 8q-3$ and order $|G| = 36q^2(q-1) =$

$\frac{9}{128}(d+3)^2(d-5)$. It remains to prove that the diameter of $C(G,X)$ is equal to 3. We have

$$(x_1,x_2,0,0) = a(x_1x_2u)a(x_2u)^{-1} \text{ where } u=(x_1-1)^{-1},\, x_1\neq 1 \text{ and } x_2\neq 0,$$
$$(x_1,0,0,0) = c(x_1)c(1)^{-1} \text{ for any } x_1\in F^*.$$

Since $(1,x_2,0,0)$ is in X for $x_2\in F^*$, we can obtain any element $(x_1,x_2,0,0)$ where $x_1\in F^*$ and $x_2\in F^+$ as a product of at most 2 generators of X. Similarly, it is possible to show that any element $(x_1,0,x_3,0)$, $x_1\in F^*$, $x_3\in F^+$ can be obtained as a product of 2 generators of X. This helps us to show that we can express any element (x_1,x_2,x_3,s) of G for $s=0,1,3$ or 18 as a product of at most 3 generators of X. We have

$$(x_1,x_2,x_3,0) = (x_1,0,x_3,0)d(x_2) \text{ if } x_2\neq 0,$$
$$(x_1,x_2,x_3,1) = (x_1x_2^{-1},0,x_3,0)a(x_2) \text{ if } x_2\neq 0,$$
$$(x_1,0,x_3,1) = (x_1,0,x_3,0)a_0,$$
$$(x_1,x_2,x_3,3) = (x_1x_3^{-1},x_2,0,0)b(x_3) \text{ if } x_3\neq 0,$$
$$(x_1,x_2,0,3) = (x_1,x_2,0,0)b_0,$$
$$(x_1,x_2,x_3,18) = (x_1,x_2,0,0)e(x_3).$$

Now we consider the other elements of G. Let $i,j\in\{-1,1\}$.

$$(x_1,x_2,x_3,3+i) = a(x_1^ix_3^{-i})^ib(x_3)d(x_2-ix_1^ix_3^{-i}) \text{ if } x_3\neq 0 \text{ and } x_2\neq ix_1^ix_3^{-i},$$
$$(x_1,x_2,0,3+i) = a(x_1^i)^ib_0d(x_2-ix_1^i) \text{ if } x_2\neq ix_1^i,$$
$$(x_1,ix_1^ix_3^{-i},x_3,3+i) = a(x_1^ix_3^{-i})^ib(x_3) \text{ if } x_3\neq 0,$$
$$(x_1,ix_1^i,0,3+i) = a(x_1^i)^ib_0,$$

$$(x_1,x_2,x_3,9+i+3j) = a(ix_2)^ib(jx_3)^jc(ijx_1x_2^{-i}x_3^{-j}) \text{ if } x_2,x_3\neq 0,$$
$$(x_1,x_2,0,9+i+3j) = a(ix_2)^ib_0^jc(ix_1x_2^{-i}) \text{ if } x_2\neq 0,$$
$$(x_1,0,x_3,9+i+3j) = a_0^ib(jx_3)^jc(jx_1x_3^{-j}) \text{ if } x_3\neq 0,$$
$$(x_1,0,0,9+i+3j) = a_0^ib_0^jc(x_1),$$

$$(x_1,x_2,x_3,9+3j) = b(jx_3)^jc(jx_1x_3^{-j})d(x_2) \text{ if } x_2,x_3\neq 0,$$
$$(x_1,x_2,0,9+3j) = b_0^jc(x_1)d(x_2) \text{ if } x_2\neq 0,$$
$$(x_1,0,x_3,9+3j) = b(jx_3)^jc(jx_1x_3^{-j}) \text{ if } x_3\neq 0,$$
$$(x_1,0,0,9+3j) = b_0^jc(x_1),$$

$$(x_1,x_2,x_3,9+i) = a(ix_2)^ic(ix_1^{-1}x_2^i)^{-1}e(x_3) \text{ if } x_2\neq 0,$$
$$(x_1,0,x_3,9+i) = a_0^ic(x_1^{-1})^{-1}e(x_3),$$

$(x_1, x_2, x_3, 9) = c(x_1^{-1})^{-1}d(x_2)e(x_3)$ if $x_2 \neq 0$,
$(x_1, 0, x_3, 9) = c(x_1^{-1})^{-1}e(x_3)$,
$(x_1, x_2, x_3, 15+i) = a(ix_2)^i b(ix_1^{-1}x_2^i)^{-1}e(ix_1^{-1}x_2^i + x_3)$ if $x_2 \neq 0$,
$(x_1, 0, x_3, 15+i) = a_0^i b(x_1^{-1})^{-1}e(x_1^{-1} + x_3)$
$(x_1, x_2, x_3, 15) = b(x_1^{-1})^{-1}d(x_2)e(x_1^{-1} + x_3)$ if $x_2 \neq 0$,
$(x_1, 0, x_3, 15) = b(x_1^{-1})^{-1}e(x_1^{-1} + x_3)$,
$(x_1, x_2, x_3, 17) = a(x_1^{-1})^{-1}d(x_1^{-1} + x_2)e(x_3)$ if $x_2 \neq -x_1^{-1}$,
$(x_1, -x_1^{-1}, x_3, 17) = a(x_1^{-1})^{-1}e(x_3)$.

We showed that any element (x_1, x_2, x_3, s) where $x_1 \in F^*$, $x_2, x_3 \in F^+$ and $0 \leq s \leq 18$ can be obtained as a product of at most 3 generators of X. Since X is closed under taking inverses, elements $(x_1, x_2, x_3, -s)$ can be expressed similarly. It is easy to see that it is not possible to express any element of G with the last coordinate $9 + i + 3j$ as a product of fewer than 3 generators of X, therefore the diameter of $C(G, X)$ is exactly 3. $\square$

It would be desirable to have a similar result for cyclic Cayley graphs of diameter 3. Unfortunately we have not been able to obtain a construction of cyclic Cayley graphs of diameter 3 and order close to $\frac{9}{128}d^3$ for an infinite set of degrees d, hence this remains an open problem for future research.

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